J. DIFFERENTIAL GEOMETRY
60 (2002) 55-101

THE GEOMETRY OF FINITE TOPOLOGY BRYANT SURFACES QUASI-EMBEDDED IN A HYPERBOLIC MANIFOLD

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Abstract

We prove that a finite topology properly embedded Bryant surface in a complete hyperbolic 3-manifold has finite total curvature. This permits us to describe the geometry of the ends of such a Bryant surface. Our theory applies to a larger class of Bryant surfaces, which we call quasi-embedded. We give many examples of these surfaces and we show their end structure is modelled on the quotient of a ruled Bryant catenoid end by a parabolic isometry. When the ambient hyperbolic 3-manifold is hyperbolic 3-space, the theorems we prove here were established by Collin, Hauswirth and Rosenberg, 2001.

1. Introduction

We are interested in Bryant (mean curvature one) surfaces M properly immersed in a hyperbolic manifold N^3.

Our main result is that Bryant surfaces M with finite topology properly embedded in N^3 have finite total curvature. To prove this we will study the lift $\widetilde{M}$ of M to the universal cover $\mathbb{H}^3$. We know that $N^3 = \mathbb{H}^3/\Gamma$ where Γ is a discrete subgroup of isometries of $\mathbb{H}^3$ acting properly and discontinuously on $\mathbb{H}^3$, and $\widetilde{M}$ is invariant under the action of Γ. We shall prove that each end of $\widetilde{M}$ is an annulus and by the work of P. Collin, L. Hauswirth and H. Rosenberg [5], we know that such an end is regular, asymptotic to a catenoid cousin and has finite total curvature.

Received January 31, 2002.

We extend this result to a larger class of properly immersed Bryant surfaces which we define as the quasi-embedded ones. In this class we have, for example the immersed catenoid cousin (nodoids). It is a natural class of surfaces in this theory. These surfaces satisfy, by the definition given below, a separating property analogous to the embedded ones. But in contrast with the latter, they admit “ends” that are topologically half-planes. We describe the geometry of these ends, showing that they are asymptotic to a ruled catenoid example in $\mathbb{H}^3$. Moreover we shall prove that a quasi-embedded Bryant surface has finite total curvature.

Definition 1. A Bryant surface properly immersed in a hyperbolic manifold N^3 is quasi-embedded if there exists a compact domain K such that M is properly embedded outside K and $M - K$ separates $N^3 - K$ in a collection of n connected components $(W_k)_{1\leq k\leq n}$; and each W_k can be oriented so that the orientation of $M \cap \partial W_k$ is by the mean curvature vector of M.

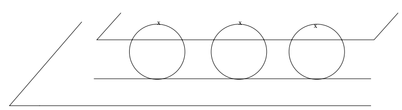
Figure 1-a.

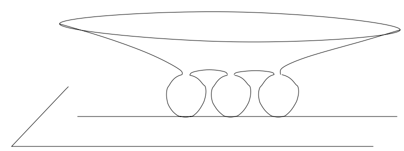
Figure 1-b.

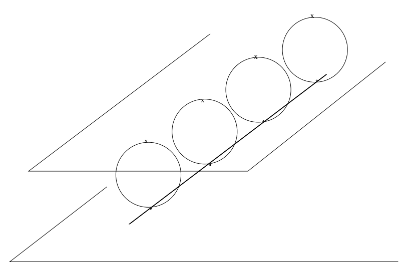
Figure 2-a.

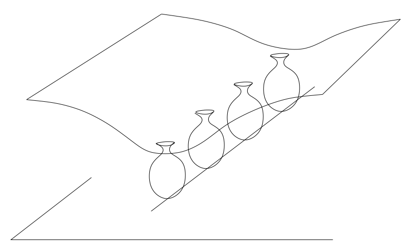
Figure 2-b.

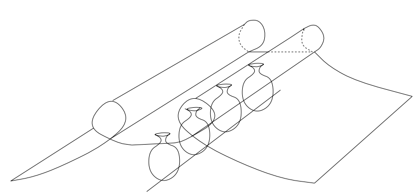
Figure 2-c.

In Section 3 we will prove that except for flat surfaces, properly embedded Bryant surfaces are quasi-embedded.

An important consequence of the work of P. Collin, L. Hauswirth and H. Rosenberg [5] is: there is no properly embedded helicoid Bryant surface in hyperbolic space. Our first motivation in this work was: Does there exist a properly embedded Scherk-type Bryant surface? The answer is no. Recently F. Pacard and F. Pimentel gave new examples of properly embedded Bryant surfaces with finite topology by desingularizing a finite set of tangent horospheres [9]. In particular, consider a finite set of disjoint horospheres in hyperbolic space as in Figure 1-a, all represented by spheres of the same radius in the upper half-space model and consider a horosphere tangent to each of them with end point at infinity i.e., a horizontal plane in this model. The theorem of F. Pacard and F. Pimentel applies. One can desingularize this situation, and all horospheres become asymptotic to catenoid ends (cf. Figure 1-b). Now we consider an infinite number of spheres with the same Euclidean radius and not intersecting each other, distributed in a periodic way and tangent along a straight line at the plane at infinity. We also consider the horizontal plane tangent to all these spheres (See Figure 2-a). If we try to desingularize this example, the horizontal horosphere gives rise to two ends homeomorphic to two periodic half-planes (actually, the horizontal horosphere becomes one topological end in the desingularized surface. It is in the quotient by the parabolic translation, that two annular ends arise. However we will refer to "two half-plane ends" in $\mathbb{H}^3$). One might expect two different behaviours for half-plane ends, in analogy with what happens for the family of catenoid cousin ends. Namely, that there are ends with height, in the upper half-space model, going to infinity (this is the embedded case, Figure 2-b), or to zero (in the non-embedded case, Figure 2-c). We prove that this situation cannot occur for ends belonging to properly embedded surfaces. More precisely, we prove in Section 4 (see Theorem 2):

Main Result 1. *Let M be a properly embedded Bryant surface with finite topology in a hyperbolic manifold N^3. Then M has finite total curvature. Each end $E \subset M$ lifts to some annular regular end $\widetilde{E}$ in $\mathbb{H}^3$ with finite total curvature. In particular $\widetilde{E}$ is asymptotic to a catenoid end, or to a horosphere end.*

As a consequence the example discussed above (see Figure 2-b) cannot be embedded. However, for the quasi-embedded class half-plane ends are quite natural. We prove that when the lift is topologically a half-plane we have (see Theorem 4 and Theorem 5 of Section 5):

Main Result 2. *Let M be a quasi-embedded Bryant surface of finite topology in a hyperbolic manifold N^3. Then M has finite total curvature. Each end $E \subset M$ lifts to some annular regular end $\widetilde{E}$ in $\mathbb{H}^3$ with finite total curvature or to a one-periodic end topologically a half-plane. In this last case, the end E in the quotient space has finite total curvature, and is regular in a sense defined in Section 2. One-periodic ends are asymptotic to ruled catenoid ends.*

We have to mention that there exists examples properly embedded with two half-plane ends but not periodic [8]. Consider a finite set of horospheres distributed along a line at infinity in the upper half space model. Consider the horizontal plane P tangent to each of them and apply the F. Pacard and F. Pimentel theorem. We get a finite topology properly embedded Bryant surface M with catenoid cousin ends. One of these ends (the top end) is a graph over P. Now take a finite number of horospheres distributed along the same line, tangent to M at points of the top end (see Figure 3-a). One can desingularize M with this set of horospheres with the same technique (see Figure 3-b). Iterating this procedure gives the example at the limit.

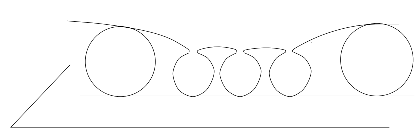

Figure 3-a.

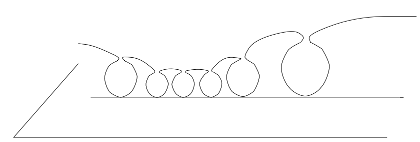

Figure 3-b.

2. Bryant surfaces in hyperbolic manifolds

2.1 The hyperbolic space

Let $\mathcal{L}^4$ be Minkowski 4-space with the Lorentzian metric of signature $(-,+,+,+)$. Hyperbolic 3-space can be represented as

$$\mathbb{H}^3 = \left\{ (y_0, y_1, y_2, y_3) \in \mathcal{L}^4; \sum_{i=1}^{3} y_i^2 - y_0^2 = -1,\ y_0 > 0 \right\}$$

with the metric induced from $\mathcal{L}^4$.

It is useful to identify $\mathcal{L}^4$ with the space of 2×2 hermitian matrices: a point (y_0, y_1, y_2, y_3) corresponds to

$$A = \begin{pmatrix} y_0 + y_3 & y_1 + iy_2 \\ y_1 - iy_2 & y_0 - y_3 \end{pmatrix}$$

Then $\mathbb{H}^3 = \{aa^*; a \in Sl(2, \mathbb{C})\}$, where $a^* =^t \overline{a}$.

The group of orientation preserving isometries of $\mathbb{H}^3$ is isomorphic to $\mathrm{PSL}(2, \mathbb{C})$. Every T in this group can be written up to a sign as an element of $\mathrm{SL}(2, \mathbb{C})$. The action of T on the matrix A is $A \longrightarrow TAT^*$. We recall that such isometries can be classified in terms of its fixed point properties. T is called (see [11]):

- *elliptic*, if it fixes points of $\mathbb{H}^3$,
- *parabolic*, if it fixes no point of $\mathbb{H}^3$, and one point at $\partial_\infty \mathbb{H}^3$,
- *hyperbolic*, if it fixes no point of $\mathbb{H}^3$, and two points at $\partial_\infty \mathbb{H}^3$.

In the upper half-space model a parabolic isometry ϕ is written as $\phi(q) = \xi\tau\xi^{-1}(q)$, where ξ is any isometry of $\mathbb{H}^3$ and τ is a fixed point free Euclidean isometry of the plane $x_3 = 0$. (See [11], Theorem 4.7.2, p. 142.) Thus τ has to be a translation in the plane $x_3 = 0$. Note that this doesn't happen for $\mathbb{H}^n$, $n > 3$. So after an isometry of $\mathbb{H}^3$, namely the map ξ mentioned above, we may consider ϕ to be in fact a horizontal translation and the fixed point in $\partial_\infty \mathbb{H}^3$ as the point at infinity.

Now we describe matrices representing parabolic translations:

Lemma 1. *The matrices $\pm T \in \mathrm{SL}(2, \mathbb{C})$ associated to a horizontal translation in the upper half-space model by $\tau \in \mathbb{C}$ are given by*

$$\pm T = \pm \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}.$$

Proof. The Hermitian matrix A corresponds in the upper half-space the point:

$$x_1 + ix_2 = \frac{y_1 + iy_2}{y_0 - y_3},$$
$$x_3 = \frac{1}{y_0 - y_3}.$$

The image of the isometry represented by $\pm T$ is given by $TAT^* = \overline{A}$, and by direct computation

$$\overline{A} = \begin{pmatrix} y_0 + y_3 + \tau(y_1 - iy_2) + \overline{\tau}(y_1 + iy_2) + |\tau|^2 (y_0 - y_3) & y_1 + iy_2 + \tau(y_0 - y_3) \\ (y_1 - iy_2) + \overline{\tau}(y_0 - y_3) & y_0 - y_3 \end{pmatrix}.$$

and the corresponding point is:

$$\bar{x}_1 + i\bar{x}_2 = \frac{\bar{y}_1 + i\bar{y}_2}{\bar{y}_0 - \bar{y}_3} = \frac{y_1 + iy_2 + \tau(y_0 - y_3)}{y_0 - y_3} = x_1 + ix_2 + \tau$$
$$\bar{x}_3 = \frac{1}{\bar{y}_0 - \bar{y}_3} = x_3.$$

Conversely the correspondence between A, $A \in \mathrm{PSL}(2, \mathbb{C})$ and the group of orientation preserving isometries of $\mathbb{H}^3$ is an isomorphism. q.e.d.

2.2 The Bryant representation

Let M be a simply connected Riemann surface and $F : M \to Sl(2, \mathbb{C})$ a holomorphic immersion satisfying:

$$dAdD - dBdC = 0,$$

where $F = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

Then $f = FF^* : M \to \mathbb{H}^3$ is a conformal immersion of mean curvature one. If $H \in \mathrm{SU}(2)$ then $f = F_1 F_1^*$ where $F_1 = FH$.

Conversely any mean curvature one surface in $\mathbb{H}^3$ is given locally by such an F. The reader should consult [3] and [15] for the details.

The Weierstrass data of the minimal cousin in $\mathbb{R}^3$ is given by:

$$F^{-1}dF = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega.$$

Then one obtains

$$g = -\frac{B'}{A'} = -\frac{D'}{C'}, \quad \omega = AC' - A'C, \quad G = \frac{A'}{C'}.$$

In particular the holomorphic maps A and C are solutions of

$$X'' - \frac{\omega'}{\omega} X' - \omega g' X = 0 \tag{E.1}$$

and maps B and D are solutions of

$$X'' - \frac{(g^2\omega)'}{(g^2\omega)} X' - \omega g' X = 0. \tag{E.2}$$

In the upper half-space model of $\mathbb{H}^3$, one can express the immersion in terms of F.

$$\begin{aligned}(x_1 + ix_2)(z) &= \frac{A\overline{C} + B\overline{D}}{|C|^2 + |D|^2}(z) \\ x_3(z) &= \frac{1}{|C|^2 + |D|^2}(z).\end{aligned}$$

2.3 Periodic Bryant surfaces

We consider hyperbolic manifolds $N^3 = \mathbb{H}^3/\Gamma$ where Γ is a discrete group of isometries of $\mathbb{H}^3$ acting properly and discontinuously. Let $\pi : \mathbb{H}^3 \longrightarrow N^3$ denote the usual covering map. If M is immersed in N^3, we can lift it by $\widetilde{M} = \pi^{-1}(M)$ in $\mathbb{H}^3$, where Γ is a group of isometries leaving the Bryant surfaces $\widetilde{M}$ invariant. (Note that $\widetilde{M}$ is not necessarily connected.) If E is an annular end of M, the lift $\widetilde{E}$ can be an annulus in $\mathbb{H}^3$ or it can be homeomorphic to a half-plane, with an isometry T of Γ leaving $\widetilde{E}$ invariant.

Definition 2. A Bryant end $\widetilde{E}$, topologically a half-plane, invariant by an isometry T is called a one-periodic end (or a T-periodic end).

Then if $F : \widetilde{E} \longrightarrow Sl(2, \mathbb{C})$ is the immersion, there exists a map $\sigma : \widetilde{E} \longrightarrow \widetilde{E}$ such that $F\sigma = TF$ and $F^{-1}(\sigma)dF(\sigma) = F^{-1}dF$. The Weierstrass data (g, ω) pass to the quotient and they are well-defined on E. If E is conformally a punctured disk and g is meromorphic, E has finite total curvature.

Usually, in $\mathbb{H}^3$, ends having a meromorphic hyperbolic Gauss map G are said to be regular. In the case of a one-periodic end, G is not well-defined. We shall then adopt Bryant's original definition (see Proposition 6, p. 344 in [3]) and consider an end to be regular if and only if it's Hopf quadratic differential form $Q = \omega dg$ has a pole of order less than two. With this condition, the associated differential equations (E.1) and (E.2) are regular in the O.D.E's theory.

3. The separating property of properly embedded Bryant surfaces

Properly embedded surfaces separate simply connected ambient spaces in exactly two connected components. When N^3 is not simply connected we prove:

Proposition 1. *Let M be a properly embedded Bryant surface in N^3 a hyperbolic manifold. Then M separates N^3 in two connected components* (*one is mean convex*) *or M lifts to a set of concentric horospheres in $\mathbb{H}^3$* (*we say that M is horospherical*).

Proof. Assume M does not separate N so there is a loop γ intersecting M at only one point p. We shall prove that M is flat and $\widetilde{M}$ is a set of concentric horospheres. Orient γ so the tangent vector of γ at p coincides with the mean curvature vector of M. Consider $\widetilde{\gamma} \in \pi^{-1}(\gamma)$ with end points $\{\widetilde{p_1}, \widetilde{p_2}\} \in \pi^{-1}(p)$ and such that $\widetilde{\gamma} \cap \widetilde{M} = \{\widetilde{p_1}, \widetilde{p_2}\}$. At these end points, the tangent vector of $\widetilde{\gamma}$ coincides with the mean curvature vector of $\widetilde{M}$. (This comes from the local properties of the covering map π.) If $\widetilde{M}$ were connected, it would separate $\mathbb{H}^3$ in two connected components and the tangent vector $\widetilde{\gamma}$ could not coincide with the mean curvature vector at $\widetilde{p_1}$ and $\widetilde{p_2}$. This fact implies that $\widetilde{M}$ has more than one connected component. Let $\widetilde{M_1}$ and $\widetilde{M_2}$ be the connected components containing respectively $\widetilde{p_1}$ and $\widetilde{p_2}$. One of them (say $\widetilde{M_2}$) is contained in the mean convex component of the other component (say $\widetilde{M_1}$). We consider $W \subset \mathbb{H}^3 - (\widetilde{M_1} \cup \widetilde{M_2})$ the connected component which contains $\widetilde{\gamma}$ and we solve a Plateau problem in W. We construct a complete stable Bryant surface Σ properly immersed in W. It is well-known that a such Σ is a horosphere and by construction Σ is included in the mean convex component of $\mathbb{H}^3 - \widetilde{M_1}$. The Half-space Theorem [12] in $\mathbb{H}^3$ implies that $\widetilde{M_1}$ is itself a horosphere and therefore $\widetilde{M}$ is a set of concentric horospheres (M is horospherical).

We now construct Σ. Let Γ be a cycle on $\widetilde{M_1}$, bounding a compact domain $S \subset \widetilde{M_1}, \Gamma = \partial S$. We consider compact 3-dimensional domains $Q \subset W$ with $\partial Q = E \cup S$, where E is a compact surface in W with $\partial E = \Gamma$. We define the functional

$$F(Q) = A(E) + 2V(Q)$$

where $A(E)$ and $V(Q)$ denote the area of E and the volume of Q respectively. We consider a geodesic ball B_R of $\mathbb{H}^3$ centered at $\widetilde{p_1}$ of radius R containing $\widetilde{\gamma}$, Γ and S in its interior. Let W_R be the component of

$W \cap B_R$ that contains $\widetilde{p}_1$. If S is stable then $(Q, S) = (\emptyset, S)$ is a minimun local for F. Otherwise we claim F has a minimum (Q, E) with E a smooth compact Bryant surface $E \neq S$. To see this, we will show that $\widetilde{M}_1 \cup \widetilde{M}_2 \cup \partial B_R$ is a barrier for F with respect to W; i.e., if (Q_n, E_n) is a minimizing sequence for F with $Q_n \subset W$ then Q_n does not cross $\widetilde{M}_1 \cup \widetilde{M}_2 \cup \partial B_R$. Since S is unstable, there exists a small deformation (Q, E) of S such that $\partial E = \Gamma$, $E \cap \widetilde{M}_1 = \Gamma$ and $F(Q, E) < F(S) = A(S)$ (the first eigenfunction of the stability operator associated to the second variation of F is strictly positive with eigenvalue negative).

Let $T_t = \{p \in W_R / \mathrm{dist}_{\mathbb{H}^3}(p, \partial W_R) < t\}$. For $t > 0$ small enough, $\partial T_t \cap W$ is an embedded geodesic graph over ∂W_R. We remove from T_t the part which is a normal geodesic graph on S and we define with the unit normal vector field N on S,

$$N_t = T_t - \{p \in W_R / p = q + \mathrm{Exp}(sN(q)), \forall q \in S, |s| < t\}.$$

We note ∂N_t^1 and ∂N_t^2 part of ∂N_t which are respectively geodesic graph on $\widetilde{M}_1 \cup B_R$ and $\widetilde{M}_2$.

We assume that there is (Q, E) with $E \cap \partial W_R \neq \emptyset$, then ∂N_{t_0} separates Q into two regions Q_1, Q_2 for some $t_0 > 0$ small enough with $Q_1 = N_{t_0} \cap Q$ and $Q_2 = (W_R - N_{t_0}) \cap Q$. Let $E_{t_0} = \partial N_{t_0} \cap Q$, $E_1 = E \cap N_{t_0}$ and E_2 the complement of E_1 in E. Then $Q = Q_1 \cup Q_2$, $\partial Q_1 = E_{t_0} \cup E_1$ and $\partial Q_2 = E_{t_0} \cup E_2$.

If a surface M is oriented by an unit normal vector field N and the mean curvature vector is $H = hN$ where h is a positive real-function on the surface, we can consider the geodesic parallel surface $M(a) = M + \mathrm{Exp}(aN)$, with $a \in \mathbb{R}$ and then direct computation gives us (see, e.g., [14] for a reference):

$$h(a) = \frac{-\sinh 2a + \cosh 2a h(0) - \sinh a \cosh a k(0)}{\cosh 2a - \sinh 2a h(0) + \sinh^2 a k(0)}$$

where $k(0)$ and $h(0)$ are respectively the Gauss and the mean curvature on M. Then on ∂N_{t_0}, we have mean curvature vector pointing into Q_2 with a value $h(t) \geq 1$ along $E_{t_0} \cap \partial N_{t_0}^1$ and a mean curvature vector pointing into Q_1 with a value $h(t) \leq 1$ along $E_{t_0} \cap \partial N_{t_0}^2$ with in this case $a = -t < 0$ (see Figure 4).

Now we have to prove that the functional on (Q, E) is strictly greater than its value on (Q_2, E_2).

$$\begin{aligned} F(Q_2) &= A(E) + A(E_{t_0}) - A(E_1) + 2V(Q) - 2V(Q_1) \\ &= F(Q) + A(E_{t_0})) - A(E_1) - 2V(Q_1). \end{aligned}$$

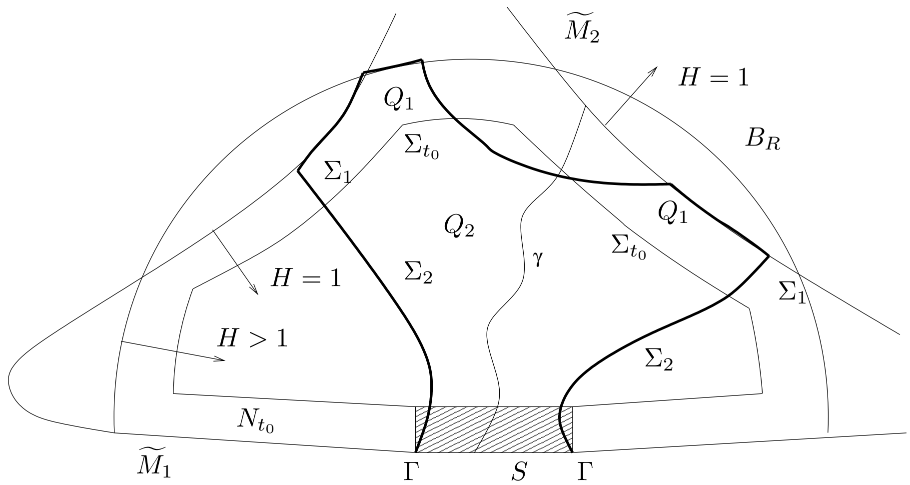

Figure 4.

Let Y be the unit vector field normal to the leaves of the foliation ∂N_t which points into the interior of W along ∂W. We have $\operatorname{div} Y = -2\langle H_t, Y\rangle$ at any point q of ∂N_t, where H_t is the mean curvature vector of the leaf ∂N_t at q and div is the divergence operator. Then by Stokes' Theorem:

$$-2\int_{Q_1} \langle H(t), Y\rangle = \int_{Q_1} \operatorname{div} Y = \int_{\partial Q_1} \langle Y, \vec{n}\rangle = A(E_{t_0}) - A(E_1)$$

where $\vec{n}(t)$ is the outer conormal vector along ∂Q_1. When $H(t)$ is pointing into Q_1, the mean curvature value is less than one and we get

$$2V(Q_1) \geq -2\int_{Q_1} \langle H(t), Y\rangle = A(E_{t_0}) - A(E_1).$$

When $H(t)$ is pointing into Q_2, the mean curvature value is greater than one and

$$A(E_{t_0}) - A(E_1) \leq 0$$

which gives us the result

$$F(Q_2) \leq F(Q).$$

Then a minimizing sequence (Q_n, E_n) can always be modified in such a way that $E_n \cap \partial W_R = \Gamma$ and then a minimizer of F is a smooth Bryant surface.

Now if we consider an exhaustion of $\widetilde{M_1}$ by compact domains $S(R)$ (consider the connected component of $\widetilde{p}_1$ in $\widetilde{M_1} \cap B_R$ and let R go to infinity) and solve the Plateau problem $(Q(R), E(R))$ for each $S(R)$, then a subsequence of these stable Bryant surfaces $E(R_n)$ converges to a complete stable Bryant surface Σ. q.e.d.

Corollary 1. *Let M be a properly embedded Bryant surface in N^3, then M separates N^3 (hence M is quasi-embedded) or M is horospherical.*

The separating property of M on N^3 allows one to apply Theorem 7, 8 and 9 of P. Collin, L. Hauswirth and H. Rosenberg in the quasi-embedded case. For any quasi-embedded Bryant surface M in N^3 with finite topology, we associate the compact K and a connected component W_k of $N^3 - K$ (see Definition 1). We can choose, if necessary, K large enough such that all connected components of $M - K$ are topologically annuli in M. Let $E \subset M$, be an end and W be the connected component of $N^3 - (M \cup K)$ mean convex along E. We consider the surface $\Sigma = \overline{W} \cap K$ (then $\partial W \subset \Sigma \cup (M - K)$. The surface Σ is not of mean curvature one but ∂W is a piecewise smooth embedded surface. We consider $\widetilde{E}, \widetilde{W}, \partial\widetilde{W}, \widetilde{\Sigma}$ their associate lifts in $\mathbb{H}^3$.

Proposition 2. *If $\gamma \subset E$ is a proper non compact arc that separates ∂W, or if $\gamma \subset E$ is a Jordan curve not null homotopic in E, then a connected component in the lift $\widetilde{\gamma}$ can have at most one point at infinity in $\mathbb{H}^3$.*

Proof. This proposition is Theorem 10 in [5] for annular ends in $\mathbb{H}^3$. Assume that E lifts to a T-periodic end $\widetilde{E}$. We consider one connected component of $\widetilde{E}, \widetilde{W}, \partial\widetilde{W}$ and $\widetilde{\Sigma}$. For such a component $\partial\widetilde{W}$ separates hyperbolic space $\mathbb{H}^3$ in two connected components. We consider a Jordan curve γ not null homotopic in E, then this curve separates ∂W in two connected components M_1, M_2. One, say M_1, is a proper sub-annulus of E and lifts to a half-plane in $\mathbb{H}^3$. The Jordan curve γ lifts to a non compact T-periodic proper arc in $\mathbb{H}^3$. We claim that M_2 cannot be compact. If not the lift of M_2 is contained in a solid cylinder C which contains a lift of the Jordan curve. By considering a compact arc α linking this cylinder in a non trivial way, the component of $\partial\widetilde{W}$ does not separate $\mathbb{H}^3$, a contradiction (this arc is only intersecting the half-plane; see Figure 5).

Suppose some connected component of $\widetilde{\gamma}$ has exactly two points p_1 and p_2 at infinity. By using M_1, M_2 as barriers we construct two non compact surfaces Σ_1, Σ_2 properly embedded with mean curvature one outside a compact set K_1 of N^3. These surfaces will verify the properties listed in Theorem 8 of [5] in N^3:

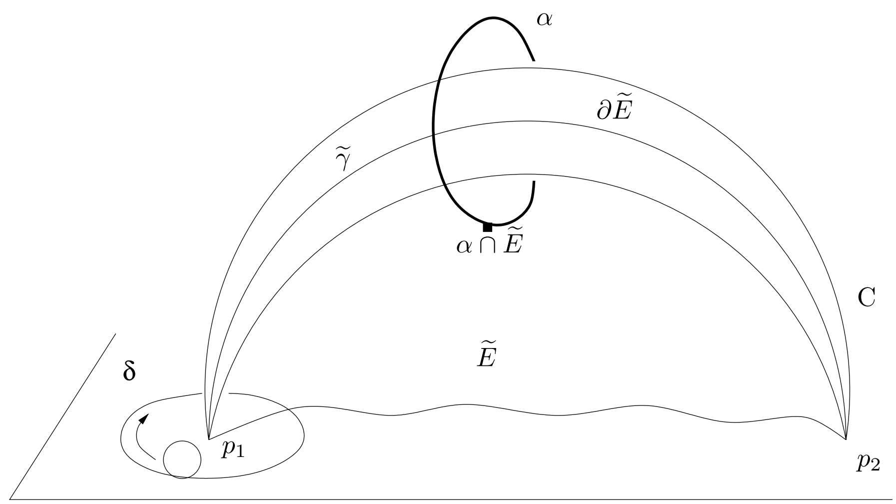

Figure 5.

a) Σ_1 and Σ_2 are stable, $\partial\Sigma_1 = \partial\Sigma_2 = \gamma$, $\Sigma_1 \cap \Sigma_2 = \gamma$.

b) $\Sigma_1 \cup \Sigma_2$ bounds a domain R mean convex outside the compact K_1 and contained in W.

c) $\Sigma_1 \cup M_1$ separates N^3 and $\Sigma_2 \cup M_2$ as well.

If $\Sigma \subset \partial K$ has mean curvature strictly greater than one and the mean curvature vector of Σ points into W, we can apply Theorem 8 of [5] to construct $\Sigma_1 \cup \Sigma_2$. In the general case, we can always change the metric in a compact domain of N^3 to obtain a barrier Σ which is mean convex in the new metric $\bar{g}$ (see Theorem 10 in [5]).

The surfaces Σ_1 and Σ_2 will have mean curvature one outside a compact domain but we are only interested in their behavior at infinity. The lift of K is contained in a cylinder C which is globally invariant by the action of T and has two points at infinity. This cylinder C is a finite distance from $\widetilde{\gamma}$ in $\mathbb{H}^3$. Then $\widetilde{\gamma}$ bounds two mean curvature one surfaces outside the cylinder C containing the curve $\widetilde{\gamma}$ and $\widetilde{K}$. We consider a path δ in the plane at infinity linking the cylinder C in a non trivial way and a family of small horospheres having points at infinity along

δ as in Figure 5 (note that in Figure 5, Σ_1 and Σ_2 are not represented but Σ_1 is homologous to $\widetilde{E}$). Using the maximum principle at infinity with this family of horospheres and a foliation by catenoid cousins as in Theorem 9 of [5], we conclude that such surfaces cannot exist. Thus a Jordan curve not null homotopic in E can lift to curves having at most one point at infinity.

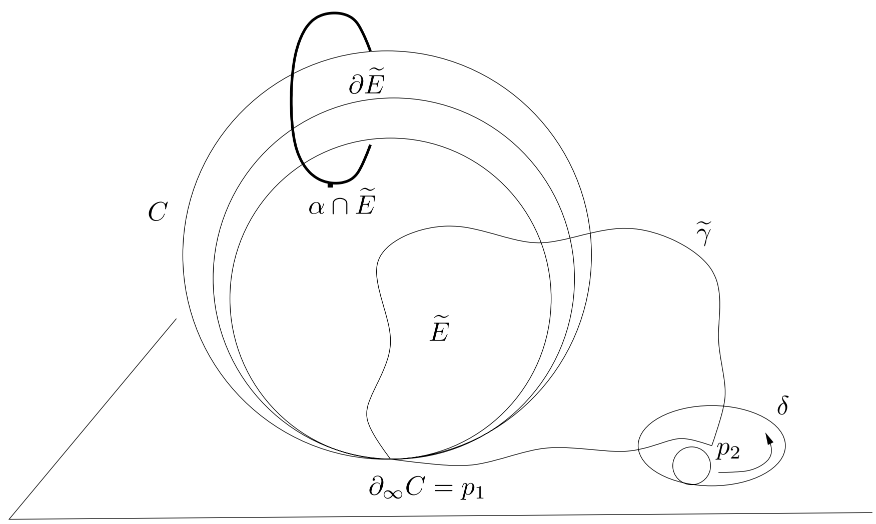

Figure 6.

In case γ is a non compact proper arc on E, one can apply the same arguments. The arc γ separates E into two connected components. Let M_1, M_2 be defined by $\partial W = M_1 \cup M_2$, $E = M_1 \cup (M_2 \cap E)$ and $\partial M_1 = \partial M_2 = \gamma$. The proper arc γ lifts to a set of non compact proper curves in $\widetilde{E}$. One of them, say $\widetilde{\gamma}$, has two different points p_1, p_2 at infinity (by hypothesis). An arc α linking the cylinder C (see Figure 6), as in the previous case, proves that $(M_2 - E)$ cannot be compact. Now we can construct stable surfaces Σ_1, Σ_2 bounded by γ in N^3. Σ_1 and Σ_2 have mean curvature one outside a compact of N^3 and lift in $\mathbb{H}^3$ to mean curvature one surfaces outside the cylinder C. One connected component of $\widetilde{\Sigma}_1$ is bounded by $\widetilde{\gamma}$. In $\partial_\infty\mathbb{H}^3$, the set of points at infinity of the lift of γ is a set of isolated points which can accumulate only at $\partial_\infty\widetilde{E}$, (the unique fixed point of T in $\mathbb{H}^3 \cup \partial_\infty\mathbb{H}^3$). Then we can apply the maximum principle at infinity with a set of small horospheres distributed along a curve δ linking $\widetilde{\gamma}$ at infinity around p_2 a point of $\partial_\infty\widetilde{\gamma}$ different from $\partial_\infty C$ (see Figure 6). q.e.d.

4. The geometry of $\widetilde{M}$ in $\mathbb{H}^3$

In this section we prove our main result for properly embedded Bryant surfaces. It comes as a consequence of Theorem 1 below on the one-periodic ends. We prove that if $\widetilde{E}$ is a one-periodic end, then it must be a bounded vertical graph in the upper half-space model (after an ambiant isometry). We then show that the entire surface is vertically bounded (Theorem 2). By the Half-space Theorem in $\mathbb{H}^3$ [12], such a surface is a horosphere. Thus every end $E \subset M$ must lift to an annulus $\widetilde{E}$ in $\mathbb{H}^3$. By the result of P. Collin, L. Hauswirth and H. Rosenberg [5] these ends have finite total curvature and are regular.

We will now prove Proposition 3:

Proposition 3. *Let E be an end on a quasi-embedded Bryant surface M. If a lift $\widetilde{E} \subset \widetilde{M}$ is a T-periodic end, then T is a parabolic isometry.*

Proof. For a compact K large enough in N^3, ∂E is a nontrivial loop in ∂W. E has genus zero and ∂E separates ∂W into two connected components. We can apply Proposition 2: $\partial \widetilde{E}$ has exactly one point at infinity. Since $\partial \widetilde{E}$ is globally invariant by T, points at infinity of $\partial \widetilde{E}$ are the fixed points of T and therefore T is parabolic. q.e.d.

We now gather some results coming from geometrical techniques introduced in [5] that will enable us to control the geometry at infinity of M. It consists of analyzing the intersection of M with its tangent horosphere at a point. In the analysis that follows we'll be using the half-space model of $\mathbb{H}^3$ with coordinates (x_1, x_2, x_3), $x_3 > 0$. In this model T is a Euclidean horizontal translation. From now on $\widetilde{E}$ will denote a T-periodic end and the period will be assumed to be parallel to the $x_2 - axis$. $\partial \widetilde{E}$ is a periodic curve contained in a horizontal cylinder.

For $q \in M$ let $H(q)$ denote the tangent horosphere at q, i.e., the horosphere tangent to M at q whose mean curvature vector has the same direction as that of M at q, and let $G(q)$ denote the image of the hyperbolic Gauss map at q, i.e., the point of $H(q)$ in $\partial_\infty \mathbb{H}^3$. The local intersection of M and $H(q)$ at q is an analytic curve with isolated singularities, at q there are $2k + 2$ smooth branches meeting at equal angles, where k is an integer at least one. This local picture is as the local intersection of a minimal surface in $\mathbb{R}^3$ with its tangent plane. We'll also use the notation $H^+(q)$ to denote the mean convex component bounded by $H(q)$, and $H_t(q)$ to denote the leaves of the foliation of $\mathbb{H}^3$ by equidistant horospheres at distance t from $H(q)$. For $t > 0$, $H_t(q)$

will be inside $H(q)$ and outside $H(q)$ for $t < 0$. Then we have the following.

Proposition 4. *Let E be an end on a quasi-embedded Bryant surface M that lifts to a T-periodic end $\widetilde{E}$. Let $q \in \widetilde{E}$ with $G(q) \neq \{\infty\}$. Then the connected component at q of $\widetilde{E} \cap H(q)$ has to be compact, by at q we mean a connected component of $\widetilde{E} \cap H(q)$ containing q in its boundary.*

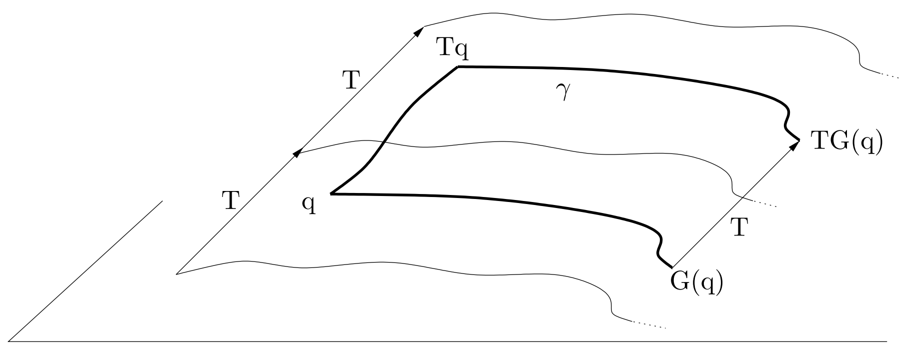

Figure 7.

Proof. If this were not true, there would be a proper arc in $\widetilde{E} \cap H(q)$ starting at q and converging to $G(q)$. Since $\widetilde{E}$ is periodic, there is also a proper arc starting at $T(q)$ and converging to $TG(q)$. Without loss of generality we may assume that these curves do not intersect each other. By connecting q and $T(q)$ by a compact arc, we obtain a proper arc that separates a lift of ∂W into two connected components, with $G(q)$ and $TG(q)$ at infinity. Notice that $G(q) \neq TG(q)$ since $G(q) \neq \{\infty\}$ by hypothesis. We apply Proposition 2 to this situation which proves the Proposition 4 (see Figure 7). q.e.d.

The next proposition comes from [5].

Proposition 5. *Let E be an end on a quasi-embedded Bryant surface M that lifts to a T-periodic end $\widetilde{E}$. Suppose that for $q \in \widetilde{E}$ we have $\partial\widetilde{E} \cap H(q) = \varnothing$. Then there is at most one compact component at q of $\widetilde{E} \backslash H(q)$, whose boundary is in $H(q)$, by "at q" we mean a connected component of $\widetilde{E} \backslash H(q)$ containing q in its boundary.*

Proof. First we show that a connected component $\widetilde{E}_{\mathrm{Out}}$ of $\widetilde{E} \backslash H(q)$ that is outside $H(q)$ cannot be compact. If $\widetilde{E}_{\mathrm{Out}}$ were compact we would have in particular $\partial\widetilde{E}_{\mathrm{Out}} \subset H(q)$, and we could then consider $t < 0$, $|t|$ large enough such that $\widetilde{E}_{\mathrm{Out}}$ is inside $H_t(q)$. Then, by decreasing $|t|$ we would find a negative t_0 such that $H_{t_0}(q)$ touches $\widetilde{E}_{\text{Out}}$ for the first time at an interior point and this can't happen by the maximum principle. Now consider a compact connected component $\widetilde{E}_1$ of $\widetilde{E}\backslash H(q)$, note that $\partial\widetilde{E}_1 \subset H(q)$, let $D_1 \subset H(q)$ be a compact domain with $\partial D_1 = \partial\widetilde{E}_1$ and Q_1 a compact domain in $H^+(q)$ with $\partial Q_1 = D_1 \cup \widetilde{E}_1$, we claim that Q_1 is mean convex along $\widetilde{E}_1$. Indeed, for a t big enough we have $H_t(q) \subset H^+(q)\backslash Q_1$, then by decreasing t there would be a positive t_0 for which $H_{t_0}(q)$ touches Q_1 for the first time. By the maximum principle the mean curvature vector at this point has to point into Q_1.

Finally, suppose we had two compact connected components $\widetilde{E}_1$ and $\widetilde{E}_2$ at q of $\widetilde{E}\backslash H(q)$, let Q_1, Q_2 be the corresponding domains as considered above. Since $\widetilde{E}$ is a graph over $H(q)$ near q, the mean curvature vectors, noted by $\overrightarrow{H}$, of $\widetilde{E}_1$ and $\widetilde{E}_2$ point into the same connected component C of $H^+(q)\backslash(\widetilde{E}_1 \cup \widetilde{E}_2)$ near q.

If C is compact, then it is contained in Q_1 or Q_2, say Q_1. So $\overrightarrow{H}(\widetilde{E}_1)$ points into $C \cap Q_1$ and $\overrightarrow{H}(\widetilde{E}_2)$ as well. Now $\widetilde{E}_2 \subset Q_1$ hence $Q_2 \subset Q_1$, so along $\widetilde{E}_2$, $\overrightarrow{H}(\widetilde{E}_2)$ would point into the non compact component of $H^+(q)\backslash Q_2$, contradicting the fact that Q_2 is mean convex along $\widetilde{E}_2$.

If C is not compact, then $\overrightarrow{H}(q)$ points into C, so along $\widetilde{E}_1$, $\overrightarrow{H}$ points into C as well. But along $\widetilde{E}_1$ it must point into Q_1, a contradiction. q.e.d.

Now we prove the central Theorem 1:

Theorem 1. *Let E be an end on a quasi-embedded Bryant surface M that lifts to a T-periodic end $\widetilde{E}$ in $\mathbb{H}^3$. Then there exists a sub-end $E' \subset E$ and a domain $\Omega = \{(x_1, x_2) \in \mathbb{R}^2/x_1 \geq C_0\}$ such that $\widetilde{E'}$ is a bounded vertical graph $(x_1, x_2, u(x_1, x_2))$ where $u : \Omega \longrightarrow \mathbb{R}^+$ is a bounded function:*

$$\sup_{z\in\Omega} u \leq \sup_{z\in\partial\Omega} u \leq C_1.$$

Proof. After an isometry T is parallel to the vector $\mathbf{e_2}$. Fix a constant C_1, such that $C_1 = \sup_{q\in\partial\widetilde{E}} x_3(q)$ and consider the horosphere $\{x_3 = C_1\}$. Let $C_2 > 0$ and consider domains of the half-space $\mathbb{R}^{3+}$ defined by $A = \{(x_1, x_2, x_3) \in \mathbb{R}^{3+}/0 \leq x_3 \leq C_1 \text{ and } |x_1| \geq C_2\}$, $B = \{(x_1, x_2, x_3) \in \mathbb{R}^{3+}/0 \leq x_3 \leq C_1 \text{ and } |x_1| \leq C_2\}$ and $C = \{(x_1, x_2, x_3) \in \mathbb{R}^{3+}/x_3 \geq C_1\}$. First we prove that there is no constant C_2, such that $A \cap \widetilde{E} = \varnothing$, unless $\widetilde{E}$ is part of a horosphere. Then we will use a sequence of points in $A \cap \widetilde{E}$ to prove the theorem.

If there exists such a constant C_2, then $\widetilde{E} \subset (B \cup C)$. Now consider the part $D = B \cup \{x_3 \leq C_3\}$ with small positive constant C_3; C_3 chosen so that $\partial\widetilde{E} \subset B - D$. Since E is a proper annulus in $\mathbb{H}^3/T$, there exists a subend E' which is contained in C/T or in D/T. We will prove in Lemma 2 below, that when $E' \subset C/T$, then E' is part of a horosphere. But before proving Lemma 2, we show E' can not be contained in D/T.

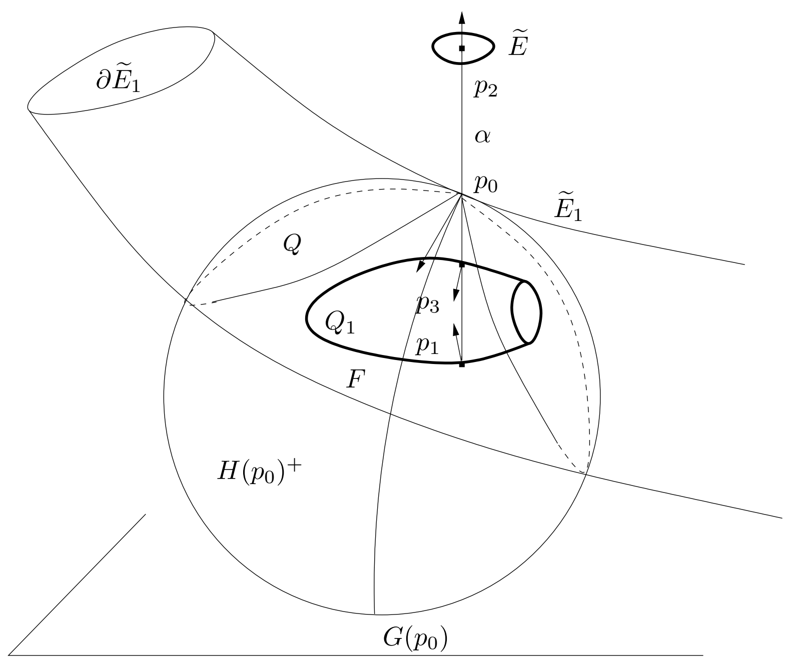

Figure 8.

Assume the contrary: $E' \subset D/T$. First observe that the mean curvature vector at points of E' must point into the northern hemisphere, otherwise, the tangent horosphere at a point $q \in E'$, where $\vec{H}(q)$ points down, would be below $x_3 = 2C_3$, hence disjoint from $\partial\widetilde{E}$. But then $H(q) \cap E'$ is compact, which contradicts Proposition 5. So E' is a local graph with $\vec{H}$ pointing up along E'.

We claim that E' is a global graph and by the proper hypothesis this graph is diverging to $\partial_\infty\mathbb{H}^3$. Then one can construct a path γ similar to the path constructed in Proposition 4 and such an end cannot exist.

If E' is not a global graph, there exists two disks in $\widetilde{E}$ vertical graphs over the same domain at infinity. We consider a segment α of a vertical line intersecting $\widetilde{E}$ at p_1 and p_2. The mean curvature vector is pointing up at these points and M is quasi-embedded hence α intersects $\partial\widetilde{W}$ at a first point p_0 above p_1. The mean curvature vector is pointing down at p_0. Then p_0 is a point of an other annulus or half-plane $\widetilde{E}_1$ having boundary in $\widetilde{\Sigma}$ and $H(p_0) \cap \widetilde{\Sigma} = \varnothing$, since the mean curvature vector points down at p_0. If $\widetilde{E}_1$ is a half-plane end, Proposition 5 implies that $H(p_0) \cap \widetilde{E}_1$ is not compact and in this case one can construct the path γ as in Proposition 4. If $\widetilde{E}_1$ is an annulus, we know from [5] that $\widetilde{E}_1$ is asymptotically a catenoid cousin. Such an end is regular, the hyperbolic Gauss map extends meromorphically at the puncture and in a neighborhood of the puncture the Gauss map is injective and converges to $\partial_\infty \widetilde{E}_1$ (see Corollary 2 in [5]). Then $H(p_0) \cap \widetilde{E}_1$ is compact and by Proposition 1 in [5], the point p_1 is in the bounded component Q of the horoball $H^+(p_0) - \widetilde{E}_1$ that contains p_0.

Since $\widetilde{E}$ is simply connected and proper, the component of $\widetilde{E}$ in the closure of Q that contains p_1 (call it F) is compact. F together with a compact disk on $H(p_0)$, bounds a compact domain $Q_1 \subset Q$, and Q_1 is mean convex along F (one see's this by expanding small horospheres inside $H(p_0)$–from the point at infinity of $H(p_0)$–until they first touch Q_1). Now α must enter Q_1 at p_1 since $\vec{H}(p_1)$ is pointing up, and ∂Q_1 separates $\mathbb{H}^3$ so there is a first point p_3 of F where α leaves Q_1. But then $\vec{H}(p_3)$ points down (since ∂Q_1 is mean convex along F). This contradicts the mean curvature vector of $\widetilde{E}$ pointing up in D.

To prove the fact that $A \cap \widetilde{E} \neq \varnothing$ it remains to prove the Lemma 2 below. Then we will prove Lemmas 3 and 4, to establish Theorem 1.

Lemma 2. *A properly embedded one-periodic end $\widetilde{E}$ such that $\partial\widetilde{E} \subset H$, where H is a horizontal horosphere, cannot lie above H, except if $\widetilde{E}$ itself is part of a horosphere.*

Proof. If this happened to be false the mean convex region of $\mathbb{H}^3$ defined by H would be divided into two components by $\widetilde{E}$. Consider the component into which the mean curvature vector of $\widetilde{E}$ points. Choose a horocycle in the part of H in the above mentioned component, parallel to the period and disjoint from the trace of $\widetilde{E}$ in H.

Now we use a family of equidistant surfaces to a hyperbolic plane P. Let us describe them in the half-space model. A hyperbolic plane P is a half-sphere bounded by a great circle C at infinity or a vertical half-plane bounded by a straight line at infinity. P has mean curvature zero. The equidistant surface $\Sigma_c = \{p \in \mathbb{H}^3 | \mathrm{dist}_{\mathbb{H}^3}(p, P) = c\}$ is a spherical cap (see Figure 9-a) or an oblique plane (see Figure 9-b) bounded by $\partial_\infty P$, in both cases. The angle α between the tangent plane of Σ_c and the plane $\{x_3 = 0\}$ is constant along $\partial_\infty \Sigma_c$ and is determined by the distance c. Σ_c has constant mean curvature $H(c)$, $0 < H(c) < 1$, directed to the connected component of $\mathbb{H}^3$ which contains P.

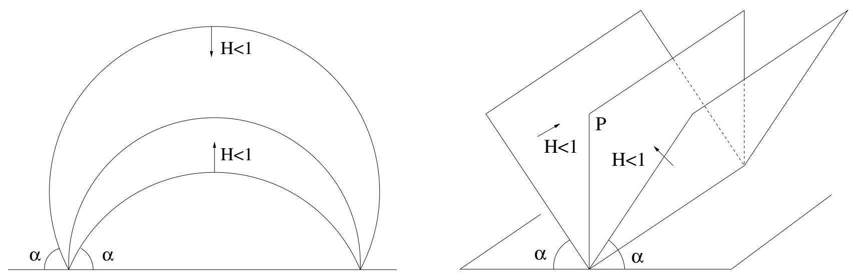

Figure 9-a. Figure 9-b.

We choose an equidistant Σ containing this horocycle, with $\widetilde{E} \cap \Sigma \neq \varnothing$ and such that a foliation by equidistants converging to Σ on the mean convex region defined by Σ is disjoint from the trace of $\widetilde{E}$ in H.

Note that such Σ and the corresponding foliation exist, for otherwise $\widetilde{E}$ would be part of a horosphere. So by considering leaves of the foliation, it is clear that equidistants with height smaller than the height of H are disjoint from $\widetilde{E}$, by construction there would be a first interior point of contact between $\widetilde{E}$ and a leaf of the foliation. But note that at this point the mean curvature vector of the leaf would point into the mean convex region defined by $\widetilde{E}$ and H, and this is impossible by the maximum principle. The equidistants' mean curvature is strictly less than one (see Figure 10). q.e.d.

This lemma implies in particular that there is no constant C_2, such that $A \cap \widetilde{E} = \varnothing$, unless $\widetilde{E}$ is part of a horosphere. For if we had such an end E we would have a subend $E' \subset E$ giving rise to a lift $\widetilde{E}'$ satisfying the above lemma's hypothesis.

Lemma 3. *There is a constant $C_2 > 0$, such that $\widetilde{E} \cap A$ is a vertical graph and the mean curvature vector of $\widetilde{E} \cap A$ points up.*

Proof. Suppose the mean curvature vector pointed down at q (i.e., into the southern hemisphere), then a simple Euclidean calculation (after a translation if necessary) gives

$$\inf_{p \in H(q)} |x_1(p)| \geq |x_1(q)| - 2C_1.$$

If $|x_1(q)| > 2C_1 + \sup_{p \in \partial \widetilde{E}} |x_1(p)|$, then

$$\inf_{p \in H(q)} |x_1(p)| > \sup_{p \in \partial \widetilde{E}} |x_1(p)|,$$

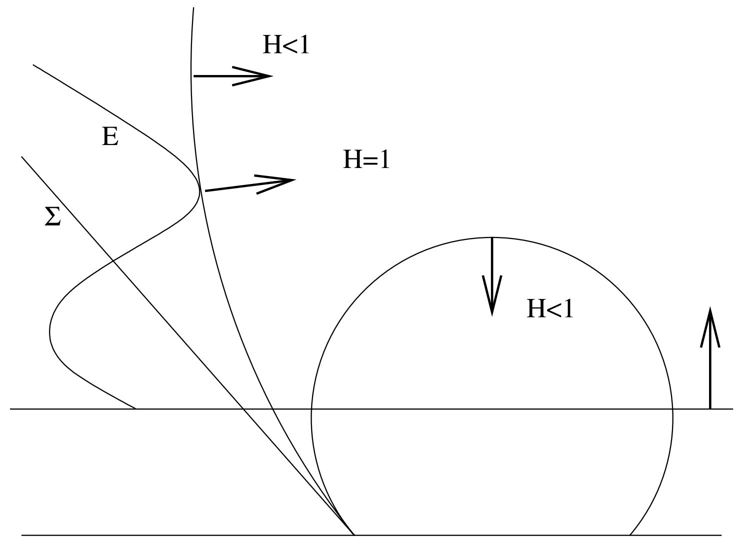

Figure 10.

so $\partial\widetilde{E} \cap H(q) = \varnothing$. Note that the topological picture of $\widetilde{E} \cap H(q)$ is, in a neighborhood of q, an intersection of at least two curves. By the geometrical situation we know that these curves cannot reach $\partial\widetilde{E}$, and by Proposition 4 they're compact. This implies they must enclose at least two compact regions on $\widetilde{E}$, contradicting Proposition 5. So the mean curvature vector points up. q.e.d.

Now we consider the constant C_2 of the last lemma to define A. The following lemma proves the Theorem 1.

Lemma 4. *Let $\{q_n\} \subset \widetilde{E} \cap A$ be a sequence of points such that $|x_1(q_n)| \to \infty$ as $n \to \infty$. Then there is a sub-end $E' \subset E$ that lifts to $\widetilde{E}' \subset A$ and $\widetilde{E}'$ is bounded above.*

Proof. We define a right circular cylinder C with axis parallel to the period such that its axis is $\{x_3 = x_1 = 0\}$ and $\sup_{p\in C} x_3(p) = 2C_1$. $\widetilde{E}$ is properly embedded in $\mathbb{R}^{3+}$ and T-periodic, then $\widetilde{E}/T \cap C/T$ is compact. There is a sub-end $E' \subset E$ that lifts to $\widetilde{E}' \subset \widetilde{E}$ with $\partial\widetilde{E}' \subset C$. Now $\partial\widetilde{E}'$ is homotopic to each ray of $C \cap \{x_3 = 0\}$ i.e; $\widetilde{E}'$ is homotopic to $\{(x_1, x_2, x_3) \in \mathbb{R}^{3+}/x_3 = 0, x_1 \geq C_1\}$ and $\{(x_1, x_2, x_3) \in \mathbb{R}^{3+}/x_3 = 0, x_1 \leq -C_1\}$. If $D \subset \mathbb{R}^3$ is the mean convex component bounded by C, we have $\widetilde{E}'$ that separates $\mathbb{R}^{3+}\backslash D$ into two non compact connected components (one is mean convex along $\widetilde{E}$) and some finite number of compact components (coming from some finite number of topological disks of $\widetilde{E}$ intersecting D).

For $q \in \widetilde{E}'$ let γ be the minimizing geodesic from q to C. Assume $|x_1(q)|$ is big enough so that the highest point of γ is not in D. We parametrize γ by arclength so that $\gamma(0)$ is the highest point and $q = \gamma(t_0)$, $t_0 < 0$.

Let $P(t)$ be the family of hyperbolic planes orthogonal to γ at $\gamma(t)$.

For t very negative $P(t) \cap \widetilde{E}' = \varnothing$, since $\partial_\infty \widetilde{E}' = \{\infty\}$, so there is a first $t_1 \leq t_0$ such that $P(t_1) \cap \widetilde{E}' \neq \varnothing$.

We do Alexandrov reflection of $\widetilde{E}'$ with planes $P(t)$ as t increases from t_1 to 0. Let $S(t)$ be the symmetry of $\mathbb{H}^3$ through $P(t)$, $\widetilde{E}'^{+}(t)$ the part of $\widetilde{E}'$ on the side of $P(t)$ not containing C, and $\widetilde{E}'^{*}(t) = S(t)(\widetilde{E}'^{+}(t))$.

For t slightly larger than t_1, $\widetilde{E}'^{+}(t)$ is a graph over a part of $P(t)$, $int(\widetilde{E}'^{*}(t)) \subset H_1$ where H_1 is the connected component of $\mathbb{R}^{3+} \backslash (D \cup \widetilde{E}')$ mean convex along $\widetilde{E}'$ and the angle between $P(t)$ and $\widetilde{E}'^{+}(t)$ along $\partial \widetilde{E}'^{+}(t)$ is never $\frac{\pi}{2}$. These properties continue to hold until the first t, t_2 say, such that $\widetilde{E}'^{*}(t_2)$ touches C, for if one of these properties failed to hold at some earlier t, $P(t)$ would be a plane of symmetry of $\widetilde{E}'$. Then $\widetilde{E}'$ would be part of a properly embedded compact constant mean curvature one surface with no boundary, a contradiction.

Note that $t_2 < 0$ when $x_3(q) < \sup_{p \in C} x_3(p)$ and $|x_1(q)|$ large enough. In fact, the symmetry of q through $P(0)$, a vertical plane, is lower that $\sup_{p \in C} x_3(p)$, so there is some $t < 0$ such that the symmetry of q through $P(t)$ meets C.

Thus there is some point $\hat{q} \in \widetilde{E}'^{+}(t_2)$ such that $S(t_2)(\hat{q}) \in C$.

Now consider the sequence $\{q_n\}$ such that $x_3(q_n) < C_1$ and $|x_1(q_n)| \to \infty$ as $n \to \infty$, and the corresponding $\hat{q}_n$ associated to the first accident of the Alexandrov reflection. Let $\hat{q}_n^*$ be the reflection of $\hat{q}_n$ on the critical plane, i.e., the plane for which the first accident occurs.

Without loss of generality, since $\widetilde{E}'$ is periodic, we may choose $\{q_n\}$ such that $|x_2(\hat{q}_n^*)| \leq |T|$, where T is the period.

We now show that $|x_1(\hat{q}_n)| \to \infty$ and $x_3(\hat{q}_n) < C_1$ as $n \to \infty$. In fact, as the critical plane of reflection is not vertical we have $|x_1(\hat{q}_n)| > \frac{1}{2}|x_1(q_n)|$, since $|x_1(q_n)| \to \infty$ as $n \to \infty$ it follows that $|x_1(\hat{q}_n)| \to \infty$. Note also that since the critical plane occurs " before" the vertical plane, $x_3(\hat{q}_n^*)$ is bigger than $x_3(\hat{q}_n)$. But since $\hat{q}_n^*$ touches C, $x_3(\hat{q}_n^*)$ is smaller than $2C_1$, so

$$x_3(\hat{q}_n) < 2C_1.$$

Now consider points q_n with $|x_1(\hat{q}_n)|$ big enough such that $\widetilde{E}'$ is a graph for points p_n with $x_3(p_n) < 2C_1$ and $|x_1(p_n)| \geq \frac{1}{2}|x_1(\hat{q}_n)|$. By Theorem 7 in Appendix A, we get for a graph $(x_1, x_2, u(x_1, x_2))$ with $u \leq C_1$:

$$|\nabla u|^2 \leq \frac{2u}{R-2u} \leq \frac{2u}{R-4C_1} \leq \frac{4C_1}{R-2C_1}$$

where $W = \sqrt{1+|\nabla u|^2}$ and R is the Euclidean radius of the tangent horosphere at a point.

From this gradient estimate, as $|x_1(\hat{q}_n)| \to \infty$ we obtain

$$|\nabla u(\hat{q}_n)| \leq \epsilon_n^2,$$

with $\epsilon_n \to 0$, as $n \to \infty$, and $\epsilon_n \sim |x_1(\hat{q}_n)|^{-\frac{1}{4}}$.

Then the maximum oscillation of u on the horizontal Euclidean disk D of radius $\frac{x_3(\hat{q}_n)}{\epsilon_n}$, centered at $\hat{q}_n$, is $2\epsilon_n x_3(\hat{q}_n)$. To check this, notice that $|\nabla u|$ is at most $|x_1(\bar{q}_n)|^{-\frac{1}{2}}$, where $\bar{q}_n$ is the point of D with the minimal value for $|x_1|$. As $|x_1(\hat{q}_n)| = |x_1(\bar{q}_n)| + \frac{x_3(\hat{q}_n)}{\epsilon_n}$ we have $|x_1(\hat{q}_n)| \sim |x_1(\bar{q}_n)| + x_3(\hat{q}_n)\,|x_1(\hat{q}_n)|^{\frac{1}{4}}$. So for n big enough we may write

$$|x_1(\bar{q}_n)| \geq \frac{1}{2}\,|x_1(\hat{q}_n)|\,.$$

Then $|x_1(\bar{q}_n)|^{-\frac{1}{2}} \leq \sqrt{2}\,|x_1(\hat{q}_n)|^{-\frac{1}{2}}$ and the oscillation on D is at most

$$\frac{x_3(\hat{q}_n)}{\epsilon_n}\,|x_1(\bar{q}_n)|^{-\frac{1}{2}} \leq \frac{x_3(\hat{q}_n)}{\epsilon_n}\sqrt{2}\,|x_1(\hat{q}_n)|^{-\frac{1}{2}} \leq 2x_3(\hat{q}_n)\epsilon_n.$$

Now define $D_n = D + (0, 0, x_3(\hat{q}_n))$, a horizontal disk above the graph of u over D. Since $x_3(\hat{q}_n) < 2C_1$ and $\widetilde{E}$ is a graph for points p_n with $x_3(p_n) < 4C_1$ and $|x_1(p_n)| \geq \frac{1}{2}|x_1(\hat{q}_n)|$, we conclude that D_n lies in H_1. Furthermore note that the hyperbolic distance between D_n and the graph of u over D is bounded by $\ln 2$ and the hyperbolic radius of D_n is $\frac{1}{2}\epsilon_n^{-1}$, going to infinity as $n \to \infty$.

Finally, let $t_n < 0$ denote the first time that $S(t_n)(\hat{q}_n)$ touches C. We have $F_n = S(t_n)(D_n) \subset H_1$ and the distance to C is at most $\ln 2$. Thus, by our choice of $\{q_n\}$, there is a fixed compact set of $\mathbb{H}^3$, intersecting all F_n. As $n \to \infty$, the hyperbolic radius of the elements of $\{F_n\}$ goes to infinity, and therefore there is a subsequence of $\{F_n\}$ converging to a horizontal horosphere F which must be in H_1. Thus $\widetilde{E}$ lies below F.

q.e.d.

By the maximun principle $\widetilde{E}$ is bounded and the theorem is proved. q.e.d.

Now we prove our main theorem on properly embedded Bryant surfaces.

Theorem 2. *Let M be a properly embedded Bryant surface with finite topology in a hyperbolic manifold N^3. Then M has finite total curvature. Each end $E \subset M$ lifts to some annular regular end $\widetilde{E}$ in $\mathbb{H}^3$ with finite total curvature. In particular $\widetilde{E}$ is asymptotic to a catenoid cousin end or M is horospherical.*

Proof. To prove the theorem we have to show that each end $E \subset M$ lifts to an annular end $\widetilde{E}$ in $\mathbb{H}^3$ and the result follows from [5]. Since M has finite topology it follows that $C(M) < \infty$. Suppose the theorem is not true, we would then have $\widetilde{E} \subset \widetilde{M}$ a one-periodic end. We'll show that if this happened we would have $\sup_{q\in M} x_3(q) < \infty$ in the upper half-space model with $\partial_\infty \widetilde{E}$ at infinity, and the mean curvature vector of M pointing in a direction that would contradict the Half-space Theorem in $\mathbb{H}^3$ ([12]). By Proposition 1, $\widetilde{M}$ has only one connected component that separate $\mathbb{H}^3$ into two connected components. By considering the Alexandrov reflection of Lemma 4 we obtain a horosphere in the mean convex component of $\mathbb{H}^3 \backslash M$. Indeed, if an accident occured before touching the cylinder C, there would be a first point of contact with $\widetilde{M} \backslash \widetilde{E}$, with an orientation contradicting the maximum principle. But then this horosphere would bound the surface from above, contradicting the Half-space Theorem. q.e.d.

5. The regularity and the finite total curvature of one-periodic ends

In this section we consider properly embedded one-periodic ends, graphs over the domain $\Omega = \left\{(x_1, x_2) \in \mathbb{R}^2/x_1 \geq 0 \text{ and } 0 \leq x_2 \leq |T|\right\}$ that are graphs of functions $u(x_1, x_2)$ over the half-strip. The pair $(\mathbf{e}_1, \mathbf{e}_2)$ will denote the canonical basis of the plane (x_1, x_2), the period T being parallel to $\mathbf{e}_2$.

As trivial examples we have horosphere ends $\{x_3 = cte > 0\}$. The other example we want to consider comes from a one-periodic Bryant surface immersed in $\mathbb{H}^3$, parametrized in the upper half-space model, for $(x,y) \in \mathbb{R}^2$ by:

$$\begin{aligned} x_1 &= x - 2\tanh x, \\ x_2 &= y, \\ x_3 &= \frac{2}{\cosh x}. \end{aligned}$$

This surface may be constructed by lifting, from $\mathbb{C}^*$ to $\mathbb{C}$ via the exponential map, the Weierstrass data of the ruled catenoid cousin. More precisely one takes $g(z) = e^z, \omega(z) = \frac{-e^{-z}}{4} dz, z \in \mathbb{C}$, see Section 6 for details and Figure 11.

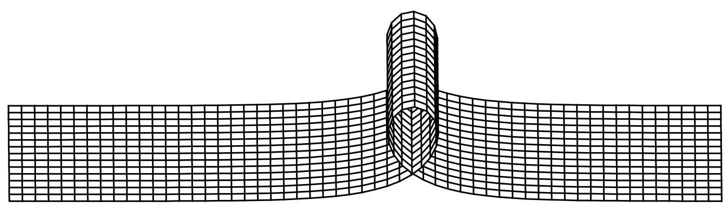

Figure 11 ([13]).

The graph we'll be interested in, coming from this surface, can be described as follows. After a horizontal translation parallel to the $x_1 - axis$ we may place the surface in a manner such that it has a vertical tangent plane for $x_1 = 0$ and the self-intersection lies in the region $x_1 > 0$. The horocycle intersecting the vertical plane $\{x_1 = 0\}$ divides the translated surface, as an abstract surface, into two parts. We then consider a half-plane end consisting of the part of the translated surface lying in the region $x_1 \geq 0$. We denote such an end by $\widetilde{E}(1)$ and call it the standard end.

In the sequel it will be convenient to consider an isometric family of ruled ends consisting of isometric copies of $\widetilde{E}(1)$, obtained by homothety. More precisely we define $\widetilde{E}(t), t \in \mathbb{R}^+$, to be the end $\{\frac{1}{t}(x_1, x_2, x_3)/(x_1, x_2, x_3) \in \widetilde{E}(1)\}$. Note that the profile curve of $\widetilde{E}(1)$ decreases as $4e^{-x_1}$ and $\widetilde{E}(t)$ decreases as $\frac{4}{t}e^{-tx_1}$. We call t the growth of $\widetilde{E}(t)$ (see Section 6 for details).

Our next theorem shows that a one-periodic graph end lies either in the region bounded by two horizontal horospheres or in the region bounded by two ruled ends with the same growth. We will then prove that graph ends have finite total curvature and are regular.

Theorem 3. *A one-periodic end $\widetilde{E}$ is contained in the region bounded either by two horizontal horospheres $H_1 = \{x_3 = \inf_{\partial\Omega} u\}$ and $H_2 = \{x_3 = \sup_{\partial\Omega} u\}$, or by two ends of ruled surfaces $\widetilde{E}(\alpha)$ and $\widetilde{E}(\alpha) + \lambda \mathbf{e}_1, \lambda \in \mathbb{R}$.*

Proof. By a reasoning analogous to the one used in Lemma 2, we know that a Bryant graph over Ω with $\sup_{p\in\partial\Omega} u(p) = C_1 < \infty$ is bounded from above by C_1. In fact, by considering the same foliation as in Lemma 2, starting with an equidistant surface intersecting a point with height greater than C_1, we would obtain a first interior point of contact, which is impossible by the maximum principle.

First we show that one can find a ruled end $E(s)$, with $\partial E(s)$ strictly above ∂E and $E(s) \cap E = \gamma$, γ a compact curve non-homologous to zero in $\mathbb{H}^3/T$. After knowing how E intersects with $E(s)$, we will use the family of ruled ends together with the maximum principle to obtain the desired results.

We consider the family $\widetilde{E}(t)$ of ends of ruled surfaces, graphs over Ω, with the rulings parallel to $\mathbf{e}_2$ and tangent to the vertical plane $\{x_1 = 0\}$ at $\partial\widetilde{E}(t)$. The family $\widetilde{E}(t)$ foliates $\Omega \times \mathbb{R}^+$, and if we write $\widetilde{E}(t)$ as $(x_1, x_2, u_t(x_1, x_2))$, there exists positive constants C_1 and C_2 such that

$$C_1 \frac{e^{-tx_1}}{t} \leq u_t(x_1, x_2) \leq C_2 \frac{e^{-tx_1}}{t}.$$

Let $E(t) = \widetilde{E}(t)/T$. If $E(t) \cap E \neq \varnothing$, with $\partial E(t)$ strictly above ∂E, then by the Theorem 8 in Appendix B, $E(t) \cap E = \gamma$ is compact in $\mathbb{H}^3/T$ (we remark that the hypothesis iii) in Theorem 8 is the gradient estimate of Theorem 7). If γ were non compact $\widetilde{E}$ would have to be a ruled end. Moreover, γ can't be homologous to zero in E, for otherwise we would have a compact domain N on E with $\partial N \subset E(t)$. By letting t vary to infinity there would be a last point of contact with N implying $E = E(t)$ by the maximum principle. So γ is a Jordan curve that generates $\pi_1(E)$ and it bounds an end $E(t)$ lying below E.

If we had t' such that $E(t') \cap E = \varnothing$, and $\partial E(t')$ strictly above ∂E, then let $t > t'$ grow to infinity. As $E(t)$ is vertical over $\partial\Omega \times \mathbb{R}^+$ and E is a graph with bounded gradient, there exists $s > t'$, such that $\partial E(s)$ lies above ∂E and $E(s) \cap E \neq \varnothing$. But then we're precisely in the preceding situation where $E(s) \cap E \neq \varnothing$ and $\partial E(s)$ strictly above E.

Let $E(s)$ be as described above with $E(s) \cap E = \gamma$, a compact curve not homologous to zero in E. Note that $E(s)$ is below E outside a compact set. We translate $E(s)$ horizontally by $\lambda_s \mathbf{e}_1$, with $\lambda_s < 0$ to

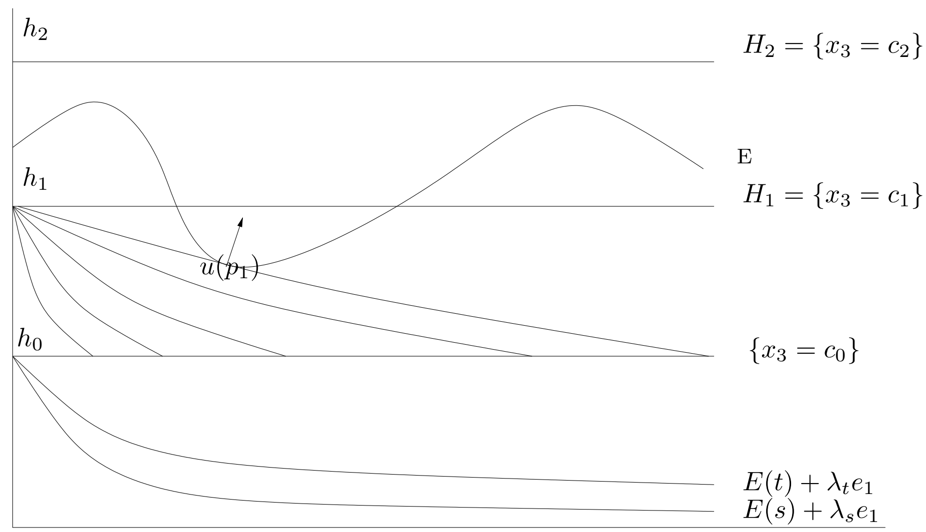

Figure 12.

obtain a Bryant graph below E, that is, $(E(s) + \lambda_s \mathbf{e_1}) \cap E = \varnothing$ and $(E(s)+\lambda_s \mathbf{e_1}) \cap (\partial\Omega \times \mathbb{R}^+) = h_0$, where h_0 is a horocycle contained in the horosphere $\{x_3 = c_0\}$, with $c_0 < \inf_{p\in\partial\Omega} u(p)$. For every $t \in [0, s)$, we may translate horizontally the corresponding ends $E(t)$, in such a way that $(E(t) + \lambda_t \mathbf{e}_1)$ passes through h_0. In particular, this family obtained by translations, foliates the region bounded by $(E(s) + \lambda_s \mathbf{e}_1)$ and $\{x_3 = c_0\}$. Each leaf has h_0 as boundary, and is given by $(E(t) + \lambda_t e_1)$, for $t \in [0, s)$, where $E(0) = \{x_3 = c_0\}$. We now analyze $(E(t)+\lambda_t \mathbf{e_1}) \cap E$ for $t \in [0, s)$. There are two possible cases, either $(E(t)+\lambda_t \mathbf{e_1}) \cap E = \varnothing$ for every $t \in [0, s)$ or there exists $\alpha \in (0, s)$ such that $(E(\alpha)+\lambda_\alpha \mathbf{e_1}) \cap E = \varnothing$ and $(E(t) + \lambda_t \mathbf{e_1}) \cap E \neq \varnothing$ for every $t < \alpha$. The first case implies that E lies in the region bounded by the horospheres $H_1 = \{x_3 = \inf_{\partial\Omega} u\}$ and $H_2 = \{x_3 = \sup_{\partial\Omega} u\}$, while the second case implies that E lies in the region bounded by $E(\alpha)$ and $(E(\alpha) + \lambda_\alpha \mathbf{e}_1)$.

So first suppose that $(E(t) + \lambda_t \mathbf{e_1}) \cap E = \varnothing$ for every $t \in [0, s)$, then $\widetilde{E}$ lies above the horosphere $\{x_3 = c_0\}$ and we show that $\inf_{p\in\partial\Omega} u(p) \leq \inf_{p\in\Omega} u(p)$. In fact, suppose there exists $p_1 \in \Omega$ such that $u(p_1) < \inf_{p\in\partial\Omega} u(p)$ and consider the horosphere $\{x_3 = c_1\}$, with $c_0 < c_1 < \inf_{p\in\partial\Omega} u(p)$, and such that $\{x_3 = c_1\} \cap E \neq \varnothing$. We foliate the region below this horosphere by translations of $E(t)$ as above, with all leaves passing through the horocycle $h_1 = \{x_3 = c_1\} \cap \{x_1 = 0\}$. This family foliates the region bounded by $\{x_3 = c_1\}$ and $\{x_3 = c_0\}$, and by varying

t, there would be a translation of $E(t)$ in this family with a first interior point of contact with $\widetilde{E}$, contradicting the maximum principle. Thus $\widetilde{E}$ would be contained in the region bounded by $H_1 = \{x_3 = \inf_{\partial\Omega} u\}$ and $H_2 = \{x_3 = \sup_{\partial\Omega} u\}$ (see Figure 12).

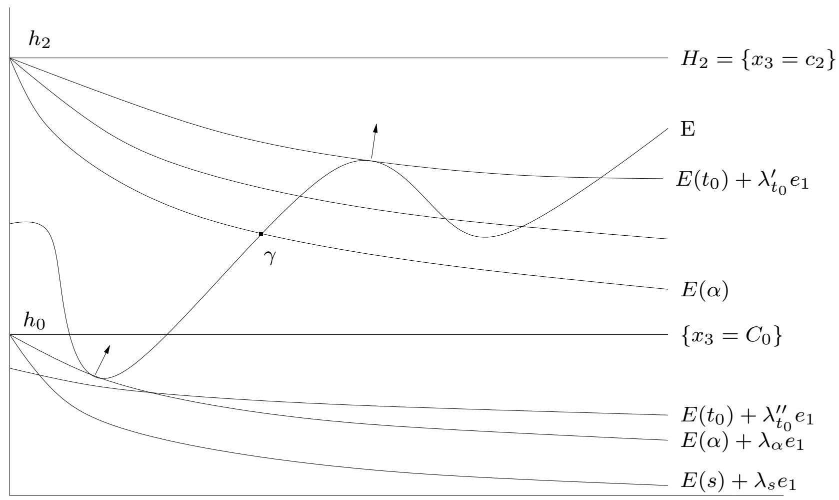

Figure 13.

Now we consider the second case. Let $\alpha \in (0, s)$ be such that $(E(\alpha)+\lambda_\alpha \mathbf{e_1}) \cap E = \varnothing$ and $(E(t) + \lambda_t \mathbf{e_1}) \cap E \neq \varnothing$ for all $t < \alpha$. As $\alpha < s$, we have $E(\alpha)$ above $E(s)$, $\partial E(\alpha)$ above ∂E. We'll show that $E(\alpha) \cap E = \varnothing$, and this implies that E lies in the region bounded from above by $E(\alpha)$ and from below by $E(\alpha) + \lambda_\alpha \mathbf{e_1}$ (see Figure 13).

So suppose $E(\alpha) \cap E = \gamma \neq \varnothing$, then γ is a compact curve, not homologous to zero and E is above $E(\alpha)$ outside a compact set. By a horizontal translation we consider the family $E(t)+\lambda'_t \mathbf{e_1}$ passing through $h_2 = E(\alpha) \cap \{x_1 = 0\}$ and that foliates the region bounded by $E(\alpha)$ and the horizontal horosphere containing h_2. There exists then $t_0 < \alpha$, close enough to α, such that $(E(t_0) + \lambda'_{t_0} \mathbf{e_1}) \cap E \neq \varnothing$. After a horizontal translation of this end we have, for a $\lambda''_{t_0} < 0$, $(E(t_0) + \lambda''_{t_0} \mathbf{e_1}) \cap E = \varnothing$, and $\partial(E(t_0) + \lambda''_{t_0} \mathbf{e_1})$ is below h_0.

Finally, by the asymptotic behavior of the ruled ends, their x_3 coordinate decreases as e^{-tx_1}, we conclude that $(E(t) + \lambda_t \mathbf{e_1})$ is below $(E(t_0) + \lambda''_{t_0} \mathbf{e_1})$ at infinity, for all $t \in (t_0, \alpha)$ and E is above $(E(t_0) + \lambda''_{t_0} \mathbf{e_1})$. This shows that $(E(t) + \lambda_t \mathbf{e_1})$ is below E outside a compact set for $t \in (t_0, \alpha)$. But note also that, by the definition of α, we have $(E(t) + \lambda_t \mathbf{e_1}) \cap E \neq \varnothing$. The asymptotic behavior described above together with $(E(t) + \lambda_t \mathbf{e_1}) \cap E \neq \varnothing$ contradict the maximum principle.

More precisely, it would imply the existence of a first interior point of contact between E and $(E(t) + \lambda_t \mathbf{e_1})$ as we vary t in (t_0, α) (see Figure 13). q.e.d.

Proposition 6. *The end E that lift to a one-periodic bounded graph has the conformal type of the punctured disk.*

Proof. Suppose we knew that $\widetilde{E}$ is conformal to an open half-plane. Then, since t is in fact the exponential map, E would be conformal to the punctured disc, see [6]. To determine the conformal type of $\widetilde{E}$ we'll show that the map from the open half-plane plane $x_3 = 0$, $x_1 > 0$, to $\widetilde{E}_{\text{Eucl}}$, i.e., $\widetilde{E}$ equipped with the Euclidean metric, defined by

$$(x_1, x_2) \to (x_1, x_2, u(x_1, x_2))$$

is quasiconformal. A classical result, see [10], then assures that $\widetilde{E}$ is conformally the half-plane. Pfluger's result says that if we have a quasiconformal map between two Riemann surfaces then they have the same conformal type. Finally, as the Euclidean and hyperbolic metrics are conformal, it then follows that $\widetilde{E}$ is conformal to an open half-plane. To show quasiconformality we calculate the dilatation of the above mentioned map. Recall that a diffeomorphism is quasiconformal when its dilatation is bounded. Following the notation in [1], we write the metric in $\widetilde{E}_{\text{Eucl}}$ in terms of x_1, x_2 as

$$E dx_1^2 + 2F dx_1 dx_2 + G dx_2^2,$$

where

$$\begin{aligned} E &= 1 + u_{x_1}^2, \\ F &= u_{x_1} u_{x_2}, \\ G &= 1 + u_{x_2}^2. \end{aligned}$$

The dilatation is then given by

$$\left(\frac{(E+G) + \sqrt{(E-G)^2 + 4F^2}}{2(EG - F^2)^{\frac{1}{2}}} \right)^{\frac{1}{2}} = \sqrt{1 + |\nabla u|^2}.$$

Since we know that $|\nabla u| \to 0$ as $x_1 \to \infty$ (by Theorem 7) and is periodic in the x_2-coordinate, the dilatation is bounded and the map is then quasiconformal. q.e.d.

Now we study one-periodic ends contained between two horospheres.

Theorem 4. *Let $\widetilde{E}$ be a one-periodic end contained between*

$$H_1 = \left\{ x_3 = \inf_{q \in \partial \widetilde{E}} x_3(q) \right\} \quad \text{and} \quad H_2 = \left\{ x_3 = \sup_{q \in \partial \widetilde{E}} x_3(q) \right\}.$$

Then in $\mathbb{H}^3/\Gamma$, E has finite total curvature and is regular.

Proof. From the previous lemma, we know that E is conformally D^*. Since $x_3 \geq c_0$, we have $|C|$ and $|D|$ bounded (see Section 2.2). By Lemma 1, we have

$$T = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}$$

and then $F(y + 2\pi i) = TF(y)$. $|C|$ and $|D|$ are well-defined and by Lemma 7 of [5], $C = z^{\alpha} f_1(z)$ and $D = z^{\beta} f_2(z)$ for some real α, β and some holomorphic functions f_1, f_2 on D^* that extend meromorphically to the puncture by the bounded property. Since $g = \frac{C'}{D'}$, the end E has finite total curvature. Now $|C| \to c$ and $|D| \to d$, not both zero ($c \neq 0$). Since the end is properly embedded, we have $y_0 = |A|^2 + |B|^2 + |C|^2 + |D|^2 \to \infty$ (in the Lorentzian space). Since $B = \frac{AD-1}{C}$ and $|C|$ and $|D|$ are bounded it follows that $|B| \leq c_1|A| + c_2$ so we have that $|A| \to \infty$.

On the other hand $A(z + 2\pi i) = A(z) + \tau C(z)$ implies that $\frac{A(z)}{C(z)} - \frac{\tau \ln z}{2\pi i} = \phi$ is a well defined meromorphic function on D^*. If ϕ had an essential singularity, then it would assume all values in any neighborhood of zero in D^*. In particular $\left|\phi(z_n) + \frac{\tau \ln z_n}{2\pi i}\right| \leq C_0$ for a sequence z_n converging to zero, contradicting $|A| \to \infty$. Then $A(z) = C(z)\left(\frac{\tau \ln z}{2\pi i} + \phi(z)\right)$ implies that the Schwartzian derivative $S(G) = S\left(\frac{A'}{C'}\right) = S(g) + Q(z)$ has at most a pole of order two (G is not well-defined but it's Schwarzian derivative is a meromorphic function). Since $S(g)$ has at most a pole of order two (g extends meromorphically to the puncture), the end is regular (Q has a pole at most of order two). q.e.d.

Now we will consider a graph end $E \subset \mathbb{H}^3/T$ contained in a region bounded by $E(\alpha)$ and $E(\alpha) + \lambda \mathbf{e}_1$, $\lambda \in \mathbb{R}$. For these ends, we use elliptic P.D.E. techniques to understand the asymptotic behavior of the hyperbolic Gauss map G. Recall that for a point p belonging to a surface in $\mathbb{H}^3$, $G(p)$ is the point in $\partial_\infty \mathbb{H}^3$ touched by the tangent horosphere at p.

More precisely, writing the end as $(x_1, x_2, u(x_1, x_2))$, we will show that without loss of generality one may write the hyperbolic Gauss Map

$G = (G_1, G_2)$ in terms of $x_1, x_2, u(x_1, x_2)$ and its derivatives as

$$G(x_1, x_2) = \left(x_1 - (1+W)\frac{uu_{x_1}}{|\nabla u|^2}, x_2 - (1+W)\frac{uu_{x_2}}{|\nabla u|^2} \right) \tag{1}$$

and we show that $G_1 - x_1$ is bounded. Finally, we use this result together with the fact that the graph lies above a ruled end to prove finite total curvature and the regularity. This will be done by considering the Bryant representation.

To be able to write G as in (1) we must rule out the point at infinity from its image. In other words, we need the following lemma.

Lemma 5. *Possibly restricting ourselves to a subend of E, we may assume that $|\nabla u| \neq 0$.*

Proof. In fact, if we had a point p arbitrarily far from $\partial\widetilde{E}$ such that $|\nabla u(p)| = 0$ then the intersection of the tangent horosphere $H(p)$ at p with $\partial\widetilde{E}$ would be empty ($H(p)$ is a horizontal plane) By looking at the trace of $H(p)/\Gamma$ in E, which we may visualize in the punctured disk, there are at least four branches issuing from p. Note first that none of these can be noncompact (the end is below $E(\alpha)$). Indeed, from the geometrical situation we know that, in a neighborhood of the puncture, E lies outside the mean convex region determined by $H(p)/\Gamma$. This implies the existence of a compact component at p of $E/H(p)$. From the maximum principle this compact component must lie in the interior of $H(p)$. Since we have a graph we know that the mean curvature vector points up contradicting the maximum principle. q.e.d.

Now, assuming $|\nabla u| \neq 0$, we derive the desired expression for G.

Lemma 6. *The hyperbolic Gauss map G of the end $\widetilde{E}$ is given in the (x_1, x_2) coordinates by the following expression.*

$$G(x_1, x_2) = \left(x_1 - (1+W)\frac{uu_{x_1}}{|\nabla u|^2}, x_2 - (1+W)\frac{uu_{x_2}}{|\nabla u|^2} \right),$$

where $W = \sqrt{1 + |\nabla u|^2}$.

Proof. Let $\mathbf{o} = (G_1, G_2, R)$ denote the Euclidean center of the tangent horosphere at a point $\mathbf{p} = (x_1, x_2, u(x_1, x_2))$, where G_1, G_2 are the coordinates of G and R is the Euclidean radius of the horosphere. The normal to the surface, oriented by the mean curvature, is given by

$$\mathbf{N} = \frac{(-u_{x_1}, -u_{x_2}, 1)}{W},$$

where $W = \sqrt{1 + |\nabla u|^2}$. From Euclidean geometry we have

$$\mathbf{o} - R\mathbf{N} = \mathbf{p},$$

so that

$$G_1 + \frac{Ru_{x_1}}{W} = x_1,$$

$$G_2 + \frac{Ru_{x_2}}{W} = x_2,$$

$$R\left(1 - \frac{1}{W}\right) = u.$$

From the last of the above equations $\frac{R}{W} = \frac{u}{W-1}$, substitution in the other two equations gives us the desired result. q.e.d.

To prove the regularity, we want to show that $G_1 - x_1$ is bounded. As $|\nabla u| \to 0$ as $x_1 \to \infty$, by Theorem 7, we have

$$\left|(1+W)\frac{uu_{x_1}}{|\nabla u|^2}\right| \leq \left|(2 + |\nabla u|)\frac{u}{|\nabla u|}\right| \leq \frac{3u}{|\nabla u|}.$$

In this way we are led to study the behavior of $\frac{u}{|\nabla u|}$ or, in other words, to obtain a gradient bound for the function $v = \ln u$. The P.D.E. analysis that follows is inspired by the work of P. Collin and R. Krust [4].

Lemma 7. *Let u be a solution of the mean curvature equation ($H = 1$) in Ω. Then $v = \ln u$ satisfies*

$$L_u v = \frac{|\nabla u|^4}{u^2}\left(1 - \frac{W^2}{(1+W)^2}\right) \tag{2}$$

where

$$L_u v = v_{x_1x_1}(1 + u_{x_2}^2) + v_{x_2x_2}(1 + u_{x_1}^2) - 2u_{x_1}u_{x_2}v_{x_1x_2},$$

and $W = \sqrt{1 + |\nabla u|^2}$. The operator L_u defined above is uniformly elliptic in Ω, and the smallest eigenvalue of the quadratic form associated to it is constant.

Proof. From Lemma 9, in the Appendix, v satisfies

$$\operatorname{div}\left(\frac{\nabla v}{W}\right) = \frac{-|\nabla u|^4}{u^2 W(1+W)^2}.$$

Now

$$\begin{aligned}\operatorname{div}\left(\frac{\nabla v}{W}\right) &= \frac{v_{x_1x_1}}{W} + \frac{v_{x_2x_2}}{W} - \frac{v_{x_1}}{W^3}(u_{x_1}u_{x_1x_1} + u_{x_2}u_{x_1x_2}) \\ &\quad - \frac{v_{x_2}}{W^3}(u_{x_1}u_{x_1x_2} + u_{x_2}u_{x_2x_2}).\end{aligned}$$

As

$$\begin{aligned} u_{x_i} &= uv_{x_i}, \\ u_{x_ix_j} &= u\left(v_{x_ix_j} + \frac{u_{x_i}u_{x_j}}{u^2}\right), \end{aligned}$$

we obtain

$$\begin{aligned} & v_{x_1x_1}W^2 + v_{x_2x_2}W^2 \\ &\quad - v_{x_1}\left(u_{x_1}u\left(v_{x_1x_1} + \frac{u_{x_1}^2}{u^2}\right) + u_{x_2}u\left(v_{x_1x_2} + \frac{u_{x_1}u_{x_2}}{u^2}\right)\right) \\ &\quad - v_{x_2}\left(u_{x_1}u\left(v_{x_1x_2} + \frac{u_{x_1}u_{x_2}}{u^2}\right) + u_{x_2}u\left(v_{x_2x_2} + \frac{u_{x_2}^2}{u^2}\right)\right) \\ &= \frac{-W^2|\nabla u|^4}{u^2(1+W)^2} \end{aligned}$$

or

$$\begin{aligned} & v_{x_1x_1}(1+u_{x_2}^2) + v_{x_2x_2}(1+u_{x_1}^2) - 2u_{x_1}u_{x_2}v_{x_1x_2} \\ &= \frac{-W^2|\nabla u|^4}{u^2(1+W)^2} + \frac{u_{x_1}^4}{u^2} + \frac{2u_{x_1}^2u_{x_2}^2}{u^2} + \frac{u_{x_2}^4}{u^2}. \end{aligned}$$

So finally we have

$$L_u v = \frac{-W^2|\nabla u|^4}{u^2(1+W)^2} + \frac{|\nabla u|^4}{u^2} = \frac{|\nabla u|^4}{u^2}\left(1 - \frac{W^2}{(1+W)^2}\right).$$

To show that L_u is uniformly elliptic we note that the eigenvalues of the quadratic form

$$\begin{bmatrix} 1+u_{x_2}^2 & -u_{x_1}u_{x_2} \\ -u_{x_1}u_{x_2} & 1+u_{x_1}^2 \end{bmatrix}$$

are 1 and 1+ $|\nabla u|^2$. So the smallest eigenvalue is a positive constant and as $|\nabla u|$ is bounded, by Theorem 7, L_u is uniformly elliptic. q.e.d.

Remark 1. The graph end given by a function u lying between graph ends $E(\alpha)$ and $E(\alpha)+\lambda\mathbf{e}_1$ satisfies

$$C_1 e^{-\alpha x_1} \le u(x_1,x_2) \le C_2 e^{-\alpha x_1},$$

with C_1, C_2 positive constants. To see this one may write $E(\alpha)$ and $E(\alpha)+\lambda\mathbf{e}_1$ respectively as

$$(x_1, x_2, u_\alpha(x_1,x_2)),$$
$$(x_1, x_2, u_\alpha(x_1-\lambda,x_2)).$$

Furthermore, the function u_α satisfies

$$K_1 e^{-\alpha x_1} \le u_\alpha(x_1,x_2) \le C_2 e^{-\alpha x_1},$$

where K_1, C_2 are positive constants. For $\lambda < 0$, and with $C_1 = K_1 e^{\alpha\lambda}$ we then have

$$C_1 e^{-\alpha x_1} \le u_\alpha(x_1-\lambda,x_2) \le u(x_1,x_2) \le u_\alpha(x_1,x_2) \le C_2 e^{-\alpha x_1}.$$

From these lemmas we get the estimate:

Lemma 8. *Let $\widetilde{E}$ be a one-periodic graph $(x_1,x_2,u(x_1,x_2))$ contained between two standard ends $\widetilde{E}(\alpha)$ and $\widetilde{E}(\alpha)+\lambda\mathbf{e_1}, \lambda\in\mathbb{R}$, then*

$$\left|\frac{u(x_1,x_2)}{|\nabla u(x_1,x_2)|} - \frac{1}{\alpha}\right| \to 0 \text{ as } x_1 \to +\infty$$

and $G(x_1,x_2)-(x_1,x_2)$ is bounded.

Proof. We consider the function $\phi = \ln(ue^{\alpha x_1}) = v + \alpha x_1$, note that ϕ verifies (2) and by the preceding remark it is bounded from below and above. Note also that the right hand side of (2), as $x_1 \to \infty$ is smaller than $\frac{\widetilde{C}}{x_1^2}$, $\widetilde{C} > 0$, this is due to the estimate $\frac{|\nabla u|^4}{u^2} \le \frac{C}{x_1^2}$ in Theorem 7.

We apply Theorem 12.4 of [7], with ϕ a bounded solution of

$$L_u\phi = f,$$

where $f = \dfrac{|\nabla u|^4}{u^2}\left(1 - \dfrac{W^2}{(1+W)^2}\right)$, which gives for any sub-domain $\widetilde{\Omega} \subset\subset \Omega$

$$\sup_{z\in\widetilde{\Omega}} |\nabla\phi| \le C_0 R^{-1}\left(|\phi|_{C^0} + \sup_{z\in\widetilde{\Omega}} |f| d_z^2\right)$$

with $d_z = \mathrm{dist}(z, \Omega)$ and $R = inf_{z \in \widetilde{\Omega}} \mathrm{dist}(z, \partial\Omega)$.

We may choose $\widetilde{\Omega} = \{(x_1, x_2) \in \mathbb{R}^2 / x_1 \geq \widetilde{R}\}$ (recall that $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 / x_1 \geq 0\}$). As ϕ is bounded, $|\phi|_{C^0}$ is finite. Moreover $|f(z)| d_z^2 = |f(x_1, x_2)| x_1^2 \leq \widetilde{C}$. Thus we may conclude that

$$\left| \frac{\nabla u(x_1,x_2)}{u(x_1, x_2)} + \alpha \right| \leq \frac{C}{x_1},$$

so that $\dfrac{u(x_1,x_2)}{|\nabla u(x_1, x_2)|} \to \dfrac{1}{\alpha}$ as $x_1 \to +\infty$. q.e.d.

We can now state and prove our second theorem.

Theorem 5. *Let $\widetilde{E}$ be a one-periodic end contained between two standard ends $\widetilde{E}(\alpha)$ and $\widetilde{E}(\alpha) + \lambda \mathbf{e_1}$, then in $\mathbb{H}^3/\Gamma$, E has finite total curvature and is regular.*

Proof. If the functions f_1, f_2 appearing in the multi-valued expressions $C = z^\alpha f_1$ and $D = z^\beta f_2$ extend meromorphically to the puncture then the end has finite total curvature. Indeed, as $g = \frac{C'}{D'}$, the function appearing in the multi-valued expression for g will extend meromorphically to the puncture and this implies finite total curvature.

The hyperbolic Gauss map changes by $i\tau$ as we do a 2π rotation in D^* so the map e^G changes by a phase factor, $e^{2\pi\tau i}$. Then we may write this map as a multivalued function on D^* as $e^G(z) = z^\mu h(z)$, where $\mu \in \mathbb{R}$ and h is a single-valued function holomorphic on D^*. We claim that $h(z)$ extends meromorphically to the puncture. From Lemma 8, $G_1 - x_1$ is bounded, so the image of e^G misses a half-plane. If h had an essential singularity at the puncture it would miss at most 2 points, so $z^\mu h(z)$ wouldn't miss a half-plane.

In this way we know $\left|e^G\right|$ grows slower than some power of z, and we may write

$$\left|e^G\right| = e^{G_1} \leq \frac{1}{|z|^p},$$

or

$$G_1 \leq -p \ln |z| .$$

Now write $C(z) = z^\alpha f_1(z)$ and suppose f_1 has an essential singularity, then there is a sequence $\{z_n\}$ in D^* with $z_n \to 0$ such that $|C(z)|^2$ grows faster than $|z|^{-q}$, $q \in \mathbb{N}$ and we write

$$|C(z_n)|^2 \geq \frac{1}{|z_n|^q}.$$

From the expression for x_3 we have

$$\frac{1}{x_3(z_n)} \geq |C(z_n)|^2 \geq \frac{1}{|z_n|^q}.$$

As the end lies above a ruled end with growth α

$$x_3(z_n) \geq C_1 e^{-\alpha x_1(z_n)}.$$

So finally we obtain

$$|z_n|^q \geq C_1 e^{-\alpha x_1(z_n)},$$

or

$$\ln C_1 - \frac{q}{\alpha} \ln |z| \leq x_1(z_n).$$

So if q is big enough, say $q > p\alpha$, we obtain a contradiction since our estimate showed that $x_1(z_n) - G_1(z_n)$ is bounded. The same argument can be applied for D, so that g extends meromorphically to the puncture and the end has finite total curvature.

Since G change by a factor $i\tau$ its Schwartzian derivative is well defined and $S(G)$, as $S(g)$, has at most a pole of order two. Then $Q = S(G) - S(g)$ has a pole of order at most two and the end is regular.

q.e.d.

6. Asymptotic geometry of one-periodic ends

In $\mathbb{H}^3$, we obtain a ruled catenoid example (see Figure 11) by integrating the Weierstrass data (see [13] for details):

$$g(z) = e^z \text{ and } \omega(z) = \frac{-e^{-z}}{4} dz \text{ on } \mathbb{C}.$$

We obtain from this, the immersion $\widetilde{R}(1)(x + iy)$ given by:

$$\widetilde{R}(1) = \begin{cases} x_1 + ix_2 &= x + iy - 2\tanh x \\ \\ x_3 &= \dfrac{2}{\cosh x}. \end{cases}$$

Up to isometries of $\mathbb{H}^3$, this example is unique. Since $\widetilde{R}(1)$ is ruled by horocycles parallel to the x_2 direction, it can be looked as a periodic surface for all T (belonging to the real numbers). In other words, for any $T \in \mathbb{R}$ we have $x_1+ix_2(z+iT) = x_1+ix_2(z)+iT$ and $x_3(z+iT) = x_3(z)$. Homotheties are isometries of $\mathbb{H}^3$ but do not induce an isometry in a quotient spaces $\mathbb{H}^3/T$. A homothety centered at the origin in the plane $\{x_3 = 0\}$ with rapport t gives us an example $\widetilde{R}(t) = t\widetilde{R}(1)$. Thus they generate a family of ruled examples in a fixed quotient.

With $w = e^{z/t} \in \mathbb{C} - \{0\}$, we can obtain this family by integrating Weierstrass data. Here $t > 0, t \in \mathbb{R}$ will represent the parameter in the family with a period T fixed in the x_2 direction. In the variable w we have:

$$g(w) = w^t \text{ and } \omega(w) = \frac{-t}{4}w^{-1-t}dw \text{ with } w \in \mathbb{C} - \{0\}.$$

Now, A, C are solutions of (see Section 2.2)

$$X'' + \frac{(1+t)}{w}X' + \frac{t^2}{4w^2}X = 0 \tag{E.1}$$

and B, D are solutions of

$$Y'' + \frac{(1-t)}{w}Y' + \frac{t^2}{4w^2}Y = 0. \tag{E.2}$$

The related indicial equations are:

$$\delta_1^2 + t\delta_1 + \frac{t^2}{4} = 0 \tag{e.1}$$

$$\delta_2^2 - t\delta_2 + \frac{t^2}{4} = 0. \tag{e.2}$$

Then by fixing the period iT with $T \in \mathbb{R}$, one can integrate these equations and find a solution (see Theorem 6 for details):

$$F = \begin{cases} A = z^{-t/2}\left(a + \dfrac{Tc}{2\pi}\ln(z)\right) & \text{and} \quad C = cz^{-t/2} \\ B = z^{t/2}\left(b + \dfrac{Td}{2\pi}\ln(z)\right) & \text{and} \quad D = dz^{t/2}. \end{cases}$$

Here a, b, c and d are constants, $ad - bc = 1$. We can choose an element $H \in \mathrm{SU}(2)$ such that FH is a solution of the same form with $c \in \mathbb{R}^+$. Now from $g = -\frac{D'}{C'} = -\frac{B'}{A'}$ we have $d = c$ and $b = -a$. From $\omega =$

$AC' - A'C$ we get $c^2 = \frac{\pi t}{2T}$ and from $ad - bc = 1$ we have $a = 1/2c$. Then

$$R(t) = \begin{cases} x_1 + ix_2 &= \dfrac{T}{2\pi}\ln(z) + \dfrac{T(r^{-t} - r^t)}{\pi t(r^{-t} + r^t)} \\ x_3 &= \dfrac{2T}{\pi t(r^{-t} + r^t)}. \end{cases}$$

The surface $R(t)$ has two ends that we note $E(t)$ standard ends. Now we prove that properly embedded T-periodic ends are asymptotic to standard ends.

Theorem 6. *If $E \subset \mathbb{H}^3/\Gamma$ is a regular T-periodic end vertical graph on a half-plane which has finite total curvature then E is uniformly asymptotic to a standard end.*

Proof. First we recall how M. Umehara and K. Yamada [15] integrate Weierstrass data. From [3], we can consider the following Weierstrass data:

$$\begin{aligned} g(z) &= z^\mu h(z) \qquad \text{where} \quad h(0) \neq 0, \mu \in \mathbb{R} \\ \omega(z) &= z^\nu f(z)dz \quad \text{where} \quad f(0) \neq 0, \nu \in \mathbb{R}. \end{aligned}$$

From the fact that the metric ds^2 is complete we have $\min\{2\mu + \nu, \nu\} \leq -1$ and from the regularity $\nu + \mu \geq -1$ (Q has a pole of order at most two). Now, A, C are solutions of

$$X'' - \frac{\omega'}{\omega}X' - \omega g' X = 0 \tag{E.1}$$

and B, D are solutions of

$$X'' - \frac{(g^2\omega)'}{(g^2\omega)}X' - \omega g' X = 0. \tag{E.2}$$

Indicial equations are:

$$\delta_1^2 - (\nu + 1)\delta_1 - q = 0 \tag{e.1}$$

$$\delta_2^2 - (2\mu + \nu + 1)\delta_2 - q = 0 \tag{e.2}$$

where

$$\begin{aligned} q &= 0 \qquad\qquad\quad \text{if} \quad \nu + \mu \geq 0 \\ q &= \mu f(0)h(0) \quad \text{if} \quad \mu + \nu = -1. \end{aligned}$$

If λ_1 and $\lambda_1 - m_1$ and λ_2, $\lambda_2 - m_2$ are solutions of (e.1) and (e.2), a fundamental system of solutions can be written as

$$\begin{array}{ll} X_1 = z^{\lambda_1} f_1(z) & X_2 = z^{\lambda_1 - m_1} f_2(z) + k_1 X_1 \ln(z) \\ Y_1 = z^{\lambda_2} g_1(z) & Y_2 = z^{\lambda_2 - m_2} g_2(z) + k_2 Y_1 \ln(z) \end{array}$$

then we write

$$\begin{array}{l} A = a_1 X_1 + a_2 X_2 \\ C = c_1 X_1 + c_2 X_2. \end{array}$$

By the expression for $G = \frac{A'}{C'} = \frac{B'}{D'}$ and knowing G has a period iT we have

$$\begin{aligned} G(ze^{2\pi n i}) &= G(z) + inT \\ &= \frac{A'(z) + a_2(e^{-m_1 2\pi n i} - 1)(z^{\lambda_1 - m_1} f_2)' + 2\pi n i k_1 a_2 X_1'(z)}{C'(z) + c_2(e^{-m_1 2\pi n i} - 1)(z^{\lambda_1 - m_1} f_2)' + 2\pi n i k_1 c_2 X_1'(z)}. \end{aligned}$$

Now letting $n \to \infty$ we get $c_2 = 0, a_2 \neq 0$ and $k_1 = \frac{Tc_1}{2\pi a_2}$. From

$$G = \frac{a_1}{c_1} + \frac{a_2 X_2'(z)}{c_1 X_1'(z)}$$

we derive easily that m_1 is an integer. We have similar result for B and D and we can suppose that m_1 and m_2 are positive by exchanging the role of λ_i and $\lambda_i - m_i$. Now we will write:

$$F = \begin{cases} A = & a_1 z^{\lambda_1} f_1 + a_2 z^{\lambda_1 - m_1} f_2 + z^{\lambda_1} f_1 \frac{Tc_1}{2\pi} \ln(z) \\ B = & b_1 z^{\lambda_2} g_1 + b_2 z^{\lambda_2 - m_2} g_2 + z^{\lambda_2} g_1 \frac{Td_1}{2\pi} \ln(z) \\ C = & c_1 z^{\lambda_1} f_1 \\ D = & d_1 z^{\lambda_2} g_1. \end{cases}$$

From this we have

$$x_3 = \frac{1}{|z|^{2\lambda_1} |c_1 f_1|^2 + |z|^{2\lambda_2} |d_1 g_1|^2}.$$

We remark that from Lemma 2, we cannot have $\lambda_1 > 0$ and $\lambda_2 > 0$. It would produce an end in the mean convex part of a horosphere $(x_3 \to \infty)$.

In the case where $\mu = 0$, we have from the hypothesis on ν that $\nu = -1$, $q = 0$ and then $m_1 = m_2 = \lambda_1 = \lambda_2 = 0$. Then one can prove easily that E is asymptotic to the quotient of the horosphere $\{x_3 = \frac{1}{|c_1 f_1(0)|^2+|d_1 g_1(0)|^2}\}$. Then we are looking at curves, for ρ fixed

$$\begin{aligned} x_1 + ix_2(\rho e^{i\theta}) &= \frac{A\overline{C}+B\overline{D}}{|C|^2+|D|^2} \\ &= \frac{T}{2\pi}\ln(z) + \frac{a_1\bar{c}_1\rho^{2\lambda_1}|f_1|^2 + b_1\bar{d}_1\rho^{2\lambda_2}|g_1|^2}{\rho^{2\lambda_1}|c_1 f_1|^2 + \rho^{2\lambda_2}|d_1 g_1|^2} \\ &\quad + \frac{a_2\bar{c}_1\rho^{2\lambda_1}z^{-m_1}\overline{f}_1 f_2 + b_2\bar{d}_1\rho^{2\lambda_2}z^{-m_2}\overline{g}_1 g_2}{\rho^{2\lambda_1}|c_1 f_1|^2 + \rho^{2\lambda_2}|d_1 g_1|^2}. \end{aligned}$$

If $\mu \neq 0$, we have $2\lambda_1 - m_1 \neq 2\lambda_2 - m_2$. Then the third term in the equation above cannot be zero. Let $z = \rho e^{i\theta}$, then for some $\rho > 0$ fixed and close to zero, we can find curves $x_1 + ix_2(z)$ arbitrarily close to

$$C(\rho,\theta) = A_1 + iA_2\theta + \rho^m e^{im\theta} A_3$$

for given constants positive A_1, A_2, A_3. These curves are not embedded for $|m| \geq 2$. In the case where $|m| = 1$, we can consider a covering of the end to get $|m| \geq 2$. Then we can deduce from this that $m_1 = m_2 = 0$. From the indicial equations we have $\mu + \nu = -1$ and then $\lambda_1 = -\frac{\mu}{2}$ and $\lambda_2 = \frac{\mu}{2}$. Now it is not hard to conclude and the details are left to the reader. q.e.d.

7. Appendix A

In this section elementary Euclidean geometry gives a useful gradient estimate for a graph.

Theorem 7. *Let E be a graph over the domain $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_1 \geq 0\}$ of a bounded function $|u| \leq C_1$. For $\overline{\Omega} = \{(x_1, x_2) \in \Omega \,|\, x_1 \geq 8C_1\}$, we have for a positive constant C*

$$\frac{|\nabla u|^2}{u} \leq \frac{2}{R-2u} \leq \frac{C}{x_1}$$

where R is the radius of the tangent horosphere at the point $q \in E$.

Proof. The proof is given in Theorem 2 of [5]. We give the argument here.

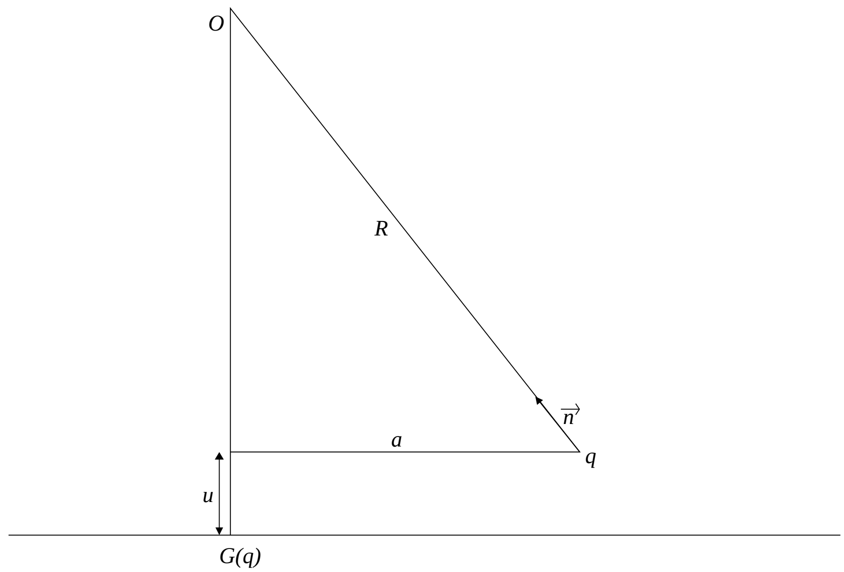

Figure 14 ([5]).

Consider the vertical plane Q containing the unit normal vector $\vec{n}$ to E at q. $G(q)$ is also in this plane and we have Figure 14 in the plane Q.

Here O is the center of $H(q)$ and R is the radius of $H(q)$. The topological picture of $E \cap H(q)$ is an intersection of at least two curves and $E \cap H(q)$ is compact. By proposition 5, $\partial E \cap H(q) \neq \emptyset$ and $R > \frac{x_1}{2} \geq 2C_1$ where $x_1(q) \geq 4C_1$. Then

$$\overrightarrow{n} = \frac{1}{W}(-u_{x_1}, -u_{x_2}, 1), W = \sqrt{1 + |\nabla u|^2}.$$

We have $a = \left| \frac{R}{W}(-u_{x_1}, -u_{x_2}) \right| = \frac{|\nabla u|}{W} R$, and $a^2 = R^2 - (R-u)^2 = u(2R-u)$. Hence

$$\frac{a^2}{R^2} = \frac{|\nabla u|^2}{W^2} = \frac{u(2R-u)}{R^2},$$

$$\frac{|\nabla u|^2}{uW^2} = \frac{2R-u}{R^2} \leq \frac{2}{R}.$$

which implies $\frac{|\nabla u|^2}{u} \leq \frac{2}{R-2u} \leq \frac{4}{x_1 - 4C_1} \leq \frac{8}{x_1}$ if $x_1 \geq 8C_1$. q.e.d.

8. Appendix B

Theorem 8. *Let Ω be a non compact domain in a half-strip $B = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 \geq 0$ and $|x_2| \leq T\}$ with at least one component of $\partial\Omega$ noncompact. Let u_1, u_2 be defined on Ω with their graphs solutions of the mean curvature equation ($H = 1$) in $\mathbb{H}^3$. Suppose the following conditions are satisfied:*

i) $u_2 \leq u_1 \leq 1$ *on* Ω, $u_1 = u_2$ *on* $\partial\Omega$.

ii) $C_1 e^{-\alpha x} \leq u_2 \leq C_2 e^{-\alpha x}$, *for some positive constants* C_1, C_2, α (u_2 *is the graph of a standard end*).

iii) $\dfrac{|\nabla u_i|^2}{u_i} \leq \dfrac{C_3}{x_1^2}, i = 1, 2$ *for some* $C_3 > 0$.

It then follows that $u_1 = u_2$ *on* Ω.

Proof. To prove the theorem we need two lemmas. The first one is the Lemma 8 in [5]:

Lemma 9. *Let u be a solution of the mean curvature equation ($H = 1$) on Ω. Then $v = \ln u$ satisfies:*

$$\operatorname{div}\left(\frac{\nabla v}{W}\right) = \frac{-|\nabla u|^4}{u^2 W (1+W)^2}, \ \text{where } W = \sqrt{1 + |\nabla u|^2}.$$

Before proving the next lemma we recall the family of standard ends in Theorem 3. Let $E(t)$, $t \in [0, \infty)$, be the isometric family of standard catenoid ends. Fix a horocycle h_0 contained in the horosphere $\{x_3 = c_0\}$ and in the vertical plane $\{x_1 = 0\}$. We then consider the end $E(s)$ having h_0 as boundary and a vertical tangent plane for $x_1 = 0$. For every $t \in [0, s)$, we may translate horizontally the corresponding ends $E(t)$, in such a way that $E'(t) = (E(t) + \lambda_t \mathbf{e}_1)$, $\lambda_t < 0$, passes through h_0. In particular, this family obtained by translations, foliates the region bounded by $E(s)$ and $\{x_3 = c_0\}$. Each leaf has h_0 as boundary, and is given by $E'(t) = (E(t) + \lambda_t e_1)$, for $t \in [0, s)$, where $E(0) = \{x_3 = c_0\}$. We will write u_t for the graph solution of the family $E'(t)$, where t indicates the growth at ∞; i.e., $u_t(x_1, x_2) \simeq e^{-t x_1}$, for x_1 big enough.

Lemma 10. *Let $\Omega(X) = \{(x_1, x_2) \in \Omega \,/\, x_1 \leq X\}$, $C(X) = \Omega(X) \cap \{x_1 = X\}$. Define $v = \ln u_1 - \ln u_2$ (u_1, u_2 as in Theorem 8), and $M(X) = \sup\{|v(x_1, x_2)|\,; x_1 = X\}$. Then if $v \neq 0$ there is a $\beta < \alpha$ such that $M(X) \geq c_1(\alpha - \beta)X$, for X sufficiently large, and some constant $c_1 > 0$.*

Proof. Consider a family of standard ends as discussed above with the horocycle h_0 contained in the horosphere $\{x_3 = 1\}$ and in the vertical plane defined by ∂B in such a manner that $E(s)$ has growth $s = \alpha$. As we vary the growth for ends in the family $E'(t)$, starting with growth zero, we are initially above u_2.

As $t \to \alpha$ we can't have $u_t \geq u_1$ for every t for otherwise $u_1 = u_2$, so there is a $\beta < \alpha$ such that the graph of u_β intersects the graph of u_1. By the maximum principle the intersection cannot be compact. Thus $u_1 \geq u_\beta$ on a noncompact domain. We then have on this domain

$$\ln u_1 - \ln u_2 \geq \ln u_\beta - \ln u_\alpha \geq c_1(\alpha - \beta)x_1,$$

and

$$M(X) \geq c_1(\alpha - \beta)X,$$

for X sufficiently large and some constant $c_1 > 0$. q.e.d.

Let $v = \ln u_1 - \ln u_2 = v_1 - v_2$, $v \geq 0$ on Ω and $v = 0$ on $\partial\Omega$. Note that for x_1 large we have $v \leq \gamma x_1$, $\gamma > 0$ (γ could be α). Indeed, $v = \ln u_1 - \ln u_2 \leq -\ln u_2$, and for x_1 big enough we have $-\ln u_2 \simeq \alpha x_1$. Suppose $v \neq 0$, we will find a contradiction. By Stokes' Theorem

$$\int_{\Omega(X)} \operatorname{div}\left(\frac{v\nabla v_1}{W_1}\right) - \operatorname{div}\left(\frac{v\nabla v_2}{W_2}\right) = \int_{\partial\Omega(X)} v\left\langle \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2}, N\right\rangle,$$

where N is the outer conormal along $\partial\Omega(X)$. For $v = v_1 - v_2$ we have

$$\begin{aligned}(1)\qquad & \int_{\Omega(X)} \left\langle \nabla v_1 - \nabla v_2, \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2}\right\rangle \\ & + \int_{\Omega(X)} v\left(\operatorname{div}\left(\frac{\nabla v_1}{W_1}\right) - \operatorname{div}\left(\frac{\nabla v_2}{W_2}\right)\right) \\ & = \int_{\partial\Omega(X)} v\left\langle \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2}, N\right\rangle.\end{aligned}$$

Now we estimate the following integrals:

$$\begin{aligned}\left|\int_{X_0}^{X}\int_{C(x_1)} v \operatorname{div}\left(\frac{\nabla v_i}{Wi}\right)\right| &\leq \left|\int_{X_0}^{X}\int_{C(x_1)} v\frac{|\nabla u_i|^4}{u_i^2 W_i(1+W_i)^2}\right| \\ &\leq C_3^2\gamma\left|\int_{X_0}^{X}\int_{C(x_1)} \frac{x_1}{x_1^4}\frac{1}{W_i(1+W_i)^2}\right|,\end{aligned}$$

the first inequality is justified by Lemma 9, the second follows from hypothesis i) and iii), i.e., $\frac{|\nabla u_i|^2}{u_i} \leq \frac{C_3}{x_1^2}$, and from $v \leq \gamma x_1$. So when $X \to \infty$ the integrals above converge. Therefore we may rewrite (1) as follows:

(2)

$$A + \int_{\Omega(X)} \left\langle \nabla v_1 - \nabla v_2, \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right\rangle \leq \int_{\partial\Omega(X)} v \left\langle \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2}, N \right\rangle$$

For $X_1 > 0$, we define

$$\mu(X_1) = \int_{\Omega(X_1)} \left\langle \nabla v_1 - \nabla v_2, \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right\rangle.$$

We have

$$\begin{aligned} &\left\langle \nabla v_1 - \nabla v_2, \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right\rangle \\ &= W_1 \left| \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right|^2 + (W_1 - W_2) \left\langle \frac{\nabla v_2}{W_2}, \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right\rangle. \end{aligned}$$

Also

$$\begin{aligned} &\left| (W_1 - W_2) \left\langle \frac{\nabla v_2}{W_2}, \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right\rangle \right| \\ &= \left| \frac{\nabla u_2}{u_2 W_2} \right| \left| \frac{|\nabla u_1|^2 - |\nabla u_2|^2}{W_1 + W_2} \right| \left| \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right| \\ &\leq \frac{c_2}{x_1^2} \left| \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right|. \end{aligned}$$

Note that $\left| \frac{\nabla u_2}{u_2 W_2} \right|$ is equivalent to α, and

$$\left| |\nabla u_1|^2 - |\nabla u_2|^2 \right| \leq \frac{|\nabla u_1|^2}{u_1} + \frac{|\nabla u_2|^2}{u_2} \leq \frac{2C_3}{x_1^2}.$$

So we rewrite (2) as

$$\begin{aligned} &A + \mu(X_1) + \int_{X_1}^{X} \int_{C(x_1)} W_1 \left| \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right|^2 \\ &\quad - \int_{X_1}^{X} \int_{C(x_1)} \frac{c_2}{x_1^2} \left| \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right| \\ &\leq \int_{C(X)} v \left\langle \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2}, N \right\rangle. \end{aligned} \tag{3}$$

We define

$$\eta(x_1) = \int_{C(x_1)} \left| \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right| .$$

By Cauchy-Schwartz we have

$$\eta^2(x_1)T^{-1} \leq \int_{C(x_1)} \left| \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right|^2 \leq \int_{C(x_1)} W_1 \left| \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right|^2 ,$$

where T is the width of the half-strip B.

Using the definition of $M(X)$ and $\left\langle \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2}, N \right\rangle \leq \left| \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right|$, (3) then implies:

$$A + \mu(X_1) + \int_{X_1}^{X} \eta^2(X)T^{-1} - \int_{X_1}^{X} \frac{c_2 \eta(x_1)}{x_1^2} \leq M(X)\eta(X). \tag{4}$$

Now $\partial\Omega$ is not compact, $v = 0$ on $\partial C(x_1)$ and $M(x_1)$ is the maximun of v on $C(x_1)$ so that

$$M(x_1) \leq \int_{C(x_1)} |\nabla v| .$$

Next

$$\left| \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right| \geq \frac{1}{W_1} |\nabla v_1 - \nabla v_2| - |\nabla v_2| \left| \frac{1}{W_2} - \frac{1}{W_1} \right| ,$$

and $\frac{1}{W_1} \geq c_3 > 0$, $|\nabla v_2| \leq c_4 \alpha$, $\left| \frac{1}{W_2} - \frac{1}{W_1} \right| \leq 2$, so that

$$\eta(x_1) \geq c_3 M(x_1) - 2c_4 T\alpha.$$

By Lemma 10 we conclude that $\eta(x_1) \to \infty$ as $x_1 \to \infty$ unless $v = 0$. Then there is a constant $c_5 > 0$ and a $X_0 \geq 0$ such that for $x_1 \geq X_0$,

$$\eta^2(x_1)T^{-1} - \frac{c_2 \eta(x_1)}{x_1^2} \geq c_5 \eta^2(x_1).$$

Thus, for $X_1 \geq X_0$, (4) may be replaced by

$$A + \mu(X_1) + c_5 \int_{X_1}^{X} \eta^2(x_1) \leq M(X)\eta(X). \tag{5}$$

Now we will show that for X_1 greater or equal to some (other) X_0 we have

$$\widetilde{\mu}(X_1) = A + \mu(X_1) > 0.$$

Indeed, by definition

$$\widetilde{\mu}(X_1) = A + \int_{\Omega(X_1)} W_1 \left| \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right|^2 + \int_{\Omega(X_1)} (W_1 - W_2) \left\langle \frac{\nabla v_2}{W_2}, \frac{\nabla v_1}{W_1} - \frac{\nabla v_2}{W_2} \right\rangle.$$

The module of the second integral is at most

$$\int_0^{X_1} \frac{c_2}{x_1^2} \eta(x_1)$$

and $W_1 \geq 0$ so that $\widetilde{\mu}(X_1) \geq A + \displaystyle\int_0^{X_1} T^{-1}\eta^2(x_1) - \frac{c_2}{x_1^2}\eta(x_1)$, which diverges since $\eta(x_1) \to \infty$.

Now $\widetilde{\mu}(X_1) \geq \widetilde{\mu}(X_0) + c_5 \int_{X_0}^{X_1} \eta^2(x_1)$. By Lemma 10 and the comparison between $\eta(x_1)$ and $M(x_1)$ we conclude $\widetilde{\mu}(X_1)$ grows at least as fast as X_1^3 for some X_1 big enough:

$$\widetilde{\mu}(X_1) \geq c_5 \int_{X_0}^{X_1} \eta^2(x_1) \geq c_6 \int_{X_0}^{X_1} M^2(x_1) \geq c_7 \int_{X_0}^{X_1} x_1^2 \geq c_8 X_1^3.$$

We write equation (5) as

$$\widetilde{\mu}(X_1) + c_5 \int_{X_1}^{X} \eta^2(x_1) \leq S\eta(X), \tag{6}$$

for $X \in [X_1, X_2]$, where $S = \sup\{M(X); X_1 \leq X \leq X_2\}$.

Let ξ be the function defined on the interval $J = \left[X_1, X_1 + \frac{2S^2}{c_5 \widetilde{\mu}(X_1)}\right)$, by

$$\frac{c_5}{S}(X - X_1) = \frac{2S}{\widetilde{\mu}(X_1)} - \frac{1}{\xi(X)}.$$

So $\xi(X)$ satisfies

$$\frac{\widetilde{\mu}(X_1)}{2} + c_5 \int_{X_1}^{X} \xi^2(x_1) = S\xi(X).$$

The connected component of $\{X \in J \cap [X_1, X_2]; \xi(X) < \eta(X)\}$ that contains X_1 is open ($\xi(X_1) = \frac{\widetilde{\mu}(X_1)}{2S}$ while $\eta(X_1) \geq \frac{\widetilde{\mu}(X_1)}{S}$), and by (6) we verify that it is also closed. So the connected component is in fact the interval $J \cap [X_1, X_2]$. Since $\xi(x_1) \to \infty$ when $x_1 \to X_1 + \dfrac{2S^2}{c_5\widetilde{\mu}(X_1)}$ and η is bounded we conclude that $X_2 \in J$, so $X_2 \leq X_1 + \dfrac{2S^2}{c_5\widetilde{\mu}(\widetilde{X}_1)}$.

Finally, as $S \leq \gamma X_2$ it follows that

$$\sqrt{\frac{c_5\widetilde{\mu}(X_1)}{2}(X_2 - X_1)} \leq \gamma X_2.$$

However this contradicts our estimate for the growth of $\widetilde{\mu}(X_1)$. Take for instance $X_2 = 2X_1$, this implies

$$\sqrt{\frac{c_5\widetilde{\mu}(X_1)}{2}X_1} \leq 2X_1\gamma,$$

so

$$\frac{c_5\widetilde{\mu}(X_1)}{2}X_1 \leq 4X_1^2\gamma^2$$

and $\widetilde{\mu}(X_1) \leq \dfrac{8\gamma^2}{c_5}X_1$, contradicting $\widetilde{\mu}(X_1) \geq c_8X_1^3$. q.e.d.

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