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ISOPERIMETRIC INEQUALITY FOR THE SECOND EIGENVALUE OF A SPHERE

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Abstract

We prove Hersch's type isoperimetric inequality for the second positive eigenvalue on a two dimensional sphere.

1. Introduction

Let (S^2, g) be a Riemannian manifold diffeomorphic to the two-dimensional sphere. Assume that the area of (S^2, g) is equal to the area of a unit sphere in $\mathbb{R}^3$:

$$\text{Area}(S^2, g) = 4\pi.$$

Denote by

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$$

the spectrum of the Laplacian on (S^2, g). The classical isoperimetric inequality of Hersch states that

$$\lambda_1 \leq 2,$$

and equality is attained only when (S^2, g) is a standard sphere in $\mathbb{R}^3$ (see [1]).

The goal of this paper is to prove a similar inequality for λ_2.

Theorem. *Let g be a metric on S^2 such that* $\text{Area}(S^2, g) = 4\pi$. *Then*

$$\lambda_2 < 4.$$

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In the theorem, the inequality turns into equality if we break the sphere (S^2, g) into two round spheres, both of area 2π. The inequality is isoperimetric, i.e., there exists a sequence of metrics on S^2 of areas 4π for which λ_2 tends to 4. To get such metrics, one can simply connect two round spheres of area 2π by a thin passage.

It is an interesting question to get sharp upper bounds for all the eigenvalues λ_n in terms of $\mathrm{Area}(S^2, g)$. In [2], Korevaar proved that there exists a constant C such that

$$\lambda_n \leq \frac{1}{\mathrm{Area}(S^2, g)} C \cdot n.$$

We expect that the last inequality holds for $C = 8\pi$, with equality when the sphere is broken into n equal round spheres.

2. Proof of the theorem

(1) Let $(ds)^2$ be the standard metric on the round sphere (S^2, can). We may assume without loss of generality that the metric g is conformally equivalent to $(ds)^2$, i.e., that $g = \omega(ds)^2$. There exists a unique point $e \in \mathbb{R}^3$, $|e| < 1$, so that, for the Möbius map

$$\mu : S^2 \to S^2$$

given by the formula

$$\mu(x) = \mu_e(x) = \frac{(1 - |e|^2)x - (1 - 2\langle e, x\rangle + |x|^2)e}{1 - 2\langle e, x\rangle + |e|^2|x|^2}, \quad x \in \mathbb{R}^3,$$

we have the orthogonality relations

$$\int_{S^2} X_i \circ \mu dg = 0, \quad i = 1, 2, 3,$$

where the X_i are the coordinate functions in $\mathbb{R}^3$ ([1], [3]). The same result is also true for any nonsingular measure ω on S^2.

For $X \in \mathbb{R}^3$ and $s \in S^2$, put $X_s = \langle X, s\rangle$. Let s_0 maximize the integral

$$\int_{S^2} X_s^2 \circ \mu dg$$

over all $s \in S^2$. Put

$$u = u_\mu = X_{s_0} \circ \mu.$$

Then since $X_1^2 + X_2^2 + X_3^2 = 1$ on S^2

$$\int_{S^2} u^2 dg \geq \frac{1}{3} \int_{S^2} dg$$

and

$$\int_{S^2} |\nabla u|^2 dg = \int_{S^2} |\nabla u|^2 ds = \int_{S^2} |\nabla X_i|^2 ds = \frac{8\pi}{3}.$$

Hence

$$\lambda := \int_{S^2} |\nabla u|^2 dg \Big/ \int_{S^2} u^2 dg \leq 2.$$

We call the function u a quasieigenfunction and λ a quasieigenvalue of the metric g. If there is only one choice of the point s_0 which maximizes the above integral, then we say that the quasieigenfunction u is simple. The nodal set of u is a circle on S^2, which we denote by $N = N_u = N_g$. The center of N will be $-e$.

(2) Denote by M the set of all spherical caps on S^2. Take $a \in M$. We denote by a^* the adjacent cap to a, namely $S^2 \setminus \overline{a}$, and by $B(a) \subseteq S^2$ the boundary circle of a. Recall that $B(a) = B(a^*)$. For each $a \in M$, there exists a unique conformal reflection $C_a : S^2 \to S^2$ which changes the orientation of S^2 and is the identity on $B(a)$. We have that $C_a = C_{a^*}$ and $C_a(a) = a^*$. Let $g_a = \omega_a (ds)^2$ be the image of the metric g under C_a. Put

$$G_a = \begin{cases} (\omega + \omega_a)(ds)^2 & \text{on } a, \\ 0 & \text{on } a^*. \end{cases}$$

Then $\text{Area}(S^2, G_a) = \text{Area}(a, G_a) = 4\pi$.

(3) Take $a \in M$ and let $u = u_a$ be a quasieigenfunction of G_a. Define

$$U = \begin{cases} u & \text{on } a, \\ u \circ C_a & \text{on } a^*. \end{cases}$$

Then

$$\int_{S^2} U dg = 0,$$
$$\int_{S^2} U^2 dg = \int_{S^2} u^2 dG_a$$

and

$$\int_{S^2} |\nabla U|^2 ds = 2 \int_a |\nabla u|^2 ds.$$

Since

$$\int_{S^2} |\nabla u|^2 ds \Big/ \int_{S^2} u^2 dG_a \leq 2,$$

we get

$$\int_{S^2} |\nabla U|^2 ds \Big/ \int_{S^2} U^2 dg < 4.$$

Thus, if the quasieigenfunction u is non-simple, then we get a two-dimensional space of functions U on S^2 for which the last inequality will hold. Hence, by the variational principle for eigenvalues, it follows that the second eigenvalue of (S^2, g) will be less than 4. Therefore, we may assume without loss of generality that, for all $a \in M$, the quasieigenfunction u_a of the metric G_a is simple. Then, for any $a \in M$, the function u_a is determined up to a sign. The circle $n(a) := N_{u_a}$ depends continuously on a. Denote by $\Omega \subseteq M$ the set of all a such that $n(a) \cap B(a) = \varnothing$.

(4) Since $M \cong S^2 \times I$, we have $\pi_1(M) = 0$. Thus, by a suitable choice of sign of u_a, we may assume that the functions u_a are continuously dependent on the parameter $a \in M$. Hence, if we denote by $s(a) \in S^2$ the point where u_a attains its supremum, then we get a continuous map

$$s : M \to S^2.$$

Denote by $p(a) \in S^2$ the center of the cap a. Denote by $g(a)$ the orthogonal projection of the vector $s(a)$ on the plane tangent to S^2 at $p(a)$, namely, $T_{p(a)}S^2$. Note that $g(a) = 0$ if and only if $g(a^*) = 0$.

There exists a finite collection of sets $E_i \subseteq M$, $i = 1, \ldots, N$, such that each E_i is diffeomorphic to a ball and $\bigcup E_i = M$. By Sard's theorem for any $\varepsilon > 0$ there exists a smooth map $\varphi_1 : M \to TS^2$ with $\varphi_1(a) \in T_{p(a)}S^2$ for all $a \in M$ such that $|\varphi_1| < \varepsilon$; and, if we define $F_1 = f + \varphi_1$, then $F_1^{(-1)}(0) \cap E_1$ is a nonsingular one-dimensional set. We can define inductively a sequence of φ_i and $F_i = F_{i-1} + \varphi_{i-1}$ such that $|\varphi_i| < \varepsilon$ and $F_i^{(-1)}(0) \cap (E_1 \cup \cdots \cup E_i)$ is a nonsingular one-dimensional set. If we put $F = F_N$, then $F^{(-1)}(0)$ is a union of nonsingular curves and $|f - F| < N\varepsilon$ Since, for all $a \in M$, $n(a)$ does not coincide with $B(a)$, for sufficiently small $\varepsilon > 0$ we have $F^{(-1)}(0) \subseteq \Omega$.

(5) Let Γ be the zero set of F. As discussed above, we assume that $\varepsilon > 0$ is so small that $\Gamma \subseteq \Omega$. Then Γ is a finite union of smooth curves, without intersection.

Let $Q \subseteq M$ be the set of all points $q \in M$ such that $B(q)$ is a big circle on S^2. Then $Q \cong S^2$ and p maps Q into S^2.

If we restrict the map F to Q, then we get a vector field on S^2 which we denote by v. Put $Z = Q \cap \Gamma$. Then the critical points of v are precisely the image of Z under p. By taking a small variation of F, we may assume without loss of generality that Γ intersects Q transversally at Z.

Let $\gamma \subseteq \Gamma$ be a curve. Assume that the intersection $\gamma \cap Q$ has more than one point. Let z_1 and z_2 be subsequent points on γ of $\gamma \cap Q$. Let $\xi_t \subseteq M$, $t \in [1, 2]$, be a continuous family of small loops around γ such that $\xi_1 \subseteq Q$, $\xi_2 \subseteq Q$ and $p(\xi_i)$ is a loop around $p(z_i)$, $i = 1, 2$. The orientations of ξ_1 and ξ_2 on Q are clearly opposite. Therefore, the orientations of $p(\xi_1)$ and $p(\xi_2)$ on S^2 are also opposite. On the other hand, the rotation of the vector $F(a)$ as the point a goes around the loop ξ_t is independent of t. Therefore, the indices of v at the points $p(z_1)$ and $p(z_2)$ are opposite.

Therefore, there exists a curve $\zeta \subseteq \Gamma$ which has odd points of intersection with Q. Such a curve ζ is necessarily unclosed, and hence the endpoints of ζ are on ∂M. Let us parametrize ζ by $\zeta = \{\zeta_t, t \in [0, 1]\}$, so that $\zeta_0, \zeta_1 \in \partial M$. By the oddness of $\zeta \cap Q$, ζ_t tends to a point as $t \to 0$ and to S^2 as $t \to 1$. Consequently,

$$\begin{aligned} &\mathrm{Area}(\zeta_t, g) \to 0 \quad \text{as } t \to 0, \\ &\mathrm{Area}(\zeta_t^*, g) \to 0 \quad \text{as } t \to 1. \end{aligned}$$

Denote $u_t := u_{\zeta_t}$, $N_t := N_{u_t}$. Then

$$\int_{\zeta_t} u_t^2 dg \to 0 \quad \text{as } t \to 0$$

and

$$\int_{\zeta_t^*} u_t^2 dg \to 0 \quad \text{as } t \to 1.$$

Thus there exists $t_0 \in (0, 1)$ so that

$$\int_{\zeta_{t_0}} u_{t_0}^2 dg = \int_{\zeta_{t_0}^*} u_{t_0}^2 dg.$$

Define

$$U = \begin{cases} u_{t_0} & \text{on } \zeta_{t_0}, \\ u_{t_0} \circ C_{\zeta_{t_0}} & \text{on } \zeta^*_{t_0}. \end{cases}$$

Since $\zeta_{t_0} \in \Omega$, $N_{t_0} \cap N^*_{t_0} = \emptyset$. Hence the function U has three nodal domains: $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ such that $\partial\mathcal{D}_1 = N_{t_0}$, $\partial\mathcal{D}_2 = N^*_{t_0}$ and $\mathcal{D}_3 = S^2 \setminus (\mathcal{D}_1 \cup \mathcal{D}_2)$. Since

$$\int_{\mathcal{D}_1} U^2 dg = \int_{\mathcal{D}_2} U^2 dg,$$

we have for $i = 1, 2, 3$ that

$$\frac{\int_{\mathcal{D}_i} |\nabla U|^2 dg}{\int_{\mathcal{D}_i} U^2 dg} = 2\lambda,$$

where λ is the quasieigenvalue of the quasieigenfunction $u_{\zeta_{t_0}}$. Since $\lambda < 2$, the theorem follows from the variational principle for the second eigenvalue of (S^2, g).

References

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