J. DIFFERENTIAL GEOMETRY
61 (2002) 397-451

CHEEGER MANIFOLDS AND THE CLASSIFICATION OF BIQUOTIENTS

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Abstract

We classify the biquotient manifolds which are either rational homology spheres or Cheeger manifolds (connected sums of two rank-one symmetric spaces). Also: there are only finitely many 2-connected biquotient manifolds in each dimension.

A closed manifold is called a biquotient if it is diffeomorphic to $K \setminus G/H$ for some compact Lie group G with closed subgroups K and H such that K acts freely on G/H. Every biquotient has a Riemannian metric of nonnegative sectional curvature. In fact, almost all known manifolds of nonnegative curvature are biquotients. The only known closed manifolds of nonnegative sectional curvature which were not defined in this way are those found by Cheeger [6] in 1973, and Grove and Ziller [16] in 2000.

In order to understand these constructions better, this paper analyzes which of the Cheeger and Grove-Ziller manifolds are actually diffeomorphic to biquotients. In the process, we develop a general approach to the classification of biquotient manifolds. A priori, it is hard to determine whether a given manifold is a biquotient, because there is no obvious upper bound on the dimension of the groups involved. We give a procedure which allows us to reduce the dimension of the groups needed to describe a given manifold as a biquotient. A surprising application is that Gromoll-Meyer's example of an exotic 7-sphere which is a biquotient [15] is the only exotic sphere of any dimension which is a biquotient. More generally, we classify all biquotients which are simply connected rational homology spheres of any dimension (Theorem 6.1). The classification of biquotients which are simply connected

Received 09/18/2002.

rational homology spheres has been given independently by Kapovitch and Ziller [17]. More generally, they classify all simply connected biquotients whose rational cohomology ring is generated by one element.

Another application of our general classification theory for biquotients is the precise determination of which Cheeger manifolds, the connected sums of two rank-one symmetric spaces with any orientations, are diffeomorphic to biquotients (Theorem 2.1). Some of the Cheeger manifolds were known to be biquotients, such as $\mathbf{CP}^2\# -\mathbf{CP}^2$ (the nontrivial S^2-bundle over S^2), but we find that many of the other Cheeger manifolds are also biquotients, such as $\mathbf{CP}^2\#\mathbf{CP}^2$. On the other hand, the Cheeger manifold $\mathbf{CP}^4\#\mathbf{HP}^2$ is not diffeomorphic (or even homotopy equivalent) to a biquotient. The positive result that some Cheeger manifolds such as $\mathbf{CP}^2\#\mathbf{CP}^2$ are biquotients implies that they have nonnegatively curved Riemannian metrics with various good properties that were not clear from Cheeger's construction. First, the new metrics are real analytic. Further, the new metrics determine a natural complex-analytic structure on the whole tangent bundle, since Aguilar [2] showed that biquotients have this property; my paper [26] has a weaker result which applies to all the Cheeger manifolds. Finally, the geodesic flow for the new metrics on the Cheeger manifolds has zero topological entropy. Paternain [23] proved the latter property for Cheeger's metric on $\mathbf{CP}^2\#\mathbf{CP}^2$, but now we know it for some metric on a large class of the Cheeger manifolds.

Finally, our general classification results imply that there are only finitely many diffeomorphism classes of 2-connected biquotients in any given dimension (Theorem 4.9). This fails for biquotients that are only simply connected already in dimension 6, as I showed in [27]. Also, the finiteness of 2-connected biquotients shows the distance between biquotients and general manifolds of nonnegative sectional curvature, since Grove and Ziller have constructed nonnegatively curved metrics on all S^3-bundles over S^4, giving infinitely many homotopy types of 2-connected 7-manifolds with nonnegative sectional curvature [16].

I would like to thank Gabriel Paternain for many useful conversations. Also, thanks to Wolfgang Ziller for several references to earlier work.

1. Notation

We begin with the equivalence of several definitions of biquotient manifolds, pointed out by Eschenburg ([9], [11]).

Lemma 1.1. *The following properties of a closed smooth manifold M are equivalent:*

(1) $M = K\backslash G/H$ *for some compact Lie group* G *with closed subgroups* K *and* H *such that* K *acts freely on* G/H.

(2) $M = G/H$ *for some compact Lie groups* G *and* H *together with a homomorphism* $H \to G \times G$ *such that* H *acts freely on* G *by left and right translation,* $(g_1, g_2)(g) := g_1 g g_2^{-1}$.

(3) $M = G/H$ *for some compact Lie groups* G *and* H *together with a homomorphism* $H \to (G \times G)/Z(G)$ *such that* H *acts freely on* G.

In (3) we are using that the center $Z(G)$, imbedded diagonally in $G \times G$, acts trivially on G by left and right translation.

Proof. Clearly (1) implies (2) and (2) implies (3). We prove that (3) implies (1). The point is that $(G \times G)/Z(G)$ acts transitively on G with stabilizer at $1 \in G$ equal to the diagonal subgroup $G/Z(G)$. So, if $M = G/H$ as in (3), then we can also describe M as in (1) by

$$M = \big(G/Z(G)\big) \backslash \big((G \times G)/Z(G)\big)/H.$$

q.e.d.

In the paper, we use definition (3) of biquotients; that is, "a biquotient G/H" will mean that G and H are compact Lie groups and we are given a homomorphism $H \to (G \times G)/Z(G)$ such that H acts freely on G. When G and H are simply connected, such a homomorphism lifts uniquely to $G \times G$.

We call a connected compact Lie group *simple* if it is nonabelian and every proper normal subgroup is finite. We define a *simple factor* of a connected compact Lie group H to be the universal cover of a simple normal subgroup of H. We write $\mathrm{Sp}(2a)$ for the simply connected simple group of type C_a, which topologists usually call $\mathrm{Sp}(a)$. The reason for the name $\mathrm{Sp}(2a)$ is that we think of each compact classical group using its standard complex representation: $\mathrm{SU}(n)$ is a subgroup of $\mathrm{GL}(n, \mathbf{C})$, $\mathrm{SO}(n)$ is a subgroup of $\mathrm{GL}(n, \mathbf{C})$, and (with the notation here) $\mathrm{Sp}(2a)$ is a subgroup of $\mathrm{GL}(2a, \mathbf{C})$.

By definition, the *Dynkin index* of a homomorphism $H \to G$ of simply connected simple groups is the integer corresponding to the homomorphism $\pi_3 H \to \pi_3 G$, both groups being canonically isomorphic to $\mathbf{Z}$. Dynkin computed the Dynkin index in many cases ([8], Chapter I, Section 2). Finally, we write $\mathrm{UT}(S^n)$ for the unit tangent bundle of the n-sphere, $\mathrm{UT}(S^n) = \mathrm{Spin}(n+1)/\mathrm{Spin}(n-1)$.

2. The Cheeger manifolds that are diffeomorphic to biquotients

By definition, a Cheeger manifold is the connected sum of any two rank-one symmetric spaces with any orientations. The rank-one symmetric spaces, besides the sphere which is not interesting for this purpose, are the real, complex, and quaternionic projective spaces, together with the Cayley plane associated to the octonions. The Cheeger manifolds are the only known examples of connected sums, with neither summand a homotopy sphere and at least one summand not a rational homology sphere, which admit metrics of nonnegative sectional curvature. In fact, the conjecture that manifolds of nonnegative sectional curvature are elliptic in the sense of Félix, Halperin, and Thomas [12] would imply that any connected sum which admits a metric of nonnegative sectional curvature must be roughly of the Cheeger type. Precisely, a connected sum $M_1 \# M_2$ of simply connected manifolds which is elliptic, and such that neither M_1 nor M_2 is a k-homology sphere for a field k, must have the rings $H^*(M_1, k)$ and $H^*(M_2, k)$ both generated by a single element, as follows from Lambrechts [18], Theorem 3. Combining this with Adams and Atiyah's results on the Hopf invariant problem [1] shows that if a connected sum of simply connected manifolds is elliptic, with neither summand a homotopy sphere and at least one summand not a rational homology sphere, then both summands have the integral cohomology ring of $\mathbf{CP}^n$, $\mathbf{HP}^n$, or $\mathbf{CaP}^2$.

Theorem 2.1. *The following Cheeger manifolds are diffeomorphic to biquotients. First,* $\mathbf{CP}^n \# -\mathbf{CP}^n$, $\mathbf{HP}^n \# -\mathbf{HP}^n$, *and* $\mathbf{CaP}^2 \# -\mathbf{CaP}^2$. *Next,* $\mathbf{RP}^n \# \mathbf{RP}^n$, $\mathbf{RP}^{2n} \# \mathbf{CP}^n$, $\mathbf{RP}^{4n} \# \mathbf{HP}^n$, *and* $\mathbf{RP}^{16} \# \mathbf{CaP}^2$. *Here* $\mathbf{RP}^n$ *is non-orientable for n odd and has an orientation-reversing diffeomorphism for n even, so the orientations of the summands do not matter in these cases. Next,* $\mathbf{CP}^{2n} \# -\mathbf{HP}^n$, $\mathbf{HP}^4 \# -\mathbf{CaP}^2$, *and* $\mathbf{CP}^8 \# -\mathbf{CaP}^2$. *Finally,* $\mathbf{CP}^n \# \mathbf{CP}^n$, $\mathbf{HP}^n \# \mathbf{HP}^n$, *and* $\mathbf{CP}^{4e+2} \# \mathbf{HP}^{2e+1}$.

The remaining Cheeger manifolds are not even homotopy equivalent to biquotients. These are $\mathbf{CaP}^2 \# \mathbf{CaP}^2$, $\mathbf{CP}^8 \# \mathbf{CaP}^2$, $\mathbf{HP}^4 \# \mathbf{CaP}^2$, *and* $\mathbf{CP}^{4e} \# \mathbf{HP}^{2e}$.

Proof. In this section, we prove only the first, positive, statement. We will prove the negative statement in Sections 7 and 8, after setting up a general classification theory of biquotients.

We begin with the cases which are straightforward generalizations of Cheeger's observation that $\mathbf{CP}^2\# -\mathbf{CP}^2$ is a biquotient [6]. Let A_k denote the standard algebra ($\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$, or the octonions $\mathbf{Ca}$) of dimension k over $\mathbf{R}$, where $k = 1$, 2, 4, or 8. Then, for $k = 1$, 2, or 4, $A_k\mathbf{P}^n\# - A_k\mathbf{P}^n$ is an S^k-bundle over $A_k\mathbf{P}^{n-1}$, namely the biquotient manifold $(S^{nk-1}\times S^k)/S^{k-1}$, where S^{k-1} is a group for $k = 1$, 2, or 4, acting freely on S^{nk-1} and by rotations on S^k. For $k = 8$, S^7 is not a group, but we can still describe $\mathbf{CaP}^2\# - \mathbf{CaP}^2$ as a biquotient,

$$(\mathrm{Spin}(9)\times S^8)/\mathrm{Spin}(8),$$

where $\mathrm{Spin}(8)\to\mathrm{Spin}(9)$ is the standard inclusion and $\mathrm{Spin}(8)$ acts on $S^8\subset\mathbf{R}^9$ by the direct sum of an 8-dimensional real spin representation and the trivial representation. Since $\mathrm{Spin}(9)/\mathrm{Spin}(8) = S^8$, this description exhibits $\mathbf{CaP}^2\# - \mathbf{CaP}^2$ as an S^8-bundle over S^8.

Next, we check that the connected sum of $\mathbf{RP}^n$ with any rank-one symmetric space is diffeomorphic to a biquotient. As mentioned in the theorem, orientations do not matter in this case, because $\mathbf{RP}^n$ is non-orientable for n odd and has an orientation-reversing diffeomorphism for n even. For any closed n-manifold M, the connected sum $\mathbf{RP}^n\# M$ is doubly covered by $M\# -M$. This suggests a way to view $\mathbf{RP}^{nk}\# A_k\mathbf{P}^n$ as a biquotient: we replace S^k in the above description of $A_k\mathbf{P}^n\# - A_k\mathbf{P}^n$ by $\mathbf{RP}^k$. That is:

$$\begin{aligned}
\mathbf{RP}^n\#\mathbf{RP}^n &= (S^{n-1}\times S^1)/\mathbf{Z}/2\\
\mathbf{RP}^{2n}\#\mathbf{CP}^n &= (S^{2n-1}\times\mathbf{RP}^2)/S^1\\
\mathbf{RP}^{4n}\#\mathbf{HP}^n &= (S^{4n-1}\times\mathbf{RP}^4)/S^3\\
\mathbf{RP}^{16}\#\mathbf{CaP}^2 &= (\mathrm{Spin}(9)\times\mathbf{RP}^8)/\mathrm{Spin}(8)
\end{aligned}$$

The manifolds $\mathbf{CP}^{2n}$, $\mathbf{HP}^{2n}$, and $\mathbf{CaP}^2$ have natural orientations, corresponding to the highest power of any generator of H^2, H^4, or H^8, respectively. In fact, $\mathbf{HP}^n$ has a natural orientation for all $n\geq 2$, but this takes more care to define. We use that the mod 3 Steenrod operation P^1 on $H^*(\mathrm{BSU}(2),\mathbf{F}_3)$ acts by $P^1c_2 = c_2^2$, as one checks by restricting to the torus $S^1\subset\mathrm{SU}(2)$. Since $H^4(\mathbf{HP}^n,\mathbf{Z})$ has a generator which is c_2 of an SU(2)-bundle over $\mathbf{HP}^n$, this generator c_2 satisfies $P^1c_2 = c_2^2$ in $H^8(\mathbf{HP}^n,\mathbf{F}_3)$. It follows that (for $n\geq 2$) there is a unique generator z of $H^4(\mathbf{HP}^n,\mathbf{Z})$ such that $P^1z = -z^2$ in $H^8(\mathbf{HP}^n,\mathbf{F}_3)$, namely $z = -c_2$. We define the natural orientation on $\mathbf{HP}^n$ to be the one corresponding to the highest power z^n of this generator z.

Using this orientation, we can define one more class of Cheeger manifolds as bundles. The manifold $\mathbf{CP}^{2n}\# -\mathbf{HP}^n$ is the $\mathbf{CP}^2$-bundle over

$\mathbf{HP}^{n-1}$ associated to the rank-3 complex vector bundle $E \oplus \mathbf{C}$, where E is the tautological rank-2 complex vector bundle over $\mathbf{HP}^{n-1}$. Thus $\mathbf{CP}^{2n}\# - \mathbf{HP}^n$ is a biquotient of the form $(\mathbf{CP}^2 \times S^{4n-1})/\mathrm{SU}(2)$. We might try to imitate this construction by viewing $\mathbf{HP}^4\# - \mathbf{CaP}^2$ as an $\mathbf{HP}^2$-bundle over S^8 and $\mathbf{CP}^8\# - \mathbf{CaP}^2$ as a $\mathbf{CP}^4$-bundle over S^8, but that turns out to be impossible. For these constructions, we would need a complex vector bundle E over S^8 with c_4E equal to plus or minus the class of a point in $H^8(S^8, \mathbf{Z})$, whereas in fact every complex vector bundle on S^8 has c_4E a multiple of $(4-1)! = 6$, by Bott periodicity.

Instead, we construct $\mathbf{HP}^4\# - \mathbf{CaP}^2$ as the quotient of an S^{11}-bundle over S^8 by a free $\mathrm{SU}(2)$-action. Let $S^- : \mathrm{Spin}(8) \to \mathrm{SO}(8)$ denote one of the two spin representations. We also write S^- for the associated $\mathrm{Spin}(9)$-equivariant real vector bundle of rank 8 over $\mathrm{Spin}(9)/\mathrm{Spin}(8) = S^8$. Let N be the S^{11}-bundle over S^8 defined as the unit sphere bundle $S(S^- \oplus \mathbf{R}^4)$. Define a homomorphism $\mathrm{SU}(2) \to \mathrm{Spin}(9)$ by $\mathrm{SU}(2) \cong \mathrm{Spin}(3) \subset \mathrm{Spin}(9)$. Using this homomorphism, $\mathrm{SU}(2)$ acts on S^8 and acts compatibly on the vector bundle S^- over S^8. Let $\mathrm{SU}(2)$ act on N by the given action on S^8 and on the vector bundle S^-, and by the standard faithful representation $V_{\mathbf{R}}$ of $\mathrm{SU}(2)$ on $\mathbf{R}^4$.

This action of $\mathrm{SU}(2)$ on N is free. To check this, it suffices to check that $\mathrm{SU}(2)$ acts freely on the S^7-bundle $S(S^-)$ over S^8 and on the S^3-bundle $S(\mathbf{R}^4) = S^3 \times S^8$ over S^8. The second statement is clear by the choice of $\mathrm{SU}(2)$-action on $\mathbf{R}^4$. To prove the first statement, first note that $\mathrm{Spin}(9)$ acts on $S(S^-) \cong S^{15}$ by the spin representation of $\mathrm{Spin}(9)$. Then use that the restriction of any spin representation of $\mathrm{Spin}(n)$ to $\mathrm{Spin}(n-1)$ is a sum of spin representations of $\mathrm{Spin}(n-1)$. Thus the action of $\mathrm{SU}(2) \cong \mathrm{Spin}(3)$ on $S(S^-) = S^{15}$ is by a sum of copies of the 4-dimensional real spin representation $V_{\mathbf{R}}$. It follows that $\mathrm{SU}(2)$ acts freely on S^{15}.

Let M be the quotient manifold $N/\mathrm{SU}(2)$, of dimension 16. Clearly M is a biquotient of the form $(\mathrm{Spin}(9) \times S^{11})/(\mathrm{Spin}(8) \times \mathrm{SU}(2))$. By construction, M is the union of a disc bundle over $S(S^-)/\mathrm{SU}(2) = S^{15}/\mathrm{SU}(2) = \mathbf{HP}^3$ and a disc bundle over $S(\mathbf{R}^4)/\mathrm{SU}(2) = (S^8 \times S^3)/\mathrm{SU}(2) = S^8$ along their common boundary, S^{15}. So M is diffeomorphic to the connected sum of $\mathbf{HP}^4$ and $\mathbf{CaP}^2$, with some orientations. To pin down the orientations, it is convenient to compute the cohomology ring of M. Since N is an $\mathrm{SU}(2)$-equivariant S^{11}-bundle over S^8, the quotient M is the total space of a fibration $S^{11} \to M \to S^8//\mathrm{SU}(2)$ up to homotopy, where $S^8//\mathrm{SU}(2) = (S^8 \times E\mathrm{SU}(2))/\mathrm{SU}(2)$ denotes the homotopy quotient via the given homomorphism $\mathrm{SU}(2) \to \mathrm{Spin}(9)$.

We compute that

$$H^*(S^8//\mathrm{SU}(2),\mathbf{Z}) = \mathbf{Z}[y,\chi(S^-)]/((\chi(S^-)+y^2)^2=0),$$

where y in H^4 is c_2 of the standard representation V of $\mathrm{SU}(2)$ and $\chi(S^-)$ in H^8 is the Euler class of the $\mathrm{SU}(2)$-equivariant vector bundle S^- over S^8, with some orientation. To check this, note that the spectral sequence for the fibration $S^8 \to S^8//\mathrm{SU}(2) \to \mathrm{BSU}(2)$ degenerates, since all the cohomology is in even degrees. This implies that the cohomology of $S^8//\mathrm{SU}(2)$ has the above form, for some relation of the form $\chi(S^-)^2 + b\chi(S^-)y^2 + cy^4 = 0$ with $b, c \in \mathbf{Z}$. Furthermore, we have $TS^8 \oplus \mathbf{R} \cong \mathbf{R}^9$ as $\mathrm{SU}(2)$-equivariant vector bundles on S^8. It follows that

$$\chi(TS^8)^2 = c_8(TS^8 \otimes_{\mathbf{R}} \mathbf{C}) = 0$$

in $H^{16}(S^8//\mathrm{SU}(2),\mathbf{Z})$. Also, $\chi(TS^8)$ has degree 2 on S^8 while $\chi(S^-)$ has degree ± 1 on S^8, so we must have $\chi(TS^8) = \pm 2\chi(S^-) + dy^2$ in $H^8(S^8//\mathrm{SU}(2),\mathbf{Z})$ for some integer d. So $(\pm\chi(S^-) + (d/2)y^2)^2 = 0$ in $H^{16}(S^8//\mathrm{SU}(2),\mathbf{Q})$. By what we know about the form of the relation, $d/2$ must be an integer a, and we have

$$(\pm\chi(S^-) + ay^2)^2 = 0$$

in $H^{16}(S^8//\mathrm{SU}(2),\mathbf{Z})$. Finally, the action of $\mathrm{SU}(2) = \mathrm{Spin}(3) \subset \mathrm{Spin}(9)$ on S^8 preserves a 2-sphere, and so we have an inclusion $\mathrm{B}S^1 \simeq S^2//\mathrm{SU}(2) \to S^8//\mathrm{SU}(2)$. Let x be the standard generator of the polynomial ring $H^*(\mathrm{B}S^1,\mathbf{Z})$ in degree 2. We compute that the restriction map takes $y = c_2V$ to $-x^2$ and $\chi(S^-)$ to $\pm x^4$, using that the restriction of the spin representation S^- to $S^1 = \mathrm{Spin}(2) \subset \mathrm{Spin}(8)$ is the direct sum of 4 copies of the standard 2-dimensional real representation of S^1. Therefore, for a suitable orientation on S^-, the relation in $H^{16}(S^8//\mathrm{SU}(2),\mathbf{Z})$ must be

$$(\chi(S^-) + y^2)^2 = 0.$$

Since M is an S^{11}-bundle over $S^8//\mathrm{SU}(2)$, we have one more relation in $H^{12}(M,\mathbf{Z})$, saying that the Euler class of this S^{11}-bundle is zero. The S^{11}-bundle is $S(S^- \oplus V_{\mathbf{R}})$, and so its Euler class is $\chi(S^- \oplus V_{\mathbf{R}}) = \chi(S^-)y$ in $H^{12}(S^8//S^1,\mathbf{Z})$. Thus M has cohomology ring

$$\begin{aligned} H^*(M,\mathbf{Z}) &= \mathbf{Z}[y,\chi(S^-)]/((\chi(S^-)+y^2)^2 = 0, \chi(S^-)y = 0) \\ &= \mathbf{Z}[y,\chi(S^-)]/(\chi(S^-)^2 + y^4 = 0, \chi(S^-)y = 0). \end{aligned}$$

This is the cohomology ring of $\mathbf{HP}^4\# - \mathbf{CaP}^2$, not of $\mathbf{HP}^4\#\mathbf{CaP}^2$. So the biquotient M is diffeomorphic to $\mathbf{HP}^4\# - \mathbf{CaP}^2$.

The proof that $\mathbf{CP}^8\# - \mathbf{CaP}^2$ is diffeomorphic to a biquotient M is completely analogous: it is the quotient of the S^9-bundle $S(S^- \oplus \mathbf{R}^2)$ over S^8 by a free S^1-action. Here S^1 acts on S^8 and on the vector bundle S^- by the homomorphism $S^1 \cong \mathrm{Spin}(2) \subset \mathrm{Spin}(9)$, and on $\mathbf{R}^2$ by the standard real representation of S^1. Thus M is a biquotient of the form $(\mathrm{Spin}(9) \times S^9)/(\mathrm{Spin}(8) \times S^1)$.

Finally, we come to the most surprising examples of biquotients, starting with $\mathbf{CP}^n\#\mathbf{CP}^n$. The manifold $\mathbf{CP}^n$ has an orientation-reversing diffeomorphism from n odd, and so $\mathbf{CP}^n\#\mathbf{CP}^n$ for n odd is diffeomorphic to $\mathbf{CP}^n\# - \mathbf{CP}^n$. For n even, we will show that $\mathbf{CP}^n\#\mathbf{CP}^n$ is diffeomorphic to a biquotient $(S^3 \times S^{2n-1})/(S^1)^2$, for a free isometric action of $(S^1)^2$ on $S^3 \times S^{2n-1}$. The action is defined by the following homomorphism from $(S^1)^2$ to the maximal torus $(S^1)^2 \times (S^1)^n$ of $\mathrm{SO}(4) \times \mathrm{SO}(2n)$:

$$(x, y) \mapsto ((x, y), (xy^{-1}, xy, \ldots, xy)).$$

It is straightforward to check that this action of $(S^1)^2$ on $S^3 \times S^{2n-1}$ is free. Let M be the quotient manifold. We use that the action of $(S^1)^2$ on S^3 has cohomogeneity one, with trivial generic stabilizer group and stabilizers at the two special orbits equal to the two factors S^1. It follows that M is the union of a 2-disc bundle over $S^{2n-1}/S^1 = \mathbf{CP}^{n-1}$ (corresponding to the action of the first factor S^1) and a 2-disc bundle over $S^{2n-1}/S^1 = \mathbf{CP}^{n-1}$ (corresponding to the second S^1) along their common boundary, which is a sphere S^{2n-1}. So M is the connected sum of two copies of $\mathbf{CP}^n$, with some orientations. To work out the orientations, it is convenient to compute the cohomology ring of M.

The quotient manifold M fits into a fibration

$$S^3 \times S^{2n-1} \to M \to (\mathrm{B}S^1)^2.$$

Here $(\mathrm{B}S^1)^2$ has cohomology ring $\mathbf{Z}[u, v]$ with u and v in H^2. From the description of the $(S^1)^2$-action, we read off that the Euler classes of the S^3-bundle and S^{2n-1}-bundle over $(\mathrm{B}S^1)^2$ are uv and $(u-v)(u+v)^{n-1}$. Since these form a regular sequence in the polynomial ring $H^*((\mathrm{B}S^1)^2, \mathbf{Z})$, we have

$$\begin{aligned} H^*(M, \mathbf{Z}) &= \mathbf{Z}[u, v]/(uv = 0, (u-v)(u+v)^{n-1} = 0) \\ &= \mathbf{Z}[u, v]/(uv = 0, u^n = v^n). \end{aligned}$$

This is the cohomology ring of $\mathbf{CP}^n\#\mathbf{CP}^n$, and (for n even) not that of $\mathbf{CP}^n\# -\mathbf{CP}^n$. Therefore the biquotient M is diffeomorphic to $\mathbf{CP}^n\#\mathbf{CP}^n$.

We next show that $\mathbf{HP}^n\#\mathbf{HP}^n$ is diffeomorphic to a biquotient. Imitating the construction for $\mathbf{CP}^n\#\mathbf{CP}^n$ might suggest identifying $\mathbf{HP}^n\#\mathbf{HP}^n$ with a biquotient $(S^7 \times S^{8n-1})/\mathrm{SU}(2)^2$. That works for n odd, but not for n even. For n even, we have to consider a more general type of biquotient, $(\mathrm{Sp}(4) \times S^{4n-1})/\mathrm{SU}(2)^3$. Here the group $\mathrm{SU}(2)^3$ acts on $\mathrm{Sp}(4)$ by a homomorphism $\mathrm{SU}(2)^3 \to \mathrm{Sp}(4)^2$. To define this, first let V_i for $i = 1, 2, 3$ denote the standard 2-dimensional complex representation of the ith factor of $\mathrm{SU}(2)$. Then the homomorphism $\mathrm{SU}(2)^3 \to \mathrm{Sp}(4)^2$ is defined by $(V_1 \oplus V_2, V_3 \oplus \mathbf{C}^2)$. Also, let $W_{12} : \mathrm{SU}(2)^2 \to \mathrm{SO}(4)$ be the natural double covering, viewed as a 4-dimensional real representation of the first two copies of $\mathrm{SU}(2)$, and let $(V_3)_{\mathbf{R}} : \mathrm{SU}(2) \to \mathrm{SO}(4)$ be the natural real faithful representation of the third copy of $\mathrm{SU}(2)$. We define the action of $\mathrm{SU}(2)^3$ on S^{4n-1} by the homomorphism $\mathrm{SU}(2)^3 \to \mathrm{SO}(8n)$ defined by $(V_3)_{\mathbf{R}}^{\oplus n-1} \oplus W_{12}$.

The action of $\mathrm{SU}(2)^3$ on $\mathrm{Sp}(4)$ has cohomogeneity one, with trivial generic stabilizer group and with stabilizers at the two special orbits both isomorphic to $\mathrm{SU}(2)$, the first conjugate to the subgroup $\{(x, 1, x)\}$ in $\mathrm{SU}(2)^3$ and the second conjugate to $\{(1, x, x)\}$. These two subgroups act freely on S^{4n-1}, and so $\mathrm{SU}(2)^3$ acts freely on $\mathrm{Sp}(4) \times S^{4n-1}$. Let M be the quotient manifold. Using the description of the $\mathrm{SU}(2)^3$-action on $\mathrm{Sp}(4)$, we see that M is the union of two disc bundles over $S^{4n-1}/\mathrm{SU}(2) = \mathbf{HP}^{n-1}$ along their common boundary, S^{4n-1}. Therefore M is the connected sum of two copies of $\mathbf{HP}^n$, with some orientations.

To determine the relevant orientations, it suffices for n even to compute the cohomology ring of M. For n odd, the manifolds $\mathbf{HP}^n\#\mathbf{HP}^n$ and $\mathbf{HP}^n\# -\mathbf{HP}^n$ have isomorphic cohomology rings. In that case we will also need a mod 3 Steenrod operation to see that M is diffeomorphic to $\mathbf{HP}^n\#\mathbf{HP}^n$ rather than $\mathbf{HP}^n\# -\mathbf{HP}^n$. First, we compute the cohomology ring of the homotopy quotient $\mathrm{Sp}(4)//\mathrm{SU}(2)^3$, with respect to the above action of $\mathrm{SU}(2)^3$ on $\mathrm{Sp}(4)$. Write $z_i = -c_2V_i$, for $i = 1, 2, 3$, which are generators in H^4 of the polynomial ring $H^*(\mathrm{BSU}(2)^3, \mathbf{Z})$. The generators c_2 and c_4 of $H^*(\mathrm{BSp}(4), \mathbf{Z})$ pull back under the left homomorphism $\mathrm{SU}(2)^3 \to \mathrm{Sp}(4)$ to $-z_1 - z_2$ and z_1z_2, and under the right homomorphism $\mathrm{SU}(2)^3 \to \mathrm{Sp}(4)$ to $-z_3$ and 0. Therefore, using the Eilenberg-Moore spectral sequence as suggested by Singhof [25], the cohomology of $\mathrm{Sp}(4)//\mathrm{SU}(2)^3$ is

$$\mathbf{Z}[z_1, z_2, z_3]/(z_1 + z_2 = z_3, z_1 z_2 = 0).$$

The manifold M is an S^{4n-1}-bundle over $\mathrm{Sp}(4)//\mathrm{SU}(2)^3$, up to homotopy. The Euler class of this bundle, $(V_3)_{\mathbf{R}}^{\oplus n-1} \oplus W_{12}$, is $\pm(-z_3)^{n-1}(-z_1 + z_2)$, and so M has cohomology ring

$$\begin{aligned}&\mathbf{Z}[z_1, z_2, z_3]/(z_1 + z_2 = z_3, z_1 z_2 = 0, z_3^{n-1}(z_1 - z_2) = 0)\\ &= \mathbf{Z}[z_1, z_2]/(z_1 z_2 = 0, z_1^n = z_2^n).\end{aligned}$$

This is the cohomology ring of $\mathbf{HP}^n \# \mathbf{HP}^n$, and for n even it is not the cohomology ring of $\mathbf{HP}^n \# {-\mathbf{HP}^n}$. So M is diffeomorphic to $\mathbf{HP}^n \# \mathbf{HP}^n$ for n even. For n odd, $n \geq 3$, we also need to observe that the classes z_i are distinguished from their negatives by the fact that $P^1 z_i = -z_i^2$ in $H^8(M, \mathbf{F}_3)$. By definition of the natural orientation on $\mathbf{HP}^n$ for $n \geq 2$, this means that M is diffeomorphic to $\mathbf{HP}^n \# \mathbf{HP}^n$ for all $n \geq 2$, not to $\mathbf{HP}^n \# - \mathbf{HP}^n$. The case $n = 1$ is trivial, since $\mathbf{HP}^n \# \mathbf{HP}^n = S^4 \# S^4 = S^4$.

The last Cheeger manifolds which are diffeomorphic to biquotients are the manifolds $\mathbf{CP}^{4e+2} \# \mathbf{HP}^{2e+1}$. The relevant biquotient M has the form $M = (S^5 \times S^{8e+3})/(S^1 \times \mathrm{SU}(2))$. To be more explicit, let L be the standard 1-dimensional complex representation of S^1, and let V be the standard 2-dimensional complex representation of $\mathrm{SU}(2)$. Then we let $S^1 \times \mathrm{SU}(2)$ act on S^5 as the unit sphere in $V \oplus L$, and on S^{4n-1} as the unit sphere in $(V \otimes_{\mathbf{C}} L)^{\oplus 2e+1}$. The action of $S^1 \times \mathrm{SU}(2)$ on S^5 has cohomogeneity one, with trivial generic stabilizer and with stabilizers at the two special orbits conjugate to the two factors S^1 and $\mathrm{SU}(2)$, respectively. Both of these subgroups act freely on S^{8e+3}, and so $S^1 \times \mathrm{SU}(2)$ acts freely on $S^5 \times S^{8e+3}$. Let M be the quotient manifold.

From the action of $S^1 \times \mathrm{SU}(2)$ on S^5, we see that M is the union of a disc bundle over $S^{8e+3}/S^1 = \mathbf{CP}^{4e+1}$ and a disc bundle over $S^{8e+3}/\mathrm{SU}(2) = \mathbf{HP}^{2e}$ along their common boundary, S^{8e+3}. It follows that M is the connected sum of $\mathbf{CP}^{4e+2}$ and $\mathbf{HP}^{2e+1}$ with some orientations.

To show that M is diffeomorphic to $\mathbf{CP}^{4e+2} \# \mathbf{HP}^{2e+1}$ rather than to $\mathbf{CP}^{4e+2} \# - \mathbf{HP}^{2e+1}$, we compute the cohomology ring and a mod 3 Steenrod operation on M. Since $M = S(V \oplus L) \times S((V \otimes L)^{\oplus 2e+1})/(S^1 \times \mathrm{SU}(2))$, we can view M as an $(S^5 \times S^{8e+3})$-bundle over $\mathrm{B}S^1 \times \mathrm{BSU}(2)$. Here $\mathrm{B}S^1 \times \mathrm{BSU}(2)$ has cohomology ring $\mathbf{Z}[x, z]$, where we let $x = c_1 L$ and $z = -c_2 V$. The vector bundles $V \oplus L$ and $(V \otimes L)^{\oplus 2e+1}$ on

$BS^1 \times BSU(2)$ have Euler classes $(-z)x$ and $(x^2 - z)^{2e+1}$. So M has cohomology ring

$$\begin{aligned} &\mathbf{Z}[x,z]/(-xz = 0, (x^2 - z)^{2e+1} = 0) \\ &= \mathbf{Z}[x,z]/(xz = 0, x^{4e+2} = z^{2e+1}). \end{aligned}$$

Here the element z in $H^4(M, \mathbf{Z})$ is distinguished from its negative, for $e \geq 1$, by the property that $P^1 z = -z^2$ in $H^8(M, \mathbf{F}_3)$. By definition of the natural orientation on $\mathbf{HP}^{2e+1}$ for $e \geq 1$, M is diffeomorphic to $\mathbf{CP}^{4e+2} \# \mathbf{HP}^{2e+1}$, not to $\mathbf{CP}^{4e+2} \# -\mathbf{HP}^{2e+1}$.

By contrast, the analogous calculation of the cohomology ring shows that the biquotient

$$(S(V \oplus L) \times S((V \otimes L)^{\oplus 2e}))/(S^1 \times \mathrm{SU}(2))$$

is diffeomorphic to $\mathbf{CP}^{4e} \# -\mathbf{HP}^{2e}$, not to $\mathbf{CP}^{4e} \# \mathbf{HP}^{2e}$. This will be used in Section 8.

3. Simplifying the description of a given biquotient manifold

Here is an elementary but essential beginning to our simplification of the description of a given biquotient manifold. (Throughout the paper, G and H will denote compact Lie groups.)

Lemma 3.1. *Let M be a simply connected biquotient manifold. Then we can write $M = G/H$ for some simply connected group G and connected group H acting on G by a homomorphism $H \to (G \times G)/Z(G)$. If M is 2-connected, then H is simply connected and H acts on G by a homomorphism $H \to G \times G$.*

Proof. Since M is connected, we can write M as a biquotient G/H with G connected. Since M is simply connected, the long exact sequence of the fibration $H \to G \to M$,

$$\pi_1 H \to \pi_1 G \to \pi_1 M \to \pi_0 H \to \pi_0 G,$$

shows that H is connected and $\pi_1 H \to \pi_1 G$ is surjective. Let C be the kernel of $\pi_1 H \to \pi_1 G$, a finitely generated abelian group.

Let $\widetilde{G}$ and $\widetilde{H}$ be the universal covers of G and H. We can identify $\pi_1 H$ with the kernel of $\widetilde{H} \to H$, and so C is a central subgroup of $\widetilde{H}$. We have $\widetilde{G} \cong K_G \times \mathbf{R}^a$ and $\widetilde{H} \cong K_H \times \mathbf{R}^b$ for some simply connected compact Lie groups K_G and K_H. The given homomorphism $H \to$

$(G \times G)/Z(G)$ lifts to a homomorphism $\widetilde{H} \to \widetilde{G} \times \widetilde{G}$. The resulting action of $\widetilde{H}$ on $\widetilde{G}$ is trivial on the subgroup C, and $\widetilde{H}/C$ acts freely on $\widetilde{G}$ with $M = \widetilde{G}/(\widetilde{H}/C)$.

Here $\widetilde{G} = K_G \times \mathbf{R}^a$, and the action of $\widetilde{H}/C$ on $\widetilde{G}$ is the product of an action on K_G and an action on $\mathbf{R}^a$. Furthermore, since the group $\mathbf{R}^a$ is abelian, $\widetilde{H}/C$ acts on $\mathbf{R}^a$ by translations (left or right translations being the same). Since $\widetilde{H}/C$ is a connected group and the quotient is compact, $\widetilde{H}/C$ must act transitively on $\mathbf{R}^a$. Let L be the kernel of the action of $\widetilde{H}/C$ on $\mathbf{R}^a$. Then L must act freely on K_G, with $M = K_G/L$. Here K_G is a simply connected compact Lie group. Also, L must be connected by the long exact sequence

$$\pi_1 M \to \pi_0 L \to \pi_0 K_G.$$

Finally, if M is 2-connected, then the same long exact sequence shows that L is simply connected. In this case, the homomorphism $L \to (K_G \times K_G)/Z(G)$ lifts uniquely to a homomorphism $L \to K_G \times K_G$. q.e.d.

Since we intend to study simply connected biquotient manifolds, Lemma 3.1 tells us that we can assume G is simply connected, and thus a product of simply connected simple groups. Thus, much of the complexity of more general compact Lie groups is avoided.

The main method of simplifying the description of a given biquotient is the following easy observation.

Lemma 3.2. *Let H be a compact Lie group acting on manifolds X_1 and X_2 such that H acts transitively on X_1 and H acts freely on $X_1 \times X_2$. Let $K \subset H$ be the stabilizer of some point in X_1. Then the quotient manifold $(X_1 \times X_2)/H$ is diffeomorphic to X_2/K, where K acts freely on X_2 by the restriction of the action of H.*

Proof. This is clear by identifying X_1 with H/K. q.e.d.

Applying this method of simplification to biquotients gives the following fundamental result.

Lemma 3.3. *Let M be a simply connected biquotient manifold. Then we can write $M = G/H$ such that G is simply connected, H is connected, and H does not act transitively on any simple factor of G.*

Proof. By Lemma 3.1, we can write M as a biquotient G/H with G simply connected and H connected. If H acts transitively on some simple factor of G, then Lemma 3.2 allows us to remove that factor of G

while replacing H by a subgroup. The exact sequence $\pi_1 M \to \pi_0 H \to \pi_0 G$ shows that the new subgroup H is still connected. By induction on the number of simple factors of G, we can arrange that H does not act transitively on any simple factor of G. q.e.d.

Convention 3.4. *From now on, we will only consider simply connected biquotient manifolds, and in writing $M = G/H$ we will assume that G and H satisfy the properties listed in Lemma* 3.3.

4. Bounding G in terms of M

As mentioned in Convention 3.4, for the rest of the paper we only consider simply connected biquotient manifolds. We can and do assume that every such manifold M is written $M = G/H$ with G simply connected, H connected, and H not acting transitively on any simple factor of G. In this section, we will show how these properties determine G up to finitely many possibilities in terms of the rational homotopy groups of M.

The following lemma will not be needed in this section, but is included here for later use.

Lemma 4.1. *Let $M = G/H$ be a simply connected biquotient manifold. If in addition the rational homotopy group $\pi_3 M_{\mathbf{Q}}$ is zero, then each simple factor H_1 of H acts trivially on all factors of G isomorphic to H_1.*

Proof. By the long exact sequence of the fibration $H \to G \to M$, since $\pi_3 M_{\mathbf{Q}} = 0$, the homomorphism $\pi_3 H_{\mathbf{Q}} \to \pi_3 G_{\mathbf{Q}}$ is surjective. In particular, for each simple factor G_1 of G, $\pi_3 H_{\mathbf{Q}} \to \pi_3(G_1)_{\mathbf{Q}}$ is surjective. Suppose that some simple factor H_1 of H which is isomorphic to G_1 acts nontrivially on G_1; we will derive a contradiction. Here, a priori, H_1 can act by left and right translation on G_1. If H_1 acts nontrivially on only one side of G_1, thus by a nontrivial homomorphism $H_1 \to G_1$, then this homomorphism must be an isomorphism, since H_1 is simple. In particular, H_1 acts transitively on G_1, which contradicts Convention 3.4. Therefore, H_1 must act nontrivially on both sides of G_1. Then the two homomorphisms from H_1 to G_1 must both be isomorphisms, using simplicity of H_1 again. We now use that any automorphism of a simply connected simple group G_1 acts trivially on $\pi_3(G_1) \cong \mathbf{Z}$, as follows from the proof of this isomorphism using the Killing form. The homomorphism $\pi_3(H_1)_{\mathbf{Q}} \to \pi_3(G_1)_{\mathbf{Q}}$ is the difference of the homomorphisms given by the two homomorphisms $H_1 \to G_1$, and so it is zero. Furthermore, since the two homomorphisms $H_1 \to G_1$ both have only finite centralizer, there is no room for the rest of H to act on G_1; in other words, H acts on G_1 through a quotient group isogenous to H_1. It follows that the homomorphism $\pi_3 H_{\mathbf{Q}} \to \pi_3(G_1)_{\mathbf{Q}}$ is zero, contradicting what we know from $\pi_3 M_{\mathbf{Q}} = 0$. Thus, each simple factor H_1 of H must act trivially on simple factors of G isomorphic to H_1. q.e.d.

To find more precise information on G, we need the classification of the simple compact Lie groups. In particular, we use that a simple group G has rational homotopy groups concentrated in odd degrees $2d-1$, and we call the numbers d that occur the *degrees* of G. Particularly important for us is the *maximal degree* of G, which we call $d(G)$. It is also called the *Coxeter number* of G. The degrees of G are well-known in many contexts: we can also say that $H^*(G, \mathbf{Q})$ is an exterior algebra with generators in degrees $2d-1$ where d runs over the degrees of G, or that the degrees of G are the degrees of the generators of the ring of invariants of the Weyl group acting on its reflection representation. We tabulate the degrees of the simple groups here, following Bourbaki [5] or Gorbatsevich-Onishchik ([14], Table 1, p. 127). To avoid repetitions, one can assume that A_l has $l \geq 1$, B_l has $l \geq 3$, C_l has $l \geq 2$, and D_l has $l \geq 4$.

Table 4.2

$$
\begin{aligned}
A_l &: 2, 3, \ldots, l+1 \\
B_l &: 2, 4, 6, \ldots, 2l \\
C_l &: 2, 4, 6, \ldots, 2l \\
D_l &: 2, 4, 6, \ldots, 2l-2; l \\
G_2 &: 2, 6 \\
F_4 &: 2, 6, 8, 12 \\
E_6 &: 2, 5, 6, 8, 9, 12 \\
E_7 &: 2, 6, 8, 10, 12, 14, 18 \\
E_8 &: 2, 8, 12, 14, 18, 20, 24, 30.
\end{aligned}
$$

Using Table 4.2 and the known low-dimensional representations of each group, Onishchik proved the following result [20]. He later gave a more systematic proof, using reflection groups [21].

Lemma 4.3. *Let $H \to G$ be a nontrivial homomorphism of simply connected simple groups. Then the maximal degrees satisfy* $d(H) \leq$

$d(G)$. Moreover, if $d(H) = d(G)$, then either $H \to G$ is an isomorphism or G/H is one of the following homogeneous spaces. On the right we show the degrees of G and H.

$\mathrm{Spin}(2n)/\mathrm{Spin}(2n-1)$ $= S^{2n-1}, n \geq 4$	$2, 4, 6, \ldots, 2n-2; n$	$2, 4, 6, \ldots, 2n-2$
$\mathrm{SU}(2n)/\mathrm{Sp}(2n), n \geq 2$	$2, 3, 4, \ldots, 2n$	$2, 4, 6, \ldots 2n$
$\mathrm{Spin}(7)/G_2 = S^7$	$2, 4, 6$	$2, 6$
$\mathrm{Spin}(8)/G_2 = S^7 \times S^7$	$2, 4, 4, 6$	$2, 6$
E_6/F_4	$2, 5, 6, 8, 9, 12$	$2, 6, 8, 12.$

In all these cases except $\mathrm{Spin}(8)/\mathrm{Spin}(7) = S^7$*, there is a unique conjugacy class of nontrivial homomorphisms $H \to G$; in the case* $\mathrm{Spin}(8)/\mathrm{Spin}(7)$*, there are three conjugacy classes which are equivalent under outer automorphisms of* $\mathrm{Spin}(8)$*. Also, in all the above cases, the centralizer of H in G is finite.*

The following result shows how to apply Lemma 4.3 to biquotients $M = G/H$, although it is only a step on the way to the more precise Theorem 4.8. For a simple factor G_1 of G, we say that a degree d of G_1 is *killed by H* if the homomorphism $\pi_{2d-1}H_{\mathbf{Q}} \to \pi_{2d-1}(G_1)_{\mathbf{Q}}$ associated to the action of H on G_1 is nonzero.

Lemma 4.4. *Let $M = G/H$ be a simply connected biquotient, written according to Convention* 3.4*. Let G_1 be a simple factor of G such that the maximal degree of G_1 is killed by H. Then either there is a simple factor H_1 of H such that H_1 acts nontrivially on exactly one side of G_1 by one of the homomorphisms in Lemma* 4.3*, so that G_1/H_1 is one of* $\mathrm{Spin}(2n)/\mathrm{Spin}(2n-1) = S^{2n-1}$*,* $\mathrm{SU}(2n)/\mathrm{Sp}(2n)$*,* $\mathrm{Spin}(7)/G_2 = S^7$*,* $\mathrm{Spin}(8)/G_2 = S^7 \times S^7$*, or E_6/F_4; or G_1 is isomorphic to* $\mathrm{SU}(2n+1)$ *for some n and there is a simple factor H_1 of H also isomorphic to* $\mathrm{SU}(2n+1)$ *which acts on G_1 by $h(g) = hgh^t$. The* $\mathrm{SU}(2n+1)$ *case cannot occur if $\pi_3 M_{\mathbf{Q}} = 0$.*

Proof. We are given a simple factor G_1 of G such that the maximal degree of G_1 is killed by H. It follows that there must be a simple factor H_1 of H which kills the maximal degree d of G_1. Since H_1 is simply connected, the action of H_1 on G_1 is given by a homomorphism $H_1 \to G_1 \times G_1$, and the resulting linear map $\pi_{2d-1}(H_1)_{\mathbf{Q}} \to \pi_{2d-1}(G_1)_{\mathbf{Q}} \cong \mathbf{Q}$ is nonzero. This linear map is the difference of the two linear maps associated to the two homomorphisms $H_1 \to G_1$ (on the left and right), so at least one of those two linear maps is nonzero. By Lemma 4.3, either H_1 is isomorphic to G_1 or (G_1, H_1) is one of the pairs $(\mathrm{Spin}(2n), \mathrm{Spin}(2n-1))$, $(\mathrm{SU}(2n), \mathrm{Sp}(2n))$, $(\mathrm{Spin}(7), G_2)$, $(\mathrm{Spin}(8), G_2)$, or (E_6, F_4).

Suppose first that (G_1, H_1) is one of these 5 pairs. If H_1 acts nontrivially on both sides of G_1, then in all cases except $(\mathrm{Spin}(8), \mathrm{Spin}(7))$, Lemma 4.3 implies that the two homomorphisms $H_1 \to G_1$ are conjugate, so the resulting map $\pi_*(H_1)_{\mathbf{Q}} \to \pi_*(G_1)_{\mathbf{Q}}$ (the difference of the left and right maps) is 0, contradicting the fact that H_1 kills the top degree of G_1. Even in the case $(\mathrm{Spin}(8), \mathrm{Spin}(7))$, we compute that the outer automorphism group of Spin(8) acts trivially on the top degree, 6, of Spin(8), that is, on $\pi_{11}\mathrm{Spin}(8)_{\mathbf{Q}}$. It follows that if Spin(7) acts nontrivially on both sides of Spin(8), then the left and right homomorphisms $\mathrm{Spin}(7) \to \mathrm{Spin}(8)$ give the same linear map into the top degree of Spin(8), and so the action of Spin(7) of Spin(8) cannot kill the top degree of Spin(8), a contradiction. Thus, in all these cases, H_1 acts nontrivially on only one side of G_1, by one of the homomorphisms in Lemma 4.3. The lemma is proved in this case.

The remaining case is where H_1 is isomorphic to G_1. Clearly this cannot occur if $\pi_3 M_{\mathbf{Q}} = 0$, by Lemma 4.1. In general, if H_1 acts nontrivially on only one side of G_1, then it must act by an isomorphism $H_1 \to G_1$. So H acts transitively on G_1, contrary to Convention 3.4. Therefore H_1 must act nontrivially on both sides of G_1, clearly by two isomorphisms $H_1 \to G_1$. We are given that H_1 kills the top degree of G_1, so these two isomorphisms $H_1 \to G_1$ must give *different* linear maps into the top degree of $\pi_*(G_1)_{\mathbf{Q}}$. So the outer automorphism group of G_1 must act nontrivially on the top degree of G_1. Of the simply connected simple groups, only A_n, D_n, and E_6 have nontrivial outer automorphism group, and only in the case $G_1 = \mathrm{SU}(2n+1) = A_{2n}$ does the outer automorphism group act nontrivially on the top degree of G_1. (The outer automorphism group $\mathbf{Z}/2$ of $\mathrm{SU}(n)$, $n \geq 3$, acts by the identity on the even degrees of $\mathrm{SU}(n)$ and by -1 on the odd degrees. A topological way to see this is to identify $\pi_* G_{\mathbf{Q}}$ with the dual to the vector space $H^{>0}(\mathrm{B}G, \mathbf{Q})/(H^{>0} \cdot H^{>0})$, and use in the case $G = \mathrm{SU}(n)$ that the outer automorphism $E \mapsto E^*$ acts on Chern classes by $c_i \mapsto (-1)^i c_i$.) So we must have $G_1 = \mathrm{SU}(2n+1)$, with $H_1 = \mathrm{SU}(2n+1)$ acting on G_1 by the identity on one side and by the outer automorphism $x \mapsto (x^t)^{-1}$ on the other. q.e.d.

In order to have strong restrictions on the simple factors of G in terms of the rational homotopy groups of $M = G/H$, we need to analyze more completely the way H acts on simple factors of G isomorphic to

Spin$(2n)$, SU$(2a)$, Spin(7), Spin(8), or E_6. In analogy with Lemmas 4.3 and 4.4, we will first formulate a general statement on the subgroups of these groups, and then apply it to biquotients. Before that, we need some simple topological results.

Lemma 4.5. *Let H and G be connected compact Lie groups, and let $f : H \to G$ be any continuous map. If $f_* : \pi_* H_{\mathbf{Q}} \to \pi_* G_{\mathbf{Q}}$ is surjective, then f is surjective.*

Proof. Both H and G have the rational homotopy type of products of odd-dimensional spheres. So $H^*(H, \mathbf{Q})$ and $H^*(G, \mathbf{Q})$ are both exterior algebras, and the assumption means that the homomorphism

$$f^* : H^{>0}(G, \mathbf{Q})/(H^{>0} \cdot H^{>0}) \to H^{>0}(H, \mathbf{Q})/(H^{>0} \cdot H^{>0})$$

is injective. Thus $H^*(H, \mathbf{Q})$ is the exterior algebra generated by the generators of $H^*(G, \mathbf{Q})$ together with some other generators. So $H^*(G, \mathbf{Q}) \to H^*(H, \mathbf{Q})$ is injective. Since G is a closed orientable manifold, it follows that $H \to G$ is surjective. q.e.d.

Corollary 4.6. *Let H and G be connected compact Lie groups, with an action of H on G by a homomorphism $H \to (G \times G)/Z(G)$. If the associated homomorphism $\pi_* H_{\mathbf{Q}} \to \pi_* G_{\mathbf{Q}}$ is surjective, then H acts transitively on G.*

Proof. We are given that the homomorphism associated to the orbit map $f : H \to G$ of some point in G is surjective on rational homotopy groups. By Lemma 4.5, $H \to G$ is surjective. That is, H acts transitively on G. q.e.d.

We now apply Corollary 4.6 to get information on the subgroups of the groups occurring in Lemma 4.3. (We also use the classification of simple Lie groups in the following proof, but it is more pleasant to use Corollary 4.6 when possible.)

Lemma 4.7. *Let $\varphi : H \to \mathrm{Spin}(2n)$ be a homomorphism from a simply connected simple group such that the homomorphism*

$$\pi_{2n-1} H_{\mathbf{Q}} \to \pi_{2n-1} \mathrm{Spin}(2n)_{\mathbf{Q}} / \pi_{2n-1} \mathrm{Spin}(2n-1)_{\mathbf{Q}} \cong \mathbf{Q}$$

is not zero. Then H acts transitively on the sphere $\mathrm{Spin}(2n)/\mathrm{Spin}(2n-1) = S^{2n-1}$.

Next, let G/K be one of the other homogeneous spaces from Lemma 4.3: $\mathrm{SU}(2n)/\mathrm{Sp}(2n)$ *with $n \geq 2$,* $\mathrm{Spin}(7)/G_2 = S^7$*,* $\mathrm{Spin}(8)/G_2 =$

$S^7 \times S^7$, or E_6/F_4. Let $H \to G$ be a homomorphism from a simply connected compact Lie group H which kills the second-largest degree of G. (In the case $G/K = \mathrm{Spin}(8)/G_2$, where $\mathrm{Spin}(8)$ has degrees $2, 4, 4, 6$, we assume that both degrees 4 of $\mathrm{Spin}(8)$ are killed by H.) Then H acts transitively on G/K.

Proof. In the cases where G/K is one of $\mathrm{Spin}(2n)/\mathrm{Spin}(2n-1) = S^{2n-1}$, $\mathrm{Spin}(7)/G_2 = S^7$, or $\mathrm{Spin}(8)/G_2 = S^7 \times S^7$, the assumption implies that the action of $H \times K$ on G is surjective on rational homotopy groups. By Corollary 4.6, $H \times K$ acts transitively on G. Equivalently, H acts transitively on G/K.

Next, let $G/K = \mathrm{SU}(2n)/\mathrm{Sp}(2n)$, $n \geq 2$, and suppose that H kills the second-largest degree, $2n-1$, of $\mathrm{SU}(2n)$. We can replace H by one of its simple factors without changing this property. If H is isomorphic to G, then the homomorphism $H \to G$ is an isomorphism, and so H acts transitively on G/K. Otherwise, by Lemma 4.3, H has maximal degree at most that of $\mathrm{SU}(2n)$, which is $2n$, and if equality holds then $H = \mathrm{Sp}(2n)$. But $\mathrm{Sp}(2n)$ does not kill the degree $2n-1$ of $\mathrm{SU}(2n)$. So H must have maximal degree $2n-1$. By Table 4.2, the only simple group with maximal degree an odd number is $H = \mathrm{SU}(2n-1)$. Since $n \geq 2$, any nontrivial homomorphism $\mathrm{SU}(2n-1) \to \mathrm{SU}(2n)$ is equivalent to the standard inclusion by some automorphism of $\mathrm{SU}(2n-1)$. Then $\mathrm{SU}(2n-1)$ acts transitively on $\mathrm{SU}(2n)/\mathrm{Sp}(2n)$, because $\mathrm{Sp}(2n)$ acts transitively on $\mathrm{SU}(2n)/\mathrm{SU}(2n-1) = S^{4n-1}$.

Finally, suppose that $G/K = E_6/F_4$ and H kills the second-largest degree, 9, of E_6. We can replace H by one of its simple factors without changing this property. Then H has rank at most the rank 6 of E_6 and has 9 as a degree. By Table 4.2, it follows that H is isomorphic to E_6. So the given homomorphism $H \to E_6$ must be an isomorphism, and in particular H acts transitively on E_6/F_4. q.e.d.

We now apply this result on subgroups to deduce strong information on the classification of biquotient manifolds. For a given biquotient $M = G/H$, we say that a given simple factor G_i of G *contributes degree d_i to M* if the homomorphism $\pi_{2d_i-1}H_{\mathbf{Q}} \to \pi_{2d_i-1}(G_i)_{\mathbf{Q}}$ is not surjective. If $G = \prod G_i$ and every factor G_i contributes some degree d_i to M, then $\pi_{2d-1}M_{\mathbf{Q}} = \pi_{2d-1}G_{\mathbf{Q}}/\pi_{2d-1}H_{\mathbf{Q}}$ has dimension at least equal to the number of simple factors G_i with $d_i = d$.

Theorem 4.8. *Let $M = G/H$ be a simply connected biquotient manifold, written using Convention 3.4. Let G_1 be any simple factor of G. Then at least one of the following holds:*

(1) *G_1 contributes its maximal degree to M.*

(2) *G_1 contributes its second-largest degree to M, and there is a simple factor H_1 of G_1 which acts nontrivially on exactly one side of G_1, with G_1/H_1 isomorphic to one of the homogeneous spaces* $\mathrm{SU}(2n)/\mathrm{Sp}(2n)$ *with* $n \geq 2$, $\mathrm{Spin}(7)/G_2 = S^7$, $\mathrm{Spin}(8)/G_2 = S^7 \times S^7$, *or E_6/F_4. The second-largest degree of G_1 is, respectively,* $2a-1$, 4, 4, *or* 9. (*In the case* $G_1 = \mathrm{Spin}(8)$, *which has degrees* $2, 4, 4, 6$, *the claim is only that G_1 contributes at least one degree 4 to M.*)

(3) *$G_1 \cong \mathrm{Spin}(2n)$ with $n \geq 4$ contributes its degree n to M, and there is a simple factor $H_1 \cong \mathrm{Spin}(2n-1)$ which acts nontrivially on exactly one side of G_1, by the standard inclusion, with* $\mathrm{Spin}(2n)/\mathrm{Spin}(2n-1) = S^{2n-1}$.

(4) *$G_1 \cong \mathrm{SU}(2n+1)$ contributes degrees $2, 4, 6, \ldots, 2n$ to M, and there is a simple factor $H_1 \cong \mathrm{SU}(2n+1)$ of G_1 which acts on G_1 by $h(g) = hgh^t$.*

Proof. Let G_1 be a simple factor of G. Suppose that (1) does not hold, in other words that the maximal degree of G_1 is killed by H. By Lemma 4.4, there is a simple factor H_1 of H such that either H_1 acts nontrivially on exactly one side of G_1 by one of the homomorphisms listed in (2) or (3) above, or $G_1 = H_1 = \mathrm{SU}(2n+1)$ and H_1 acts on G_1 as in (4). The remaining point is to show that G_1 contributes the degrees to M that we have claimed.

In cases (2) and (3), H_1 has finite centralizer in G_1 by Lemma 4.3, so the rest of H can act on G_1 only on the other side from H_1. Since H_1 and the rest of H together do not act transitively on G_1, by Convention 3.4, Lemma 4.7 shows that G_1 contributes its degree n to M if $G_1/H_1 = \mathrm{Spin}(2n)/\mathrm{Spin}(2n-1) = S^{2n-1}$, or its second-largest degree to M if G_1/H_1 is one of $\mathrm{SU}(2n)/\mathrm{Sp}(2n)$, $\mathrm{Spin}(7)/G_2 = S^7$, $\mathrm{Spin}(8)/G_2 = S^7 \times S^7$, or E_6/F_4.

In case (4), since $H_1 = \mathrm{SU}(2n+1)$ acts with finite centralizer on both sides of $G_1 = \mathrm{SU}(2n+1)$, no other factor of H can act on G_1. So the image of $\pi_* H_{\mathbf{Q}} \to \pi_*(G_1)_{\mathbf{Q}}$ is equal to the image of $\pi_*(H_1)_{\mathbf{Q}} \to \pi_*(G_1)_{\mathbf{Q}}$. This homomorphism is the difference of the identity map on $\pi_*(G_1)_{\mathbf{Q}}$ with the map given by the outer automorphism $g \mapsto (g^t)^{-1}$, which acts by 1 on the even degrees and by -1 on the odd degrees (as shown in the proof of Lemma 4.4). So the image of $\pi_{2d-1}(H_1)_{\mathbf{Q}} \to$

$\pi_{2d-1}(G_1)_{\mathbf{Q}}$ is zero for d even. That is, G_1 contributes all its even degrees $2, 4, \ldots, 2n$ to M. q.e.d.

Theorem 4.8 implies the following important qualitative statement on the classification of biquotients. The analogous result for homogeneous spaces is easy and probably well-known. It is perhaps surprising that the following statement requires the detailed classification work we have done in this section, but that seems to be true, at least for now.

Theorem 4.9. *There are only finitely many diffeomorphism classes of* 2*-connected biquotient manifolds of a given dimension.*

Theorem 4.9 is vaguely reminiscent of the Petrunin-Tuschmann theorem, which says in particular that for any number C, there are only finitely many diffeomorphism classes of 2-connected closed Riemannian manifolds with curvature $0 \leq K \leq C$ and diameter 1 [24]. But there is probably no way to actually deduce Theorem 4.9 from the Petrunin-Tuschmann theorem, since there is no obvious upper bound on the curvature of 2-connected biquotients until one goes through the proof of Theorem 4.9. In fact, the discussion after Theorem 1.1 in my paper [27] shows that there can be no upper bound on the curvature of simply connected biquotients of dimension 6, if one fixes their diameter to be 1.

Proof of Theorem 4.9. Any biquotient manifold M is rationally elliptic; that is, all but finitely many of the rational homotopy groups of M are zero. So, writing n for the dimension of M, the odd-degree rational homotopy groups $\pi_{2d-1}M_{\mathbf{Q}}$ are zero for $d > n$, and the total dimension of the odd-degree rational homotopy groups of M is at most n, by Friedlander and Halperin ([13], p. 434). Since M is simply connected, we can write M as a biquotient G/H according to Convention 3.4. In particular, G is simply connected and, since M is 2-connected, H is also simply connected. The essential point is Theorem 4.8, which implies that each simple factor of G contributes at least one degree to M, and that one such degree is at least half the maximal degree of G. Therefore the number of simple factors of G is at most the total dimension of $\pi_{\text{odd}}M_{\mathbf{Q}}$ and hence at most n, and the maximal degree of each simple factor of G is at most $2n$. Thus G, being a product of simply connected simple groups, is determined up to finitely many possibilities by n. Also, H is a simply connected group of dimension at most that of G, so H and the homomorphism $H \to (G \times G)/Z(G)$ (up to conjugacy) are determined up to finitely many possibilities. q.e.d.

5. Further general results on the classification of biquotients

In this section we continue the previous section's method: we classify subgroups of compact Lie groups with certain properties, and apply the results to the classification of biquotients. The main result of this section is Theorem 5.3, which describes the possible simple factors of G in a biquotient $M = G/H$ which contribute only their top degree to M. (This terminology is defined before Theorem 4.8.)

We begin by stating a classification of certain subgroups of $\mathrm{Spin}(2n)$.

Lemma 5.1. *Let $\varphi : H \to \mathrm{Spin}(2n)$, $n \geq 4$, be a homomorphism from a simply connected simple group H such that the linear map*

$$\pi_{2n-1} H_{\mathbf{Q}} \to \pi_{2n-1}\mathrm{Spin}(2n)_{\mathbf{Q}}/\pi_{2n-1}\mathrm{Spin}(2n-1)_{\mathbf{Q}} \cong \mathbf{Q}$$

is not zero. Then φ is either an isomorphism or one of the following homomorphisms, up to the standard $\mathbf{Z}/2$ group of outer automorphisms of $\mathrm{Spin}(2n)$*:* $\mathrm{SU}(n) \hookrightarrow \mathrm{Spin}(2n)$, $\mathrm{Sp}(2a) \hookrightarrow \mathrm{Spin}(4a)$*, the spin representation* $\mathrm{Spin}(7) \hookrightarrow \mathrm{Spin}(8)$*, or the spin representation* $\mathrm{Spin}(9) \hookrightarrow \mathrm{Spin}(16)$.

This is straightforward to prove using the known degrees and low-dimensional representations of all the simple groups. Alternatively, we can deduce it from Borel's classification of groups which act linearly and transitively on the sphere, in the odd-dimensional case [4]. Namely, the hypothesis of Lemma 5.1 means that the action of H on S^{2n-1} associated to the homomorphism $H \to \mathrm{Spin}(2n)$ has orbit map $H \to S^{2n-1}$ such that $\pi_{2n-1} H_{\mathbf{Q}} \to \pi_{2n-1} S^{2n-1}_{\mathbf{Q}} \cong \mathbf{Q}$ is not zero. It follows that $H \to S^{2n-1}$ is surjective, in other words that H acts transitively on S^{2n-1}. Then Lemma 5.1 follows from Borel's classification.

The subgroups we classify next are the simple subgroups H of any simple group G such that the maximal degree of H is at least the second-largest degree of G. This is closely related to two classifications by Onishchik. First, he classified the simple subgroups $H \subset G$ such that $\dim_{\mathbf{Q}} \pi_{\mathrm{odd}}(G/H)_{\mathbf{Q}} = 1$ ([14], Table 3, p. 185), that is, H kills all but one degree of G; these make up the first part of the list in Lemma 5.2. Next, he classified the simple subgroups $H \subset G$ such that $d(H) \geq d(G) - 2$ ([22], Table 7.2, p. 195); it turns out that these include all the subgroups on the second part the list in Lemma 5.2.

Lemma 5.2. *Let $H \to G$ be a nontrivial homomorphism of simply connected simple groups such that the maximal degree of H is at least the second-largest degree of G and is less than the maximal degree of G. Then G/H is isomorphic to one of the following homogeneous spaces:*

On the left is the Dynkin index of the homomorphism $H \to G$. On the right are shown the degrees of G not occurring in H, and the degrees of H not occurring in G, in both cases with multiplicities. In the last column is the centralizer of H in G, written modulo finite groups.

1	$\mathrm{SU}(n)/\mathrm{SU}(n-1)$ $= S^{2n-1}, n \geq 3$	n		S^1
1	$\mathrm{Sp}(2n)/\mathrm{Sp}(2n-2)$ $= S^{4n-1}, n \geq 2$	$2n$		A_1
1	$\mathrm{Spin}(2n+1)/\mathrm{Spin}(2n)$ $= S^{2n}, n \geq 3$	$2n$	n	1
1	$\mathrm{Spin}(2n+1)/\mathrm{Spin}(2n-1)$ $= \mathrm{UT}(S^{2n}), n \geq 3$	$2n$		S^1
2	$\mathrm{Sp}(4)/\mathrm{SU}(2) = \mathrm{UT}(S^4)$	4		S^1
10	$\mathrm{Sp}(4)/\mathrm{SU}(2)$	4		1
4	$\mathrm{SU}(3)/\mathrm{SO}(3)$	3		1
1	$\mathrm{Spin}(9)/\mathrm{Spin}(7) = S^{15}$	8		1
1	$G_2/\mathrm{SU}(3) = S^6$	6	3	1
1	$G_2/\mathrm{SU}(2) = \mathrm{UT}(S^6)$	6		A_1
3	$G_2/\mathrm{SU}(2)$	6		A_1
4	$G_2/\mathrm{SO}(3)$	6		1
28	$G_2/\mathrm{SO}(3)$	6		1
1	$F_4/\mathrm{Spin}(9) = \mathbf{CaP}^2$	12	4	1
1	$\mathrm{Spin}(2n)/\mathrm{Spin}(2n-2)$ $= \mathrm{UT}(S^{2n-1}), n \geq 4$	$n, 2n-2$	$n-1$	S^1
1	$\mathrm{Spin}(2n)/\mathrm{Spin}(2n-3), n \geq 4$	$n, 2n-2$		A_1
1	$\mathrm{SU}(2n+1)/\mathrm{Sp}(2n), n \geq 2$	$3, 5, \ldots, 2n+1$		S^1
2	$\mathrm{SU}(2n+1)/\mathrm{SO}(2n+1), n \geq 2$	$3, 5, \ldots, 2n+1$		1
1	$\mathrm{Spin}(10)/\mathrm{Spin}(7)$	$5, 8$		S^1
2	$\mathrm{SU}(7)/G_2$	$3, 4, 5, 7$		1
1	$\mathrm{Spin}(9)/G_2$	$4, 8$		S^1
1	$\mathrm{Spin}(10)/G_2$	$4, 5, 8$		A_1.

The proof is straightforward for G classical, using the known low-dimensional representations of each simple group, or alternatively the results by Onishchik mentioned above. For G exceptional, more than enough information is provided by Dynkin's paper on the subgroups of the exceptional groups [8]. For example, Dynkin's Table 16 classifies the A_1 subgroups of G_2. Notice that the listings for $\mathrm{Spin}(9)/\mathrm{Spin}(7) = S^{15}$ and $\mathrm{Spin}(10)/\mathrm{Spin}(7)$ refer to the spin representation of $\mathrm{Spin}(7)$;

these spaces are different from the spaces $\mathrm{Spin}(2n+1)/\mathrm{Spin}(2n-1) = \mathrm{UT}(S^{2n})$ for $n = 4$ and $\mathrm{Spin}(2n)/\mathrm{Spin}(2n-3)$ for $n = 5$. Also, there are three conjugacy classes of nontrivial homomorphisms $\mathrm{SU}(2) \to \mathrm{Sp}(4)$, where we write V for the standard representation of $\mathrm{SU}(2)$: $V \oplus \mathbf{C}^2$, where $\mathrm{Sp}(4)/\mathrm{SU}(2) = \mathrm{Sp}(4)/\mathrm{Sp}(2) = S^7$, which has Dynkin index 1; $V \oplus V$, where $\mathrm{Sp}(4)/\mathrm{SU}(2) = \mathrm{Spin}(5)/\mathrm{Spin}(3) = \mathrm{UT}(S^4)$, which has Dynkin index 2; and S^3V, which has Dynkin index 10.

We now apply Lemma 5.2 to the classification of biquotients. Together with Theorem 4.8, the following theorem will be our most important tool in the classification of biquotients.

Theorem 5.3. *Let $M = G/H$ be a simply connected biquotient, written according to Convention* 3.4. *Let G_1 be a simple factor of G which contributes only its top degree to M. Then at least one of the following holds:*

(1) *G_1 is isomorphic to* $\mathrm{SU}(2)$*, which has degree* 2.

(2) *G_1 is a rank-*2 *group* $\mathrm{SU}(3)$*,* $\mathrm{Sp}(4)$*, or G_2, with top degree* 3, 4, 6 *respectively, and there is a simple factor $H_1 \cong \mathrm{SU}(2)$ of H which acts nontrivially on G_1.*

(3) *There is a simple factor H_1 of H such that H_1 acts nontrivially on exactly one side of G_1 and G_1/H_1 is one of the homogeneous spaces in the first part of Lemma* 5.2*'s list.*

(4) *There are two simple factors H_1 and H_2 of H which act nontrivially on the two sides of G_1 in one of the following ways, up to switching H_1 and H_2: $G_1 = \mathrm{Spin}(2n)$, $n \geq 4$, H_1 is $\mathrm{Spin}(2n-2)$ or $\mathrm{Spin}(2n-3)$, H_2 is $\mathrm{SU}(n)$, or $\mathrm{Sp}(2a)$ with $n = 2a$, or $\mathrm{Spin}(9)$ with $n = 8$; here G_1 has top degree $2n-2$. Or $G_1 = \mathrm{SU}(2n+1)$, $n \geq 3$, H_1 is $\mathrm{Sp}(2n)$ or $\mathrm{SO}(2n+1)$, and $H_2 = \mathrm{SU}(2n-1)$; here G_1 has top degree $2n+1$. Or $H_1 \backslash G_1/H_2$ is one of $G_2 \backslash \mathrm{SU}(7)/\mathrm{SU}(5)$, $G_2 \backslash \mathrm{Spin}(9)/\mathrm{SU}(4)$, $G_2 \backslash \mathrm{Spin}(9)/\mathrm{Sp}(4)$, or $G_2 \backslash \mathrm{Spin}(10)/\mathrm{SU}(5)$; here G_1 has top degree* 7, 8, 8, 8, *respectively.*

Proof. If $G_1 = \mathrm{SU}(2)$, we have conclusion (1). So we can assume that G_1 has rank at least 2. Equivalently, G_1 has at least two degrees. There must be a simple factor H_1 of H which kills at least one second-largest degree of G_1. (By Table 4.2, G_1 has a unique second-largest degree except when G_1 is $\mathrm{Spin}(8)$, which has degrees $2, 4, 4, 6$.)

Suppose that H_1 is isomorphic to G_1. If H_1 acts nontrivially on only one side of G_1, then it acts by an isomorphism on that side of

G_1, and so H_1 acts transitively on G_1, contrary to Convention 3.4. So H_1 must act by an isomorphism on both sides of G_1. Since all automorphisms of G_1 act as the identity on $\pi_3 G_1 = \mathbf{Z}$, the resulting homomorphism $\pi_3(H_1)_{\mathbf{Q}} \to \pi_3(G_1)_{\mathbf{Q}}$ is zero. Also, since H_1 is acting with finite centralizer on both sides of G_1, the rest of H cannot act on G_1, and so the whole homomorphism $\pi_3 H_{\mathbf{Q}} \to \pi_3(G_1)_{\mathbf{Q}}$ is zero. That is, G_1 contributes its degree 2 to M. Since G_1 is not isomorphic to SU(2), this contradicts our assumption that G_1 contributes only its top degree to M.

Thus H_1 is not isomorphic to G_1. We know that $d(H_1) \leq d(G_1)$ by Lemma 4.3. Suppose that $d(H_1) = d(G_1)$. By Lemma 4.3, (G_1, H_1) is one of the pairs $(\mathrm{Spin}(2n), \mathrm{Spin}(2n-1))$, $(\mathrm{SU}(2n), \mathrm{Sp}(2n))$, $(\mathrm{Spin}(7), G_2)$, $(\mathrm{Spin}(8), G_2)$, or (E_6, F_4). In all these cases except $(\mathrm{Spin}(8), \mathrm{Spin}(7))$, there is a unique conjugacy class of nontrivial homomorphisms $H_1 \to G_1$; moreover, even in the case $(\mathrm{Spin}(8), \mathrm{Spin}(7))$, all nontrivial homomorphisms are equivalent under automorphisms of Spin(8) and so they all give the same homomorphism $\pi_3 H_1 \to \pi_3 G_1$. Also, the centralizer of H_1 in G_1 is finite in all cases. So, in all cases, if H_1 acts nontrivially on both sides of G_1, then no other factor of H acts on G_1, and the homomorphism $\pi_3 H_{\mathbf{Q}} \to \pi_3(G_1)_{\mathbf{Q}}$ is zero. That is, G_1 contributes its degree 2 to M, contrary to our assumption that G_1 contributes only its top degree to M. Therefore H_1 must act only on one side of G_1. But that implies, in all these cases, that H_1 kills the top degree of G_1. This contradicts our assumption that G_1 contributes its top degree to M.

So we must have $d(H_1) < d(G_1)$. Since H_1 kills at least one second-largest degree of G_1, $d(H_1)$ must be at least the second-largest degree of G_1. Therefore (G_1, H_1) must be one of the pairs listed in Lemma 5.2. If H_1 is isomorphic to SU(2), then G_1 has rank 2 and we have conclusion (2). We can now assume that H_1 is not isomorphic to SU(2).

Next, we will show that H_1 acts nontrivially on only one side of G_1 in all the remaining cases. Suppose that H_1 acts nontrivially on both sides of G_1. By Lemma 5.2, the centralizer of H_1 on each side of G_1 is at most finite by A_1. So any simple factor of H other than H_1 which acts nontrivially on G_1 is isomorphic to SU(2). Thus H_1 by itself must kill all the degrees of G_1 greater than 2 and less than the maximal degree $d(G_1)$. This is clearly impossible for the pairs (G_1, H_1) on the second part of Lemma 5.2's list, since in these cases G_1 contains at least one degree in the interval $(2, d(G_1))$ with greater multiplicity than H_1 does.

So (G_1, H_1) is on the first part of Lemma 5.2's list. Since H_1 is not isomorphic to SU(2), the list shows that all nontrivial homomorphisms

$H_1 \to G_1$ have Dynkin index 1. Since H_1 acts nontrivially on both sides of G_1, it follows that the associated homomorphism $\pi_3(H_1)_{\mathbf{Q}} \to \pi_3(G_1)_{\mathbf{Q}}$ is zero. That is, H_1 does not kill the degree 2 of G_1. Since the whole group H does kill the degree 2 of G_1, at least one of the two nontrivial homomorphisms $H_1 \to G_1$ must have centralizer containing an A_1 subgroup. By the first part of Lemma 5.2's list, it follows that the pair (G_1, H_1) is $(\mathrm{Sp}(2n), \mathrm{Sp}(2n-2))$ for some $n \geq 3$.

But for (G_1, H_1) equal to $(\mathrm{Sp}(2n), \mathrm{Sp}(2n-2))$ with $n \geq 3$, there is a unique conjugacy class of nontrivial homomorphisms $H_1 \to G_1$. Since H_1 acts nontrivially on both sides of G_1, it follows that the associated homomorphism $\pi_*(H_1)_{\mathbf{Q}} \to \pi_*(G_1)_{\mathbf{Q}}$ is zero. In particular, H_1 does not kill the second-largest degree, $2n-2$, of G_1. This is a contradiction. We have thus completed the proof that H_1 acts nontrivially on only one side of G_1.

For spaces G_1/H_1 on the first part of Lemma 5.2's list, this completes the proof of conclusion (3). It remains to consider spaces G_1/H_1 on the second part of Lemma 5.2's list. To prove conclusion (4), we need to identify another simple factor of H which acts on G_1.

In several cases on the second part of Lemma 5.2's list, G_1 is isomorphic to $\mathrm{Spin}(2n)$, $n \geq 4$. Here H_1 is either $\mathrm{Spin}(2n-2)$, $\mathrm{Spin}(2n-3)$, or, for $n = 5$, $\mathrm{Spin}(7)$ (with a different homomorphism to $\mathrm{Spin}(10)$) or G_2. In all these cases, the degree n of G_1 is not killed by H_1, meaning that $\pi_{2n-1}(H_1)_{\mathbf{Q}} \to \pi_{2n-1}(G_1)_{\mathbf{Q}}$ is not surjective. So there must be another simple factor H_2 of H which acts on G_1 such that $\pi_{2n-1}(H_2)_{\mathbf{Q}} \to \pi_{2n-1}(G_1)_{\mathbf{Q}}/\pi_{2n-1}(H_1)_{\mathbf{Q}}$ is not zero. If n is odd, this just means that $\pi_{2n-1}(H_2)_{\mathbf{Q}} \to \pi_{2n-1}(G_1)_{\mathbf{Q}}$ is not zero. On the other hand, if n is even, then we can always assume, after automorphisms of H_1 and G_1, that $H_1 \to G_1$ is the standard inclusion. So, for n odd or even, we can say that

$$\pi_{2n-1}(H_2)_{\mathbf{Q}} \to \pi_{2n-1}\mathrm{Spin}(2n)_{\mathbf{Q}}/\pi_{2n-1}\mathrm{Spin}(2n-1)_{\mathbf{Q}} \cong \mathbf{Q}$$

is not zero. Here H_2 cannot be $\mathrm{SU}(2)$ since $n \geq 4$, so H_2 must act only on the other side of G_1 from H_1, since the centralizer of H_1 in G_1 is at most finite by A_1. By Lemma 5.1, the space G_1/H_2 must be one of $\mathrm{Spin}(2n)/\mathrm{SU}(n)$, $\mathrm{Spin}(4a)/\mathrm{Sp}(2a)$, $\mathrm{Spin}(8)/\mathrm{Spin}(7)$, or $\mathrm{Spin}(16)/\mathrm{Spin}(9)$. If $G_1/H_2 = \mathrm{Spin}(8)/\mathrm{Spin}(7)$, then we can replace H_1 by H_2 and we have conclusion (3); so we can exclude that case. We have thus proved conclusion (4) for $G_1 = \mathrm{Spin}(2n)$.

Next, there are several cases in the second part of Lemma 5.2's list where $G_1 = \mathrm{SU}(2n+1)$, $n \geq 2$. Here H_1 is either $\mathrm{Sp}(2n)$, $\mathrm{SO}(2n+1)$, or, for $n = 3$, G_2. In all these cases, H_1 has only even degrees, and in particular it does not kill the degree $2n - 1$ of G_1. So there must be another simple factor H_2 of H which kills the degree $2n-1$ of G_1. Here H_2 must act nontrivially only on the other side of G_1 from H_1, because the centralizer of H_1 in G_1 is at most S^1 by finite in these cases. It is easy to read from Table 4.2 that any simple group H_2 which maps nontrivially to $G_1 = \mathrm{SU}(2n + 1)$, $n \geq 2$, and has $2n - 1$ as a degree is either $\mathrm{SU}(2n - 1)$, $\mathrm{SU}(2n)$, or $\mathrm{SU}(2n + 1)$. Here $H_2 = \mathrm{SU}(2n + 1)$ is excluded by Convention 3.4, which says that H does not act transitively on G_1. If $H_2 = \mathrm{SU}(2n)$, then we can replace H_1 by H_2 and we have conclusion (3). The remaining possibility is $H_2 = \mathrm{SU}(2n-1)$. We have proved conclusion (4) for $G_1 = \mathrm{SU}(2n + 1)$.

The last case, from the second part of Lemma 5.2's list, is where G_1 is Spin(9) and H_1 is the exceptional group G_2. Here the degree 4 of G_1 is not killed by H_1, so it must be killed by some other simple factor H_2 of H. The centralizer of H_1 in G_1 is finite by S^1, so H_2 must act nontrivially only on the other side of G_1 from H_1. From Table 4.2, since H_2 has a degree 4 and maps nontrivially to Spin(9), H_2 is one of Sp(4), SU(4), Spin(7), Spin(8), or Spin(9). The case $H_2 = \mathrm{Spin}(9)$ is excluded by Convention 3.4, which says that H does not act transitively on G_1. If H_2 is Spin(7) or Spin(8), then we can replace H_1 by H_2 and we have conclusion (3). The remaining possibilities for H_2 are Sp(4) and SU(4). We have proved conclusion (4) for $G_1 = \mathrm{Spin}(9)$. q.e.d.

6. Biquotients which are rational homology spheres

As an application of Theorems 4.8 and 5.3, we now classify all biquotients which are simply connected rational homology spheres. The result seems surprising. In particular, the Gromoll-Meyer exotic 7-sphere which is a biquotient [15] is the only such example in any dimension. As mentioned in the introduction, Theorem 6.1 was found at the same time by Kapovitch and Ziller [17].

Theorem 6.1. *Any biquotient which is a simply connected rational homology sphere is either a homogeneous manifold, the Gromoll-Meyer exotic* 7*-sphere which is a biquotient* $\mathrm{Sp}(4)/\mathrm{SU}(2)$*, or a certain* 4*-connected* 11*-manifold with the integral homology groups of* $\mathrm{UT}(S^6)$ *which is a biquotient* $G_2/\mathrm{SU}(2)$*.*

The homogeneous manifolds which are simply connected rational homology spheres are the sphere S^n*, the unit tangent bundle* $\mathrm{UT}(S^{2n})$*, the*

Wu 5-manifold SU(3)/SO(3) [7], *the Berger 7-manifold* Sp(4)/SU(2) *with* $\pi_3 M$ *isomorphic to* $\mathbf{Z}/10$ [3], *the 11-manifold* G_2/SU(2) *with* $\pi_3 M$ *isomorphic to* $\mathbf{Z}/3$, *and the two 11-manifolds* G_2/SO(3) *with* $\pi_3 M$ *isomorphic to* $\mathbf{Z}/4$ *or* $\mathbf{Z}/28$.

The homogeneous spaces in Theorem 6.1 were classified by Onishchik ([14], Table 3, p. 185). The nontrivial biquotient $G_2/\mathrm{SU}(2)$ was first discovered by Eschenburg ([10], pp. 166-170). Kapovitch and Ziller have shown that this biquotient $G_2/\mathrm{SU}(2)$ is diffeomorphic to the connected sum of $\mathrm{UT}(S^6)$ with some homotopy 11-sphere [17]. It is not known whether $G_2/\mathrm{SU}(2)$ is actually diffeomorphic to $\mathrm{UT}(S^6)$; one hopes for a negative answer, which would be more interesting.

Before starting the proof of Theorem 6.1, we assemble some elementary facts about biquotients $\mathrm{Sp}(4)/H$. In the following lemma, we consider free actions of a group H on Sp(4) given by a homomorphism $H \to \mathrm{Sp}(4)^2/Z(\mathrm{Sp}(4))$.

Lemma 6.2.

(1) *There is no free action of* SO(3) *on* Sp(4). *Any free action of* SU(2) *on* Sp(4) *is either trivial on one side of* Sp(4), *so that* Sp(4)/SU(2) *is* S^7, $\mathrm{UT}(S^4)$, *or the Berger* 7*-manifold, or given by the homomorphisms* $(V \oplus \mathbf{C}^2, V^{\oplus 2})$ *up to switching the two sides of* Sp(4), *so that* Sp(4)/SU(2) *is the Gromoll-Meyer exotic* 7*-sphere* [15]. *Here* V *denotes the standard representation of* SU(2).

(2) *Any free action of* $\mathrm{SU}(2)^2$ *on* Sp(4) *is given, up to switching the two* SU(2) *factors and switching the two sides of* Sp(4), *by the homomorphisms* $(V_1 \oplus V_2, \mathbf{C}^4)$ *or* $(V_1 \oplus \mathbf{C}^2, V_2^{\oplus 2})$. *Here* V_1 *and* V_2 *are the standard representations of the two* SU(2) *factors. In both cases, the quotient* $\mathrm{Sp}(4)/\mathrm{SU}(2)^2$ *is diffeomorphic to* S^4.

Proof. We only prove (1) here, the calculation for (2) being similar. There are three conjugacy classes of nontrivial homomorphisms $\mathrm{SU}(2) \to \mathrm{Sp}(4)$, $V \oplus \mathbf{C}^2$, $V^{\oplus 2}$, and S^3V, by Lemma 5.2. If SU(2) acts trivially on one side of Sp(4), then Sp(4)/SU(2) is one of the three homogeneous spaces mentioned in (1). So we can assume that SU(2) acts nontrivially on both sides of Sp(4). Let H be SU(2) or $\mathrm{SU}(2)/\{\pm 1\} \cong \mathrm{SO}(3)$. The group H acts freely on Sp(4) if and only if the two images of each nontrivial element of H in Sp(4) are not conjugate. In particular, the two homomorphisms $\mathrm{SU}(2) \to \mathrm{Sp}(4)$ must be non-conjugate. If one homomorphism is $V \oplus \mathbf{C}^2$ and the other is

$V^{\oplus 2}$, then Gromoll and Meyer showed that SU(2) acts freely on Sp(4) and that the quotient manifold is an exotic 7-sphere [15]. Otherwise, SU(2) must act by S^3V on one side and either $V \oplus \mathbf{C}^2$ or $V^{\oplus 2}$ on the other. Then neither SU(2) nor SU(2)/$\{\pm 1\}$ acts freely on Sp(4), since the images of the diagonal matrix (ζ_3, ζ_3^{-1}) in SU(2) under S^3V and $V \oplus \mathbf{C}^2$ are conjugate, and likewise the images of the diagonal matrix (ζ_4, ζ_4^{-1}) in SU(2) under S^3V and $V^{\oplus 2}$ are conjugate. (Here ζ_n denotes a primitive nth root of unity.) This proves (1). q.e.d.

Proof of Theorem 6.1. We write $M = G/H$ according to Convention 3.4. Since M is a rational homology sphere, the odd-dimensional rational homotopy of M has dimension 1. In more detail, S^{2n-1} has 1-dimensional rational homotopy in dimension $2n-1$ and zero otherwise, while S^{2n} has 1-dimensional rational homotopy in dimensions $2n$ and $4n-1$, and zero otherwise. Since each simple factor of G contributes at least dimension 1 to $\pi_{\mathrm{odd}} M_{\mathbf{Q}}$ by Theorem 4.8, G must be simple. This already makes the situation much more understandable; it would not be clear at all without the analysis leading to Theorem 4.8.

If the simply connected rational homology sphere M has dimension r at most 4, then it is automatically a homotopy sphere. It is therefore not surprising to find that M is diffeomorphic to S^r for $r \leq 4$. First, for $r = 2$ we can just use that every homotopy 2-sphere is diffeomorphic to S^2. For $r \geq 3$, we have $\pi_2 M_{\mathbf{Q}} = 0$, and so H has finite fundamental group; equivalently, H is semisimple. For $r = 3$, $\pi_3 M \cong \mathbf{Z}$ and $\pi_4 M_{\mathbf{Q}} = 0$, which implies that G has one more simple factor than H does. Since G is simple, $H = 1$. Since M is a homotopy 3-sphere, G has degree 2 only, and so G is SU(2). Thus M is diffeomorphic to SU(2), that is, to S^3.

For $r = 4$, $\pi_2 M = \pi_3 M = 0$ and $\pi_4 M \cong \mathbf{Z}$, which implies that H is simply connected and has one more simple factor than G does. Since G is simple, H has two simple factors. By the rational homotopy groups of M, G must contribute degree 4 to M while H contributes degree 2, and nothing else. By Theorems 4.8 and 5.3, there is a simple factor H_1 of H such that either G/H_1 is a homogeneous space diffeomorphic to S^7, or G/H_1 is $\mathrm{Spin}(8)/G_2 = S^7 \times S^7$, or (G, H_1) is $(\mathrm{Sp}(4), \mathrm{SU}(2))$. Let H_2 denote the other simple factor of H. If $G/H_1 = S^7$, then H_2 has degree 2 only, so H_2 is isomorphic to SU(2). In this case M is the quotient of S^7 by SU(2) acting freely by a homomorphism $\mathrm{SU}(2) \to \mathrm{O}(8)$. Such a homomorphism is unique up to conjugacy, and so M is the standard quotient S^4. Next, if G/H_1 is $\mathrm{Spin}(8)/G_2 = S^7 \times S^7$, then H_2 has degrees

2 and 4, and so H_2 is isomorphic to Sp(4). But there is in fact no free isometric action of Sp(4) on $S^7 \times S^7$, because the restriction of any action of Sp(4) on S^7 to the subgroup SU(2) = Sp(2) $\subset$ Sp(4) has a fixed point. (Either the associated 8-dimensional complex representation of Sp(4) has a trivial summand, or it is the sum of two copies of the standard representation of Sp(4) and hence restricts on SU(2) to $V^{\oplus 2} \oplus \mathbf{C}^4$.)

For $r = 4$, it remains to consider the case $(G, H_1) = (\mathrm{Sp}(4), \mathrm{SU}(2))$. Here H_2 has degree 2 only, and so H_2 is isomorphic to SU(2). Thus M is a biquotient $\mathrm{Sp}(4)/\mathrm{SU}(2)^2$. By Lemma 6.2, M is diffeomorphic to S^4.

We proceed to the case $r \geq 5$. As mentioned earlier, we have $\pi_2 M_{\mathbf{Q}} = 0$, and so H is semisimple. Since $\pi_3 M_{\mathbf{Q}} = \pi_4 M_{\mathbf{Q}} = 0$, H has the same number of simple factors as G. Since G is simple, so is H.

One of the cases (1) to (4) in Theorem 4.8 must hold. Here (4) is excluded since G is simple and H acts freely on G. In case (3), M is the homogeneous space $\mathrm{Spin}(2n)/\mathrm{Spin}(2n-1) = S^{2n-1}$. In case (2), M is one of the homogeneous spaces $\mathrm{SU}(2a)/\mathrm{Sp}(2a)$ with $a \geq 2$, $\mathrm{Spin}(7)/G_2 = S^7$, $\mathrm{Spin}(8)/G_2 = S^7 \times S^7$, or E_6/F_4. Since M is a rational homology sphere, considering the degrees of these homogeneous spaces shows that M is either $\mathrm{SU}(4)/\mathrm{Sp}(4) = \mathrm{Spin}(6)/\mathrm{Spin}(5) = S^5$ or $\mathrm{Spin}(7)/G_2 = S^7$.

There remains case (1) of Theorem 4.8, where G contributes its top degree to M. In this case, one of conclusions (1) to (4) in Theorem 5.3 must hold. Here case (4) of Theorem 5.3 is excluded since H is simple. In case (3), M is one of the homogeneous spaces listed in the first part of Lemma 5.2. Since M is a rational homology sphere, the possibilities are as listed in Theorem 6.1.

Case (1), $G = \mathrm{SU}(2)$, of Theorem 5.3 is excluded because we are considering biquotients M of dimension $r \geq 5$. So there remains only case (2). That is, H_1 is SU(2), H is either $H_1 = \mathrm{SU}(2)$ or $H_1/\{\pm 1\} \cong \mathrm{SO}(3)$, and M is a biquotient $\mathrm{SU}(3)/H$, $\mathrm{Sp}(4)/H$, or G_2/H, of dimension 5, 7, or 11, respectively. The homogeneous spaces of this type are listed in Lemma 5.2. So we can assume that H_1 acts nontrivially on both sides of G. Biquotients of this type (a rank-2 group divided by a rank-1 group) were classified by Eschenburg ([10], pp. 166–170), but I will give my own proof since Eschenburg's paper is not widely available.

First, suppose $G = \mathrm{SU}(3)$. Then there are two conjugacy classes of nontrivial homomorphisms $\mathrm{SU}(2) \to \mathrm{SU}(3)$, $V \oplus \mathbf{C}$ and S^2V, where V denotes the standard representation of SU(2). Since H (which is SU(2) or SO(3)) acts freely on G, the two images in SU(3) of any nontrivial element of H are not conjugate in SU(3). So SU(2) must act by $V \oplus \mathbf{C}$

on one side of SU(3) and by S^2V on the other. But, if we write ζ_a for a primitive ath root of unity, the diagonal matrix (ζ_3, ζ_3^{-1}) in SU(2) has images under both homomorphisms to SU(3) which are conjugate to $(\zeta_3, 1, \zeta_3^{-1})$. So neither SU(2) nor SU(2)$/\{\pm 1\}$ can be acting freely. Thus, the case $G = \mathrm{SU}(3)$ does not occur.

The case $G = \mathrm{Sp}(4)$ is covered by Lemma 6.2. Since SU(2) acts nontrivially on both sides of Sp(4), M is the Gromoll-Meyer exotic 7-sphere.

The last case to consider is where G is the exceptional group G_2. Here, as described in Lemma 5.2, there are 4 conjugacy classes of non-trivial homomorphisms $\mathrm{SU}(2) \to G_2$, which we identify by their Dynkin index, 1, 3, 4, or 28. Using these Dynkin indices, we compute that the composed representation $\mathrm{SU}(2) \to G_2 \to \mathrm{U}(7)$ must be: $W_1 = V^{\oplus 2} \oplus \mathbf{C}^3$, $W_3 = S^2V \oplus V^{\oplus 2}$, $W_4 = (S^2V)^{\oplus 2} \oplus \mathbf{C}$, and $W_{28} = S^6V$. In particular, the homomorphisms W_4 and W_{28} from SU(2) to G_2 are trivial on $\{\pm 1\}$ in SU(2), and the other two are not.

We are assuming that SU(2) acts nontrivially on both sides of G_2. Since the action is free for either SU(2) or SU(2)$/\{\pm 1\}$, the two homomorphisms $\mathrm{SU}(2) \to G_2$ must be non-conjugate. There are now 6 cases to consider, corresponding to the 6 unordered pairs of distinct homomorphisms W_i. Notice that the action of SU(2) on G_2 is trivial on $\{\pm 1\}$ if and only if the two homomorphisms send -1 to the same element of the center of G_2. The center of G_2 is trivial, so the two homomorphisms must both be trivial on -1. That is, SU(2) acts on G_2 through its quotient SO(3) if and only if the two homomorphisms are (W_4, W_{28}).

Given that SU(2) acts on G_2 by W_i on one side and W_j on the other, the action of SU(2) is free if and only if every nontrivial element of SU(2) has non-conjugate images in G_2 under the two homomorphisms. Every element of SU(2) is conjugate to a diagonal element (x, x^{-1}) with $x \in S^1$, so it suffices to consider those elements. Furthermore, it is convenient to observe that two elements of G_2 are conjugate if and only if their images in U(7) are conjugate.

Using this, we can check whether each pair (W_i, W_j) of homomorphisms $\mathrm{SU}(2) \to G_2$ gives a free action of SU(2) on G_2 or not. Here the image of $x \in S^1 \subset \mathrm{SU}(2)$ in U(7) under the homomorphism W_i is conjugate to the diagonal matrix:

$$\begin{aligned} W_1 &: (x, x^{-1}, x, x^{-1}, 1, 1, 1) \\ W_3 &: (x, x^{-1}, x, x^{-1}, x^2, 1, x^{-2}) \\ W_4 &: (x^2, x^{-2}, x^2, x^{-2}, 1, 1, 1) \\ W_{28} &: (x^{-6}, x^{-4}, x^{-2}, 1, x^2, x^4, x^6). \end{aligned}$$

The result is that 5 of the 6 unordered pairs (W_i, W_j) do not give free actions of SU(2) (or SO(3), in the case (W_4, W_{28})) on G_2. Indeed, the following nontrivial elements $x \in S^1$ have conjugate images (in U(7), hence in G_2) under the two representations.

$$\begin{aligned} (W_1, W_3) &: x = -1 \\ (W_1, W_4) &: x = \zeta_3 \\ (W_1, W_{28}) &: x = \zeta_3 \\ (W_3, W_{28}) &: x = \zeta_5 \\ (W_4, W_{28}) &: x = \zeta_3 \end{aligned}$$

If SU(2) acts on G_2 by (W_3, W_4), however, then we compute from the above formulas that the two images of any nontrivial element $x \in S^1 \subset \mathrm{SU}(2)$ are not conjugate in U(7), and hence not conjugate in G_2. So this is a free action of SU(2) on G_2. Since the Dynkin indices 3 and 4 differ by 1, the resulting biquotient $M = G_2/\mathrm{SU}(2)$ has $\pi_3 M = 0$. Using the known homology of G_2, we compute that the 11-manifold M is 4-connected and has the integral homology groups of $\mathrm{UT}(S^6)$.

Thus, the only biquotient G_2/H with H isomorphic to SU(2) or SO(3) and H acting nontrivially on both sides is the 4-connected 11-manifold $G_2/\mathrm{SU}(2)$, with SU(2) acting by (W_3, W_4). q.e.d.

7. Three Cheeger manifolds which are not homotopy equivalent to biquotients

We now return to the proof of Theorem 2.1. Roughly in order of increasing difficulty, we show in this section that the 16-manifolds $\mathbf{CaP}^2\#\mathbf{CaP}^2$, $\mathbf{CP}^8\#\mathbf{CaP}^2$, and $\mathbf{HP}^4\#\mathbf{CaP}^2$ are not homotopy equivalent to biquotients.

Suppose that the 16-manifold $\mathbf{CaP}^2\#\mathbf{CaP}^2$ is homotopy equivalent to a biquotient $M = G/H$. As throughout the paper, we assume that

M is written as a biquotient G/H which satisfies Convention 3.4. Here $H^*(M, \mathbf{Z}) \cong \mathbf{Z}[x, y]/(xy = 0, x^2 = y^2)$, $|x| = |y| = 8$. This is a complete intersection ring with 2 generators in degree 8 and 2 relations in degree 16. It follows that the rational homotopy groups of M are isomorphic to $\mathbf{Q}^2$ in dimensions 8 and 15, and otherwise 0 (the same as for $S^8 \times S^8$). So the map $\pi_{2d-1}H_{\mathbf{Q}} \to \pi_{2d-1}G_{\mathbf{Q}}$ must have 2-dimensional cokernel for $d = 8$, 2-dimensional kernel for $d = 4$, and otherwise is an isomorphism. Equivalently, we say that G contributes two degrees 8 to M, while H contributes two degrees 4, and nothing else.

Each simple factor of G contributes at least one degree to M, by Theorem 4.8, so G has at most two simple factors. Suppose first that G is simple. We know that G contributes two degrees 8 to M, and nothing else. One of the four cases in Theorem 4.8 must hold. Here (1) cannot hold: if G contributes its maximal degree to M, that would have to be 8, but the maximal degree of each simple Lie group occurs only with multiplicity 1. The remaining cases of the theorem are incompatible with the fact that G contributes only degree 8 to M. Thus we have a contradiction from the assumption that G is simple.

So G has two simple factors, $G = G_1 \times G_2$. (Here G_2 does not denote the exceptional group G_2.) Each factor must contribute one degree 8 to M, and nothing else. We can apply Theorem 4.8 to each factor G_i. From the degrees, it is clear that only cases (1) and (3) can occur. That is, for $1 \le i \le 2$, either G_i contributes its maximal degree to M, which must be 8, or else $G_i = \mathrm{Spin}(16)$ and there is a simple factor $H_i \cong \mathrm{Spin}(15)$ of H which acts nontrivially on exactly one side of G_i, with $G_i/H_i = S^{15}$. If G_i contributes its maximal degree to M, we can apply Theorem 5.3 to G_i. Cases (1) and (2) cannot arise, since the maximal degree of G_i is 8. Therefore, by cases (3) and (4), there is a simple factor H_i of H which acts nontrivially on exactly one side of G_i such that G_i/H_i is one of the homogeneous spaces $\mathrm{Spin}(9)/\mathrm{Spin}(8) = S^8$, $\mathrm{SU}(8)/\mathrm{SU}(7) = \mathrm{Sp}(8)/\mathrm{Sp}(6) = \mathrm{Spin}(9)/\mathrm{Spin}(7) = S^{15}$, $\mathrm{Spin}(9)/\mathrm{Spin}(7) = \mathrm{UT}(S^8)$, $\mathrm{Spin}(10)/\mathrm{Spin}(8)$, $\mathrm{Spin}(10)/\mathrm{Spin}(7)$, $\mathrm{Spin}(9)/G_2$, or $\mathrm{Spin}(10)/G_2$.

Since neither G nor H contributes any degree 2 to M, and each simple group has exactly one degree 2 (that is, every simple group has $\pi_3 = \mathbf{Z}$), H must have the same number of simple factors as G. Thus H has two simple factors.

Let G_1 and G_2 denote the two simple factors of G. We know that there is a simple factor H_1 of H such that H_1 acts nontrivially on exactly one side of G_1, with G_1/H_1 equal to either $\mathrm{Spin}(16)/\mathrm{Spin}(15) = S^{15}$ or one of the other homogeneous spaces listed above. We also know the analogous statement for G_2, but a priori it is possible that the same simple factor H_1 of H plays the same role for both G_1 and G_2. But then $G = G_1 \times G_2$ has two degrees 14 (if G_1 and G_2 are Spin(16)) or 7 (if G_1 and G_2 are SU(8)) or 6 (if G_1 and G_2 are Spin(9) or Spin(10)), both of which must be killed by H, whereas the group H_1 has only one degree 14 or 6 or 7. So we can order the two simple factors of H in such a way that H_i kills the relevant degree of G_i, both for $i = 1$ and for $i = 2$. By the proof of Theorem 5.3, it follows that H_i acts nontrivially on exactly one side of G_i, with G_i/H_i equal to one of the above homogeneous spaces, for $i = 1$ and for $i = 2$.

Since M has dimension only 16, both G_1/H_1 and G_2/H_2 must be the homogeneous space Spin(9)/Spin(8) $= S^8$. If H_1 acts trivially on G_2, or also if H_2 acts trivially on G_1, then M is an S^8-bundle over S^8 and hence has signature zero, contradicting the fact that M is homotopy equivalent to $\mathbf{CaP}^2\#\mathbf{CaP}^2$. So H_1 acts nontrivially on G_2 and H_2 acts nontrivially on G_1. Since H_i has finite centralizer in G_i for $i = 1, 2$, the action of H_1 on G_2 must be given by a nontrivial homomorphism $H_1 \to G_2$ on the other side of G_2 from H_2, and likewise for the action of H_2 on G_1. Any nontrivial homomorphism Spin(8) $\to$ Spin(9) has Dynkin index 1. So the homomorphism $\pi_3 H \to \pi_3 G$, $\mathbf{Z}^2 \to \mathbf{Z}^2$, is given by a 2×2 matrix with both rows equal to $(1, -1)$ or $(-1, 1)$. Such a matrix has zero determinant, and so $\pi_3 M_{\mathbf{Q}}$ is not zero. Again, this contradicts the fact that M is homotopy equivalent to $\mathbf{CaP}^2\#\mathbf{CaP}^2$. This completes the proof that $\mathbf{CaP}^2\#\mathbf{CaP}^2$ is not homotopy equivalent to a biquotient. In fact, the proof shows that $\mathbf{CaP}^2\#\mathbf{CaP}^2$ is not even rationally homotopy equivalent to a biquotient.

We now prove that $\mathbf{CP}^8\#\mathbf{CaP}^2$ is not homotopy equivalent to a biquotient $M = G/K$. Since G is simply connected, the boundary map $\mathbf{Z} \cong \pi_2 M \to \pi_1 K$ in the long exact sequence is an isomorphism.

So, if we let H be the commutator subgroup of K, then H is simply connected and K/H is isomorphic to S^1. Let N be the 17-manifold G/H. Since N is the natural S^1-bundle over $M \simeq \mathbf{CP}^8\#\mathbf{CaP}^2$, we compute that N has the integral cohomology ring of $S^8 \times S^9$. To go further, we use Wu's theorem that the Stiefel-Whitney classes of the tangent bundle of M are invariants of the homotopy type of M ([19], Theorem 11.14). Because the 8-sphere in the Cayley plane has self-intersection 1, the Stiefel-Whitney class $w_8(\mathbf{CaP}^2) \in H^8(\mathbf{CaP}^2, \mathbf{F}_2)$ is not zero. Therefore $w_8(\mathbf{CP}^8\#\mathbf{CaP}^2)$ is not in the subgroup $(H^2)^4$ of H^8. By Wu's theorem, the same holds for the manifold M which is homotopy equivalent to $\mathbf{CP}^8\#\mathbf{CaP}^2$. Since N is an S^1-bundle over M, it follows that $w_8(N)$ is not zero. We will use this later.

The cohomology ring of M is a complete intersection ring. The degrees of the generators and relations determine the rational homotopy groups of M, and hence of N. The result is that G contributes degrees 5 and 8 to N, while H contributes 4, and nothing else. Since every simple factor of G contributes at least one degree to N, G has at most 2 simple factors. Also, since neither G nor H contributes degree 2, G and H have the same number of simple factors.

Suppose first that G is simple. It follows that H is simple, too. We can apply Theorem 4.8, and cases (2), (3), (4) are excluded since G contributes degrees 5 and 8 to N and no more. Therefore case (1) must hold; that is, G contributes its maximal degree to N. So G has maximal degree 8. There is no simple group whose highest two degrees are 5 and 8, so the second-largest degree of G must be killed by H. Since G has exactly one degree 8, H must have maximal degree less than 8. So, regardless of how H acts on G, (G, H) must be one of the pairs in Lemma 5.2. From the list there, since we know that G adds degrees 5 and 8 to M and H adds degree 4 and no more, we must have $G = \mathrm{Spin}(10)$ and $H = \mathrm{Spin}(8)$. All nontrivial homomorphisms $H \to G$ have Dynkin index 1. So if H acts nontrivially on both sides of G, then G would contribute degree 2 to M, a contradiction. So H acts on only one side of G and G/H is isomorphic to the homogeneous space $\mathrm{Spin}(10)/\mathrm{Spin}(8) = \mathrm{UT}(S^9)$.

Indeed, there is a free S^1-action on $\mathrm{UT}(S^9)$, with quotient the 8-dimensional complex quadric $Q^8_{\mathbf{C}}$. This quadric has the rational homotopy type of $\mathbf{CP}^8 \# \mathbf{CaP}^2$. But no S^1-quotient of $N = \mathrm{UT}(S^9)$ can have the homotopy type of $\mathbf{CP}^8 \# \mathbf{CaP}^2$. The point is that N is an S^8-bundle over S^9. It follows that there is a sphere S^8 in N which represents a generator of $H_8(N, \mathbf{F}_2)$ and which has trivial normal bundle. Therefore $w_8(N)|S^8 = 0$, and hence $w_8(N) = 0$, contradicting our earlier calculation that the Stiefel-Whitney class $w_8(N)$ is not zero.

So G must have two simple factors. Since G and H have the same number of simple factors, H also has two simple factors. Each simple factor of G must contribute exactly one degree to N; we can assume that G_1 contributes degree 8 and G_2 contributes degree 5. By Theorems 4.8 and 5.3, there is a simple factor H_1 acting on exactly one side of G_1 such that G_1/H_1 is one of the homogeneous spaces $\mathrm{Spin}(9)/\mathrm{Spin}(8) = S^8$, $\mathrm{Spin}(16)/\mathrm{Spin}(15) = \mathrm{SU}(8)/\mathrm{SU}(7) = \mathrm{Sp}(8)/\mathrm{Sp}(6) = \mathrm{Spin}(9)/\mathrm{Spin}(7) = S^{15}$, $\mathrm{Spin}(9)/\mathrm{Spin}(7) = \mathrm{UT}(S^8)$, $\mathrm{Spin}(10)/\mathrm{Spin}(8)$, $\mathrm{Spin}(10)/\mathrm{Spin}(7)$, $\mathrm{Spin}(9)/G_2$, or $\mathrm{Spin}(10)/G_2$. Also, by the same theorems, there is a simple factor H_2 acting on exactly one side of G_2 such that G_2/H_2 is one of the homogeneous spaces $\mathrm{Spin}(10)/\mathrm{Spin}(9) = \mathrm{SU}(5)/\mathrm{SU}(4) = S^9$ or $\mathrm{SU}(6)/\mathrm{Sp}(6)$.

If H_1 and H_2 are the same factor of H, this factor must be isomorphic to $\mathrm{Sp}(6)$, and N is a biquotient of the form $(\mathrm{Sp}(8)\times\mathrm{SU}(6))/(\mathrm{Sp}(6)\times X)$. Here X must be a simple group with degrees $2,3,4,4,6$, but there is no such group. So H_1 and H_2 are the two different simple factors of H.

From the degrees of N (or just by its dimension, which is only 17), the homogeneous space G_1/H_1 must be $\mathrm{Spin}(9)/\mathrm{Spin}(8) = S^8$ and the homogeneous space G_2/H_2 must be either $\mathrm{Spin}(10)/\mathrm{Spin}(9) = S^9$ or $\mathrm{SU}(5)/\mathrm{SU}(4) = S^9$. The 17-manifold N is not an S^8-bundle over S^9, because we know that $w_8(N)$ is not zero. So $H_1 = \mathrm{Spin}(8)$ must act nontrivially on the second factor G_2 of G. It follows that G_2/H_2 is $\mathrm{Spin}(10)/\mathrm{Spin}(9) = S^9$, not $\mathrm{SU}(5)/\mathrm{SU}(4) = S^9$. Furthermore, $H_2 = \mathrm{Spin}(9)$ must act trivially on $G_1 = \mathrm{Spin}(9)$, by Lemma 4.1. Thus N is a biquotient of the form $(\mathrm{Spin}(9)\times\mathrm{Spin}(10))/(\mathrm{Spin}(8)\times\mathrm{Spin}(9))$ with $\mathrm{Spin}(9)$ acting trivially on $\mathrm{Spin}(9)$ and $\mathrm{Spin}(8)$ acting nontrivially on $\mathrm{Spin}(10)$. That is, N is a nontrivial S^9-bundle over S^8.

There are three conjugacy classes of nontrivial homomorphisms from $\mathrm{Spin}(8)$ to $\mathrm{Spin}(10)$. They have the form $W \oplus \mathbf{R}^2$, where W is either the standard 8-dimensional real representation V of $\mathrm{Spin}(8)$ or one of the two spin representations S^- and S^+. Thus N is the S^9-bundle $S(W \oplus \mathbf{R}^2)$ over S^8, where W is the real vector bundle over $\mathrm{Spin}(9)/\mathrm{Spin}(8) = S^8$ corresponding to the representation W of $\mathrm{Spin}(8)$. If W is the standard representation V of $\mathrm{Spin}(8)$, then the corresponding vector bundle on S^8 is the tangent bundle, which has zero Stiefel-Whitney classes. It follows that the sphere bundle $S(V \oplus \mathbf{R}^2)$ has zero Stiefel-Whitney classes, contrary to what we know about N. So W must be one of the two spin representations of $\mathrm{Spin}(8)$. Without loss of generality, we can assume that $W = S^-$; if instead $W = S^+$, we can apply an automorphism of order 2 of $\mathrm{Spin}(8)$ which switches the isomorphism classes of the representations S^- and S^+ and does not change the isomorphism class of the representation V and hence of the standard inclusion $\mathrm{Spin}(8) \to \mathrm{Spin}(9)$. Once that is done, N is the S^9-bundle $S(S^- \oplus \mathbf{R}^2)$ over S^8.

Finally, we need to analyze the free S^1-action on N that gives $M = N/S^1$. By computing the centralizers of the homomorphisms that define N, we see that the S^1-action on N is defined by a homomorphism $S^1 \to \mathrm{Spin}(9)$ together with a homomorphism from S^1 to S^1, the identity component of the centralizer of $\mathrm{Spin}(8)$ in $\mathrm{Spin}(10)$. The homomorphism

$S^1 \to \mathrm{Spin}(9)$ defines an action of S^1 on S^8 and also on the Spin(9)-equivariant vector bundle S^- over S^9, while the homomorphism $S^1 \to S^1$, of the form $z \mapsto z^b$ for some integer b, defines the action of S^1 on the trivial bundle $\mathbf{R}^2$ over S^8. Since S^1 is acting freely on $N = S(S^- \oplus \mathbf{R}^2)$, it must act freely on both $S(S^-) = S^{15}$ and on $S(\mathbf{R}^2) = S^1 \times S^8$. Since the S^1-action on S^8 must have a fixed point, the freeness of the S^1-action on $S(\mathbf{R}^2)$ means that $b = \pm 1$. We compute that there is a unique conjugacy class of homomorphisms $S^1 \to \mathrm{Spin}(9)$ which give a free action of S^1 on $S(S^-) = S^{15}$, namely the subgroup $S^1 = \mathrm{Spin}(2) \subset \mathrm{Spin}(9)$. We can assume that $b = 1$, since changing $b = -1$ to $b = 1$ clearly does not change the diffeomorphism class of the quotient manifold $M = N/S^1$. Thus we have uniquely described the biquotient M.

But the biquotient M we have described is exactly the one which we proved to be diffeomorphic to $\mathbf{CP}^8 \# -\mathbf{CaP}^2$ in Section 2. In particular, it is not homotopy equivalent to $\mathbf{CP}^8 \# \mathbf{CaP}^2$. This completes the proof that $\mathbf{CP}^8 \# \mathbf{CaP}^2$ is not homotopy equivalent to a biquotient.

We now show that $\mathbf{HP}^4 \# \mathbf{CaP}^2$ is not homotopy equivalent to a biquotient $M = G/H$. Since M is 2-connected, H is simply connected, by Lemma 3.1. The ring $H^*(M, \mathbf{Z})$ is a complete intersection, and so the degrees of its generators and relations determine the rational homotopy groups of M. The result is that G contributes degrees 6 and 8 to M, while H contributes degrees 2 and 4, and no others. Since every simple factor of G contributes at least one degree to M, G has at most 2 simple factors. Since H contributes exactly one degree 2, H has one more simple factor than G has.

Suppose first that G is simple. We can apply Theorem 4.8, and cases (2), (3), (4) are excluded because G contributes degrees 6 and 8 to M and no more. Therefore case (1) must hold; that is, G contributes its maximal degree to M. So G has maximal degree 8. By Table 4.2, G must be one of Sp(8), Spin(9), Spin(10), or SU(8), which have degrees respectively $2, 4, 6, 8$, $2, 4, 6, 8$, $2, 4, 5, 6, 8$, or $2, 3, 4, 5, 6, 7, 8$. It follows that H has, correspondingly, degrees $2, 2, 4, 4$, $2, 2, 4, 4$, $2, 2, 4, 4, 5$, or $2, 2, 3, 4, 4, 5, 7$. But there is no semisimple group H with the last two sets of degrees: one simple factor of H would have to have maximal degree equal to an odd number d (5 or 7), hence would be isomorphic to $\mathrm{SU}(d)$, and hence would have all degrees from 2 to d, which is not the case here. Therefore G is isomorphic to Sp(8) or Spin(9), and H has degrees $2, 2, 4, 4$. Since H is simply connected, Table 4.2 shows that H is isomorphic to $\mathrm{Sp}(4)^2$.

We compute that there is no free action of $\mathrm{Sp}(4)^2$ on Spin(9), and that the only free action of $\mathrm{Sp}(4)^2$ on Sp(8) is the one-sided action given by the standard inclusion $\mathrm{Sp}(4)^2 \subset \mathrm{Sp}(8)$. So M is the homogeneous space $\mathrm{Sp}(8)/\mathrm{Sp}(4)^2$. This homogeneous space has the rational homotopy type of $\mathbf{HP}^4\#\mathbf{CaP}^2$, but not the homotopy type, because we compute that $w_4(\mathrm{Sp}(8)/\mathrm{Sp}(4)^2) = 0$ whereas $w_4(\mathbf{HP}^4\#\mathbf{CaP}^2) \neq 0$, since $w_4(\mathbf{HP}^4) \neq 0$. To compute these Stiefel-Whitney classes of the homogeneous spaces $\mathrm{Sp}(8)/\mathrm{Sp}(4)^2$ and $\mathbf{HP}^4$, we can use Singhof's approach, which works more generally for biquotients [25]. Namely, say for $X = \mathrm{Sp}(8)/\mathrm{Sp}(4)^2$, the tangent bundle is

$$TX = sp(8) - sp(4)_1 - sp(4)_2$$

in the Grothendieck group of real vector bundles on X. The groups $H^i(\mathrm{BSp}(2n), \mathbf{F}_2)$ are zero for $0 < i < 4$, and so

$$w_4X = w_4(sp(8)) - w_4(sp(4)_1) - w_4(sp(4)_2).$$

This is zero because $w_4(sp(4)_1) + w_4(sp(4)_2)$ is clearly some $\mathbf{F}_2$-multiple of the sum of the generators of $H^4(\mathrm{BSp}(4)_1, \mathbf{F}_2)$ and of $H^4(\mathrm{BSp}(4)_2, \mathbf{F}_2)$, while the generator of $H^4(\mathrm{BSp}(8), \mathbf{F}_2)$ pulls back to the sum of the generators of $H^4(\mathrm{BSp}(4)_1, \mathbf{F}_2)$ and $H^4(\mathrm{BSp}(4)_2, \mathbf{F}_2)$ and also pulls back to zero in $H^4(X, \mathbf{F}_2)$.

Therefore G is not simple. It must have two simple factors, each of which contributes exactly one degree to M. We know that H has one more simple factor than G, so H has three simple factors. We can write $G = G_1 \times G_2$ where G_1 contributes degree 6 to M and G_2 contributes degree 8 to M, and nothing else. By Theorems 4.8 and 5.3 applied to G_1, there is a simple factor H_1 of H such that either G_1 is the exceptional group G_2, H_1 is SU(2), and H_1 acts nontrivially on G_1, or H_1 acts nontrivially on exactly one side of G_1 and G_1/H_1 is isomorphic to one of the homogeneous spaces $\mathrm{Spin}(7)/\mathrm{Spin}(6) = G_2/\mathrm{SU}(3) = S^6$, $\mathrm{Spin}(12)/\mathrm{Spin}(11) = \mathrm{SU}(6)/\mathrm{SU}(5) = \mathrm{Sp}(6)/\mathrm{Sp}(4) = S^{11}$, $\mathrm{Spin}(7)/\mathrm{Spin}(5) = \mathrm{UT}(S^6)$, $\mathrm{Spin}(8)/\mathrm{Spin}(6)$, or $\mathrm{Spin}(8)/\mathrm{Spin}(5)$. Likewise, by Theorems 4.8 and 5.3 applied to the second factor G_2, there is a simple factor H_2 of H which acts nontrivially on exactly one side of G_2 with G_2/H_2 isomorphic to one of the homogeneous spaces $\mathrm{Spin}(9)/\mathrm{Spin}(8) = S^8$, $\mathrm{Spin}(16)/\mathrm{Spin}(15) = \mathrm{SU}(8)/\mathrm{SU}(7) = \mathrm{Sp}(8)/\mathrm{Sp}(6) = \mathrm{Spin}(9)/\mathrm{Spin}(7) = S^{15}$, $\mathrm{Spin}(9)/\mathrm{Spin}(7) = \mathrm{UT}(S^8)$, $\mathrm{Spin}(10)/\mathrm{Spin}(8)$, $\mathrm{Spin}(10)/\mathrm{Spin}(7)$, $\mathrm{Spin}(9)/G_2$, or $\mathrm{Spin}(10)/G_2$.

We see that in all the above cases G_1/H_1 and G_2/H_2, H_1 is never isomorphic to H_2, so in particular H_1 is not the same simple factor of H

as H_2. So, given G_1/H_1 and G_2/H_2, the known degrees of M determine the degrees of the remaining simple factor of H, H_3. Let us say that a given set of factors of G and H, for example G_1 and H_1, *adds* a degrees d to M if the integer a is the number of degrees d in the given factors of G minus the number of degrees d in the given factors of H. From the above list, G_1/H_1 always adds exactly one degree 6 to M, while G_2/H_2 adds none; so, by the known degrees of M, H_3 has no degree 6. The only simple group with two degrees 4 is Spin(8), which also has a degree 6; so H_3 has at most one degree 4.

Also, from the above list, G_2/H_2 adds no degree 3 to M, so the cases where G_1/H_1 adds -1 degree 3 to M cannot occur (otherwise H_3 would have -1 degrees 3). Thus G_1/H_1 is not $\mathrm{Spin}(7)/\mathrm{Spin}(6) = S^6$, $G_2/\mathrm{SU}(3) = S^6$, or $\mathrm{Spin}(8)/\mathrm{Spin}(6) = \mathrm{UT}(S^7)$. Thus G_1/H_1 and G_2/H_2 both add zero degrees 3 to M, and so H_3 has no degree 3. Next, we observe that G_1/H_1 adds ≥ 0 degrees 4 to M, so the cases where G_2/H_2 adds one degree 4 to M cannot occur (otherwise H_3 would have at least two degrees 4). Thus G_2/H_2 is not $\mathrm{Spin}(9)/G_2$ or $\mathrm{Spin}(10)/G_2$. Next, G_1/H_1 adds zero degrees 5 to M, so in the cases where G_2/H_2 adds degree 5 to M, H_3 must have a degree 5. Since H_3 has no degree 6, it follows that H_3 is isomorphic to SU(5), contradicting the fact that H_3 has no degree 3. Thus G_2/H_2 is not Spin(10)/Spin(8) or Spin(10)/Spin(7).

Finally, if G_1/H_1 is $\mathrm{Spin}(8)/\mathrm{Spin}(5) = \mathrm{Spin}(8)/\mathrm{Sp}(4)$, then G_1/H_1 adds one degree 4 to M, and so, since H_3 has at most one degree 4, G_2/H_2 must add -1 degrees 4 to M; that is, G_2/H_2 is $\mathrm{Spin}(9)/\mathrm{Spin}(8) = S^8$. Then H_3 has degrees $2, 4$ and so H_3 is isomorphic to Sp(4). Here M is a biquotient $(\mathrm{Spin}(8) \times \mathrm{Spin}(9))/(\mathrm{Sp}(4) \times \mathrm{Spin}(8) \times \mathrm{Sp}(4))$. Since $H_2 = \mathrm{Spin}(8)$ has finite centralizer in $G_2 = \mathrm{Spin}(9)$, the two Sp(4) factors of H can only act on Spin(9) on the other side from Spin(8); so $\mathrm{Sp}(4)^2$ acts on S^8. Looking at the low-dimensional orthogonal representations of $\mathrm{Sp}(4)^2$ shows that this action must have a fixed point. Since H acts freely on G, it follows that there is a subgroup of H isomorphic to $\mathrm{Sp}(4)^2$ which acts freely on Spin(8). We compute, however, that there is no free action of $\mathrm{Sp}(4)^2$ on Spin(8). This contradiction shows that G_1/H_1 is not $\mathrm{Spin}(8)/\mathrm{Spin}(5) = \mathrm{Spin}(8)/\mathrm{Sp}(4)$.

The only remaining possibilities for G_1/H_1 are those which add degree 6 to M and nothing else: a homogeneous space $\mathrm{Spin}(12)/\mathrm{Spin}(11) = \mathrm{SU}(6)/\mathrm{SU}(5) = \mathrm{Sp}(6)/\mathrm{Sp}(4) = S^{11}$ or $\mathrm{Spin}(7)/\mathrm{Spin}(5) = \mathrm{UT}(S^6)$ or $(G_1, H_1) = (G_2, A_1)$ (the exceptional group G_2). Also, G_2/H_2 is either $\mathrm{Spin}(9)/\mathrm{Spin}(8) = S^8$, which adds one degree 8 and subtracts one degree 4 from M, or else a homogeneous space which adds degree 8 to M and nothing else: $\mathrm{Spin}(16)/\mathrm{Spin}(15) = \mathrm{SU}(8)/\mathrm{SU}(7) = \mathrm{Sp}(8)/\mathrm{Sp}(6) = \mathrm{Spin}(9)/\mathrm{Spin}(7) = S^{15}$ or $\mathrm{Spin}(9)/\mathrm{Spin}(7) = \mathrm{UT}(S^8)$. From the known degrees of M, it follows that H_3 has only degree 2 if G_2/H_2 is S^8 and has degrees $2, 4$ otherwise. Therefore H_3 is isomorphic to $\mathrm{SU}(2)$ if G_2/H_2 is S^8 and to $\mathrm{Sp}(4)$ otherwise.

Suppose that H_3 is isomorphic to $\mathrm{Sp}(4)$; we will derive a contradiction. First, we can easily exclude the possibility that the first factor G_1 is the exceptional group G_2. The point is that, in this case, neither $H_3 = \mathrm{Sp}(4)$ nor H_2 (from the list, above) has a nontrivial homomorphism to the first factor G_1. Since $H_2 \times H_3 \subset H$ must act freely on $G_1 \times G_2$, it follows that $H_2 \times H_3$ acts freely on the second factor G_2. This is impossible because the list of possible spaces G_2/H_2 shows that H_2 has rank 1 less than the second factor G_2, and so $H_2 \times H_3 = H_2 \times \mathrm{Sp}(4)$ has rank 1 greater than the second factor G_2. The impossibility here follows from the fact that a finite-dimensional elliptic space X has the alternating sum of its rational homotopy groups $\chi_\pi(X) \leq 0$, by Halperin ([13], p. 434). So the first factor G_1 is not the exceptional group G_2.

We continue to assume that H_3 is $\mathrm{Sp}(4)$. We know that G_1/H_1 is a homogeneous space $\mathrm{Spin}(12)/\mathrm{Spin}(11) = \mathrm{SU}(6)/\mathrm{SU}(5) = \mathrm{Sp}(6)/\mathrm{Sp}(4) = S^{11}$ or $\mathrm{Spin}(7)/\mathrm{Spin}(5) = \mathrm{UT}(S^6)$, and G_2/H_2 is a homogeneous space $\mathrm{Spin}(16)/\mathrm{Spin}(15) = \mathrm{SU}(8)/\mathrm{SU}(7) = \mathrm{Sp}(8)/\mathrm{Sp}(6) = \mathrm{Spin}(9)/\mathrm{Spin}(7) = S^{15}$ or $\mathrm{Spin}(9)/\mathrm{Spin}(7) = \mathrm{UT}(S^8)$. For each possible G_1/H_1 and G_2/H_2, we check immediately that either H_1 must act trivially on G_2 or H_2 must act trivially on G_1. Here we use in particular that a simple factor of H must act trivially on a simple factor of G isomorphic to it, by Lemma 4.1. It follows that the biquotient $(G_1 \times G_2)/(H_1 \times H_2)$ is either a G_1/H_1-bundle over G_2/H_2 or a G_2/H_2-bundle over G_1/H_1. Now G_1/H_1 is either S^{11} or $\mathrm{UT}(S^6)$, so it has the 3-local homotopy type of S^{11}; and likewise G_2/H_2 is either S^{15} or $\mathrm{UT}(S^8)$, so it has the 3-local homotopy type of S^{15}. Therefore the biquotient $(G_1 \times G_2)/(H_1 \times H_2)$ is 3-locally 11-connected. It follows that the natural map

$$M = (G_1 \times G_2)/(H_1 \times H_2 \times \mathrm{Sp}(4)) \to \mathrm{BSp}(4)$$

is 3-locally 11-connected. Up to this point, any odd prime number would serve in place of 3, but we now derive a contradiction by a 3-local calculation. Since M is homotopy equivalent to $\mathbf{HP}^4 \# \mathbf{CaP}^2$, the map $\pi_8 M \otimes H^8(M, \mathbf{Z}) \to \mathbf{Z}$ is surjective. On the other hand, the map $\pi_8 \mathrm{BSp}(4) \times H^8(\mathrm{BSp}(4), \mathbf{Z}) \to \mathbf{Z}$ has image $6\mathbf{Z}$. Indeed, $H^8(\mathrm{BSp}(4), \mathbf{Z})$

is generated by c_2^2 and c_4, and c_2^2 is clearly zero for any Sp(4)-bundle over S^8, while c_4 of such a bundle is a multiple of $(4-1)! = 6$ by Bott periodicity. Since the above map $M \to \mathrm{BSp}(4)$ is 3-locally 11-connected, we have a contradiction.

That shows that H_3 is not isomorphic to Sp(4). Therefore H_3 is isomorphic to SU(2). In this case, we know that G_2/H_2 is Spin(9)/Spin(8) $= S^8$. Also, G_1/H_1 is either a homogeneous space Spin(12)/Spin(11) $=$ SU(6)/SU(5) $=$ Sp(6)/Sp(4) $= S^{11}$ or Spin(7)/Spin(5) $= \mathrm{UT}(S^6)$, or else G_1 is the exceptional group G_2 and H_1 is SU(2) acting nontrivially on G_1, perhaps on both sides.

Since $M = G/H$ is homotopy equivalent to $\mathbf{HP}^4\#\mathbf{CaP}^2$, it has the same Stiefel-Whitney classes as $\mathbf{HP}^4\#\mathbf{CaP}^2$, by Wu ([19], Theorem 11.14). In particular, the sphere S^8 in $\mathbf{HP}^4\#\mathbf{CaP}^2$ has self-intersection number 1, and so w_8M is not in the subgroup $H^4(M, \mathbf{F}_2)^2$ of $H^8(M, \mathbf{F}_2)$. The biquotient $N := (G_1 \times G_2)/(H_1 \times H_2)$ is a principal SU(2)-bundle over M, so its stable tangent bundle is the pullback of that of M. It follows that $w_8N \in H^8(N, \mathbf{F}_2)$ is not zero.

Suppose that G_1 is the exceptional group G_2. Then M is a biquotient of the form $(G_2 \times \mathrm{Spin}(9))/(\mathrm{SU}(2) \times \mathrm{Spin}(8) \times \mathrm{SU}(2))$. Here Spin(8) must act trivially on G_2, and so we can write

$$M = (G_2 \times S^8)/\mathrm{SU}(2)^2.$$

Any torus acting by isometries on an even-dimensional sphere has a fixed point, since every orthogonal representation of a torus is a sum of 2-dimensional representations and trivial representations. In particular, a maximal torus in $\mathrm{SU}(2)^2$ has a fixed point on S^8. Since $\mathrm{SU}(2)^2$ acts freely on $G_2 \times S^8$, this torus acts freely on G_2. Since every element of $\mathrm{SU}(2)^2$ is conjugate to an element of the torus, $\mathrm{SU}(2)^2$ acts freely on G_2. Thus M is a fiber bundle

$$S^8 \to M \to G_2/\mathrm{SU}(2)^2.$$

Since S^8 has signature zero, it follows that M has signature zero, contradicting the fact that M is homotopy equivalent to $\mathbf{HP}^4\#\mathbf{CaP}^2$. Thus the first factor G_1 is not the exceptional group G_2.

So G_1/H_1 is either Spin(12)/Spin(11) $=$ SU(6)/SU(5) $=$ Sp(6)/Sp(4) $= S^{11}$ or Spin(7)/Spin(5) $= \mathrm{UT}(S^6)$, and we know that G_2/H_2 is Spin(9)/Spin(8) $= S^8$. But since w_8N is not zero, N is not an S^8-bundle over S^{11} or over $\mathrm{UT}(S^6)$. So $H_2 = \mathrm{Spin}(8)$ must act nontrivially on G_1. It follows that G_1/H_1 must be Spin(12)/Spin(11) $= S^{11}$.

Thus M is a biquotient of the form

$$(\mathrm{Spin}(12) \times \mathrm{Spin}(9))/(\mathrm{Spin}(11) \times \mathrm{Spin}(8) \times \mathrm{SU}(2)).$$

Here Spin(11) automatically acts trivially on Spin(9). So the biquotient

$$N := (\mathrm{Spin}(12) \times \mathrm{Spin}(9))/(\mathrm{Spin}(11) \times \mathrm{Spin}(8))$$

is completely determined by the homomorphism $\mathrm{Spin}(8) \to \mathrm{Spin}(12)$, which we know is nontrivial. (Since Spin(11) has finite centralizer on Spin(12), the action of Spin(8) on Spin(12) must be on the other side of Spin(12) from Spin(11).) There are 3 conjugacy classes of nontrivial homomorphisms $\mathrm{Spin}(8) \to \mathrm{Spin}(12)$, each of the form $W \oplus \mathbf{R}^4$, where the 8-dimensional real representation W of Spin(8) is either the standard representation V or one of the two spin representations. Thus N is the S^{11}-bundle $S(W \oplus \mathbf{R}^4)$ over $\mathrm{Spin}(9)/\mathrm{Spin}(8) = S^8$. If W is the standard representation V of Spin(8), then the associated rank-8 vector bundle over S^8 has $w_8 = 0$, and so the manifold N would have $w_8 N = 0$, contradicting what we know. So W must be one of the two spin representations of Spin(8). If necessary, we can apply an order-2 automorphism to Spin(8), not changing the conjugacy class of the standard inclusion $\mathrm{Spin}(8) \to \mathrm{Spin}(9)$, to arrange that $W = S^-$.

Thus N is the S^{11}-bundle $S(S^- \oplus \mathbf{R}^4)$ over S^8. The action of SU(2) on N is given by a homomorphism $\mathrm{SU}(2) \to \mathrm{Spin}(9)$ together with a homomorphism $\mathrm{SU}(2) \to \mathrm{Spin}(4)$, because Spin(4) is the identity component of the centralizer of Spin(8) in Spin(12). The homomorphism $\mathrm{SU}(2) \to \mathrm{Spin}(9)$ determines the action of SU(2) on S^8 and on the Spin(9)-equivariant vector bundle S^- over S^8, while the homomorphism $\mathrm{SU}(2) \to \mathrm{Spin}(4)$ determines the action of SU(2) on $\mathbf{R}^4$. Since SU(2) acts freely on $N = S(S^- \oplus \mathbf{R}^4)$, it must act freely on $S(S^-) = S^{15}$ and on $S(\mathbf{R}^4) = S^3 \times S^8$.

As in the proof that $\mathbf{CP}^8 \# \mathbf{CaP}^2$ is not homotopy equivalent to a biquotient, we compute that there is only one free SU(2)-action on $S(S^- \oplus \mathbf{R}^4)$: SU(2) must act by $\mathrm{SU}(2) \cong \mathrm{Spin}(3) \subset \mathrm{Spin}(9)$ on S^8 and on the bundle S^- over S^8, and by the standard representation $V_{\mathbf{R}}$ on $\mathbf{R}^4$. Thus we have described the manifold M uniquely up to diffeomorphism. But we showed in Section 2 that exactly this manifold is diffeomorphic to $\mathbf{HP}^4 \# -\mathbf{CaP}^2$. In particular, it is not homotopy equivalent to $\mathbf{HP}^4 \# \mathbf{CaP}^2$. This completes the proof that $\mathbf{HP}^4 \# \mathbf{CaP}^2$ is not homotopy equivalent to a biquotient.

8. The Cheeger manifold $\mathbf{CP}^{4e}\#\mathbf{HP}^{2e}$ is not homotopy equivalent to a biquotient

Finally, we will show that $\mathbf{CP}^{4e}\#\mathbf{HP}^{2e}$, $e \geq 1$, is not homotopy equivalent to a biquotient $M = G/K$, completing the proof of Theorem 2.1.

The cohomology ring of M is a complete intersection ring. The degrees of its generators and relations determine the rational homotopy groups of M. The result is that G contributes degrees 3 and $4e$ to M and K contributes degrees 1 and 2, where the 1 means that the abelianization of K is isomorphic to S^1. Let H be the commutator subgroup of K, and let N be the biquotient G/H; then $M = N/S^1$. Here G contributes degrees 3 and $4e$ to N, while H contributes degree 2. From the degree 2, it follows that H has one more simple factor than G has. Also, by Theorem 4.8, each simple factor of G contributes at least one degree to N. So G has at most two simple factors.

Furthermore, since M and $\mathbf{CP}^{4e}\#\mathbf{HP}^{2e}$ are homotopy equivalent, they have the same Stiefel-Whitney classes, by Wu ([19], Theorem 11.14). In particular, w_4M is not in the subgroup $H^2(M, \mathbf{F}_2)^2$ of $H^4(M, \mathbf{F}_2)$. Since N is an S^1-bundle over M, the stable tangent bundle of N is the pullback of that of M, and so w_4N is not zero.

Suppose first that G is simple. Since H has one more simple factor than G, H has two simple factors. We know that G contributes degrees 3 and $4e$ to N, and nothing else. In Theorem 4.8, cases (2), (3), (4) are incompatible with these degrees, and so case (1) must hold, that is, G contributes its maximal degree to N. Thus G has maximal degree $4e$. The only simple groups which have 3 as a degree are the groups $\mathrm{SU}(n)$, $n \geq 3$, and so G is isomorphic to $\mathrm{SU}(4e)$. Since G has exactly one degree $4e$, the known degrees of N imply that all simple factors of H have maximal degree less than $4e$.

Suppose that $e \geq 2$. Then the second-largest degree, $4e - 1$, of $G = \mathrm{SU}(4e)$ is killed by H. Let H_1 be a simple factor of H which kills the degree $4e - 1$ of G. We know that H_1 has maximal degree less than $4e$, and so it has maximal degree $4e-1$, which implies that H_1 is isomorphic to $\mathrm{SU}(4e - 1)$. Any nontrivial homomorphism $\mathrm{SU}(4e - 1) \to \mathrm{SU}(4e)$ has Dynkin index 1, and so H_1 must act nontrivially on only one side of G; otherwise $\mathrm{SU}(4e)$ would contribute degree 2 to N. So G/H_1 is isomorphic to the homogeneous space $\mathrm{SU}(4e)/\mathrm{SU}(4e-1) = S^{8e-1}$. But then H_1 kills the degree 3 of G, which should appear in M. Thus $e \geq 2$ leads to a contradiction, for G simple.

This leaves the case $e = 1$, with G simple. As shown above, G is isomorphic to SU(4). Since G has degrees $2, 3, 4$, the known degrees of N imply that H has degrees $2, 2$. So H is isomorphic to $\mathrm{SU}(2)^2$. Thus N is a biquotient $\mathrm{SU}(4)/\mathrm{SU}(2)^2$. Indeed, there is at least one such biquotient, the homogeneous space $\mathrm{UT}(S^5) = \mathrm{Spin}(6)/\mathrm{Spin}(4)$, which admits a free S^1-action. In that case, the quotient is the 4-dimensional complex quadric $Q^4_{\mathbf{C}}$, which has the rational homotopy type of $\mathbf{CP}^4\#\mathbf{HP}^2$. But no biquotient $N = \mathrm{SU}(4)/\mathrm{SU}(2)^2$ can have the homotopy type of an S^1-bundle over $\mathbf{CP}^4\#\mathbf{HP}^2$, by the following argument. By Singhof's description of the tangent bundle of a biquotient [25], as used in the previous section, we have

$$w_4 N = w_4(\mathrm{su}(4)) - w_4(\mathrm{su}(2)_1) - w_4(\mathrm{su}(2)_2).$$

We then make the convenient calculation that

$$w_4(\mathrm{su}(n)) = 0 \in H^4(\mathrm{BSU}(n), \mathbf{F}_2)$$

for all even n. So $w_4 N = 0$, which contradicts what we know about N. Thus we have a contradiction from the assumption that G is simple.

Thus G has two simple factors. We return to the general case, $e \geq 1$. Since H has one more simple factor than G, H has three simple factors. Each simple factor of G contributes exactly one degree to N. We can write $G = G_1 \times G_2$ such that G_1 contributes degree 3 to N and G_2 contributes degree $4e$, and nothing more. By Theorems 4.8 and 5.3 applied to G_1, there is a simple factor H_1 of H such that either G_1/H_1 is isomorphic to the homogeneous space $\mathrm{SU}(4)/\mathrm{Sp}(4) = S^5$ (which is the same as $\mathrm{Spin}(6)/\mathrm{Spin}(5)$) or (G_1, H_1) is $(\mathrm{SU}(3), \mathrm{SU}(2))$ for some nontrivial action of SU(2) on SU(3).

If (G_1, H_1) is $(\mathrm{SU}(3), \mathrm{SU}(2))$ and SU(2) acts nontrivially on both sides of SU(3), then the centralizers of both homomorphisms $\mathrm{SU}(2) \to \mathrm{SU}(3)$ are finite or finite by S^1, so no other simple factor of H can act on G_1. Moreover, the two nontrivial homomorphisms $\mathrm{SU}(2) \to \mathrm{SU}(3)$ have Dynkin indices 1 and 4, by Lemma 5.2. So the map $\pi_3 H_1 \to \pi_3 G_1$, $\mathbf{Z} \to \mathbf{Z}$, is either zero or multiplication by $1-4 = -3$ or $4-1 = 3$. In any case, it is not surjective, which contradicts the fact that $\pi_3 H \to \pi_3 G_1$ must be surjective, since $\pi_3 M = 0$. So any SU(2) factor of H acts nontrivially on at most on one side of SU(3). Moreover, SU(2) factors of H are the only ones that can act on $G_1 = \mathrm{SU}(3)$ (SU(3) factors of H are excluded by Lemma 4.1). At least one SU(2) factor must act with Dynkin index 1 rather than 4, again using that $\pi_3 H \to \pi_3 G_1$ is surjective. Thus, if G_1 is SU(3), then we can choose the simple factor H_1 isomorphic to SU(2) such that G_1/H_1 is isomorphic to the homogeneous space $\mathrm{SU}(3)/\mathrm{SU}(2) = S^5$.

Thus, G_1/H_1 is either $\mathrm{Spin}(6)/\mathrm{Spin}(5) = S^5$ or $\mathrm{SU}(3)/\mathrm{SU}(2) = S^5$. In both cases, G_1/H_1 adds degree 3 to N and nothing else. Next, we can apply Theorems 4.8 and 5.3 to the second factor G_2, giving a simple factor H_2 with certain properties. From the known degrees of $N = (G_1 \times G_2)/(H_1 \times H_2 \times H_3)$, since G_1/H_1 adds degree 3 to N and nothing else, G_2/H_2 cannot have any degrees with multiplicity < 0 (that is, any degrees which occur in H_2 more than in G_2). Given this, Theorems 4.8 and 5.3 imply that either $e = 1$ and G_2 is isomorphic to Sp(4), or there is a simple factor H_2 of H such that G_2/H_2 is isomorphic to a homogeneous space $\mathrm{Spin}(8e)/\mathrm{Spin}(8e-1) = \mathrm{SU}(4e)/\mathrm{SU}(4e-1) = \mathrm{Sp}(4e)/\mathrm{Sp}(4e-2) = S^{8e-1}$, $\mathrm{Spin}(7)/G_2 = S^7$ with $e = 1$, $\mathrm{Spin}(9)/\mathrm{Spin}(7) = S^{15}$ with $e = 2$, $\mathrm{Spin}(4e+1)/\mathrm{Spin}(4e-1) = \mathrm{UT}(S^{4e})$ with $e \geq 2$, $\mathrm{Spin}(8)/G_2 = S^7 \times S^7$ with $e = 1$, $\mathrm{Spin}(9)/G_2$ with $e = 2$, $\mathrm{Spin}(4e+2)/\mathrm{Spin}(4e-1)$ with $e \geq 2$, $\mathrm{Spin}(10)/\mathrm{Spin}(7)$ via the spin representation with $e = 2$, or $\mathrm{Spin}(10)/G_2$ with $e = 2$. In these homogeneous spaces, it is clear that the simple factor H_2 of H is different from H_1, which is SU(2) or $\mathrm{Spin}(5) = \mathrm{Sp}(4)$.

When G_2/H_2 is one of the homogeneous spaces diffeomorphic to S^{8e-1} or $\mathrm{UT}(S^{4e})$, then G_2/H_2 adds only one degree $4e$ to N and nothing else. In those cases, the third factor H_3 of H has degree 2 only, so H_3 is isomorphic to SU(2). When G_2 is isomorphic to Sp(4), the product of the two simple factors of H besides H_1, $H_2 \times H_3$, has degrees $2, 2$, and so H_2 and H_3 are both isomorphic to SU(2). When G_2/H_2 is $\mathrm{Spin}(8)/G_2 = S^7 \times S^7$ with $e = 1$ or $\mathrm{Spin}(9)/G_2$ with $e = 2$, then H_3 has degrees $2, 4$, and so H_3 is isomorphic to Sp(4). Finally, the cases where G_2/H_2 is $\mathrm{Spin}(4e+2)/\mathrm{Spin}(4e-1)$ with $e \geq 2$, $\mathrm{Spin}(10)/\mathrm{Spin}(7)$ with $e = 2$, or $\mathrm{Spin}(10)/G_2$ with $e = 2$ cannot occur. In these cases, part (4) of Theorem 5.3 implies that H must have a factor $\mathrm{SU}(2e+1)$, which must be the third factor H_3. But then H has one more degree 3 than G, contradicting the known degrees of $N = G/H$.

The cases where H_3 is Sp(4) are easier to analyze, so we consider them first. Either G_2/H_2 is $\mathrm{Spin}(8)/G_2 = S^7 \times S^7$ and $e = 1$, or G_2/H_2 is $\mathrm{Spin}(9)/G_2$ and $e = 2$. We know that G_1/H_1 is either $\mathrm{SU}(3)/\mathrm{SU}(2) = S^5$ or $\mathrm{Spin}(6)/\mathrm{Spin}(5) = S^5$. By the known low-dimensional representations of $H_3 = \mathrm{Sp}(4)$, the action of H_3 on $G_1/H_1 = S^5$ must have a fixed point. So there is a subgroup of $H_1 \times H_3$ which projects isomorphically to $H_3 = \mathrm{Sp}(4)$ and which fixes a point in G_1. Also, the exceptional group G_2 must act trivially on G_1 (which is Spin(6) or SU(3)), so we have a subgroup of H isomorphic to $\mathrm{Sp}(4) \times G_2$ which fixes a point in G_1. Since H acts freely on G, we have a free action of $\mathrm{Sp}(4) \times G_2$ on the second factor Spin(8) or Spin(9). We compute, however, that there is no such free action.

So we must have $H_3 = \mathrm{SU}(2)$. We know that G_1/H_1 is either $\mathrm{SU}(3)/\mathrm{SU}(2) = S^5$ or $\mathrm{Spin}(6)/\mathrm{Spin}(5) = S^5$. Also, either $e = 1$, G_2 is isomorphic to $\mathrm{Sp}(4)$ and H_2 is isomorphic to $\mathrm{SU}(2)$, or G_2/H_2 is a homogeneous space $\mathrm{Spin}(8e)/\mathrm{Spin}(8e-1) = \mathrm{SU}(4e)/\mathrm{SU}(4e-1) = \mathrm{Sp}(4e)/\mathrm{Sp}(4e-2) = S^{8e-1}$, $\mathrm{Spin}(7)/G_2 = S^7$ with $e = 1$, $\mathrm{Spin}(9)/\mathrm{Spin}(7) = S^{15}$ with $e = 2$, or $\mathrm{Spin}(4e+1)/\mathrm{Spin}(4e-1) = \mathrm{UT}(S^{4e})$ with $e \geq 2$.

We begin with the case where G_1/H_1 is $\mathrm{SU}(3)/\mathrm{SU}(2) = S^5$ and G_2/H_2 is $\mathrm{Spin}(8e)/\mathrm{Spin}(8e-1) = S^{8e-1}$. This turns out to be the main step of the whole proof; most other cases will reduce to this one. The group $H_2 = \mathrm{Spin}(8e-1)$ must act trivially on $G_1 = \mathrm{SU}(3)$, and so $(G_1 \times G_2)/(H_1 \times H_2)$ is an S^{8e-1}-bundle over S^5. The manifold M is the quotient of this bundle by a free action of a group $D := K/(H_1 \times H_2)$ which is isogenous to $S^1 \times \mathrm{SU}(2)$. Since $\pi_2 M$ is isomorphic to $\mathbf{Z}$, $\pi_1 D$ is isomorphic to $\mathbf{Z}$. So D is isomorphic to $S^1 \times \mathrm{SU}(2)$ or to $\mathrm{U}(2)$.

In dimensions less than $8e-1$, M clearly has the homotopy type of a homotopy quotient $S^5//D$, or equivalently an S^5-bundle over the classifying space $\mathrm{B}D$. Here D acts on S^5 through the group $\mathrm{U}(3)$, coming from the group $G_1 = \mathrm{SU}(3)$ together with the centralizer S^1 of $H_1 = \mathrm{SU}(2)$ in G_1. From the cohomology ring of M, the Euler class of the homomorphism $D \to \mathrm{U}(3)$ must be the product of a generator of $H^2(\mathrm{B}D, \mathbf{Z})$ with an element of $H^4(\mathrm{B}D, \mathbf{Z})$ that generates $H^4/(H^2)^2$. Let L be the standard 1-dimensional complex representation of S^1, V be the standard 2-dimensional representation of $\mathrm{SU}(2)$, and E the standard 2-dimensional representation of $\mathrm{U}(2)$. By inspecting the low-dimensional representations of D, it follows that the homomorphism $D \to \mathrm{U}(3)$ is isomorphic to $L^{\pm 1} \oplus L^a \otimes V$ if D is $S^1 \times \mathrm{SU}(2)$, or to $(\det E)^{\pm 1} \oplus (\det E)^a \otimes E$ if D is $\mathrm{U}(2)$, for some sign and some integer a. In particular, the subgroup $\mathrm{SU}(2) = H_3$ of D acts on $G_1/H_1 = \mathrm{SU}(3)/\mathrm{SU}(2) = S^5$ by the natural inclusion $\mathrm{SU}(2) \to \mathrm{SU}(3)$, on the other side of G_1 from H_1.

It follows that the diagonal subgroup C in $H_1 \times H_3 = \mathrm{SU}(2) \times \mathrm{SU}(2)$ has a fixed point in $\mathrm{SU}(3)$. So C must act freely on S^{8e-1}. It follows that C acts on S^{8e-1} by the real representation $(V_{\mathbf{R}})^{2e}$, where V is the standard 2-dimensional complex representation of $C = \mathrm{SU}(2)$. In particular, the associated complex representation is a direct sum of copies of V. By the Clebsch-Gordan formula, an irreducible complex representation of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ whose restriction to the diagonal subgroup is a sum of copies of V must be isomorphic to $V_1 \otimes \mathbf{C}$ or $\mathbf{C} \otimes V_2$, the standard representations of the two factors. Therefore the action of $H_1 \times H_3 = \mathrm{SU}(2) \times \mathrm{SU}(2)$ on S^{8e-1} must be given by the real representation associated to the complex representation $(V_1)^{\oplus j} \oplus (V_3)^{\oplus 2e-j}$ for some $0 \leq j \leq 2e$.

The manifold M can be written

$$M = (\mathrm{SU}(3) \times S^{8e-1})/((\mathrm{SU}(2)^2 \times \mathbf{R})/\mathbf{Z}).$$

Here the subgroup $\mathbf{Z}$ of $\mathrm{SU}(2)^2 \times \mathbf{R}$ is generated by an element of the form $(\pm 1, \pm 1, 1)$, which we write as $(e(a_0), e(b_0), 1)$ for $a_0, b_0 \in \{0, 1/2\}$, where $e(t) := e^{2\pi i t}$. The group $(\mathrm{SU}(2)^2 \times \mathbf{R})/\mathbf{Z}$ acts on $\mathrm{SU}(3) \times S^{8e-1}$ via homomorphisms to $\mathrm{SU}(3)^2/Z(\mathrm{SU}(3))$ and to $\mathrm{SO}(8e)$.

We can assume that the first factor $\mathrm{SU}(2)$ of $(\mathrm{SU}(2)^2 \times \mathbf{R})/\mathbf{Z}$ acts on $\mathrm{SU}(3)$ by the standard inclusion on the left, while the second factor $\mathrm{SU}(2)$ acts on $\mathrm{SU}(3)$ by the standard inclusion on the right. So the two homomorphisms from $\mathbf{R}$ to $\mathrm{SU}(3)$ both map into the centralizer of $\mathrm{SU}(2)$ in $\mathrm{SU}(3)$. That is, they map $t \in \mathbf{R}$ to the diagonal matrices

$$(e(at), e(at), e(-2at)), (e(bt), e(bt), e(-2bt))$$

for some a, b. Since the generator of the above subgroup $\mathbf{Z}$ in $\mathrm{SU}(2)^2 \times \mathbf{R}$ must map into $Z(\mathrm{SU}(3)) \cong \mathbf{Z}/3 \subset \mathrm{SU}(3)^2$, a and b must satisfy:

$$\begin{aligned} a + a_0 &\in \frac{1}{3}\mathbf{Z} \\ b + b_0 &\in \frac{1}{3}\mathbf{Z} \\ a + a_0 &\equiv b + b_0 \pmod{\mathbf{Z}}. \end{aligned}$$

The homomorphism $\mathrm{SU}(2)^2 \to \mathrm{SO}(8e)$ is by the real representation $(V_1)^{\oplus j} \oplus (V_2)^{\oplus 2e-j}$ for some $0 \leq j \leq 2e$. The centralizer of this homomorphism has identity component $\mathrm{Sp}(2j) \times \mathrm{Sp}(4e-2j)$, whose maximal torus is conjugate to the center $(S^1)^{2e}$ of $\mathrm{U}(2)^{2e}$. So we can assume that $\mathbf{R}$ maps into this center. Thus the homomorphism from $(\mathrm{SU}(2)^2 \times \mathbf{R})/\mathbf{Z}$ to $\mathrm{SO}(8e)$ is the direct sum of $2e$ homomorphisms to $\mathrm{U}(2)$, of the form $(A, B, t) \mapsto e(c_i t)A$ for $1 \leq i \leq j$ and $(A, B, t) \mapsto e(d_i t)B$ for $j + 1 \leq i \leq 2e$. Since this homomorphism is trivial on the subgroup $\mathbf{Z}$, the numbers c_i and d_i must satisfy $c_i + a_0 \in \mathbf{Z}$ and $d_i + b_0 \in \mathbf{Z}$.

The action of $(\mathrm{SU}(2)^2 \times \mathbf{R})/\mathbf{Z}$ on $\mathrm{SU}(3) \times S^{8e-1}$ is free. We compute that this means that, for all $1 \le i \le j$,

$$\begin{aligned} a + 2b - c_i &= \pm 1 \\ a + 2b + c_i &= \pm 1 \\ -2a + 2b &= \pm 1, \end{aligned}$$

and for all $j + 1 \le i \le 2e$,

$$\begin{aligned} 2a + b - d_i &= \pm 1 \\ 2a + b + d_i &= \pm 1 \\ -2a + 2b &= \pm 1. \end{aligned}$$

If $j \ge 1$, then the first two equations imply that either $a + 2b = 0$ and $c_i = \pm 1$, or $a + 2b = \pm 1$ and $c_i = 0$, for $1 \le i \le j$. In particular, c_i is an integer, which implies (since $c_i + a_0 \in \mathbf{Z}$) that a_0 is 0, not 1/2. Since $a + a_0 \in (1/3)\mathbf{Z}$, it follows in particular that a is 2-integral; since $-2a + 2b = \pm 1$, b is not 2-integral. Therefore b_0 is 1/2, not 0. But if $j \le 2e-1$, then we would get the opposite conclusion (that $a_0 = 1/2$ and $b_0 = 1$) from the second three equations above. So we must have either $j = 0$ or $j = 2e$. After switching the two SU(2) factors if necessary, we can assume that $j = 2e$. That is, $\mathrm{SU}(2)^2$ acts on S^{8e-1} by the real representation $(V_1)^{2e}$. Also, we have $a_0 = 0$ and $b = 1/2$, which means that the subgroup $\mathbf{Z}$ of $\mathrm{SU}(2)^2 \times \mathbf{R}$ is generated by $(1, -1, 1 \in \mathbf{R})$.

Since the second factor SU(2) acts only on SU(3), we can rewrite M as

$$M = (S^5 \times S^{8e-1})/(\mathrm{SU}(2) \times S^1).$$

Here we have used that the quotient of $(\mathrm{SU}(2)^2 \times \mathbf{R})/\mathbf{Z}$ by the second copy of SU(2) is isomorphic to $\mathrm{SU}(2) \times S^1$. The action of $\mathrm{SU}(2) \times S^1$ on S^{8e-1} is given by the complex representation $\oplus_{i=1}^{2e} V \otimes L^{c_i}$. Also, we compute that the action of $\mathrm{SU}(2) \times S^1$ on S^5 is by the complex representation $(V \otimes L^{a+2b}) \oplus L^{-2a+2b}$. As computed above, $-2a + 2b = \pm 1$. And either $a + 2b = \pm 1$ and $c_i = 0$ for all i, or $a + 2b = 0$ and $c_i = \pm 1$ for all i.

In the first case, where $c_i = 0$, we can conjugate the action of $\mathrm{SU}(2) \times S^1$ on S^5 in the orthogonal group O(6) to make $a + 2b$ and $-2a + 2b$ equal to 1, rather than -1. Thus M is the manifold

$$(S((V \otimes L) \oplus L) \times S(V^{\oplus 2e}))/(\mathrm{SU}(2) \times S^1).$$

So M is a $\mathbf{CP}^2$-bundle over $\mathbf{HP}^{2e-1}$, and so M has signature zero, contradicting that M is homotopy equivalent to $\mathbf{CP}^{4e} \# \mathbf{HP}^{2e}$. In fact, this $\mathbf{CP}^2$-bundle over $\mathbf{HP}^{2e-1}$ is the one diffeomorphic to $\mathbf{CP}^{4e}\# - \mathbf{HP}^{2e}$, as mentioned in Section 2. In the second case, where $c_i = \pm 1$, we can conjugate the homomorphisms from $\mathrm{SU}(2)\times S^1$ to the orthogonal groups $\mathrm{O}(6)$ and $\mathrm{O}(8e)$ to make $-2a+2b=1$ and $c_i = 1$ for all i. Thus, M is the manifold

$$(S(V\oplus L)\times S((V\otimes L)^{\oplus 2e}))/(\mathrm{SU}(2)\times S^1).$$

Again, we showed in Section 2 that this manifold is diffeomorphic to $\mathbf{CP}^{4e}\# - \mathbf{HP}^{2e}$. Thus it is not homotopy equivalent to $\mathbf{CP}^{4e}\#\mathbf{HP}^{2e}$. This completes the proof that we cannot have G_1/H_1 equal to $\mathrm{SU}(3)/\mathrm{SU}(2) = S^5$ and G_2/H_2 equal to $\mathrm{Spin}(8e)/\mathrm{Spin}(8e-1) = S^{8e-1}$.

We next consider the case where G_1/H_1 is $\mathrm{Spin}(6)/\mathrm{Spin}(5) = S^5$ and G_2/H_2 is $\mathrm{Spin}(8e)/\mathrm{Spin}(8e-1) = S^{8e-1}$. A first observation is that $\mathrm{Spin}(5)$ has finite centralizer in $\mathrm{Spin}(6)$, and so all factors of K except $\mathrm{Spin}(5)$ act on the other side of $\mathrm{Spin}(6)$ from $\mathrm{Spin}(5)$. Likewise, all factors of K except $\mathrm{Spin}(8e-1)$ act on the other side of $\mathrm{Spin}(8e)$ from $\mathrm{Spin}(8e-1)$. Furthermore, $\mathrm{Spin}(8e-1)$ must act trivially on $\mathrm{Spin}(6)$, and so $(G_1\times G_2)/(H_1\times H_2)$ is an S^{8e-1}-bundle over S^5, which we can write as $(\mathrm{Spin}(6)\times S^{8e-1})/\mathrm{Spin}(5)$.

The homomorphism $K\to (G\times G)/Z(G)$ which defines the action of K on G gives a homomorphism from $D := K/(H_1\times H_2)$ to $\mathrm{SO}(6)$. Here D is isogenous to $\mathrm{SU}(2)\times S^1$ and has fundamental group isomorphic to $\mathbf{Z}$, so D is isomorphic to $\mathrm{SU}(2)\times S^1$ or to $\mathrm{U}(2)$. In dimensions less than $8e-1$, M has the homotopy type of the homotopy quotient $S^5//D$ defined by the homomorphism $D\to \mathrm{SO}(6)$, or equivalently of an S^5-bundle over the classifying space $\mathrm{B}D$. From the cohomology ring of M, the Euler class of the homomorphism $D\to\mathrm{SO}(6)$ must be the product of some generator of $H^2(\mathrm{B}D,\mathbf{Z})$ with some element of $H^4(\mathrm{B}D,\mathbf{Z})$ which generates $H^4/(H^2)^2$. By inspecting the low-dimensional real representations of D, it follows that the homomorphism $D\to\mathrm{SO}(6)$ is the real representation associated to a 3-dimensional complex representation of D. For $D = S^1\times\mathrm{SU}(2)$, write L for the standard 1-dimensional complex representation of S^1 and V for the standard 2-dimensional complex representation of $\mathrm{SU}(2)$. In order to have an Euler class of the form above, the homomorphism $D\to\mathrm{SO}(6)$ must come from the complex representation $L^{\pm1} + L^b\otimes V$ for some sign and some integer b. The Euler class of this representation is $\pm xy$, where we let $x = c_1L$ and $y = c_2V + b^2x^2$. Likewise, for $D = \mathrm{U}(2)$, let E be the standard 2-dimensional complex representation of D. In order to have an Euler class of the form above, the homomorphism $D \to \mathrm{SO}(6)$ must come from the complex representation $(\det E)^{\pm 1} + (\det E)^d \otimes E$ for some sign and some integer d. The Euler class in $H^6\mathrm{BU}(2)$ of this representation is $\pm uv$, where we let $u = c_1 E$ and $v = c_2 E + (d^2 + d)u^2$.

Because the homomorphism $D \to \mathrm{SO}(6)$ factors through $\mathrm{U}(3)$, we can replace $G_1/H_1 = \mathrm{Spin}(6)/\mathrm{Spin}(5) = S^5$ by $G_1/H_1 = \widetilde{\mathrm{U}}(3)/\widetilde{\mathrm{U}}(2)$, in our description of M as a biquotient. Here $\widetilde{\mathrm{U}}(n)$ denotes the inverse image of $\mathrm{U}(n) \subset \mathrm{SO}(2n)$ in $\mathrm{Spin}(2n)$. We can then apply the proof of Lemma 3.1 to replace $\widetilde{\mathrm{U}}(3)$ by $\mathrm{SU}(3)$. Thus we have reduced to the case where G_1/H_1 is $\mathrm{SU}(3)/\mathrm{SU}(2) = S^5$ and G_2/H_2 is $\mathrm{Spin}(8e)/\mathrm{Spin}(8e-1) = S^{8e-1}$. But we have shown that the latter case cannot occur. This completes the proof that G_2/H_2 cannot be $\mathrm{Spin}(8e)/\mathrm{Spin}(8e-1) = S^{8e-1}$, either when G_1/H_1 is $\mathrm{Spin}(6)/\mathrm{Spin}(5)$ or when it is $\mathrm{SU}(3)/\mathrm{SU}(2)$.

The situation now is as follows. First, we know that G_1/H_1 is either $\mathrm{SU}(3)/\mathrm{SU}(2) = S^5$ or $\mathrm{Spin}(6)/\mathrm{Spin}(5) = S^5$. Also, either $e = 1$, G_2 is isomorphic to $\mathrm{Sp}(4)$ and H_2 is isomorphic to $\mathrm{SU}(2)$, or G_2/H_2 is a homogeneous space $\mathrm{Spin}(8e)/\mathrm{Spin}(8e-1) = \mathrm{SU}(4e)/\mathrm{SU}(4e-1) = \mathrm{Sp}(4e)/\mathrm{Sp}(4e-2) = S^{8e-1}$, $\mathrm{Spin}(7)/G_2 = S^7$ with $e = 1$, $\mathrm{Spin}(9)/\mathrm{Spin}(7) = S^{15}$ with $e = 2$, or $\mathrm{Spin}(4e+1)/\mathrm{Spin}(4e-1) = \mathrm{UT}(S^{4e})$ with $e \geq 2$. We have shown that G_2/H_2 cannot be $\mathrm{Spin}(8e)/\mathrm{Spin}(8e-1) = S^{8e-1}$, either when G_1/H_1 is $\mathrm{Spin}(6)/\mathrm{Spin}(5)$ or when it is $\mathrm{SU}(3)/\mathrm{SU}(2)$. Most other cases reduce to this one. Namely, suppose that G_2/H_2 is diffeomorphic to a sphere S^{8e-1} (noting that in all these cases $G_2 \times Z_{G_2}(H_2)$ acts by isometries in the usual metric on the sphere), and that H_2 acts trivially on G_1. Then we can simply replace G_2/H_2 by $\mathrm{Spin}(8e)/\mathrm{Spin}(8e-1) = S^{8e-1}$ without changing the biquotient M. Since G_1 is small, namely $\mathrm{Spin}(6)$ or $\mathrm{SU}(3)$, the hypothesis that H_2 acts trivially on G_1 is automatic for most of the pairs G_2/H_2. Namely this holds when there is no nontrivial homomorphism $H_2 \to G_1$, or also when H_2 is isomorphic to G_1 by Lemma 4.1.

The cases not covered by this argument are: G_1/H_1 is $\mathrm{SU}(3)/\mathrm{SU}(2) = S^5$ or $\mathrm{Spin}(6)/\mathrm{Spin}(5) = S^5$, G_2 is isomorphic to $\mathrm{Sp}(4)$, and H_2 is isomorphic to $\mathrm{SU}(2)$, where $e = 1$; G_1/H_1 is $\mathrm{Spin}(6)/\mathrm{Spin}(5)$ and G_2/H_2 is $\mathrm{SU}(4)/\mathrm{SU}(3) = S^{15}$ with H_2 acting nontrivially on G_1, where $e = 2$; or G_1/H_1 is $\mathrm{SU}(3)/\mathrm{SU}(2)$ or $\mathrm{Spin}(6)/\mathrm{Spin}(5)$ and G_2/H_2 is $\mathrm{Spin}(4e+1)/\mathrm{Spin}(4e-1) = \mathrm{UT}(S^{4e})$ with $e \geq 2$.

The last case, where G_2/H_2 is $\mathrm{UT}(S^{4e})$ with $e \geq 2$, is easy to exclude. Let $N = (G_1 \times G_2)/(H_1 \times H_2 \times H_3)$, so that $M = N/S^1$. Because the groups involved in N are simply connected, N is 2-connected, and so

N is the S^1-bundle over M corresponding to a generator of $H^2(M, \mathbf{Z})$. Since M has the integral cohomology ring of $\mathbf{CP}^{4e} \# \mathbf{HP}^{2e}$, the spectral sequence of this S^1-bundle shows that N has the integral cohomology ring of $S^5 \times \mathbf{HP}^{2e-1}$. Next, let Y be the manifold $(G_1 \times G_2)/(H_1 \times H_2)$, which is an SU(2)-bundle over N because $H_3 = \mathrm{SU}(2)$. Because $H_2 = \mathrm{Spin}(4e-1)$ and $e \geq 2$, H_2 acts trivially on G_1 (which is Spin(6) or SU(3)), and so Y is a $\mathrm{UT}(S^{4e})$-bundle over S^5. In particular, Y is 4-connected. Also, the spectral sequence computing the cohomology of Y collapses for degree reasons, and so Y has 2-torsion in its cohomology because $\mathrm{UT}(S^{4e})$ does. But Y is also an S^3-bundle over N, and because Y is 4-connected, the Euler class of this bundle must be a generator of $H^4(N, \mathbf{Z})$. So the spectral sequence of this S^3-bundle shows that Y has the integral cohomology ring of $S^5 \times S^{8e-1}$. This contradicts the fact that Y has 2-torsion. So this case, $G_2/H_2 = \mathrm{UT}(S^{4e})$ with $e \geq 2$, does not occur.

Next, we consider the case where G_1/H_1 is $\mathrm{Spin}(6)/\mathrm{Spin}(5) = S^5$ and G_2/H_2 is $\mathrm{SU}(4)/\mathrm{SU}(3) = S^{15}$, with SU(3) acting nontrivially on Spin(6). Here $e = 2$. The point is that any nontrivial action of SU(3) on S^5 is isomorphic to the standard action, and hence is transitive. Thus H acts transitively on the factor G_1 of G, contrary to Convention 3.4.

This completes the proof that $\mathbf{CP}^{4e} \# \mathbf{HP}^{2e}$ is not homotopy equivalent to a biquotient for all $e \geq 2$. Ironically, the hardest case of all is the case $e = 1$, that is, the proof that $\mathbf{CP}^4 \# \mathbf{HP}^2$ is not homotopy equivalent to a biquotient.

For $e = 1$, it remains to consider the case where G_1/H_1 is either $\mathrm{SU}(3)/\mathrm{SU}(2) = S^5$ or $\mathrm{Spin}(6)/\mathrm{Spin}(5) = S^5$, G_2 is isomorphic to Sp(4), and H_2 is isomorphic to SU(2). Thus the 9-manifold N is a biquotient of the form

$$(\mathrm{SU}(3) \times \mathrm{Sp}(4))/\mathrm{SU}(2)^3$$

or

$$(\mathrm{Spin}(6) \times \mathrm{Sp}(4))/(\mathrm{Spin}(5) \times \mathrm{SU}(2)^2),$$

where the first factor H_1 acts by the standard inclusion on one side of G_1 and trivially on the other side of G_1. Also, if H_2 or H_3 (each isomorphic to SU(2)) acts trivially on G_1, and acts trivially on one side of Sp(4) and by the standard inclusion $V \oplus \mathbf{C}^2$ on the other, then we can replace the quotient $\mathrm{Sp}(4)/\mathrm{SU}(2) = S^7$ by $\mathrm{Spin}(8)/\mathrm{Spin}(7) = S^7$ and thus reduce to an earlier case. So we can assume that neither H_2 nor H_3 has these properties.

We checked earlier that the Stiefel-Whitney class w_4N is not zero. It is convenient to observe now that the Pontrjagin class $p_1(\mathbf{HP}^2)$ is $2z$, where z is a generator of $H^4(\mathbf{HP}^2, \mathbf{Z}) \cong \mathbf{Z}$. Furthermore, Wu showed that the first Pontrjagin class of a closed manifold is a homotopy invariant modulo 12 [28]. Since M is homotopy equivalent to $\mathbf{CP}^4 \# \mathbf{HP}^2$, p_1M in $H^4M/(H^2M)^2 \cong \mathbf{Z}$ is 2 times the class of some generator, modulo 12. Therefore, the S^1-bundle N over M has p_1N equal to 2 times the class of some generator of $H^4N \cong \mathbf{Z}$, modulo 12.

Lemma 8.1. *If H_1 acts trivially on $G_2 = \mathrm{Sp}(4)$, then $H_2 \times H_3 = \mathrm{SU}(2)^2$ does not act freely on $G_2 = \mathrm{Sp}(4)$.*

Proof. Suppose that H_1 acts trivially on $\mathrm{Sp}(4)$ and that $H_2 \times H_3$ acts freely on $\mathrm{Sp}(4)$. Since H_1 acts trivially on $\mathrm{Sp}(4)$, we can enlarge G_1 and H_1 if necessary to make G_1/H_1 equal to $\mathrm{Spin}(6)/\mathrm{Spin}(5)$, rather than $\mathrm{SU}(3)/\mathrm{SU}(2)$. The quotient $\mathrm{Sp}(4)/(H_2 \times H_3)$ is diffeomorphic to S^4, by Lemma 6.2. Also, N is the S^5-bundle over S^4 associated to some homomorphism from $H_2 \times H_3$ to $\mathrm{Spin}(6)$. By our knowledge of p_1N, p_1 of this S^5-bundle in $H^4(S^4, \mathbf{Z}) \cong \mathbf{Z}$ must be 2 times some generator, modulo 12.

By Lemma 6.2, after switching H_2 and H_3 and switching the two sides of $\mathrm{Sp}(4)$ if necessary, $H_2 \times H_3$ acts on $\mathrm{Sp}(4)$ by either $(V_2 \oplus V_3, \mathbf{C}^4)$ or $(V_2 \oplus \mathbf{C}^2, (V_3)^{\oplus 2})$. First suppose that $H_2 \times H_3$ acts on $\mathrm{Sp}(4)$ by $(V_2 \oplus V_3, \mathbf{C}^4)$. The conjugacy classes of homomorphisms from $H_2 \times H_3$ to $\mathrm{Spin}(6)$ have complexifications: $\mathbf{C}^6$, $(S^2V_2)^{\oplus 2}$, $S^2V_2 + S^2V_3$, $(S^2V_3)^{\oplus 2}$, $(V_2)^{\oplus 2} \oplus \mathbf{C}^2$, $(V_3)^{\oplus 2} \oplus \mathbf{C}^2$, $V_2 \otimes V_3 \oplus \mathbf{C}^2$, $S^4V_2 \oplus \mathbf{C}$, and $S^4V_3 \oplus \mathbf{C}$. The given homomorphism from $H_2 \times H_3$ to $\mathrm{Spin}(6)$ must have first Pontrjagin class (that is, $-c_2$ of the complexification) equal to 2 times some generator of $H^4(S^4, \mathbf{Z})$ modulo 12, where both c_2V_2 and c_2V_3 represent the same generator of $H^4(S^4, \mathbf{Z})$. This only occurs when the complexification is $(V_2)^{\oplus 2} \oplus \mathbf{C}^2$ or $(V_3)^{\oplus 2} \oplus \mathbf{C}^2$. But then one of H_2 or H_3 acts trivially on $\mathrm{Spin}(6)$ and by $(V \oplus \mathbf{C}^2, \mathbf{C}^4)$ on $\mathrm{Sp}(4)$, contrary to what we arranged before the lemma.

It remains to consider the case where $H_2 \times H_3$ acts on $\mathrm{Sp}(4)$ by $(V_2 \oplus \mathbf{C}^2, (V_3)^{\oplus 2})$. In this case, c_2V_3 represents a generator of $H^4(S^4, \mathbf{Z})$, while c_2V_2 is 2 times that generator. Going through the list of possible homomorphisms from $H_2 \times H_3$ to $\mathrm{Spin}(6)$ again, we find that the only one whose Pontrjagin class in $H^4(S^4, \mathbf{Z})$ is 2 times a generator, modulo 12, is the one with complexification $(V_3)^{\oplus 2} \oplus \mathbf{C}^2$. But then H_2 acts trivially on $\mathrm{Spin}(6)$ and by $(V \oplus \mathbf{C}^2, \mathbf{C}^4)$ on $\mathrm{Sp}(4)$, contrary to what we arranged before the lemma. q.e.d.

We can now show that the case where G_1/H_1 is Spin(6)/Spin(5) does not occur. In this case, we know that H_1 acts trivially on Sp(4) (which is isomorphic to Spin(5)), by Lemma 4.1. By Lemma 8.1, $H_2 \times H_3$ does not act freely on Sp(4). On the other hand, the action of $H_2 \times H_3$ on Spin(6) is given by one of the homomorphisms listed in the proof of Lemma 8.1. In all these cases, we check immediately that the maximal torus $(S^1)^2$ in $H_2 \times H_3 = \mathrm{SU}(2)^2$ has a fixed point in $\mathrm{Spin}(6)/\mathrm{Spin}(5) = S^5$. Therefore, there is a subgroup of $H = H_1 \times H_2 \times H_3$ which projects isomorphically to $(S^1)^2$ in $H_2 \times H_3$ and which has a fixed point on Spin(6). Since H acts freely on G, this subgroup acts freely on Sp(4). Since H_1 acts trivially on Sp(4), this means that the maximal torus $(S^1)^2$ in $H_2 \times H_3$ acts freely on Sp(4). Since every element of $H_2 \times H_3$ is conjugate to an element of the maximal torus, it follows that $H_2 \times H_3$ acts freely on Sp(4), contradicting what we have shown. Thus the case where G_1/H_1 is Spin(6)/Spin(5) does not occur.

It remains to consider the case where G_1/H_1 is $\mathrm{SU}(3)/\mathrm{SU}(2) = S^5$. That is, the 9-manifold N is a biquotient

$$(\mathrm{SU}(3) \times \mathrm{Sp}(4))/\mathrm{SU}(2)^3,$$

where we know that H_1 acts on SU(3) by the standard inclusion on one side, and we arranged earlier that neither H_2 nor H_3 acts trivially on one side of Sp(4) and by the standard inclusion on the other.

Suppose first that H_2 and H_3 act trivially on G_1. Then $H_2 \times H_3 \cong \mathrm{SU}(2)^2$ must act freely on Sp(4). By Lemma 6.2, up to switching H_2 and H_3 and switching the two sides of Sp(4), $H_2 \times H_3$ acts on Sp(4) by the homomorphisms $(V_2 \oplus \mathbf{C}^2, V_3 \oplus V_3)$ or $(V_2 \oplus V_3, \mathbf{C}^4)$. Therefore, one of the factors H_2 or H_3 acts on Sp(4) trivially on one side and by the standard inclusion on the other, as well as acting trivially on SU(3). This contradicts what we arranged earlier.

So one of H_2 or H_3 acts nontrivially on G_1. Switching H_2 and H_3 if necessary, we can assume that H_2 acts nontrivially on G_1. Since the centralizer of $H_1 = \mathrm{SU}(2)$ in $G_1 = \mathrm{SU}(3)$ is only finite by S^1, H_2 acts only on the other side of G_1 from H_1. It must act on $G_1 = \mathrm{SU}(3)$ by the homomorphism $V_2 \oplus \mathbf{C}$ or S^2V_2. Thus the centralizer of H_2 in G_1 is at most finite by S^1. Since the centralizer of H_1 on the other side of G_1 is finite by S^1, $H_3 = \mathrm{SU}(2)$ must act trivially on G_1. By what we arranged earlier, it follows that H_3 does not act on Sp(4) trivially on one side and by the standard inclusion on the other.

Suppose first that H_2 acts on SU(3) by $V_2 \oplus \mathbf{C}$. Then H_2 has a fixed point on S^5. More precisely, we see that the diagonal subgroup

$\Delta_{12} \cong \mathrm{SU}(2)$ in $H_1 \times H_2$ has a fixed point in SU(3). Since H_3 acts trivially on $G_1 = \mathrm{SU}(3)$, it follows that $\Delta_{12} \times H_3 \cong \mathrm{SU}(2)^2$ acts freely on $G_2 = \mathrm{Sp}(4)$. By the classification of free actions of $\mathrm{SU}(2)^2$ on Sp(4) in Lemma 6.2, together with the fact that H_3 does not act trivially on one side of Sp(4) and by the standard inclusion on the other, $\Delta_{12} \times H_3$ must act on Sp(4) by the homomorphism $V_{12} \oplus \mathbf{C}^2$ on one side and $V_3 \oplus V_3$ on the other. Here V_{12} denotes the standard representation of $\Delta_{12} = \mathrm{SU}(2)$. By the Clebsch-Gordan formula, it follows that $H_1 \times H_2 \times H_3 = \mathrm{SU}(2)^3$ acts on Sp(4) by $V_1 \oplus \mathbf{C}^2$ or $V_2 \oplus \mathbf{C}^2$ on one side and by $V_3 \oplus V_3$ on the other. Switching H_1 and H_2 if necessary (which we can do, since they act the same way on $G_1 = \mathrm{SU}(3)$), we can assume that $H_1 \times H_2 \times H_3 = \mathrm{SU}(2)^3$ acts on Sp(4) by $V_2 \oplus \mathbf{C}^2$ on one side and by $V_3^{\oplus 2}$ on the other. Thus, H_1 acts trivially on Sp(4) and $H_2 \times H_3$ acts freely on Sp(4), which contradicts Lemma 8.1.

Therefore $H_2 = \mathrm{SU}(2)$ must act on $G_1 = \mathrm{SU}(3)$ by the homomorphism S^2V_2. By Singhof's description of the tangent bundle of a biquotient [25], w_4N is given by

$$\begin{aligned} w_4N &= w_4\mathrm{su}(3) + w_4\mathrm{sp}(4) - w_4\mathrm{su}(2)_1 - w_4\mathrm{su}(2)_2 - w_4\mathrm{su}(2)_3 \\ &= w_4\mathrm{su}(3) + w_4\mathrm{sp}(4), \end{aligned}$$

using that $w_4\mathrm{su}(2) = 0$ in $H^4(\mathrm{BSU}(2), \mathbf{F}_2)$. Furthermore, we know that H_2 acts on one side of SU(3) by S^2V_2, and no other factor of H acts on that side of SU(3). Since $c_2(S^2V_2) = 4c_2V_2$, the generator c_2 of $H^4(\mathrm{BSU}(3), \mathbf{F}_2)$ pulls back to $4c_2V_2 = 0$ in $H^4(N, \mathbf{F}_2)$. Therefore

$$w_4N = w_4\mathrm{sp}(4).$$

We know that H_3 acts trivially on SU(3), so it must act freely on Sp(4). Furthermore, we know that it does not act trivially on one side of Sp(4) and by the standard inclusion $V_3 \oplus \mathbf{C}^2$ on the other. By Lemma 6.2, H_3 must act on at least one side of Sp(4) by the homomorphism $V_3^{\oplus 2}$ or S^3V_3. These two homomorphisms have centralizers in Sp(4) which are finite by S^1 or finite, so no other simple factor of H acts on the same side of Sp(4). Since $c_2(V_3^{\oplus 2}) = 2c_2V_3$ and $c_2(S^3V_3) = 10c_2V_3$, the generator c_2 of $H^4(\mathrm{BSp}(4), \mathbf{F}_2)$ pulls back to $2c_2V_3$ or $10c_2V_3$ in $H^4(N, \mathbf{F}_2)$, thus to zero. Therefore

$$w_4N = 0,$$

contradicting what we know about N. This completes the proof that $\mathbf{CP}^4 \# \mathbf{HP}^2$ is not homotopy equivalent to a biquotient. Theorem 2.1 is proved. q.e.d.

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