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NEW LI–YAU–HAMILTON INEQUALITIES FOR THE RICCI FLOW VIA THE SPACE-TIME APPROACH

BENNETT CHOW & DAN KNOPF

Abstract

We prove Li–Yau–Hamilton inequalties that extend Hamilton's matrix inequality for solutions of the Ricci flow with nonnegative curvature operators. To obtain our extensions, we apply the space-time formalism of S.-C. Chu and one of the authors to solutions of the Ricci flow modified by a cosmological constant. Then we adjoin to the Ricci flow the evolution of a 1-form and a 2-form flowing by a system of heat-type equations. By a rescaling argument, the inequalities we obtain in this manner yield new inequalities which are reminiscent of the linear trace inequality of Hamilton and one of the authors.

1. Introduction

In [11], Hamilton determined a sharp differential Harnack inequality of Li–Yau type for complete solutions of the Ricci flow with nonnegative curvature operator. This Li–Yau–Hamilton inequality (abbreviated as LYH inequality below) is of critical importance to the understanding of singularities of the Ricci flow, as is evident from its numerous applications in [10], [12], [13], and [14]. Moreover, it has been informally claimed by Hamilton that the discovery of a LYH inequality in dimension 3 valid without any hypothesis on curvature is the main unresolved step in his program of approaching Thurston's Geometrization Conjecture by applying the Ricci flow to closed 3-manifolds. See [13] for some of the reasons why such an inequality is believed to hold. (One may also consult the survey paper [2].) Based on unpublished research of Hamilton and Hamilton–Yau, the search for such a LYH inequality appears to be an extremely complex and delicate problem. Roughly speaking, their

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approach is to start with the 3-dimensional LYH inequality for solutions with nonnegative sectional curvature and try to perturb that estimate so that it holds for solutions with arbitrary initial data. Because of an estimate of Hamilton [13] and Ivey [15] which shows that the curvature operator of 3-dimensional solutions tends in a sense to become nonnegative, there is hope that such a procedure will work. Some unpublished work of Hamilton and Yau appears close to establishing a general LYH inequality in dimension 3. However, so far no such inequality is known.

Due to the perturbational nature of the existing approaches, it is also of interest to understand how general a LYH inequality one can prove under the original hypothesis of nonnegative curvature operator. In this direction, Hamilton and one of the authors [6] obtained a linear trace LYH inequality for a system consisting of a solution of the Lichnerowicz-Laplacian heat equation for symmetric 2-tensors coupled to a solution of the Ricci flow. Since the pair of the Ricci and metric tensors of a solution to the Ricci flow forms such a system, their linear trace inequality generalizes the traced case of Hamilton's tensor (matrix) inequality. In [10] Hamilton had already observed the formal similarity between his proof of the 2-dimensional trace LYH inequality for the Ricci flow and Li and Yau's proof [16] of their Harnack inequality for the heat equation on Riemannian manifolds. In a sense, the linear trace inequality generalizes this link to higher dimensions. In dimension 2, meanwhile, the link was made stronger and more evident by the discovery [3] of a 1-parameter family of inequalities interpolating between the Li–Yau and linear trace estimates.

In another direction, one recalls that Hamilton's matrix inequality is equivalent to the positivity of a certain quadratic form. Hamilton observed [13] that the evolution equation satisfied by that quadratic suggests that his LYH inequality may be some sort of extension of nonnegative curvature operator. This was shown to be true by S.-C. Chu and one of the authors in [4]. They introduced a degenerate metric and a certain compatible connection on space and time that extends the Levi-Civita connection of a solution of the Ricci flow. They proved that Hamilton's quadratic is in fact the curvature of that connection. Because the space-time metric and connection satisfy the Ricci flow for degenerate metrics, one can then apply the methods of [8] to show that the quadratic satisfies a nice evolution equation. This fact is the starting point for the present paper.

In this paper, we prove a new differential Harnack inequality of Li–Yau–Hamilton type for the Ricci flow by generalizing the construction in

[4]. Our inequality applies to solutions of the Ricci flow coupled to a 1-form and a 2-form solving Hodge-Laplacian heat-type equations. In this sense, one may regard it as a linear-type matrix LYH inequality. In its general form, it looks quite different from Hamilton's matrix inequality — except in the Kähler case, where as a special case, one may take the 1-form to be the exterior derivative of the scalar curvature and the 2-form to be the Ricci form, thereby obtaining an inequality slightly weaker but qualitatively similar to Hamilton's. (Note that Cao [1] has extended Hamilton's LYH inequality in the Kähler case to solutions with nonnegative bisectional curvature.) We state the general form of our result (Theorem 39) as our main theorem:

Main Theorem. *Let $(\mathcal{M}, g(t))$ be a solution of the Ricci flow on a closed manifold and a time interval $[0,\Omega)$. Let A_0 be a 2-form which is closed at $t=0$, and let E_0 be a 1-form which is closed at $t=0$. Then there is a solution $A(t)$ of*

$$\frac{\partial}{\partial t}A = \Delta_d A, \qquad A(0) = A_0$$

and a solution $E(t)$ of

$$\frac{\partial}{\partial t}E = \Delta_d E - d\,|A|_g^2, \qquad E(0) = E_0$$

which exist for $t \in [0,\Omega)$, where $-\Delta_d \doteqdot d\delta + \delta d$ is the Hodge–de Rham Laplacian. Suppose that the quadratic

$$\begin{aligned}\Psi(A, E, U, W) & \\ &\doteqdot \operatorname{Rm}(U,U) - 2\langle \nabla_W A, U\rangle + |A(W)|^2 - \langle \nabla_W E, W\rangle \\ &= R_{ijk\ell}U^{ij}U^{\ell k} + 2W^j \nabla_j A_{k\ell} U^{\ell k} + \left(g^{pq}A_{jp}A_{\ell q} - \nabla_j E_\ell\right) W^j W^\ell\end{aligned}$$

is nonnegative at $t=0$ for any 2-form U and 1-form W. Then the matrix inequality $\Psi(A,E,U,W) \geq 0$ persists for all $t \in [0,\Omega)$.

The above linear-type inequality is a special case of Theorem 38 obtained by taking a limit which actually scales away part of the main highest order terms in the more general LYH matrix inequality established in Theorem 38.

Corollary A. *Under the hypotheses above, the trace inequality*

$$0 \leq \psi(A, E, W) \doteqdot \operatorname{Rc}(W,W) - 2(\delta A)(W) + |A|^2 + \delta E$$

persists for all $t \in [0,\Omega)$.

Corollary B. *Let $(\mathcal{M}, g(t))$ be a Kähler solution of the Ricci flow with nonnegative curvature operator on a closed manifold $\mathcal{M}$. Then for any 2-form U, 1-form W, and all $t > 0$ such that the solution exists, one has the matrix estimate*

$$\begin{aligned} 0 \leq & \operatorname{Rm}(U,U) - 2\langle \nabla_W \rho, U\rangle + \frac{1}{4t^2}|W|^2 + \frac{1}{t}\operatorname{Rc}(W,W) \\ & + \operatorname{Rc}^2(W,W) + \frac{1}{2}(\nabla\nabla R)(W,W), \end{aligned}$$

where ρ is the Ricci form. By setting $U = X \wedge W$ and tracing over W, this implies the trace estimate

$$0 \leq \frac{\partial}{\partial t}R + \frac{2R}{t} + \frac{n}{2t^2} + 2\langle \nabla R, X\rangle + 2\operatorname{Rc}(X,X)$$

for any 1-form X.

Although this LYH inequality is weaker than the trace inequality special case of the matrix inequality in [1], it qualitatively similar, and it arises from a much more general inequality.

Corollary C. *Let $\left(\mathcal{M}^2, g(t)\right)$ be a solution of the Ricci flow on a closed surface. If (ϕ, f) is a pair solving the system*

$$\begin{aligned} \frac{\partial}{\partial t}\phi &= \Delta\phi + R\phi \\ \frac{\partial}{\partial t}f &= \Delta f + \phi^2, \end{aligned}$$

then the trace inequality

$$0 \leq R|X|^2 + 2\langle \nabla\phi, X\rangle + \frac{\partial}{\partial t}f$$

is preserved.

This paper is structured as follows:

- In §2, we extend the methods of [4] to the case of the Ricci flow with a cosmological term μ. Only by doing so for $\mu \equiv 1/2$ are we able to display Hamilton's differential Harnack quadratic of Li-Yau type as exactly equal to the curvature of a space-time connection, and thus to provide the reader with a precise glossary between the space-time approach and the computations in [11]. A similar but less precise correspondence was earlier established in [4].

- In §3, we study all symmetric space-time connections that are compatible with the degenerate space-time metric and evolve via the Ricci flow for degenerate metrics. Because these connections are not unique, their curvature tensors yield new Li–Yau–Hamilton inequalities, which include Hamilton's matrix inequality as a special case. We then employ scaling arguments to derive a nonnegative symmetric bilinear form on space-time, which is equivalent to the quadratic Ψ described in classical language in the Main Theorem, and whose traced form yields Corollary A.

- In §4, we develop some examples in order to compare a few special cases of the new Li–Yau–Hamilton inequality with known results. In particular, we derive Corollary B (Proposition 43) and Corollary C (a result of tracing the matrix inequality in Proposition 46).

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2. The Ricci flow rescaled by a cosmological term

2.1 Self-similar solutions of the Ricci flow

In this section, we recall the equations for self-similar solutions to the Ricci flow (called *Ricci solitons* by Hamilton) in order to motivate the introduction of the Ricci flow with a cosmological term. The basic reference is §3 of [11].

Definition 1. A solution $(\mathcal{M}^n, g(t))$ of the Ricci flow

$$\frac{\partial}{\partial t} g = -2 \operatorname{Rc}$$

on a time interval $\mathcal{I}$ containing 0 is called a *homothetic Ricci soliton* if

$$g(\cdot, t) = a(t) (\phi_t^* \hat{g})(\cdot) \tag{2.1}$$

for some fixed metric $\hat{g}$ on $\mathcal{M}$, some function a of time satisfying $a(0) = 1$, and some 1-parameter family of diffeomorphisms $\{\phi_t : t \in \mathcal{I}\}$ generated by vector fields $-V(t)$ with the property that their dual 1-forms are closed:

$$\nabla_i V_j = \nabla_j V_i. \tag{2.2}$$

In this case, we say that $(\mathcal{M}^n, g(t))$ *flows along* V.

It is not obvious that the representation (2.1) is unique, because the family $\{\phi_t\}$ may contain homotheties. We put Equation (2.1) in a canonical form as follows:

Lemma 2. *Suppose g is a homothetic Ricci soliton having the form (2.1). Let $\hat{a} \doteqdot \dot{a}(0)$, and let $\{\psi_t : t \in \mathcal{I}\}$ be the 1-parameter family of diffeomorphisms generated by the vector fields*

$$-\frac{1}{1+\hat{a}t} V(0),$$

with $\psi_0 = \mathrm{id}_{\mathcal{M}}$. Then

$$g(\cdot, t) = (1 + \hat{a}t)(\psi_t^* \hat{g})(\cdot).$$

Proof. Let $G(t)$ be a smooth 1-parameter family of metrics, and let $\{\theta_t\}$ be a family of diffeomorphisms generated by vector fields $-W(t)$. Then we have

$$\begin{aligned} \frac{\partial}{\partial t}(\theta_t^* G(t)) &= \left.\frac{\partial}{\partial s}\right|_{s=0} \left(\theta_{t+s}^* G(t+s)\right) \\ &= \theta_t^* \left(\frac{\partial}{\partial t} G(t)\right) + \left.\frac{\partial}{\partial s}\right|_{s=0} \left[\left(\theta_t^{-1} \circ \theta_{t+s}\right)^* (\theta_t^* G(t))\right] \\ &= \theta_t^* \left(\frac{\partial}{\partial t} G(t)\right) - \mathcal{L}_{\left(\theta_t^{-1}\right)_* W(t)} (\theta_t^* G(t)). \\ &= \theta_t^* \left(\frac{\partial}{\partial t} G(t) - \mathcal{L}_{W(t)} G(t)\right). \end{aligned} \tag{2.3}$$

Applying (2.3) with $G(t) = a(t)\hat{g}$ and $\theta_t = \phi_t$, we get

$$\mathrm{Rc}(g) = -\frac{1}{2}\frac{\partial}{\partial t} g(t) = -\frac{1}{2}\frac{\dot{a}}{a} g + \frac{a}{2}\phi_t^* \left(\mathcal{L}_{V(t)}\hat{g}\right). \tag{2.4}$$

Now define

$$\widetilde{g}(t) \doteqdot (1 + \hat{a}t)(\psi_t^* \hat{g}).$$

Applying (2.3) with $G(t) = (1 + \hat{a}t)\hat{g}$ and $\theta_t = \psi_t$, we obtain

$$\frac{\partial}{\partial t}\widetilde{g}(t) = \psi_t^* \left(\hat{a}\hat{g} - \mathcal{L}_{\frac{1}{1+\hat{a}t}V(0)}((1+\hat{a}t)\hat{g})\right) = \psi_t^* \left(\hat{a}\hat{g} - \mathcal{L}_{V(0)}\hat{g}\right).$$

But evaluating Equation (2.4) at $t=0$ shows that

$$\operatorname{Rc}(\hat{g}) = \operatorname{Rc}(g)|_{t=0} = -\frac{1}{2}\hat{a}\hat{g} + \frac{1}{2}\mathcal{L}_{V(0)}\hat{g}.$$

Hence

$$\frac{\partial}{\partial t}\widetilde{g}(t) = \psi_t^*(-2\operatorname{Rc}(\hat{g})) = -2\operatorname{Rc}(\widetilde{g}(t)).$$

So $\widetilde{g}(t)$ is a solution of the Ricci flow with $\widetilde{g}(0) = g(0)$. Since solutions of the Ricci flow are unique, it follows that $\widetilde{g}(t) = g(t)$ for as long as both solutions exist. q.e.d.

The gist of the lemma is that in (2.1) and (2.4) we may assume $\dot{a}(t) \equiv \hat{a}$ is independent of t, so that $a(t) = 1+\hat{a}t$ and $a(t)V(t) = V(0)$.

From the point of view of motivating the differential Harnack inequality of Li-Yau type, Hamilton considered the extreme case for the function $a(t)$ in the definition of Ricci soliton. In particular, he was interested in the case where the initial metric $g(0)$ is singular (such as the metric of a cone) and the metric $g(t)$ expands as t increases. Formally, this corresponds to letting $\hat{a} = \dot{a}(0)$ tend to infinity,

$$\lim_{\hat{a}\to\infty}\frac{\dot{a}}{a} = \lim_{\hat{a}\to\infty}\frac{\hat{a}}{1+\hat{a}t} = \frac{1}{t}, \tag{2.5}$$

and motivates the following definition:

Definition 3. A solution $(\mathcal{M}^n, g(t))$ of the Ricci flow

$$\frac{\partial}{\partial t}g = -2\operatorname{Rc}$$

on a time interval $(0,\Omega)$ containing 1 is called an *expanding Ricci soliton flowing along* V if

$$g(\cdot,t) = t(\theta_t^*\hat{g})(\cdot) \tag{2.6}$$

for some fixed metric $\hat{g}$ on $\mathcal{M}$ and some 1-parameter family of diffeomorphisms $\{\theta_t : t\in(0,\Omega)\}$ such that $\theta_1 = \mathrm{id}_{\mathcal{M}}$ and the dual 1-forms of the vector fields $-V(t)$ which generate θ_t are closed.

Differentiating (2.6) leads to the equation

$$R_{ij} = \nabla_i V_j - \frac{1}{2t}g_{ij}, \tag{2.7}$$

where $V(t) = \frac{1}{t} V(1)$. Notice that (2.7) can be obtained formally by passing to the limit (2.5) in Equation (2.4).

Now observe that $t^{-1} g(t) = \theta_t^* (g(1))$ evolves by diffeomorphisms. This motivates us to make the following transformation for any solution $g(t)$ to the Ricci flow:

$$\bar{g}(t) \doteqdot \frac{1}{t} g(t).$$

To get a nice equation for $\bar{g}$, we change the time variable by $\bar{t} \doteqdot \ln t$. Then $\bar{g}(\bar{t})$ is a solution to the equation

$$\frac{\partial}{\partial \bar{t}} \bar{g}_{ij} = -2 \left(\overline{R}_{ij} + \frac{1}{2} \bar{g}_{ij} \right)$$

on the time interval $(-\infty, \ln \Omega)$ containing $\bar{t} = 0$. We call this equation the *Ricci flow with cosmological constant* $1/2$. More generally, there is the following:

Definition 4. We say that $(\mathcal{M}^n, g(\bar{t}))$ is a solution of the *Ricci flow with cosmological term* $\mu(\bar{t}) \in \mathbb{R}$ on a time interval $\mathcal{I}$ if

$$\frac{\partial}{\partial \bar{t}} \bar{g}_{ij} = -2 \left(\overline{R}_{ij} + \mu \bar{g}_{ij} \right). \tag{2.8}$$

A solution of the Ricci flow with cosmological constant $\mu = 1/2$ is an expanding Ricci soliton

$$-\frac{1}{2} \frac{\partial}{\partial t} g_{ij} = R_{ij} = \nabla_i V_j - \frac{1}{2t} g_{ij}$$

if and only if $\bar{g}(\bar{t}) = e^{-\bar{t}} g\left(e^{\bar{t}}\right)$ satisfies

$$-\frac{1}{2} \frac{\partial}{\partial \bar{t}} \bar{g}_{ij} = \overline{R}_{ij} + \frac{1}{2} \bar{g}_{ij} = \overline{\nabla}_i \overline{V}_j, \tag{2.9}$$

where $\overline{V}_j(\bar{t}) \doteqdot V_j(t)$, hence if and only if $\bar{g}(\bar{t})$ is a steady Ricci soliton. Note that $\overline{V}^k(\bar{t}) = \bar{g}^{jk}(\bar{t}) \overline{V}_j(\bar{t}) = t V^k(t) = V^k(1)$ is independent of $\bar{t}$, so that

$$\frac{\partial}{\partial \bar{t}} \overline{V}^k = 0. \tag{2.10}$$

Taking the divergence of (2.9), using (2.2), and commuting derivatives, we compute

$$\frac{1}{2}\overline{\nabla}_j\overline{R} = \overline{\nabla}_i\overline{\nabla}^i\overline{V}_j = \overline{\nabla}_j\overline{\nabla}_i\overline{V}^i + \overline{R}_{jk}\overline{V}^k = \overline{\nabla}_j\left(\overline{R}+\frac{n}{2}\right) + \overline{R}_{jk}\overline{V}^k$$

and thus obtain the following useful identities, valid when $\mu \equiv \frac{1}{2}$:

$$\frac{1}{2}\overline{\nabla}\overline{R} = \overline{\Delta V} = -\overline{\mathrm{Rc}}\left(\overline{V}\right). \tag{2.11}$$

2.2 The space-time connection for the Ricci flow rescaled by a cosmological term

In this section, we show that the definition of the space-time connection for the Ricci flow in [4] may be extended to the case where there is a cosmological term $\mu\left(\overline{t}\right)$ in the flow equation. Motivated by the discussion in the previous section, we are mainly interested in the case $\mu \equiv \frac{1}{2}$. Since the relevant computations are modifications of those in [4], we shall omit many details of the proofs.

Let $\widetilde{\mathcal{M}} = \mathcal{M}^n \times \mathcal{I}$ and denote the time coordinate by $x^0 \doteqdot \overline{t}$. Recall that the degenerate space-time metric on $T^*\widetilde{\mathcal{M}}$ is defined by

$$\widetilde{g}^{ij} \doteqdot \begin{cases} \overline{g}^{ij} & \text{if } i,j \geq 1 \\ 0 & \text{if } i = 0 \text{ or } j = 0. \end{cases} \tag{2.12}$$

Modifying the definition in [4], we define a symmetric space-time connection $\widetilde{\nabla}$ by specifying its Christoffel symbols to be

$$\widetilde{\Gamma}_{ij}^k = \overline{\Gamma}_{ij}^k \tag{C1}$$

$$\widetilde{\Gamma}_{i0}^k = -\left(\overline{R}_i^k + \mu\delta_i^k\right) \tag{C2}$$

$$\widetilde{\Gamma}_{00}^k = -\frac{1}{2}\overline{\nabla}^k\overline{R} \tag{C3}$$

$$\widetilde{\Gamma}_{00}^0 = -\mu \tag{C4}$$

$$\widetilde{\Gamma}_{ij}^0 = \widetilde{\Gamma}_{i0}^0 = 0, \tag{C5}$$

where $i,j,k \geq 1$.

Lemma 5. *The connection $\widetilde{\nabla}$ is compatible with the degenerate metric $\widetilde{g}$: for all $i,j,k \geq 0$,*

$$\widetilde{\nabla}_i\widetilde{g}^{jk} = 0.$$

Proof. This is a straightforward computation using the identity

$$\widetilde{\nabla}_i \widetilde{g}^{jk} = \partial_i \widetilde{g}^{jk} + \widetilde{\Gamma}_{ip}^j \widetilde{g}^{pk} + \widetilde{\Gamma}_{ip}^k \widetilde{g}^{jp}$$

with formulas (C1), (C2), and (C5). q.e.d.

Given a time-dependent vector field $\overline{W}(\bar{t})$ on $\mathcal{M}$, we associate to it the space-time vector field

$$\widetilde{W}(\bar{t}) \doteqdot \frac{\partial}{\partial \bar{t}} + \overline{W}(\bar{t}).$$

In local coordinates, $\widetilde{W}^0 = 1$ and $\widetilde{W}^j = \overline{W}^j$ if $j \geq 1$.

Lemma 6. *The formulas for the covariant derivative of the vector field $\widetilde{W}$ are*

$$\text{(CW1)} \qquad \widetilde{\nabla}_i \widetilde{W}^j = \overline{\nabla}_i \overline{W}^j - \left(\overline{R}_i^j + \mu \delta_i^j \right)$$

$$\text{(CW2)} \qquad \widetilde{\nabla}_0 \widetilde{W}^j = \frac{\partial}{\partial \bar{t}} \overline{W}^j - \left(\overline{R}_k^j + \mu \delta_k^j \right) \overline{W}^k - \frac{1}{2} \overline{\nabla}^j \overline{R}$$

$$\text{(CW3)} \qquad \widetilde{\nabla}_0 \widetilde{W}^0 = -\mu$$

$$\text{(CW4)} \qquad \widetilde{\nabla}_i \widetilde{W}^0 = 0$$

for all $i, j, k \geq 1$.

Proof. This follows from $\widetilde{\nabla}_i \widetilde{W}^j = \partial_i \widetilde{W}^j + \sum_{p=1}^n \widetilde{\Gamma}_{ip}^j \widetilde{W}^p + \widetilde{\Gamma}_{i0}^j \widetilde{W}^0$ and all of the formulas (C1)-(C5). q.e.d.

We can now make the important observation that *the space-time of a steady soliton flowing along $\overline{V}$ has a geometric product structure.* Recall that a parallel vector field on a Riemannian manifold $\mathcal{N}$ gives a local splitting of $\mathcal{N}$ as the product of an open interval with an $(n-1)$-dimensional manifold $\mathcal{P}$. Hence the observation follows from:

Proposition 7. *If $\bar{g}(\bar{t})$ is a steady soliton of the Ricci flow with cosmological constant $\mu = \frac{1}{2}$ flowing along the vector fields $\overline{V}(\bar{t})$, then*

$$\widetilde{\nabla}_i \left(e^{\frac{1}{2}\bar{t}} \widetilde{V} \right)^j \equiv 0$$

for all $i, j \geq 0$. That is, the space-time vector field $e^{\frac{1}{2}\bar{t}} \widetilde{V}$ is parallel.

Proof. If $i = j = 0$, the formula follows from (CW3). For the case $i = 0,\ j \geq 1$, we apply Equations (CW2), (2.10), and (2.11). If $i \geq 1$ and $j = 0$, the formula follows from (CW4). And for the case $i \geq 1,\ j \geq 1$, we apply (CW1) and (2.9). q.e.d.

2.2.1 The Riemann curvature tensor

Denote the Riemann curvature tensor of the space-time connection $\widetilde{\nabla}$ by

$$\widetilde{R}\left(\widetilde{X},\widetilde{Y}\right)\widetilde{Z}=\widetilde{\nabla}_{\widetilde{X}}\widetilde{\nabla}_{\widetilde{Y}}\widetilde{Z}-\widetilde{\nabla}_{\widetilde{Y}}\widetilde{\nabla}_{\widetilde{X}}\widetilde{Z}-\widetilde{\nabla}_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z}. \tag{2.13}$$

Since $e^{\frac{1}{2}\bar{t}}\widetilde{V}$ is parallel and $\widetilde{R}\left(\widetilde{X},\widetilde{Y}\right)\widetilde{Z}$ is a tensor, we immediately get:

Corollary 8. *If $\bar{g}\left(\bar{t}\right)$ is a steady soliton flowing along $\overline{V}\left(\bar{t}\right)$ with $\mu=\frac{1}{2}$, then*

$$\widetilde{R}\left(\widetilde{X},\widetilde{Y}\right)\widetilde{V}=0 \tag{2.14}$$

for all $\widetilde{X}$ and $\widetilde{Y}$.

We shall see in the next section how this relates to the derivation of the Li–Yau–Hamilton quadratic in §3 of [9].

The formulas for the space-time Riemann curvature tensor are as follows:

Proposition 9. *If $i,j,k,\ell\geq 1$ and $a,b,c\geq 0$, then $\widetilde{\mathrm{Rm}}$ satisfies:*

$$\widetilde{R}_{ijk}^{\ell}=\overline{R}_{ijk}^{\ell} \tag{R1}$$

$$\widetilde{R}_{ij0}^{\ell}=\overline{\nabla}_j\overline{R}_i^{\ell}-\overline{\nabla}_i\overline{R}_j^{\ell} \tag{R2a}$$

$$\widetilde{R}_{0jk}^{\ell}=\overline{\nabla}^{\ell}\overline{R}_{jk}-\overline{\nabla}_k\overline{R}_j^{\ell} \tag{R2b}$$

$$\widetilde{R}_{i00}^{\ell}=\frac{\partial}{\partial\bar{t}}\overline{R}_i^{\ell}-\frac{1}{2}\overline{\nabla}_i\overline{\nabla}^{\ell}\overline{R}-\overline{R}_i^{p}\overline{R}_p^{\ell}-\mu\overline{R}_i^{\ell}+\frac{d\mu}{d\bar{t}}\delta_i^{\ell} \tag{R3}$$

$$\widetilde{R}_{abc}^{0}=0. \tag{R4}$$

Remark 10. The standard asymmetries satisfied by the curvature of any connection imply in particular that

$$\widetilde{R}_{0jk}^{\ell}+\widetilde{R}_{j0k}^{\ell}=0$$
$$\widetilde{R}_{i00}^{\ell}+\widetilde{R}_{0i0}^{\ell}=0.$$

Because $\widetilde{\nabla}$ is torsion-free, the first and second Bianchi identities take the form:

$$\widetilde{R}_{ijk}^{\ell}+\widetilde{R}_{jki}^{\ell}+\widetilde{R}_{kij}^{\ell}=0 \tag{B1}$$

$$\widetilde{\nabla}_m\widetilde{R}_{ijk}^{\ell}+\widetilde{\nabla}_i\widetilde{R}_{jmk}^{\ell}+\widetilde{\nabla}_j\widetilde{R}_{mik}^{\ell}=0 \tag{B2}$$

for all $i,j,k,\ell,m\geq 0$.

Remark 11. Using the evolution equation

$$\frac{\partial}{\partial \bar{t}} \overline{R}_i^\ell = \overline{\Delta} \overline{R}_i^\ell + 2\overline{R}_{ipq}^\ell \overline{R}^{pq} + 2\mu \overline{R}_i^\ell,$$

we may rewrite (R3) as

$$\widetilde{R}_{i00}^\ell = \overline{\Delta} \overline{R}_i^\ell - \frac{1}{2} \overline{\nabla}_i \overline{\nabla}^\ell \overline{R} + 2\overline{R}_{ipq}^\ell \overline{R}^{pq} - \overline{R}_i^p \overline{R}_p^\ell + \mu \overline{R}_i^\ell + \frac{d\mu}{d\bar{t}} \delta_i^\ell.$$

The identities in Proposition 9 are proved in a manner similar to Theorem 2.2 and 3.1 of [4], which give the corresponding equations for the case $\mu \equiv 0$.

The components of the Ricci tensor are given by $\widetilde{R}_{jk} = \Sigma_{i=0}^n \widetilde{R}_{ijk}^i = \Sigma_{i=1}^n \widetilde{R}_{ijk}^i$. Hence tracing (as in Corollary 2.4 of [4]) gives the following:

Corollary 12. *The Ricci tensor satisfies the identities:*

(Rc1) $$\widetilde{R}_{ij} = \overline{R}_{ij}$$

(Rc2) $$\widetilde{R}_{0j} = \frac{1}{2} \overline{\nabla}_j \overline{R}$$

(Rc3) $$\widetilde{R}_{00} = \frac{1}{2} \frac{\partial}{\partial \bar{t}} \overline{R} + n \frac{d\mu}{d\bar{t}}.$$

As in Lemma 3.3 of [4], we notice that:

Remark 13. The covariant derivatives of the Ricci tensor obey the symmetries

(CRc1) $$\widetilde{\nabla}_i \widetilde{R}_{j0} = \widetilde{\nabla}_j \widetilde{R}_{i0}$$

(CRc2) $$\widetilde{\nabla}_i \widetilde{R}_{00} = \widetilde{\nabla}_0 \widetilde{R}_{i0}$$

for all $i, j \geq 1$.

Proof. Using (Rc2) and (C2), we find that

$$\widetilde{\nabla}_i \widetilde{R}_{j0} = \overline{\nabla}_i \overline{\nabla}_j \overline{R} + \overline{R}_{ij}^2 + \mu \overline{R}_{ij} = \widetilde{\nabla}_j \widetilde{R}_{i0},$$

which proves Equation (CRc1). Using (Rc2) and (Rc3), we get

$$\begin{aligned} \widetilde{\nabla}_i \widetilde{R}_{00} - \widetilde{\nabla}_0 \widetilde{R}_{i0} = {} & \overline{\nabla}_i \left(\frac{1}{2} \frac{\partial}{\partial \bar{t}} \overline{R} \right) - 2\widetilde{\Gamma}_{i0}^p \widetilde{R}_{p0} \\ & - \frac{\partial}{\partial \bar{t}} \left(\frac{1}{2} \overline{\nabla}_i \overline{R} \right) + \widetilde{\Gamma}_{0i}^p \widetilde{R}_{p0} + \widetilde{\Gamma}_{00}^p \widetilde{R}_{ip} + \widetilde{\Gamma}_{00}^0 \widetilde{R}_{i0} \end{aligned}$$

where p is summed from 1 to n. Equation (CRc2) then follows by applying formulas (C2)–(C4) and (Rc1)–(Rc3). q.e.d.

Generalizing the definition in [4], we have the following:

Definition 14. A degenerate metric and compatible connection $\left(\widetilde{g}, \widetilde{\nabla}\right)$ satisfy the Ricci flow with cosmological term μ if for all $i, j, k \geq 0$,

$$\frac{\partial}{\partial \bar{t}} \widetilde{\Gamma}_{ij}^{k} = \widetilde{g}^{k\ell} \left(-\widetilde{\nabla}_i \widetilde{R}_{j\ell} - \widetilde{\nabla}_j \widetilde{R}_{i\ell} + \widetilde{\nabla}_\ell \widetilde{R}_{ij} \right). \tag{2.15}$$

Proposition 15. *The pair* $\left(\widetilde{g}, \widetilde{\nabla}\right)$ *satisfies the Ricci flow with cosmological term* μ.

Proof. If $i, j, k \geq 1$, the standard formula

$$\frac{\partial}{\partial \bar{t}} \overline{\Gamma}_{ij}^{k} = \frac{1}{2} \bar{g}^{k\ell} \left[\overline{\nabla}_i \left(\frac{\partial}{\partial \bar{t}} \bar{g}_{\ell j} \right) + \overline{\nabla}_j \left(\frac{\partial}{\partial \bar{t}} \bar{g}_{i\ell} \right) - \overline{\nabla}_\ell \left(\frac{\partial}{\partial \bar{t}} \bar{g}_{ij} \right) \right]$$

shows that (2.15) holds. If $k = 0$, the result is trivial by Equations (C4) and (C5). If $i = 0$ and $j, k \geq 1$, the observation $\widetilde{\Gamma}_{0j}^{p} \widetilde{R}_p^k = \widetilde{\Gamma}_{op}^{k} \widetilde{R}_j^p$ and identity (CRc1) together imply that

$$-\widetilde{\nabla}_0 \widetilde{R}_j^k - \widetilde{\nabla}_j \widetilde{R}_0^k + \widetilde{\nabla}^k \widetilde{R}_{0j} = -\widetilde{\nabla}_0 \widetilde{R}_j^k = -\frac{\partial}{\partial \bar{t}} \overline{R}_j^k = \frac{\partial}{\partial \bar{t}} \overline{\Gamma}_{0j}^k.$$

If $i = j = 0$ and $k \geq 1$, the observation $\widetilde{\Gamma}_{00}^{p} \widetilde{R}_p^k = \widetilde{\Gamma}_{0p}^{k} \widetilde{R}_0^p$ and identity (CRc2) imply

$$-2\widetilde{\nabla}_0 \widetilde{R}_0^k + \widetilde{\nabla}^k \widetilde{R}_{00} = -\widetilde{\nabla}_0 \widetilde{R}_0^k = -\frac{\partial}{\partial \bar{t}} \left(\frac{1}{2} \overline{\nabla}^k \overline{R} \right) = \frac{\partial}{\partial \bar{t}} \widetilde{\Gamma}_{00}^k.$$

q.e.d.

Lemma 16. *If* μ *is constant, the space-time curvature tensor satisfies the divergence identity*

$$\widetilde{g}^{pq} \widetilde{\nabla}_p \widetilde{R}_{qjk}^{\ell} = \widetilde{R}_{0jk}^{\ell} \tag{2.16}$$

between components of the $(2,1)$*-tensor on the LHS and components of the* $(3,1)$*-tensor on the RHS.*

Proof. If $j \geq 1$ and $k \geq 1$, this is just the contracted second Bianchi identity

$$\widetilde{g}^{pq} \widetilde{\nabla}_p \widetilde{R}_{qjk}^{\ell} = \overline{\nabla}^p \overline{R}_{pjk}^{\ell} = \overline{\nabla}^\ell \overline{R}_{jk} - \overline{\nabla}_k \overline{R}_j^\ell.$$

If $j \geq 1$, $k = 0$, and μ is constant, this follows from Remark 11 and the calculation

$$\widetilde{g}^{pq} \widetilde{\nabla}_p \widetilde{R}_{qj0}^{\ell} = -\overline{\Delta} \overline{R}_j^\ell + \frac{1}{2} \overline{\nabla}_j \overline{\nabla}^\ell \overline{R} - 2 \overline{R}^{pq} \overline{R}_{jpq}^{\ell} + \overline{R}_j^p \overline{R}_p^\ell - \mu \overline{R}_j^\ell.$$

If $j = 0$ and $k \geq 1$, one computes

$$\widetilde{g}^{pq}\widetilde{\nabla}_p\widetilde{R}^\ell_{q0k} = \overline{\nabla}^p\overline{\nabla}_k\overline{R}^\ell_p - \overline{\nabla}^p\overline{\nabla}^\ell\overline{R}_{pk} + \overline{R}^{pm}\overline{R}^\ell_{pmk} = 2\overline{R}^{pm}\overline{R}^\ell_{pmk} = 0.$$

Finally, if $j = k = 0$, one obtains

$$\begin{aligned}\widetilde{g}^{pq}\widetilde{\nabla}_p\widetilde{R}^\ell_{q00} &= \overline{\nabla}^p\overline{\Delta}\overline{R}^\ell_p - \frac{1}{2}\overline{\Delta}\overline{\nabla}^\ell\overline{R} + 2\overline{\nabla}^p\left(\overline{R}^\ell_{pqm}\overline{R}^{qm}\right) \\ &- \frac{1}{2}\overline{R}^\ell_p\overline{\nabla}^p\overline{R} - \overline{R}^{pq}\overline{\nabla}^\ell\overline{R}_{pq} \\ &= 0\end{aligned}$$

by a straightforward calculation. (See also Lemma 2.2 and the remark after it in [5]). q.e.d.

Remark 17. Tracing formula (2.16) and applying (B2) yields

$$\widetilde{R}^\ell_0 = \widetilde{g}^{jk}\widetilde{g}^{pq}\widetilde{\nabla}_p\widetilde{R}^\ell_{qjk} = \widetilde{g}^{pq}\widetilde{\nabla}_p\widetilde{R}^\ell_q = \frac{1}{2}\widetilde{\nabla}^\ell\widetilde{R} = \frac{1}{2}\overline{\nabla}^\ell\overline{R},$$

in agreement with (Rc2).

The evolution equation for the space-time curvature tensor is given by:

Proposition 18. *If μ is constant, then*

$$\widetilde{\nabla}_0\widetilde{R}^\ell_{ijk} = \widetilde{\Delta}\widetilde{R}^\ell_{ijk} + 2\left(\widetilde{B}^\ell_{ijk} - \widetilde{B}^\ell_{jik} - \widetilde{B}^\ell_{jki} + \widetilde{B}^\ell_{ikj}\right) + 2\mu\widetilde{R}^\ell_{ijk},$$

where

$$\widetilde{B}^\ell_{ijk} \doteqdot -\widetilde{g}^{pq}\widetilde{R}^m_{pij}\widetilde{R}^\ell_{kqm}.$$

Proof. This formula may be proved along the lines of [8]. Instead, we give an alternate proof using the space-time Bianchi and divergence identities. We note that taking the covariant derivative of identity (2.16) yields

$$\widetilde{\nabla}_i\widetilde{R}^\ell_{0jk} = \widetilde{\nabla}_i\left(\widetilde{g}^{pq}\widetilde{\nabla}_p\widetilde{R}^\ell_{qjk}\right) - \widetilde{\Gamma}^m_{i0}\widetilde{R}^\ell_{mjk}.$$

So by using (B2), substituting, and cancelling terms, we directly obtain

$$\begin{aligned}\widetilde{\nabla}_0\widetilde{R}^\ell_{ijk} &= \widetilde{\nabla}_i\widetilde{R}^\ell_{0jk} - \widetilde{\nabla}_j\widetilde{R}^\ell_{0ik} \\ &= \widetilde{g}^{pq}\left(\widetilde{\nabla}_i\widetilde{\nabla}_p\widetilde{R}^\ell_{qjk} - \widetilde{\nabla}_j\widetilde{\nabla}_p\widetilde{R}^\ell_{qik}\right) - \widetilde{\Gamma}^m_{i0}\widetilde{R}^\ell_{mjk} + \widetilde{\Gamma}^m_{j0}\widetilde{R}^\ell_{mik} \\ &= \widetilde{\Delta}\widetilde{R}^\ell_{ijk} + 2\widetilde{g}^{pq}\left(\widetilde{R}^m_{ipk}\widetilde{R}^\ell_{jqm} - \widetilde{R}^\ell_{ipm}\widetilde{R}^m_{jqk}\right) \\ &- \widetilde{g}^{pq}\widetilde{R}^m_{ijp}\widetilde{R}^\ell_{qmk} + 2\mu\widetilde{R}^\ell_{ijk},\end{aligned}$$

where $\widetilde{\Delta} \doteqdot \widetilde{g}^{pq}\widetilde{\nabla}_p\widetilde{\nabla}_q$ is the space-time Laplacian. Then using (B1) and the identity $\widetilde{B}_{ijk}^{\ell} = -\widetilde{g}^{pq}\widetilde{R}_{pji}^{m}\widetilde{R}_{kmq}^{\ell}$, we conclude

$$\begin{aligned}\widetilde{\nabla}_0\widetilde{R}_{ijk}^{\ell} &= \widetilde{\Delta}\widetilde{R}_{ijk}^{\ell} + 2\widetilde{B}_{ikj}^{\ell} - 2\widetilde{B}_{jki}^{\ell} \\ &\quad + \widetilde{g}^{pq}\left(\widetilde{R}_{pji}^{m} - \widetilde{R}_{pij}^{m}\right)\left(\widetilde{R}_{kqm}^{\ell} - \widetilde{R}_{kmq}^{\ell}\right) + 2\mu\widetilde{R}_{ijk}^{\ell} \\ &= \widetilde{\Delta}\widetilde{R}_{ijk}^{\ell} + 2\left(\widetilde{B}_{ikj}^{\ell} - \widetilde{B}_{jki}^{\ell} - \widetilde{B}_{jik}^{\ell} + \widetilde{B}_{ijk}^{\ell}\right) + 2\mu\widetilde{R}_{ijk}^{\ell}.\end{aligned}$$

q.e.d.

2.2.2 Space-time curvature as a bilinear form

We shall find it convenient to regard the curvature tensor as type $(4,0)$. Since the space-time metric is degenerate, we lower indices as follows:

$$\widetilde{R}_{ijk\ell} \doteqdot \begin{cases} \bar{g}_{\ell p}\widetilde{R}_{ijk}^{p} & \text{if } \ell \geq 1 \\ -\bar{g}_{kp}\widetilde{R}_{ij\ell}^{p} & \text{if } \ell = 0 \text{ and } k \geq 1 \\ 0 & \text{if } \ell = k = 0. \end{cases} \tag{2.17}$$

We may now consider $\widetilde{\mathrm{Rm}}$ to be a symmetric quadratic form on $\Lambda^2 T\widetilde{\mathcal{M}}$ by defining

$$\widetilde{\mathrm{Rm}}\left(\widetilde{S},\widetilde{T}\right) \doteqdot \sum_{i,j,k,\ell=0}^{n} \widetilde{R}_{ijk\ell}\widetilde{S}^{ij}\widetilde{T}^{\ell k}. \tag{2.18}$$

Note that this differs slightly from [4], where $\widetilde{\mathrm{Rm}}$ was regarded as a tensor of type $(2,2)$.

Setting $\widetilde{B}_{ijk\ell} \doteqdot -\widetilde{g}^{pq}\widetilde{R}_{pij}^{m}\widetilde{R}_{kqm\ell}$, we restate Proposition 18 in the form:

Corollary 19. *If μ is constant, then*

$$\widetilde{\nabla}_0\widetilde{R}_{ijk\ell} = \widetilde{\Delta}\widetilde{R}_{ijk\ell} + 2\left(\widetilde{B}_{ijk\ell} - \widetilde{B}_{jik\ell} - \widetilde{B}_{jki\ell} + \widetilde{B}_{ikj\ell}\right) + 2\mu\widetilde{R}_{ijk\ell}. \tag{2.19}$$

Recall that the degenerate metric $\widetilde{g}$ induces an inner product and Lie bracket on $\Lambda^2 T^*\widetilde{\mathcal{M}}$ as follows:

$$\left\langle \widetilde{S},\widetilde{T}\right\rangle = \widetilde{g}^{ik}\widetilde{g}^{j\ell}\widetilde{S}_{ij}\widetilde{T}_{k\ell} \tag{2.20}$$

$$\left[\widetilde{S},\widetilde{T}\right]_{ij} = \widetilde{g}^{k\ell}\left(\widetilde{S}_{ik}\widetilde{T}_{\ell j} - \widetilde{T}_{ik}\widetilde{S}_{\ell j}\right). \tag{2.21}$$

(Compare formulas (10) and (11) in [4].) The structure constants $C_{ij}^{ab,cd}$ defined by

$$C_{ij}^{ab,cd} dx^i \wedge dx^j \doteqdot \left[dx^a \wedge dx^b, dx^c \wedge dx^d \right]$$

for $0 \leq a < b \leq n$, $0 \leq c < d \leq n$ and $0 \leq i < j \leq n$ are then given by

$$C_{ij}^{ab,cd} = \delta_i^a \delta_j^d \widetilde{g}^{bc} - \delta_i^c \delta_j^b \widetilde{g}^{ad}.$$

In terms of the natural isomorphism between $\Lambda^2 T^* \widetilde{\mathcal{M}}$ and $\Lambda^2 T^* \mathcal{M} \oplus \Lambda^1 T^* \mathcal{M}$, formula (2.21) corresponds to

$$[X \oplus V, Y \oplus W] = [X, Y] \oplus (V \lrcorner Y - W \lrcorner X),$$

where $\lrcorner$ denotes the interior product. Analogous to [9] and the extension in [4], we define a symmetric bilinear operator # on $\Lambda^2 T^* \widetilde{\mathcal{M}} \otimes \Lambda^2 T^* \widetilde{\mathcal{M}}$ by

$$(F \# G)_{ijk\ell} \doteqdot F_{abcd} G_{pqrs} C_{ij}^{ab,pq} C_{\ell k}^{cd,rs},$$

and adopt the notational convenience $F^\# \doteqdot F \# F$. We also define the square of an element in $\Lambda^2 T^* \widetilde{\mathcal{M}} \otimes \Lambda^2 T^* \widetilde{\mathcal{M}}$ by

$$F_{ijk\ell}^2 \doteqdot \widetilde{g}^{ad} \widetilde{g}^{bc} F_{ijab} F_{cdk\ell}.$$

With these definitions, following [9] and [4], we find that (2.19) takes the form:

Lemma 20. *If μ is constant, then*

$$\widetilde{\nabla}_0 \widetilde{R}_{ijk\ell} = \widetilde{\Delta} \widetilde{R}_{ijk\ell} + \widetilde{R}_{ijk\ell}^2 + \widetilde{R}_{ijk\ell}^\# + 2\mu \widetilde{R}_{ijk\ell}. \tag{2.22}$$

We omit the long but straightforward computations.

2.3 Hamilton's quadratic for the Ricci flow

As was remarked in [4], the results above give an explanation for the surprising identities observed by Hamilton in §14 of [13]. Here, we shall exhibit a correspondence between the machinery Hamilton uses to prove his tensor inequality and the geometric structure of space-time, in order to show that his quadratic and the assumptions made in its derivation arise very naturally in the space-time context. A similar but less precise correspondence appeared earlier in [4].

Recall that Hamilton proved that for any 2-form U and 1-form W on a complete solution $(\mathcal{M}^n, g(t))$ of the Ricci flow with nonnegative curvature operator, the quadratic

$$Z = Z(U, W) \doteqdot M_{ij} W^i W^j + 2P_{ijk} U^{ij} W^k + R_{ijk\ell} U^{ij} U^{\ell k} \tag{2.23}$$

is nonnegative at all positive times, where $R_{ijk\ell} = g_{\ell m} R_{ijk}^m$,

$$M_{ij} \doteqdot \Delta R_{ij} - \frac{1}{2} \nabla_i \nabla_j R + 2R_{ipqj} R^{pq} - R_{ip} R_j^p + \frac{1}{2t} R_{ij}, \tag{2.24}$$

and

$$P_{ijk} \doteqdot \nabla_i R_{jk} - \nabla_j R_{ik}. \tag{2.25}$$

We shall now relate Hamilton's proof to our construction. In §2 of [11], tensors of type (r, s) on a Riemannian manifold $(\mathcal{M}, g)$ are regarded as $\mathrm{GL}(n, \mathbb{R})$-invariant maps from the linear frame bundle $\mathrm{GL}(\mathcal{M})$ to $\mathbb{R}^{n^{r+s}}$. For instance, if $P \in \mathcal{M}$ and $Y = (Y_1|_P, \ldots, Y_n|_P) \in \mathrm{GL}(\mathcal{M})$ is given by $Y_a = y_a^i \, \partial/\partial x^i$ in a chart $\{x^i\}$ at P, a 1-form θ may be identified with the system of component functions $\theta_a = \theta(Y_a)$ it induces on $\mathrm{GL}(\mathcal{M})$. Regarding the Levi-Civita connection of $(\mathcal{M}, g)$ as a $\mathrm{GL}(n, \mathbb{R})$-invariant choice of horizontal subspace for each $T_Y \, \mathrm{GL}(\mathcal{M})$, Hamilton takes space-like derivatives by means of the unique horizontal lift D_a of Y_a at $Y \in \mathrm{GL}(\mathcal{M})$. Hamilton then identifies the vertical vector field $\vee_a^b$ with the differential of the map $Y_a \mapsto y_b^i \left(y^{-1}\right)_i^a Y_a$; namely[1]

$$\vee_b^a = y_b^i \frac{\partial}{\partial y_a^i}.$$

Note that $\vee_b^a$ acts on a covariant tensor by

$$\vee_b^a T_{cd\ldots z} = \delta_c^a T_{bd\ldots z} + \delta_d^a T_{cb\ldots z} + \cdots + \delta_z^a T_{cd\ldots b}. \tag{2.26}$$

For a solution $(\mathcal{M}, g(t))$ to the Ricci flow on an interval $\mathcal{I}$, one considers the bundle $\mathrm{GL}(\mathcal{M}) \times \mathcal{I} \to \widetilde{\mathcal{M}} = \mathcal{M} \times \mathcal{I}$ and the sub-bundle of orthonormal frames $O\left(\widetilde{\mathcal{M}}\right) \doteqdot \cup_{t \in \mathcal{I}} \left(O(\mathcal{M}, g(t)), t\right) \to \widetilde{\mathcal{M}}$. Hamilton takes time-like derivatives by means of a vector field D_t on $\mathrm{GL}(\mathcal{M}) \times \mathcal{I}$ defined by

$$D_t \doteqdot \frac{\partial}{\partial t} + R_{ab} g^{bc} \vee_c^a. \tag{2.27}$$

[1] Hamilton writes ∇_a^b for what we denote $\vee_a^b$.

D_t is tangent to $O\left(\widetilde{\mathcal{M}}\right)$, because

$$D_t g_{ab} \equiv 0. \tag{2.28}$$

The geometric structure of space-time reveals why this construction is natural. Indeed, definition (2.27) corresponds to (C2) in the definition of the space-time connection $\widetilde{\nabla}$ for the Ricci flow without rescaling ($\mu = 0$), because

$$\widetilde{\nabla}_0 = \frac{\partial}{\partial \bar{t}} - \widetilde{\Gamma}_{0j}^{k} \vee_k^j$$

when acting on covariant tensors. Property (2.28) corresponds to the compatibility of $\widetilde{\nabla}$ with the space-time metric $\widetilde{g}$ (Lemma 5).

Now suppose $(\mathcal{M}, g(t))$ is a homothetically expanding soliton flowing along a gradient vector field V. (See Definition 3, and recall that $\left(\mathcal{M}, \bar{g}\left(\bar{t}\right)\right)$ is then a steady Ricci soliton flowing along $\overline{V} = e^{\bar{t}} V$.) In §3 of [11], such a solution is described by the equation

$$D_a V_b = D_b V_a \triangleq R_{ab} + \frac{1}{2t} g_{ab}. \tag{2.29}$$

Here and in what follows, we use the symbol $\triangleq$ to denote an identity that holds for an expanding gradient soliton. By applying formula (CW1) to $\widetilde{V}$, we note that condition (2.29) holds if and only if for all $i, j \geq 1$, one has

$$\widetilde{\nabla}_i \widetilde{V}^j = 0.$$

Hamilton next defines the quadratic Z in terms of the tensors M, P, and Rm. (Recall (2.23)–(2.25) and note that our sign convention for the Riemann curvature tensor is opposite Hamilton's.) In analogy with Theorem 2.2 of [4], we apply Proposition 9 with $\mu \equiv 1/2$ to observe that these also correspond to natural space-time objects:

Lemma 21. *Let $(\mathcal{M}, g(t))$ be a solution of the Ricci flow. Set $\bar{t} = \ln t$ and $\bar{g}\left(\bar{t}\right) = \frac{1}{t} g(t)$. Then for $i, j, k, \ell \geq 1$, one has*

$$R_{ijk\ell} = e^{\bar{t}} \widetilde{R}_{ijk\ell} \tag{2.30}$$

$$P_{\ell kj} = \widetilde{R}_{0jk\ell} = \widetilde{R}_{k\ell 0j} \tag{2.31}$$

$$M_{i\ell} = e^{-\bar{t}} \widetilde{R}_{i00\ell}. \tag{2.32}$$

Thus we arrive at the key observation that *the* LYH *quadratic may be identified with the space-time curvature tensor:*

$$Z = e^{\bar{t}} \sum_{i,j,k,\ell=0}^{n} \widetilde{R}_{ijk\ell} \widetilde{T}^{ij} \widetilde{T}^{\ell k},$$

where the space-time contravariant 2-tensor $\widetilde{T}$ is defined in terms of the natural isomorphism $\Lambda^2 T^* \widetilde{\mathcal{M}} \cong \Lambda^2 T^* \mathcal{M} \oplus \Lambda^1 T^* \mathcal{M}$ by $\widetilde{T} \doteqdot U \oplus \left(e^{-\bar{t}} W/2\right)$. In components $i, j \geq 1$,

$$\widetilde{T}^{ij} = U^{ij} \tag{2.33}$$

$$\widetilde{T}^{0j} = -\widetilde{T}^{j0} = \frac{1}{2} e^{-\bar{t}} W^j = \frac{1}{2t} W^j. \tag{2.34}$$

(See also Corollary 2.3 of [4]; a key difference from that paper is that taking $\mu = \frac{1}{2}$ accounts for the term $\frac{1}{2t} R_{i\ell}$ in $M_{i\ell}$.)

Differentiating the expanding gradient soliton equation (2.29), Hamilton obtains the following two relations:

$$P_{abc} + R_{abc}^{d} V_d \triangleq 0 \tag{2.35}$$

$$M_{ab} + P_{cab} V^c \triangleq 0. \tag{2.36}$$

Together, these equations prove the Li-Yau-Hamilton inequality is sharp. Indeed, if W is arbitrary and one sets $U_{ab} = \frac{1}{2}(V_a W_b - V_b W_a)$, a straightforward computation gives $Z(U, W) \triangleq 0$. This fact can be interpreted using the result of Corollary 8 that

$$\widetilde{R}_{ijk}^{\ell} \widetilde{V}^k \triangleq 0 \tag{2.37}$$

holds for all $i, j, \ell \geq 0$:

Lemma 22. *The identities* (2.35) *and* (2.36) *are equivalent to the fact that the space-time Riemannian curvature tensor* $\widetilde{\mathrm{Rm}}$ *vanishes in the direction of the parallel space-time vector field* $e^{\bar{t}/2} \widetilde{V}$ *when* $\mu = 1/2$.

Proof. If $i, j, \ell \geq 1$, Lemma 21 implies that

$$\widetilde{R}_{ijk\ell} \widetilde{V}^k = -e^{\bar{t}/2} \left(P_{ij\ell} + R_{ij\ell k} V^k \right)$$

and

$$\widetilde{R}_{0jk\ell} \widetilde{V}^k = -e^{3\bar{t}/2} \left(M_{j\ell} + P_{k\ell j} V^k \right).$$

Since $\widetilde{R}^0_{ijk} = 0$ for all i, j, k, it is clear that (2.37) holds if and only if both (2.35) and (2.36) do. q.e.d.

The evolution equations satisfied by the coefficients of Hamilton's quadratic are derived in Lemmas 4.2, 4.3, and 4.4 of [11]. Written in Hamilton's notation, they are

$$(D_t - \triangle)R_{abcd} = 2(B_{abcd} - B_{abdc} + B_{acbd} - B_{adbc}), \tag{2.38}$$

$$\begin{aligned}(D_t - \triangle)P_{abc} = &-2R_{de}D_dR_{abce} \\ &+ 2\left(R_{adbe}P_{dec} + R_{adce}P_{dbe} + R_{bdce}P_{ade}\right),\end{aligned} \tag{2.39}$$

and

$$\begin{aligned}(D_t - \triangle)M_{ab} = &\,2R_{cd}\left(D_cP_{dab} + D_cP_{dba}\right) + 2R_{acbd}M_{cd} \\ &+ 2P_{acd}P_{bcd} - 4P_{acd}P_{bdc} + 2R_{cd}R_{ce}R_{adbe} - \frac{1}{2t^2}R_{ab},\end{aligned} \tag{2.40}$$

where $B_{abcd} = R_{aebf}R_{cedf}$. In §2.5, we prove the following:

Proposition 23. *The evolution equations* (2.38), (2.39), *and* (2.40) *are equivalent to the evolution equation*

$$\widetilde{\nabla}_0\widetilde{R}_{ijk\ell} = \widetilde{\Delta}\widetilde{R}_{ijk\ell} + 2\left(\widetilde{B}_{ijk\ell} - \widetilde{B}_{jik\ell} - \widetilde{B}_{jki\ell} + \widetilde{B}_{ikj\ell}\right) + \widetilde{R}_{ijk\ell} \tag{2.41}$$

satisfied by $\widetilde{\mathrm{Rm}}$ *when* $\mu = 1/2$.

In computing the evolution of the quadratic Z, Hamilton makes the following assumptions on the 2-form U and the 1-form W at a given point:

$$(D_t - \Delta)\,W_a = \frac{1}{t}W_a \tag{A1}$$

$$(D_t - \Delta)\,U_{ab} = 0 \tag{A2}$$

$$D_aW_b = 0 \tag{A3}$$

$$D_aU_{bc} = \frac{1}{2}\left(R_{ab}W_c - R_{ac}W_b\right) + \frac{1}{4t}\left(g_{ab}W_c - g_{ac}W_b\right). \tag{A4}$$

(See the hypotheses in Theorem 4.1 of [11], and note that Equation (A4) is motivated by the fact that it holds on a soliton if (A3) holds and $U = V \wedge W$.) We shall now demonstrate that *the four assumptions* *above are also very natural from the space-time perspective.* Indeed, Equations (A1)–(A4) hold at a point in space-time if and only if $e^{\bar{t}}\widetilde{T}^{ij}$ satisfies the heat equation and is parallel in space-like directions at that point:

Lemma 24. *If $\mu = 1/2$, assumptions* (A1)-(A4) *are equivalent to*

$$\left(\widetilde{\nabla}_0 - \widetilde{\Delta} + 1\right)\widetilde{T}^{ij} = 0 \tag{2.42}$$

$$\widetilde{\nabla}_k \widetilde{T}^{ij} = 0, \tag{2.43}$$

for all $i, j \geq 0$ and $k \geq 1$.

Proof. For $i, j, k \geq 1$, we use (C1), (C2), (2.33), (2.34), and the fact that $\overline{R}^i_k = tR^i_k$ to compute

$$\begin{aligned}\widetilde{\nabla}_k \widetilde{T}^{ij} &= \nabla_k U^{ij} + \widetilde{\Gamma}^i_{k0}\widetilde{T}^{0j} + \widetilde{\Gamma}^j_{k0}\widetilde{T}^{i0} \\ &= \nabla_k U^{ij} - \left(tR^i_k + \frac{1}{2}\delta^i_k\right)\frac{1}{2t}W^j + \left(tR^j_k + \frac{1}{2}\delta^j_k\right)\frac{1}{2t}W^i.\end{aligned}$$

Hence (2.43) is valid for all $i, j, k \geq 1$ if and only if (A4) holds. For $i = 0$ but $j, k \geq 1$, we have

$$\widetilde{\nabla}_k \widetilde{T}^{0j} = \frac{1}{2t}\nabla_k W^j.$$

So (2.43) is valid for $ij = 0$ and all $k \geq 1$ if and only if (A3) holds. Similarly, since $\widetilde{\nabla}_q \widetilde{T}^{0j} = \frac{1}{2t}\nabla_q W^j$ and $\widetilde{\Delta}\widetilde{T}^{0j} = \widetilde{g}^{pq}\widetilde{\nabla}_p\widetilde{\nabla}_q\widetilde{T}^{0j} = \frac{1}{2}\Delta W^j$, we compute that

$$\begin{aligned}\left(\widetilde{\nabla}_0 - \widetilde{\Delta} + 1\right)\widetilde{T}^{0j} &= \frac{\partial}{\partial \bar{t}}\left(\frac{1}{2t}W^j\right) + \widetilde{\Gamma}^0_{00}\widetilde{T}^{0j} + \widetilde{\Gamma}^j_{0p}\widetilde{T}^{0p} \\ &\quad - \frac{1}{2}\Delta W^j + \frac{1}{2t}W^j \\ &= \frac{1}{2}\left(\frac{\partial}{\partial t}W^j - R^j_p W^p - \Delta W^j - \frac{1}{t}W^j\right).\end{aligned}$$

It follows easily that (2.42) is valid for $ij = 0$ if and only if (A1) holds. Finally, we use (C3) to calculate

$$\begin{aligned}\left(\widetilde{\nabla}_0 + 1\right)\widetilde{T}^{ij} &= \frac{\partial}{\partial \bar{t}}U^{ij} + \widetilde{\Gamma}^i_{0p}U^{pj} + \widetilde{\Gamma}^i_{00}\widetilde{T}^{0j} + \widetilde{\Gamma}^j_{0p}U^{ip} + \widetilde{\Gamma}^j_{00}\widetilde{T}^{i0} + U^{ij} \\ &= t\left(\frac{\partial}{\partial t}U^{ij} - R^i_p U^{pj} - R^j_p U^{ip}\right) - \frac{1}{4}W^j\nabla^i R + \frac{1}{4}W^i\nabla^j R.\end{aligned}$$

Then noting that for $i,j \geq 1$,

$$\widetilde{\nabla}_q \widetilde{T}^{ij} = \nabla_q U^{ij} + \frac{1}{2}\left(R_q^j W^i - R_q^i W^j\right) + \frac{1}{4t}\left(\delta_q^j W^i - \delta_q^i W^j\right),$$

we compute

$$\begin{aligned}
\widetilde{\Delta}\widetilde{T}^{ij} &= \widetilde{g}^{pq}\widetilde{\nabla}_p\widetilde{\nabla}_q\widetilde{T}^{ij} \\
&= tg^{pq}\left[\begin{array}{c} \nabla_p\nabla_q U^{ij} + \frac{1}{2}\nabla_p\left(R_q^j W^i - R_q^i W^j\right) \\ +\frac{1}{4t}\nabla_p\left(\delta_q^j W^i - \delta_q^i W^j\right) + \widetilde{\Gamma}_{p0}^i\widetilde{\nabla}_q\widetilde{T}^{0j} + \widetilde{\Gamma}_{p0}^j\widetilde{\nabla}_q\widetilde{T}^{i0} \end{array}\right] \\
&= t\left[\Delta U^{ij} + R_p^j\nabla^p W^i - R_p^i\nabla^p W^j + \frac{1}{4}\left(W^i\nabla^j R - W^j\nabla^i R\right)\right] \\
&\quad + \frac{1}{2}\left(\nabla^j W^i - \nabla^i W^j\right)
\end{aligned}$$

and collect terms to obtain

$$\begin{aligned}
\left(\widetilde{\nabla}_0 - \widetilde{\Delta} + 1\right)\widetilde{T}^{ij} &= t\left[\frac{\partial}{\partial t}U^{ij} - \Delta U^{ij} - R_p^i U^{pj} - R_p^j U^{ip}\right] \\
&\quad + \left(tR^{ip} + \frac{1}{2}g^{ip}\right)\nabla_p W^j - \left(tR^{jp} + \frac{1}{2}g^{jp}\right)\nabla_p W^i.
\end{aligned}$$

So if (A3) holds, then (2.42) is valid for $i,j \geq 1$ if and only if (A2) holds. q.e.d.

2.4 A generalized tensor maximum principle

In order to utilize space-time methods fully in investigating potential Li–Yau–Hamilton quadratics for the Ricci flow, one needs a version of the parabolic maximum principle for equations such as (2.22) and (3.9) Accordingly, we now derive a generalization of the tensor maximum principle originally proved in [8]. We begin with the observation that any smooth family $\{g(t) : 0 \leq t < \Omega\}$ of Riemannian metrics on $\mathcal{M}^n$ induces a nondegenerate metric $\hat{g}$ on $\mathcal{M} \times [0,\Omega)$ given in coordinates $\left(\partial/\partial t = x^0,\, x^1, \ldots, x^n\right)$ by

$$\hat{g}_{ij} = \begin{cases} g_{ij} & \text{if } 1 \leq i,j \leq n \\ \delta_{ij} & \text{if } i = 0 \text{ or } j = 0. \end{cases}$$

We denote the Levi-Civita connection of $\hat{g}$ by $\hat{\nabla}$.

Proposition 25. *Let $g(t)$ be a smooth 1-parameter family of complete metrics on $\mathcal{M}^n$, indexed by $t \in [0, \Omega)$. Let $\widetilde{\mathcal{M}} \doteqdot \mathcal{M} \times [0, \Omega)$ and let $\widetilde{g}$ be the degenerate metric defined on $T^*\widetilde{\mathcal{M}}$ by*

$$\widetilde{g}^{ij} \doteqdot \begin{cases} g^{ij} & \text{if if } 1 \leq i, j \leq n \\ 0 & \text{if } i = 0 \text{ or } j = 0. \end{cases}$$

Let $\widetilde{\nabla}$ be a compatible connection ($\widetilde{\nabla}_i \widetilde{g}^{jk} \equiv 0$), and let $\mathcal{Q}$ denote the space of symmetric bilinear forms on a tensor bundle $\mathcal{X}$ over $\widetilde{\mathcal{M}}$. Suppose $Q \in \mathcal{Q}$ is a solution of the reaction-diffusion equation

$$\widetilde{\nabla}_0 Q = \widetilde{\Delta} Q + \Phi(Q), \tag{2.44}$$

where $\Phi : \mathcal{Q} \to \mathcal{Q}$ is a (possibly nonlinear) locally Lipschitz map which satisfies the null eigenvector condition *that $\Phi(P)(X, X) \geq 0$ at any point where $P(X, \cdot)$ vanishes for $P \in \mathcal{Q}$ and $X \in \mathcal{X}$. Assume $\left|\widetilde{\nabla} - \hat{\nabla}\right|_{\hat{g}}$, $\left|\hat{\nabla}\left(\widetilde{\nabla} - \hat{\nabla}\right)\right|_{\hat{g}}$, and the Lipschitz constant for Φ are bounded on any subset $\mathcal{M} \times [0, \eta] \subset \widetilde{\mathcal{M}}$. If $\mathcal{M}$ is not compact, assume also that there exists $\rho : \mathcal{M} \to [1, \infty)$ with $\rho^{-1}([1, s])$ compact for every $s \in [1, \infty)$ and such that $|\nabla \rho|_g$ and $|\Delta \rho|$ are bounded on $\mathcal{M} \times [0, \eta]$. If $Q \geq 0$ on $\mathcal{M} \times \{0\}$. then $Q \geq 0$ on $\widetilde{\mathcal{M}}$.*

Proof. The metric $\hat{g}$ induces an inner product on $\mathcal{X}$ in the usual way; we shall abuse notation by writing $\hat{g}(X, Y)$ for $X, Y \in \mathcal{X}$. If $\mathcal{M}$ is compact, take $\rho \equiv 1$, and otherwise let $\rho : \widetilde{\mathcal{M}} \to [1, \infty)$ be the function in the statement of the theorem. (By [7] and Lemma 5.1 of [11], such a function always exists if the time derivatives $\partial g / \partial t$ of g and $\partial \Gamma / \partial t$ of the Levi-Civita connection of g are bounded, and if $\mathcal{M}$ has positive sectional curvature.)

By considering translates in time, it will suffice to prove there is $\eta > 0$ such that for every $\varepsilon > 0$, the quadratic form $\hat{Q}$ is strictly positive on $\mathcal{M} \times [0, \eta]$, where

$$\hat{Q}(x, t) \doteqdot Q(x, t) + \varepsilon (\eta + t) \rho(x) \hat{g}(x, t).$$

Suppose $\hat{Q}$ does not remain strictly positive, and let $t_0 \in [0, \eta]$ denote the infimum of all t such that $\hat{Q}(Y, Y)\Big|_{(x,t)} = 0$ for some $Y \in \mathcal{X}$ and $x \in \mathcal{M}$ with $|Y|_{\hat{g}} = 1$ at (x, t). We claim $t_0 > 0$. If not, there will be a sequence of compact sets $\mathcal{K}_j$ exhausting $\mathcal{M}$, points $x_j \in \mathcal{K}_j \backslash \mathcal{K}_{j-1}$, and times $t_j \searrow t_0 = 0$ such that the first zero of $\hat{Q}$ on $\mathcal{K}_j \times [0, \eta]$ occurs at (x_j, t_j). Since $Q \geq 0$ on $\mathcal{M} \times \{0\}$ and $\rho(x_j) \to \infty$ if $\mathcal{M}$ is not compact, this is impossible.

By the null eigenvector assumption,

$$\Phi(\hat{Q})\,(Y, Y)\Big|_{(x,t_0)} \geq 0.$$

Define a tensor field X in a space-time neighborhood $\mathcal{O}$ of (x, t_0) by taking $X = Y$ at (x, t_0) and extending X by parallel transport along radial geodesics with respect to the connection $\widetilde{\nabla}$. (It suffices to extend X first radially along all $\widetilde{\nabla}$-geodesics which start tangent to the hypersurface $\mathcal{M} \times \{t_0\}$, and then along any curve with tangent $\partial/\partial t$ at (x, t_0).) Notice that all symmetric space-like second covariant derivatives of X vanish at (x, t_0). (Compare §4 of [9].) Indeed, with respect to a $\hat{g}$-orthonormal frame $\{e_0 = \partial/\partial t,\ e_1, \ldots, e_n\}$, one observes that for $i = 1, \ldots, n$,

$$\widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_i} X\Big|_{(x,t_0)} = \widetilde{\nabla}_{e_i}\left(\widetilde{\nabla}_{e_i} X\right) - \widetilde{\nabla}_{\widetilde{\nabla}_{e_i} e_i} X = 0 - 0.$$

Hence for any $P \in \mathcal{Q}$, we compute at (x, t_0) that

$$\begin{aligned}\widetilde{\Delta}\,(P\,(X, X)) &= \widetilde{g}^{ij}\left[\begin{array}{c}(\widetilde{\nabla}_i\widetilde{\nabla}_j P)(X, X) + 4(\widetilde{\nabla}_i P)(X, \widetilde{\nabla}_j X) \\ +2P(\widetilde{\nabla}_i X, \widetilde{\nabla}_j X) + 2P(X, \widetilde{\nabla}_i\widetilde{\nabla}_j X)\end{array}\right] \\ &= (\widetilde{\Delta} P)\,(X, X)\,.\end{aligned}$$

Now consider the function F defined in $\mathcal{O}$ by

$$F\,(y, t) = \hat{Q}\,(X, X)\Big|_{(y,t)}\,.$$

Even though $\hat{g}$ may not be compatible with the connection $\widetilde{\nabla}$, we still have $|X|_{\hat{g}} \geq 1/2$ in a possibly smaller neighborhood $\mathcal{O}' \subseteq \mathcal{O}$. Hence F attains its minimum in $\mathcal{O}' \cap \mathcal{M} \times [0, t_0]$ at (x, t_0), where we therefore have

$$0 \geq \frac{\partial}{\partial t} F = \left(\widetilde{\nabla}_0 \hat{Q}\right)(X, X)\,,$$

and

$$0 = \frac{\partial}{\partial x^i} F = \left(\widetilde{\nabla}_i \hat{Q}\right)(X, X)$$

for $i = 1, \ldots, n$, and

$$0 \leq \widetilde{g}^{ij} \left(\frac{\partial^2 F}{\partial x^i \partial x^j} - \widetilde{\Gamma}_{ij}^k \frac{\partial F}{\partial x^k} \right) = \widetilde{\Delta} F.$$

To finish the proof, observe that there are constants C_1 and C_2 depending only on the bounds for $\left| \widetilde{\nabla} - \hat{\nabla} \right|_{\hat{g}}$ and $\left| \hat{\nabla} \left(\widetilde{\nabla} - \hat{\nabla} \right) \right|_{\hat{g}}$ on $\mathcal{M} \times [0, \eta]$ such that

$$2 \left(\widetilde{\nabla}_0 \hat{g} \right) (X, X) \Big|_{(x,t_0)} \geq -C_1 \left| X \right|_{\hat{g}}^2 = -C_1$$

and

$$2 \left(\widetilde{\Delta} \hat{g} \right) (X, X) \Big|_{(x,t_0)} \leq C_2 \left| X \right|_{\hat{g}}^2 = C_2.$$

There is C_3 depending only on the Lipschitz constant of Φ on $\mathcal{M} \times [0, \eta]$ such that

$$\begin{aligned} - \Phi(Q)(X, X)|_{(x,t_0)} &\leq \Phi(\hat{Q})(X, X) - \Phi(Q)(X, X) \\ &\leq C_3 \varepsilon \eta \left| X \right|_{\hat{g}}^4 = \varepsilon \eta C_3, \end{aligned}$$

and there is C_4 depending only on the bounds for $|\Delta \rho|$, $|\nabla \rho|_g$, and $\left| \widetilde{\nabla} - \hat{\nabla} \right|_{\hat{g}}$ such that

$$\begin{aligned} \left(\widetilde{\Delta} F \right) (x, t_0) &= \left(\widetilde{\Delta} Q \right) (X, X) + \varepsilon (\eta + t_0) \left[\begin{array}{c} \rho \left(\widetilde{\Delta} \hat{g} \right) (X, X) \\ +2 \left(\widetilde{\nabla}_{\widetilde{\nabla} \rho} \hat{g} \right) (X, X) \\ + (\Delta \rho) \hat{g} (X, X) \end{array} \right] \\ &\leq \left(\widetilde{\Delta} Q \right) (X, X) + \varepsilon \eta (\rho C_2 + C_4). \end{aligned}$$

Combining these estimates with Equation (2.44), we conclude that at (x, t_0),

$$\begin{aligned} 0 &\geq \frac{\partial}{\partial t} F \\ &= \left(\widetilde{\nabla}_0 Q \right) (X, X) + \varepsilon (\eta + t_0) \rho \left(\widetilde{\nabla}_0 \hat{g} \right) (X, X) + \varepsilon \rho \hat{g} (X, X) \\ &= \left(\widetilde{\Delta} Q \right) (X, X) + \Phi(Q)(X, X) + \varepsilon (\eta + t_0) \rho \left(\widetilde{\nabla}_0 \hat{g} \right) (X, X) + \varepsilon \rho \\ &\geq \widetilde{\Delta} F - \varepsilon \eta (\rho C_2 + C_4) - \varepsilon \eta C_3 - \varepsilon \eta \rho C_1 + \varepsilon \rho \\ &\geq \varepsilon \left[\rho (1 - \eta (C_1 + C_2)) - \eta (C_3 + C_4) \right]. \end{aligned}$$

Because $\rho \geq 1$ and the constants C_i cannot increase if η decreases, choosing $\eta > 0$ sufficiently small gives a contradiction. So $\hat{Q}$ remains strictly positive on $\mathcal{M} \times [0, \eta]$, and the result follows by letting $\varepsilon \searrow 0$.

q.e.d.

2.5 Evolution equations relating to Hamilton's quadratic

This section is devoted to the following:

Proof of Proposition 23. Assume $\mu \equiv 1/2$, and denote the RHS of (2.19) by $\digamma_{ijk\ell}$. Lemma 21 implies that the following identities are valid for all $i, j, k, \ell \geq 1$:

$$\widetilde{B}_{ijk0} \doteqdot -\widetilde{g}^{pq}\widetilde{R}^r_{pij}\widetilde{R}_{kqr0} = tg^{pq}R^r_{pij}P_{qkr} \tag{2.45}$$

$$\widetilde{B}_{i00\ell} \doteqdot -\widetilde{g}^{pq}\widetilde{R}^r_{pi0}\widetilde{R}_{0qr\ell} = -t^2 g^{pq}g^{rs}P_{ipr}P_{\ell sq} \tag{2.46}$$

$$\widetilde{B}_{i0\ell 0} \doteqdot -\widetilde{g}^{pq}\widetilde{R}^r_{pi0}\widetilde{R}_{\ell qr0} = t^2 g^{pq}g^{rs}P_{ipr}P_{q\ell s} \tag{2.47}$$

$$\widetilde{B}_{i\ell 00} \doteqdot -\widetilde{g}^{pq}\widetilde{R}^r_{pi\ell}\widetilde{R}_{0qr0} = -t^2 g^{pq}R^r_{pi\ell}M_{qr}. \tag{2.48}$$

Thus for $i, \ell \geq 1$, we have

$$\begin{aligned}\digamma_{i00\ell} &\doteqdot \widetilde{\Delta}\widetilde{R}_{i00\ell} + 2\left(2\widetilde{B}_{i00\ell} - \widetilde{B}_{0i0\ell} - \widetilde{B}_{00i\ell}\right) + \widetilde{R}_{i00\ell} \\ &= \widetilde{\Delta}\widetilde{R}_{i00\ell} + 2t^2 g^{pq}g^{rs}\left[R_{ipr\ell}M_{qs} - P_{ipr}\left(2P_{\ell sq} + P_{q\ell s}\right)\right] + tM_{i\ell}.\end{aligned}$$

On the other hand, we use (C2)–(C4) with Lemma 21 to compute directly that

$$\begin{aligned}\widetilde{\nabla}_0\widetilde{R}_{i00\ell} &= \frac{\partial}{\partial t}\widetilde{R}_{i00\ell} - \widetilde{\Gamma}^p_{0i}\widetilde{R}_{p00\ell} - \widetilde{\Gamma}^p_{0\ell}\widetilde{R}_{i00p} - \widetilde{\Gamma}^p_{00}\left(\widetilde{R}_{ip0\ell} + \widetilde{R}_{i0p\ell}\right) \\ &= t\frac{\partial}{\partial t}\left(tM_{i\ell}\right) + \left(tR^p_i + \mu\delta^p_i\right)\left(tM_{p\ell}\right) + \left(tR^p_\ell + \mu\delta^p_\ell\right)\left(tM_{ip}\right) \\ &\quad + \left(\frac{1}{2}t^2\nabla^p R\right)\left(P_{pi\ell} - P_{\ell pi}\right) + 2\mu\left(tM_{i\ell}\right) \\ &= t^2\left[\frac{\partial}{\partial t}M_{i\ell} + R^p_i M_{p\ell} + R^p_\ell M_{ip} + \frac{1}{2}\left(\nabla^p R\right)\left(P_{pi\ell} + P_{p\ell i}\right)\right] \\ &\quad + 3tM_{i\ell}.\end{aligned}$$

In the same way, we compute for $q, m \leq 1$ that

$$\widetilde{\nabla}_q\widetilde{R}_{i00\ell} = t\left(\nabla_q M_{i\ell} + R^m_q P_{mi\ell} + R^m_q P_{m\ell i}\right) + \mu\left(P_{qi\ell} + P_{q\ell i}\right)$$

and

$$\widetilde{\nabla}_q \widetilde{R}_{im0\ell} = \nabla_q P_{mi\ell} + R_q^r R_{imr\ell} + \frac{\mu}{t} R_{imq\ell}.$$

Then by using the divergence identity

$$\nabla^q P_{qi\ell} = M_{i\ell} - R^{pq} R_{ipq\ell} - \frac{1}{2t} R_{i\ell},$$

we can write

$$\begin{aligned}
\widetilde{\Delta}\widetilde{R}_{i00\ell} &= \bar{g}^{pq}\widetilde{\nabla}_p \widetilde{\nabla}_q \widetilde{R}_{i00\ell} \\
&= t^2 \nabla^q \left(\nabla_q M_{i\ell} + R_q^m P_{mi\ell} + R_q^m P_{m\ell i}\right) + t\mu \nabla^q \left(P_{qi\ell} + P_{q\ell i}\right) \\
&\quad + t g^{pq} \left(t R_p^m + \mu \delta_p^m\right) \left(\nabla_q P_{mi\ell} + R_q^r R_{imr\ell} + \frac{\mu}{t} R_{imq\ell}\right) \\
&\quad + t g^{pq} \left(t R_p^m + \mu \delta_p^m\right) \left(\nabla_q P_{m\ell i} + R_q^r R_{irm\ell} + \frac{\mu}{t} R_{iqm\ell}\right) \\
&= t^2 \Delta M_{i\ell} + \frac{t^2}{2} \left(\nabla^q R\right)\left(P_{qi\ell} + P_{q\ell i}\right) + 2t^2 R^{pq} \nabla_p \left(P_{qi\ell} + P_{q\ell i}\right) \\
&\quad + 2t^2 R_m^p R^{mq} R_{ipq\ell} + 2t M_{i\ell} - \frac{1}{2} R_{i\ell}.
\end{aligned}$$

Cancelling terms yields

$$\begin{aligned}
\widetilde{\nabla}_0 \widetilde{R}_{i00\ell} = \widetilde{\Delta}\widetilde{R}_{i00\ell} &+ t^2 \left(\frac{\partial}{\partial t} - \Delta\right) M_{i\ell} - 2t^2 R^{pq} \nabla_p \left(P_{qi\ell} + P_{q\ell i}\right) \\
&+ t^2 \left(R_i^p M_{p\ell} + R_\ell^p M_{ip} - 2R_m^p R^{mq} R_{ipq\ell}\right) + t M_{i\ell} + \frac{1}{2} R_{i\ell}.
\end{aligned}$$

Recalling (2.26) and (2.27), we conclude that the special case $\widetilde{\nabla}_0 \widetilde{R}_{i00\ell} = \digamma_{i00\ell}$ of Equation (2.19) holds if and only if

$$\begin{aligned}
D_t M_{i\ell} \doteqdot \frac{\partial}{\partial t} M_{i\ell} + R_p^q \vee_q^p M_{i\ell} &= \frac{\partial}{\partial t} M_{i\ell} + R_i^p M_{p\ell} + R_l^p M_{ip} \\
&= \Delta M_{i\ell} + 2R^{pq} \nabla_p \left(P_{qi\ell} + P_{q\ell i}\right) + 2R_m^p R^{mq} R_{ipq\ell} \\
&\quad + 2g^{pq} g^{rs} \left[R_{ipr\ell} M_{qs} - P_{ipr} \left(2P_{\ell sq} + P_{q\ell s}\right)\right] - \frac{1}{2t^2} R_{i\ell},
\end{aligned}$$

hence if and only if Equation (2.40) holds.

Now if $i, j, k \geq 1$, identities (2.45)–(2.48) let us write

$$\begin{aligned}
\digamma_{ijk0} &= \widetilde{\Delta}\widetilde{R}_{ijk0} + 2t g^{pq} \left(R_{pij}^r P_{qkr} - R_{pji}^r P_{qkr} - R_{pjk}^r P_{qir} + R_{pik}^r P_{qjr}\right) \\
&\quad + P_{ijk} \\
&= \widetilde{\Delta}\widetilde{R}_{ijk0} + 2t g^{pq} \left(R_{pij}^r P_{qrk} - R_{pjk}^r P_{qir} + R_{pik}^r P_{qjr}\right) + P_{ijk}.
\end{aligned}$$

On the other hand, (C2)–(C4) and Lemma 21 imply that

$$\begin{aligned}\widetilde{\nabla}_0 \widetilde{R}_{ijk0} &= \frac{\partial}{\partial \bar{t}} \widetilde{R}_{ijk0} - \widetilde{\Gamma}^p_{0i} \widetilde{R}_{pjk0} - \widetilde{\Gamma}^p_{0j} \widetilde{R}_{ipk0} - \widetilde{\Gamma}^p_{0k} \widetilde{R}_{ijp0} - \widetilde{\Gamma}^p_{00} \widetilde{R}_{ijkp} \\ &= t\left(\frac{\partial}{\partial t} P_{ijk} + R_i^p P_{pjk} + R_j^p P_{ipk} + R_k^p P_{ijp} + \frac{1}{2} R_{ijkp} \nabla^p R\right) \\ &\quad + 2P_{ijk}\end{aligned}$$

and

$$\widetilde{\nabla}_q \widetilde{R}_{ijk0} = \nabla_q P_{ijk} - \widetilde{\Gamma}^p_{q0} \widetilde{R}_{ijkp} = \nabla_q P_{ijk} + R_q^p R_{ijkp} + \frac{1}{2t} R_{ijkq}.$$

Noticing that $\nabla^q R_{ijkq} = P_{ijk}$ by the second Bianchi identity, we write the diffusion term in the form

$$\begin{aligned}\widetilde{\Delta} \widetilde{R}_{ijk0} &= t g^{pq} \widetilde{\nabla}_p \widetilde{\nabla}_q \widetilde{R}_{ijk0} \\ &= t \nabla^q \left(\nabla_q P_{ijk} + R_q^p R_{ijkp} + \frac{1}{2t} R_{ijkq}\right) \\ &\quad + g^{pq}\left(t R_p^m + \frac{1}{2}\delta_p^m\right)\left(\nabla_q R_{ijkm}\right) \\ &= t\left(\Delta P_{ijk} + \frac{1}{2} R_{ijkp} \nabla^p R + 2 R_q^p \nabla^q R_{ijkp}\right) + P_{ijk}\end{aligned}$$

and cancel terms to obtain

$$\begin{aligned}\widetilde{\nabla}_0 \widetilde{R}_{ijk0} &= \widetilde{\Delta} \widetilde{R}_{ijk0} + t\left[\left(\frac{\partial}{\partial t} - \Delta\right) P_{ijk} + R_p^q \vee_q^p P_{ijk} - 2 R_p^q \nabla_q R_{ijk}^p\right] \\ &\quad + P_{ijk}.\end{aligned}$$

Thus the special case $\widetilde{\nabla}_0 \widetilde{R}_{ijk0} = \mathcal{F}_{ijk0}$ of Equation (2.19) holds if and only if

$$D_t P_{ijk} = \Delta P_{ijk} + 2 R_p^q \nabla_q R_{ijk}^p + 2 g^{pq}\left(R_{pij}^r P_{qrk} - R_{pjk}^r P_{qir} + R_{pik}^r P_{qjr}\right),$$

hence if and only if (2.39) holds.

Finally, the equivalence of (2.38) and the case $i, j, k, \ell \geq 1$ of (2.19) is clear when we observe that

$$\mathcal{F}_{ijk\ell} = \Delta R_{ijk\ell} + 2\left(B_{ijk\ell} - B_{jik\ell} - B_{jki\ell} + B_{ikj\ell}\right) + \frac{1}{t} R_{ijk\ell}$$

and

$$\widetilde{\nabla}_0 \widetilde{R}_{ijk\ell} = t \frac{\partial}{\partial t}\left(\frac{1}{t} R_{ijk\ell}\right) + R_p^q \vee_q^p R_{ijk\ell} + \frac{2}{t} R_{ijk\ell} = D_t R_{ijk\ell} + \frac{1}{t} R_{ijk\ell}.$$

q.e.d.

3. Generalized space-time connections

In this section, we derive new matrix LYH inequalities for the Ricci flow by generalizing the definition of the space-time connection in §2.

So let $\left(\mathcal{M}^n, \bar{g}\left(\bar{t}\right)\right)$ be a solution of the Ricci flow rescaled by a cosmological constant μ:

$$\frac{\partial}{\partial \bar{t}}\bar{g} = -2\left(\overline{\mathrm{Rc}} + \mu\bar{g}\right).$$

Consider the family of symmetric connections $\widetilde{\nabla}$ defined on space-time $(\widetilde{\mathcal{M}}, \widetilde{g})$ by

$$\widetilde{\Gamma}_{ij}^{k} \doteqdot \overline{\Gamma}_{ij}^{k} \tag{GC1}$$

$$\widetilde{\Gamma}_{i0}^{k} \doteqdot -\left(\overline{R}_i^k + \mu\delta_i^k + A_i^k\right) \tag{GC2}$$

$$\widetilde{\Gamma}_{00}^{k} \doteqdot -\left(\frac{1}{2}\overline{\nabla}^k\overline{R} + B^k\right) \tag{GC3}$$

$$\widetilde{\Gamma}_{00}^{0} \doteqdot -\left(\mu + C\right) \tag{GC4}$$

$$\widetilde{\Gamma}_{ij}^{0} \doteqdot \widetilde{\Gamma}_{i0}^{0} = 0, \tag{GC5}$$

for $i, j, k \geq 1$, where A is a tensor of type $(1,1)$, B is a vector field, and C is a scalar function. We saw in §2 that the space-time connection $\widetilde{\nabla}$ has a number of useful and interesting properties when $A = B = C = 0$. Our goal here is to investigate what conditions on A, B, and C are necessary and sufficient for $\widetilde{\nabla}$ to retain certain desirable characteristics. In particular, we determine which connections of this form are both compatible with the space-time metric and satisfy the Ricci flow for degenerate metrics. Such space-time connections are worth studying, because their curvatures satisfy parabolic evolution equations and thus furnish Li–Yau–Hamilton quadratics for the Ricci flow.

Define a $(2,0)$-tensor $\overline{A}$ by

$$\overline{A}_{ij} \doteqdot A_i^p \bar{g}_{pj}.$$

Our first observation is that $\widetilde{\nabla}$ is both torsion-free and compatible with $\widetilde{g}$ exactly when $\overline{A}$ is a 2-form:

Lemma 26. *The metric $\widetilde{g}$ is parallel with respect to the symmetric connection $\widetilde{\nabla}$,*

$$\widetilde{\nabla}_i \widetilde{g}^{jk} = 0,$$

if and only if (GC1)-(GC5) *hold, where* $\overline{A}$ *is a* 2*-form,*

$$A_p^j \bar{g}^{pk} + A_p^k \bar{g}^{jp} = 0, \tag{3.1}$$

and there are no restrictions on either B *or* C*.*

Proof. For $i, j, k \geq 1$, the equation

$$0 = \widetilde{\nabla}_i \widetilde{g}^{jk} = \partial_i \widetilde{g}^{jk} + \widetilde{\Gamma}_{ip}^j \widetilde{g}^{pk} + \widetilde{\Gamma}_{ip}^k \widetilde{g}^{jp}$$

is equivalent to

$$\widetilde{\Gamma}_{ij}^k = \overline{\Gamma}_{ij}^k,$$

since $\overline{\nabla}$ is the unique torsion-free connection compatible with $\bar{g}$; this is (GC1). Assuming (GC2), the equation

$$0 = \widetilde{\nabla}_0 \widetilde{g}^{jk} = \partial_0 \widetilde{g}^{jk} + \widetilde{\Gamma}_{0p}^j \widetilde{g}^{pk} + \widetilde{\Gamma}_{0p}^k \widetilde{g}^{jp}$$

is valid for $j, k \geq 1$ if and only if

$$A_p^j \bar{g}^{pk} + A_p^k \bar{g}^{jp} = 0.$$

This says that when we lower an index, $\overline{A}_{ij} = A_i^p \bar{g}_{pj}$ is a 2-form. The equation

$$0 = \widetilde{\nabla}_i \widetilde{g}^{0k} = \partial_i \widetilde{g}^{0k} + \widetilde{\Gamma}_{ip}^0 \widetilde{g}^{pk} + \widetilde{\Gamma}_{ip}^k \widetilde{g}^{0p}$$

is valid for $i \geq 0$ and $k \geq 1$ if and only if

$$\widetilde{\Gamma}_{ip}^0 = 0$$

holds for all $i \geq 0$ and $p \geq 1$; this is (GC5). The identity $\widetilde{\nabla}_i \widetilde{g}^{00}$ is satisfied automatically for all $i \geq 0$. q.e.d.

Hence by lowering indices, we may regard $\overline{A}$ as a 2-form, $\overline{B}_i \doteqdot B^p \bar{g}_{pi}$ as a 1-form, and C as a 0-form.

3.1 The Riemann curvature tensor

The space-time Riemann curvature tensor is defined by (2.13); in components, one has

$$\widetilde{R}_{ijk}^\ell = \partial_i \widetilde{\Gamma}_{jk}^\ell - \partial_j \widetilde{\Gamma}_{ik}^\ell + \widetilde{\Gamma}_{jk}^m \widetilde{\Gamma}_{im}^\ell - \widetilde{\Gamma}_{ik}^m \widetilde{\Gamma}_{jm}^\ell. \tag{3.2}$$

By definition, we have the asymmetry $\widetilde{R}_{ijk}^\ell = -\widetilde{R}_{jik}^\ell$. The remaining formulas are as follows:

Proposition 27. *If $i,j,k,\ell \geq 1$ and $a,b,c \geq 0$, then $\widetilde{\mathrm{Rm}}$ satisfies:*

$$\text{(GR1)} \qquad \widetilde{R}_{ijk}^{\ell} = \overline{R}_{ijk}^{\ell}$$

$$\text{(GR2a)} \qquad \widetilde{R}_{ij0}^{\ell} = \overline{\nabla}_j \overline{R}_i^{\ell} - \overline{\nabla}_i \overline{R}_j^{\ell} + \overline{\nabla}_j A_i^{\ell} - \overline{\nabla}_i A_j^{\ell}$$

$$\text{(GR2b)} \qquad \widetilde{R}_{0jk}^{\ell} = \overline{\nabla}^{\ell} \overline{R}_{jk} - \overline{\nabla}_k \overline{R}_j^{\ell} + \overline{\nabla}_j A_k^{\ell}$$

$$\begin{aligned} \text{(GR3)} \qquad \widetilde{R}_{0j0}^{\ell} = & -\frac{\partial}{\partial \bar{t}} \overline{R}_j^{\ell} + (\mu - C)\, \overline{R}_j^{\ell} + \frac{1}{2} \overline{\nabla}_j \overline{\nabla}^{\ell} \overline{R} + \overline{R}_j^m \overline{R}_m^{\ell} \\ & - \frac{\partial}{\partial \bar{t}} A_j^{\ell} + A_j^m A_m^{\ell} + (\mu - C)\, A_j^{\ell} + \overline{R}_j^m A_m^{\ell} + A_j^m \overline{R}_m^{\ell} \\ & + \overline{\nabla}_j B^{\ell} - \mu C \delta_j^{\ell}. \end{aligned}$$

$$\text{(GR4)} \qquad \widetilde{R}_{abc}^{0} = 0.$$

Proof. Identities (GR1) and (GR4) follow easily from (3.2). To derive (GR2a), we use (GC2) to compute

$$\begin{aligned} \widetilde{R}_{ij0}^{\ell} &= \partial_i \widetilde{\Gamma}_{j0}^{\ell} + \widetilde{\Gamma}_{im}^{\ell} \widetilde{\Gamma}_{j0}^{m} - \partial_j \widetilde{\Gamma}_{i0}^{\ell} - \widetilde{\Gamma}_{jm}^{\ell} \widetilde{\Gamma}_{i0}^{m} \\ &= \overline{\nabla}_j \left(\overline{R}_i^{\ell} + A_i^{\ell} \right) - \overline{\nabla}_i \left(\overline{R}_j^{\ell} + A_j^{\ell} \right). \end{aligned}$$

To derive (GR2b), we recall that

$$\frac{\partial}{\partial \bar{t}} \overline{\Gamma}_{jk}^{\ell} = -\overline{\nabla}_j \overline{R}_k^{\ell} - \overline{\nabla}_k \overline{R}_j^{\ell} + \overline{\nabla}^{\ell} \overline{R}_{jk}$$

and calculate

$$\begin{aligned} \widetilde{R}_{0jk}^{\ell} &= \partial_0 \widetilde{\Gamma}_{jk}^{\ell} - \left(\partial_j \widetilde{\Gamma}_{0k}^{\ell} - \widetilde{\Gamma}_{jk}^{m} \widetilde{\Gamma}_{0m}^{\ell} + \widetilde{\Gamma}_{jm}^{\ell} \widetilde{\Gamma}_{0k}^{m} \right) \\ &= \frac{\partial}{\partial \bar{t}} \overline{\Gamma}_{jk}^{\ell} + \overline{\nabla}_j \left(\overline{R}_k^{\ell} + A_k^{\ell} \right) \\ &= \overline{\nabla}^{\ell} \overline{R}_{jk} - \overline{\nabla}_k \overline{R}_j^{\ell} + \overline{\nabla}_j A_k^{\ell}. \end{aligned}$$

Finally, to derive (GR3), we use (GC3) and (GC4) to compute

$$\begin{aligned}
\widetilde{R}_{0j0}^{\ell} &= \partial_0 \widetilde{\Gamma}_{j0}^{\ell} - \left(\partial_j \widetilde{\Gamma}_{00}^{\ell} + \widetilde{\Gamma}_{jm}^{\ell} \widetilde{\Gamma}_{00}^{m}\right) + \widetilde{\Gamma}_{j0}^{m} \widetilde{\Gamma}_{0m}^{\ell} \\
&= -\frac{\partial}{\partial \bar{t}} \left(\overline{R}_j^{\ell} + A_j^{\ell}\right) + \overline{\nabla}_j \left(\frac{1}{2} \overline{\nabla}^{\ell} \overline{R} + B^{\ell}\right) \\
&\quad - (\mu + C) \left(\overline{R}_j^{\ell} + \mu \delta_j^{\ell} + A_j^{\ell}\right) \\
&\quad + \left(\overline{R}_j^{m} + \mu \delta_j^{m} + A_j^{m}\right) \left(\overline{R}_m^{\ell} + \mu \delta_m^{\ell} + A_m^{\ell}\right) \\
&= -\frac{\partial}{\partial \bar{t}} \overline{R}_j^{\ell} + (\mu - C) \overline{R}_j^{\ell} + \frac{1}{2} \overline{\nabla}_j \overline{\nabla}^{\ell} \overline{R} + \overline{R}_j^{m} \overline{R}_m^{\ell} \\
&\quad - \frac{\partial}{\partial \bar{t}} A_j^{\ell} + A_j^{m} A_m^{\ell} + (\mu - C) A_j^{\ell} + \overline{R}_j^{m} A_m^{\ell} + A_j^{m} \overline{R}_m^{\ell} \\
&\quad + \overline{\nabla}_j B^{\ell} - \mu C \delta_j^{\ell}.
\end{aligned}$$

q.e.d.

Corollary 28. *If $i, j \geq 1$, then $\widetilde{\mathrm{Rc}}$ satisfies:*

$$\begin{aligned}
\widetilde{R}_{ij} &= \overline{R}_{ij} \\
\widetilde{R}_{0k} &= \frac{1}{2} \overline{\nabla}_k \overline{R} - \left(\bar{\delta} \overline{A}\right)_k \\
\widetilde{R}_{00} &= \frac{1}{2} \frac{\partial}{\partial \bar{t}} \overline{R} + C \left(\overline{R} + n\mu\right) + \left|\overline{A}\right|_{\bar{g}}^2 + \bar{\delta} \overline{B},
\end{aligned}$$

where

$$\left(\bar{\delta} \overline{A}\right)_k \doteqdot -\overline{\nabla}^p \overline{A}_{pk} = \overline{\nabla}_p A_k^p$$

and

$$\bar{\delta} \overline{B} \doteqdot -\overline{\nabla}^p \overline{B}_p = -\overline{\nabla}_p B^p.$$

Proof. The first two equations are easy. For the third, we substitute the formula

$$\frac{1}{2} \frac{\partial}{\partial \bar{t}} \overline{R} = \frac{1}{2} \overline{\Delta} \overline{R} + \overline{R}_{pq} \overline{R}^{pq} + \mu \overline{R}$$

into the calculation

$$\begin{aligned}
\widetilde{R}_{00} &= -\widetilde{R}_{0j0}^{j} \\
&= \frac{\partial}{\partial \bar{t}} \overline{R} + (C - \mu) \overline{R} - \frac{1}{2} \overline{\Delta} \overline{R} - \overline{R}_j^{m} \overline{R}_m^{m} - A_j^{m} A_m^{j} - \overline{\nabla}_j B^j + n\mu C
\end{aligned}$$

and cancel terms. q.e.d.

3.2 Solutions of the Ricci flow for degenerate metrics

The goal of this section is to determine necessary and sufficient conditions on A, B, and C for $\left(\widetilde{g}, \widetilde{\nabla}\right)$ to satisfy the rescaled Ricci flow for degenerate metrics. (Recall Definition 14.) Our results here are most easily stated if we introduce the 1-form

$$\overline{E} \doteqdot \overline{B} + 2\bar{\delta}\overline{A}.$$

We shall see that a particularly nice set of equations is obtained when $\overline{A}$ and $\overline{E}$ are closed initially. In this case, there is always a solution $\left(\widetilde{g}, \widetilde{\nabla}\right)$ satisfying Definition 14 for as long as $\bar{g}\left(\bar{t}\right)$ exists.

Proposition 29. *Suppose $C = \mu$. Then $\left(\widetilde{g}, \widetilde{\nabla}\right)$ satisfy the Ricci flow with cosmological term μ, namely*

$$\frac{\partial}{\partial \bar{t}}\widetilde{\Gamma}_{ij}^{k} = -\widetilde{\nabla}_i \widetilde{R}_j^k - \widetilde{\nabla}_j \widetilde{R}_i^k + \widetilde{\nabla}^k \widetilde{R}_{ij}, \tag{3.3}$$

if and only if the 2-form $\overline{A} = \overline{A}_{ij}\, dx^i \otimes dx^j$ satisfies

$$\frac{\partial}{\partial \bar{t}}\overline{A} = -d\bar{\delta}\overline{A} - 2\mu\overline{A} \tag{3.4}$$

and the 1-form $\overline{E} = \overline{E}_i dx^i$ satisfies

$$\frac{\partial}{\partial \bar{t}}\overline{E} = -d\bar{\delta}\overline{E} - 2\mu\overline{E} - d\left|\overline{A}\right|_{\bar{g}}^2 . \tag{3.5}$$

If $d\overline{A} = 0$ initially, then $d\overline{A} \equiv 0$ for as long as a solution exists; and if $d\overline{E} = 0$ initially, then $d\overline{E} \equiv 0$ for as long as a solution exists. So if $\overline{A}$ and $\overline{E}$ are closed initially, (3.3) *is valid if and only if $\overline{A}$ and $\overline{E}$ evolve by*

$$\frac{\partial}{\partial \bar{t}}\overline{A} = \overline{\Delta}_d\overline{A} - 2\mu\overline{A} \tag{3.6}$$

and

$$\frac{\partial}{\partial \bar{t}}\overline{E} = \overline{\Delta}_d\overline{E} - 2\mu\overline{E} - d\left|\overline{A}\right|_{\bar{g}}^2 \tag{3.7}$$

respectively, where $-\overline{\Delta}_d \doteqdot d\bar{\delta} + \bar{\delta}d$ is the Hodge–de Rham Laplacian.

Remark 30. When $\overline{A}$ and $\overline{E}$ are closed initially, (3.6) and (3.7) are both parabolic equations whose solutions exist as long as the solution of the Ricci flow with cosmological constant μ exists.

Remark 31. The choice $C = \mu$ is useful to obtain good evolution equation if either A or B is nonzero. But if A and B are both identically zero, taking $C = 0$ as in §2 generally yields better results.

Remark 32. If $\left(\overline{A}, \overline{E}\right)$ is a pair of initially-closed forms satisfying Equations (3.6) and (3.7), then the pair $\left(\lambda\overline{A}, \lambda^2\overline{E}\right)$ is also, for any $\lambda \in \mathbb{R}$.

Proof of Proposition 29. Let $\digamma_{ij}^{k}$ denote the RHS of (3.3). If $i, j, k \geq 1$, then formula (3.3) reduces to the standard evolution equation for $\overline{\Gamma}_{ij}^{k}$. It is easily checked that both sides of (3.3) vanish if $k = 0$, provided that μ and C are constant. If $j = 0$ but $i, k \geq 1$, then

$$\begin{aligned}
\digamma_{i0}^{k} &\doteqdot -\widetilde{\nabla}_i \widetilde{R}_0^k - \widetilde{\nabla}_0 \widetilde{R}_i^k + \widetilde{\nabla}^k \widetilde{R}_{i0} \\
&= -\overline{\nabla}_i \left[\frac{1}{2}\overline{\nabla}^k \overline{R} - \left(\bar{\delta}\overline{A}\right)^k\right] - \left(\overline{R}_i^p + \mu\delta_i^p + A_i^p\right) \overline{R}_p^k \\
&\quad - \frac{\partial}{\partial \bar{t}} \overline{R}_i^k - \left(\overline{R}_i^p + \mu\delta_i^p + A_i^p\right) \overline{R}_p^k + \left(\overline{R}_p^k + \mu\delta_p^k + A_p^k\right) \overline{R}_i^p \\
&\quad + \overline{\nabla}^k \left[\frac{1}{2}\overline{\nabla}_i \overline{R} - \left(\bar{\delta}\overline{A}\right)_i\right] + \bar{g}^{k\ell} \left(\overline{R}_\ell^p + \mu\delta_\ell^p + A_\ell^p\right) \overline{R}_{ip} \\
&= -\frac{\partial}{\partial \bar{t}} \overline{R}_i^k - 2A_i^p \overline{R}_p^k - \overline{\nabla}_i \overline{\nabla}^p A_p^k - \overline{\nabla}^k \overline{\nabla}_p A_i^p.
\end{aligned}$$

Since $\widetilde{\Gamma}_{i0}^k = -\left(\overline{R}_i^k + \mu\delta_i^k + A_i^k\right)$, it follows that (3.3) holds for $j = 0$ and $i, k \geq 1$ if and only if

$$\frac{\partial}{\partial \bar{t}} A_i^k = \overline{\nabla}_i \overline{\nabla}^p A_p^k + \overline{\nabla}^k \overline{\nabla}_p A_i^p + 2A_i^p \overline{R}_p^k,$$

hence if and only if

$$\frac{\partial}{\partial \bar{t}} \overline{A}_{ij} = \frac{\partial}{\partial \bar{t}} \left(A_i^k \bar{g}_{kj}\right) = -\left(d\bar{\delta}\overline{A}\right)_{ij} - 2\mu\overline{A}_{ij}.$$

If $i = j = 0$ but $k \geq 1$, we recall that

$$\overline{\nabla}^k \left(\frac{\partial}{\partial \bar{t}} \overline{R}\right) = \frac{\partial}{\partial \bar{t}} \left(\overline{\nabla}^k \overline{R}\right) - 2\left(\overline{R}_\ell^k \overline{\nabla}^\ell \overline{R} + \mu \overline{\nabla}^k \overline{R}\right)$$

and compute

$$
\begin{aligned}
\Gamma^k_{00} &\doteqdot -\widetilde{\nabla}_0 \widetilde{R}^k_0 - \widetilde{\nabla}_0 \widetilde{R}^k_0 + \widetilde{\nabla}^k \widetilde{R}_{00} \\
&= -2\left[\frac{\partial}{\partial \bar{t}}\left(\frac{1}{2}\overline{\nabla}^k \overline{R} - \left(\bar{\delta}\overline{A}\right)^k\right) - \widetilde{\Gamma}^p_{00}\widetilde{R}^k_p + \widetilde{\Gamma}^k_{0p}\widetilde{R}^p_0\right] \\
&\quad + \overline{\nabla}^k\left(\frac{1}{2}\frac{\partial}{\partial \bar{t}}\overline{R} + C\overline{R} + \left|\overline{A}\right|^2_{\bar{g}} + \bar{\delta}\overline{B}\right) - 2\bar{g}^{k\ell}\widetilde{\Gamma}^p_{\ell 0}\widetilde{R}_{p0} \\
&= -\frac{1}{2}\frac{\partial}{\partial \bar{t}}\left(\overline{\nabla}^k\overline{R}\right) - \overline{\nabla}^k\overline{\nabla}_p B^p - 2B^p\overline{R}^k_p \\
&\quad + 2\frac{\partial}{\partial \bar{t}}\left(\bar{g}^{k\ell}\overline{\nabla}_p A^p_\ell\right) + \overline{\nabla}^k\left|\overline{A}\right|^2_{\bar{g}} + 2\left(\mu - C\right)\left(\overline{\nabla}^p A^k_p\right) - 4\overline{R}^{kp}\overline{\nabla}_q A^q_p.
\end{aligned}
$$

Since $\widetilde{\Gamma}^k_{00} = -\frac{1}{2}\overline{\nabla}^k\overline{R} - B^k$, it follows that (3.3) holds for $i = j = 0$ and $k \geq 1$ if and only if

$$
\begin{aligned}
\frac{\partial}{\partial \bar{t}}\left(\overline{B}_j + 2\overline{\nabla}_p A^p_j\right) &= \frac{\partial}{\partial \bar{t}}\left[\bar{g}_{jk}\left(B^k + 2\bar{g}^{k\ell}\overline{\nabla}_p A^p_\ell\right)\right] \\
&= \overline{\nabla}_j\overline{\nabla}_p B^p - 2\mu\overline{B}_j - \overline{\nabla}_j\left|\overline{A}\right|^2_{\bar{g}} + 2\left(\mu + C\right)\overline{\nabla}^p\overline{A}_{pj}.
\end{aligned}
$$

When $C = \mu$, this equation is the same as

$$
\begin{aligned}
\frac{\partial}{\partial \bar{t}}\left(\overline{B} + 2\bar{\delta}\overline{A}\right) &= -d\bar{\delta}\overline{B} - 2\mu\left(\overline{B} + 2\bar{\delta}\overline{A}\right) - d\left|\overline{A}\right|^2_{\bar{g}} \\
&= -d\bar{\delta}\left(\overline{B} + 2\bar{\delta}\overline{A}\right) - 2\mu\left(\overline{B} + 2\bar{\delta}\overline{A}\right) - d\left|\overline{A}\right|^2_{\bar{g}},
\end{aligned}
$$

because $\bar{\delta}^2 = 0$.

To complete the proof, it suffices to note that

$$
\frac{\partial}{\partial \bar{t}}\left(d\overline{A}\right) = d\left(\frac{\partial}{\partial \bar{t}}\overline{A}\right) = -2\mu\left(d\overline{A}\right)
$$

and

$$
\frac{\partial}{\partial \bar{t}}\left(d\overline{E}\right) = d\left(\frac{\partial}{\partial \bar{t}}\overline{E}\right) = -2\mu\left(d\overline{E}\right),
$$

because the exterior derivative is independent of the metric and satisfies $d^2 = 0$. q.e.d.

In analogy with Remark 13, we make the following observation:

Remark 33. If $\overline{A}$ evolves according to Equation (3.4), one has the symmetry

$$
\widetilde{\nabla}_i\widetilde{R}_{j0} - \widetilde{\nabla}_j\widetilde{R}_{i0} = \widetilde{\nabla}_0\overline{A}_{ij}.
$$

Proof. Because

$$\begin{aligned}\widetilde{\nabla}_i \widetilde{R}_{j0} &= \overline{\nabla}_i \left(\frac{1}{2} \overline{\nabla}_j \overline{R} + \overline{\nabla}^p \overline{A}_{pj} \right) - \widetilde{\Gamma}_{i0}^p \overline{R}_{jp} \\ &= \frac{1}{2} \overline{\nabla}_i \overline{\nabla}_j \overline{R} + \overline{\nabla}_i \overline{\nabla}^p \overline{A}_{pj} + \overline{R}_i^p \overline{R}_{pj} + \mu \overline{R}_{ij} + \overline{A}_i^p \overline{R}_{pj},\end{aligned}$$

we observe that when (3.4) holds, we have

$$\begin{aligned}\widetilde{\nabla}_0 \overline{A}_{ij} &= \frac{\partial}{\partial \bar{t}} \overline{A}_{ij} - \widetilde{\Gamma}_{0i}^p \overline{A}_{pj} - \widetilde{\Gamma}_{0j}^p \overline{A}_{ip} \\ &= \overline{\nabla}_i \overline{\nabla}^p \overline{A}_{pj} - \overline{\nabla}_j \overline{\nabla}^p \overline{A}_{pi} + A_i^p \overline{R}_{pj} - A_j^p \overline{R}_{pi} \\ &= \widetilde{\nabla}_i \widetilde{R}_{j0} - \widetilde{\nabla}_j \widetilde{R}_{i0}.\end{aligned}$$

q.e.d.

3.3 New Li–Yau–Hamilton quadratics

We now wish to regard $\widetilde{\mathrm{Rm}}$ as the bilinear form defined on $\Lambda^2 T\widetilde{\mathcal{M}}$ by (2.17) and (2.18). To be useful as a LYH quadratic, it is desirable that a bilinear form be symmetric and positive. Fortunately, symmetry of $\widetilde{\mathrm{Rm}}$ is compatible with the other properties we wish $\widetilde{\nabla}$ to possess. In particular, we have the following:

Lemma 34. *The bilinear form $\widetilde{\mathrm{Rm}}$ has the symmetry*

$$\widetilde{R}_{ij0\ell} = \widetilde{R}_{0\ell ij}$$

for all $i, j, \ell \geq 1$ if and only if $\overline{A}$ is a closed 2-form. Moreover, $\widetilde{\mathrm{Rm}}$ has the symmetry

$$\widetilde{R}_{0j0\ell} = \widetilde{R}_{0\ell 0j}$$

for all $j, \ell \geq 1$ if $\overline{A}$ evolves by (3.4), $C = \mu$, and $\overline{E}$ is a closed 1-form.

Proof. By (GR2a) and (GR2b), we have

$$\widetilde{R}_{ij0\ell} - \widetilde{R}_{0\ell ij} = \bar{g}_{\ell p} \widetilde{R}_{ij0}^p - \bar{g}_{jp} \widetilde{R}_{0\ell i}^p = \overline{\nabla}_j \overline{A}_{i\ell} - \overline{\nabla}_i \overline{A}_{j\ell} + \overline{\nabla}_\ell \overline{A}_{ij} = \left(d\overline{A} \right)_{ji\ell}.$$

Next we observe that

$$\bar{g}_{jp} \frac{\partial}{\partial \bar{t}} \overline{R}_\ell^p - \bar{g}_{\ell p} \frac{\partial}{\partial \bar{t}} \overline{R}_j^p = 0$$

and

$$\bar{g}_{jp}\frac{\partial}{\partial\bar{t}}A_\ell^p - \bar{g}_{\ell p}\frac{\partial}{\partial\bar{t}}A_j^p = 2\frac{\partial}{\partial\bar{t}}\overline{A}_{\ell j} + 2\overline{R}_{jp}A_\ell^p - 2\overline{R}_{\ell p}A_j^p + 4\mu\overline{A}_{\ell j}. \tag{3.8}$$

Hence by (GR3),

$$\begin{aligned}\widetilde{R}_{0j0\ell} - \widetilde{R}_{0\ell0j} &= \bar{g}_{\ell p}\widetilde{R}_{0j0}^p - \bar{g}_{jp}\widetilde{R}_{0\ell0}^p \\ &= \bar{g}_{jp}\left(\frac{\partial}{\partial\bar{t}}\overline{R}_\ell^p + \frac{\partial}{\partial\bar{t}}A_\ell^p\right) - \bar{g}_{\ell p}\left(\frac{\partial}{\partial\bar{t}}\overline{R}_j^p + \frac{\partial}{\partial\bar{t}}A_j^p\right) \\ &\quad + 2(\mu - C)\overline{A}_{j\ell} + 2A_j^k\overline{R}_{k\ell} - 2A_\ell^k\overline{R}_{kj} + \left(\overline{\nabla}_j\overline{B}_\ell - \overline{\nabla}_\ell\overline{B}_j\right) \\ &= 2\frac{\partial}{\partial\bar{t}}\overline{A}_{\ell j} + 2(\mu + C)\overline{A}_{\ell j} - \left(d\overline{B}\right)_{\ell j}.\end{aligned}$$

If $\frac{\partial}{\partial\bar{t}}\overline{A} = -d\bar{\delta}\overline{A} - 2\mu\overline{A}$, this becomes

$$\widetilde{R}_{0j0\ell} - \widetilde{R}_{0\ell0j} = 2(C-\mu)\overline{A}_{\ell j} - d\left(\overline{B} + 2\bar{\delta}\overline{A}\right)_{\ell j}.$$

q.e.d.

It is also fortunate that the maximum principle applies to the curvature $\widetilde{\mathrm{Rm}}$ of a generalized connection. To see this, it will be convenient to introduce a $(1,1)$-tensor $\widetilde{A}$ defined for $i, j \geq 1$ by

$$\begin{aligned}\widetilde{A}_i^j &= A_i^j \\ \widetilde{A}_0^j &= \left(B + \bar{\delta}\overline{A}\right)^j \\ \widetilde{A}_i^0 &= 0 \\ \widetilde{A}_0^0 &= \mu.\end{aligned}$$

Proposition 35. *Let $\overline{A}$ and $\overline{E}$ be closed initially and evolve by (3.6) and (3.7), respectively. Let $C = \mu$ be constant. Then $\widetilde{\mathrm{Rm}}$ is a symmetric bilinear form which evolves by*

$$\widetilde{\nabla}_0\widetilde{\mathrm{Rm}} = \widetilde{\Delta}\widetilde{\mathrm{Rm}} + \widetilde{\mathrm{Rm}}^2 + \widetilde{\mathrm{Rm}}^\# + 2\mu\widetilde{\mathrm{Rm}} + \widetilde{A} \vee \widetilde{\mathrm{Rm}}. \tag{3.9}$$

Proof. Because $\widetilde{\nabla}$ is symmetric, one computes directly from the definition that

$$\begin{aligned}
&\frac{\partial}{\partial t}\widetilde{R}_{ijk}^{\ell} \\
&= \partial_i\left(\frac{\partial}{\partial t}\widetilde{\Gamma}_{jk}^{\ell}\right) - \partial_j\left(\frac{\partial}{\partial t}\widetilde{\Gamma}_{ik}^{\ell}\right) \\
&\quad + \left(\frac{\partial}{\partial t}\widetilde{\Gamma}_{jk}^{m}\right)\widetilde{\Gamma}_{im}^{\ell} + \widetilde{\Gamma}_{jk}^{m}\left(\frac{\partial}{\partial t}\widetilde{\Gamma}_{im}^{\ell}\right) - \left(\frac{\partial}{\partial t}\widetilde{\Gamma}_{ik}^{m}\right)\widetilde{\Gamma}_{jm}^{\ell} - \widetilde{\Gamma}_{ik}^{m}\left(\frac{\partial}{\partial t}\widetilde{\Gamma}_{jm}^{\ell}\right) \\
&= \partial_i\left(\frac{\partial}{\partial t}\widetilde{\Gamma}_{jk}^{\ell}\right) - \widetilde{\Gamma}_{ij}^{m}\left(\frac{\partial}{\partial t}\widetilde{\Gamma}_{mk}^{\ell}\right) - \widetilde{\Gamma}_{ik}^{m}\left(\frac{\partial}{\partial t}\widetilde{\Gamma}_{jm}^{\ell}\right) + \widetilde{\Gamma}_{im}^{\ell}\left(\frac{\partial}{\partial t}\widetilde{\Gamma}_{jk}^{m}\right) \\
&\quad - \partial_j\left(\frac{\partial}{\partial t}\widetilde{\Gamma}_{ik}^{\ell}\right) + \widetilde{\Gamma}_{ji}^{m}\left(\frac{\partial}{\partial t}\widetilde{\Gamma}_{mk}^{\ell}\right) + \widetilde{\Gamma}_{jk}^{m}\left(\frac{\partial}{\partial t}\widetilde{\Gamma}_{im}^{\ell}\right) - \widetilde{\Gamma}_{jm}^{\ell}\left(\frac{\partial}{\partial t}\widetilde{\Gamma}_{ik}^{m}\right) \\
&= \widetilde{\nabla}_i\left(\frac{\partial}{\partial t}\widetilde{\Gamma}_{jk}^{\ell}\right) - \widetilde{\nabla}_j\left(\frac{\partial}{\partial t}\widetilde{\Gamma}_{ik}^{\ell}\right).
\end{aligned}$$

Hence by Proposition 29 and the Ricci identities,

$$\begin{aligned}
\frac{\partial}{\partial t}\widetilde{R}_{ijk}^{\ell} &= \overline{\nabla}_j\left(\nabla_k\widetilde{R}_i^{\ell} - \widetilde{\nabla}^{\ell}\widetilde{R}_{ik}\right) - \widetilde{\nabla}_i\left(\widetilde{\nabla}_k\widetilde{R}_j^{\ell} - \widetilde{\nabla}^{\ell}\widetilde{R}_{jk}\right) \\
&\quad + \widetilde{R}_{jim}^{\ell}\widetilde{R}_k^m - \widetilde{R}_{jik}^m\widetilde{R}_m^{\ell}.
\end{aligned}$$

On the other hand, the second Bianchi identity implies that

$$\begin{aligned}
\widetilde{\Delta}\widetilde{R}_{ijk}^{\ell} \doteqdot \widetilde{g}^{pq}\widetilde{\nabla}_p\widetilde{\nabla}_q\widetilde{R}_{ijk}^{\ell} &= -\widetilde{g}^{pq}\widetilde{\nabla}_p\left(\widetilde{\nabla}_i\widetilde{R}_{jqk}^{\ell} + \widetilde{\nabla}_j\widetilde{R}_{qik}^{\ell}\right) \\
&= \widetilde{\nabla}_i\widetilde{\nabla}^{\ell}\widetilde{R}_{jk} - \widetilde{\nabla}_i\widetilde{\nabla}_k\widetilde{R}_j^{\ell} - \widetilde{\nabla}_j\widetilde{\nabla}^{\ell}\widetilde{R}_{ik} \\
&\quad + \widetilde{\nabla}_j\widetilde{\nabla}_k\widetilde{R}_i^{\ell} - \widetilde{R}_i^m\widetilde{R}_{jmk}^{\ell} - \widetilde{R}_j^m\widetilde{R}_{mik}^{\ell} \\
&\quad + \widetilde{g}^{pq}\begin{bmatrix} \widetilde{R}_{pij}^m\widetilde{R}_{mqk}^{\ell} + \widetilde{R}_{pik}^m\widetilde{R}_{jqm}^{\ell} - \widetilde{R}_{pim}^{\ell}\widetilde{R}_{jqk}^m \\ +\widetilde{R}_{pji}^m\widetilde{R}_{qmk}^{\ell} + \widetilde{R}_{pjk}^m\widetilde{R}_{qim}^{\ell} - \widetilde{R}_{pjm}^{\ell}\widetilde{R}_{qik}^m \end{bmatrix},
\end{aligned}$$

while straightforward calculations reveal that

$$\widetilde{R}_{ijk\ell}^2 = \widetilde{g}^{pq}\widetilde{g}^{rs}\widetilde{R}_{ijpr}\widetilde{R}_{sqk\ell} = -\widetilde{g}^{pq}\left(\widetilde{R}_{pij}^m\widetilde{R}_{mqk\ell} + \widetilde{R}_{pji}^m\widetilde{R}_{qmk\ell}\right)$$

and

$$\begin{aligned}
\widetilde{R}_{ijk\ell}^{\#} &= \widetilde{R}_{abcd}\widetilde{R}_{pqrs}\left(\delta_i^a\delta_j^q\widetilde{g}^{bp} - \delta_i^p\delta_j^b\widetilde{g}^{aq}\right)\left(\delta_\ell^c\delta_k^s\widetilde{g}^{dr} - \delta_\ell^r\delta_k^d\widetilde{g}^{cs}\right) \\
&= -\widetilde{g}^{pq}\left(\widetilde{R}_{qim\ell}\widetilde{R}_{pjk}^m + \widetilde{R}_{pik}^m\widetilde{R}_{jqm\ell} + \widetilde{R}_{jpm\ell}\widetilde{R}_{qik}^m + \widetilde{R}_{jqk}^m\widetilde{R}_{ipm\ell}\right).
\end{aligned}$$

Because Equation (3.9) is readily verified for $k = \ell = 0$, we may assume without loss of generality that $\ell \geq 1$. Then we can combine the identities above to obtain

$$\begin{aligned}\frac{\partial}{\partial \bar{t}}\widetilde{R}_{ijk\ell} &= \bar{g}_{\ell m}\frac{\partial}{\partial \bar{t}}\widetilde{R}_{ijk}^{m} - 2\overline{R}_{\ell m}\widetilde{R}_{ijk}^{m} - 2\mu\widetilde{R}_{ijk\ell} \\ &= \widetilde{\Delta}\widetilde{R}_{ijk\ell} + \widetilde{R}_{ijk\ell}^{2} + \widetilde{R}_{ijk\ell}^{\#} - 2\mu\widetilde{R}_{ijk\ell} - \widetilde{R}_{p}^{q} \vee_{q}^{p} \widetilde{R}_{ijk\ell},\end{aligned}$$

because $\bar{g}_{lm}\widetilde{\Delta}\widetilde{R}_{ijk}^{m} = \widetilde{\Delta}\widetilde{R}_{ijk\ell}$ and $\widetilde{R}_{\ell}^{m}\widetilde{R}_{ijkm} = \overline{R}_{\ell m}\widetilde{R}_{ijk}^{m}$. Now if we regard $\widetilde{\Gamma}_0$ as a (globally-defined) space-time $(1,1)$-tensor $\left(\widetilde{\Gamma}_0\right)_p^q = \widetilde{\Gamma}_{0p}^q$, we may write

$$\widetilde{\nabla}_0\widetilde{R}_{ijk\ell} = \frac{\partial}{\partial \bar{t}}\widetilde{R}_{ijk\ell} - \left(\widetilde{\Gamma}_0 \vee \widetilde{\mathrm{Rm}}\right)_{ijk\ell},$$

yielding

$$\widetilde{\nabla}_0\widetilde{\mathrm{Rm}} = \widetilde{\Delta}\widetilde{\mathrm{Rm}} + \widetilde{\mathrm{Rm}}^2 + \widetilde{\mathrm{Rm}}^{\#} - 2\mu\widetilde{\mathrm{Rm}} - \left(\widetilde{\Gamma}_0 + \widetilde{\mathrm{Rc}}\right) \vee \widetilde{\mathrm{Rm}}, \tag{3.10}$$

where $\widetilde{\mathrm{Rc}}$ here denotes the $(1,1)$-tensor $\widetilde{R}_p^q \doteqdot \widetilde{R}_{pm}\widetilde{g}^{mq}$. We claim Equation (3.10) is equivalent to Equation (3.9). Indeed, it follows from (GC2)–(GC4) and Corollary 28 that for $p, q \geq 1$ one has

$$\begin{aligned}\left(\widetilde{\Gamma}_0 + \widetilde{\mathrm{Rc}}\right)_p^q &= -\mu\delta_p^q - A_p^q \\ \left(\widetilde{\Gamma}_0 + \widetilde{\mathrm{Rc}}\right)_0^q &= -B^q - \left(\bar{\delta}\bar{A}\right)^q \\ \left(\widetilde{\Gamma}_0 + \widetilde{\mathrm{Rc}}\right)_p^0 &= 0 \\ \left(\widetilde{\Gamma}_0 + \widetilde{\mathrm{Rc}}\right)_0^0 &= -2\mu.\end{aligned}$$

So one need only check that if N denotes the number of space-like components of $\widetilde{R}_{ijk\ell}$, the RHS of (3.10) contains $-2 + N + 2(4 - N) = 6 - N$ terms of the form $\mu\widetilde{R}_{ijk\ell}$, while the RHS of (3.9) contains $2 + (4 - N) = 6 - N$ such terms. q.e.d.

Armed with the tools to show that $\widetilde{\mathrm{Rm}}$ remains nonnegative, we are now ready to construct Li–Yau–Hamilton quadratics.

Condition 36. Assume in the remainder of this paper that $g(t)$ is a solution of the Ricci flow on $\mathcal{M}$ for $t \in [0, \Omega)$, and that $\bar{g}(\bar{t}) \equiv e^{-\bar{t}} g\left(e^{\bar{t}}\right)$

is the associated solution of (2.8), the Ricci flow with cosmological constant $\mu = 1/2$, for $\bar{t} \in (-\infty, \ln \Omega)$. Assume further that the generalized connection on $(\widetilde{\mathcal{M}}, \widetilde{g})$ defined by (GC1)-(GC5) has $C = \mu = 1/2$.

Given any 2-form U and 1-form W on $\mathcal{M}$, we define $\widetilde{X} \doteqdot U \oplus \frac{1}{2} W$, so that for $i, j, \geq 1$,

$$\widetilde{X}^{ij} = U^{ij}$$
$$\widetilde{X}^{0j} = -\widetilde{X}^{j0} = \frac{1}{2} W^j.$$

If there exist $\overline{A}$ and $\overline{B}$ such that $\widetilde{\mathrm{Rm}}$ is symmetric, we define the forms

$$A_{ik} \doteqdot A_i^j g_{jk} = t\overline{A}_{ik}$$
$$B_k \doteqdot B^j g_{jk} = t\overline{B}_k$$
$$E_k \doteqdot \left(B^j + 2 \left(\bar{\delta} \overline{A} \right)^j \right) g_{jk} = B_k + 2t \left(\delta A \right)_k ,$$

and make the following observations:

Remark 37. Let $\overline{\Delta}_d$ and Δ_d denote the Hodge–de Rham Laplacians of $\left(\bar{g}, \overline{\nabla} \right)$ and (g, ∇), respectively. Then $\overline{A}$ is closed and evolves by

$$\frac{\partial}{\partial \bar{t}} \overline{A} = \overline{\Delta}_d \overline{A} - 2\mu \overline{A}$$

if and only if A is closed and evolves by

$$\frac{\partial}{\partial t} A = \Delta_d A + \frac{1 - 2\mu}{t} A.$$

Moreover, $\overline{E} \equiv \overline{B} + 2\bar{\delta} \overline{A}$ is closed and evolves by

$$\frac{\partial}{\partial \bar{t}} \overline{E} = \overline{\Delta}_d \overline{E} - 2\mu \overline{E} - d \left| \overline{A} \right|_{\bar{g}}^2$$

if and only if $E \equiv B + 2t\delta A$ is closed and evolves by

$$\frac{\partial}{\partial t} E = \Delta_d E + \frac{1 - 2\mu}{t} E - d \left| A \right|_g^2 .$$

If $\mu = 1/2$ and A is closed (exact) initially, then A remains closed (exact). Likewise, if $\mu = 1/2$ and E is closed (exact) initially, then E remains closed (exact).

Proof. The evolution equations are straightforward calculations. Together with Proposition 29, they imply that A and E remain closed if they are initially. To prove the assertions about exactness, suppose that $A(0) = d\alpha_0$ and $E(0) = d\varepsilon_0$. Let $\alpha(t)$ and $\varepsilon(t)$ be solutions of

$$\frac{\partial}{\partial t}\alpha = \Delta_d \alpha, \qquad \alpha(0) = \alpha_0$$

and

$$\frac{\partial}{\partial t}\varepsilon = \Delta_d \varepsilon - |A|_g^2, \qquad \varepsilon(0) = \varepsilon_0$$

respectively. Then

$$\frac{\partial}{\partial t}(d\alpha) = d\left(\frac{\partial}{\partial t}\alpha\right) = -d(d\delta + \delta d)\alpha = -(d\delta + \delta d)(d\alpha) = \Delta_d(d\alpha),$$

and similarly

$$\frac{\partial}{\partial t}(d\varepsilon) = d\left(\frac{\partial}{\partial t}\varepsilon\right) = d\left(\Delta_d \varepsilon - |A|_g^2\right) = \Delta_d(d\varepsilon) - d|A|_g^2.$$

By uniqueness of solutions to parabolic equations, we have $A(t) \equiv d\alpha(t)$ and $E(t) \equiv d\varepsilon(t)$ for as long as $g(t)$ exists. q.e.d.

Theorem 38. *Let $(\mathcal{M}^n, g(t))$ be a solution of the Ricci flow on a closed manifold and a time interval $[0, \Omega)$. Let A_0 be a 2-form which is closed at $t = 0$ and let E_0 be a 1-form which is closed at $t = 0$. Then there is a solution $A(t)$ of*

$$\frac{\partial}{\partial t}A = \Delta_d A, \qquad A(0) = A_0$$

and a solution $E(t)$ of

$$\frac{\partial}{\partial t}E = \Delta_d E - d|A|_g^2, \qquad E(0) = E_0$$

which exist for all $t \in [0, \Omega)$. Suppose that

$$0 \leq \mathrm{Rm}(U, U) + \frac{1}{4}|W|^2 + |A(W)|^2 - 2\langle \nabla_W A, U\rangle - \langle \nabla_W E, W\rangle$$

at $t = 0$ for any 2-form U and 1-form W on $\mathcal{M}$, and let $\widetilde{\mathrm{Rm}}$ be the curvature of the generalized connection on $(\widetilde{\mathcal{M}}, \widetilde{g})$ with $B \equiv E - 2t\delta A$

and $C = 1/2$. Then for all $t \in [0, \Omega)$, one has the estimate

$$
\begin{aligned}
0 &\le e^{\bar{t}} \widetilde{\mathrm{Rm}} \left(\widetilde{X}, \widetilde{X} \right) \\
&= R_{ijk\ell} U^{ij} U^{\ell k} + 2 \left[t \left(\nabla_\ell R_{jk} - \nabla_k R_{j\ell} \right) + \nabla_j A_{k\ell} \right] W^j U^{\ell k} \\
&\quad + \begin{bmatrix} t^2 \left(\Delta R_{j\ell} - \frac{1}{2} \nabla_j \nabla_\ell R + 2 R_{jpq\ell} R^{pq} - R_j^p R_{p\ell} \right) \\ + t \left(2\mu R_{j\ell} + A_j^p R_{p\ell} + A_\ell^p R_{pj} - 2 \nabla_j \nabla^k A_{k\ell} \right) \\ + \mu^2 g_{j\ell} - A_j^p A_{p\ell} - \nabla_j E_\ell \end{bmatrix} W^j W^\ell .
\end{aligned}
$$

Proof. By Remark 37, there are closed solutions $\overline{A}$ and $\overline{E}$ of (3.6) and (3.7), respectively, existing for $-\infty < \bar{t} < \log \Omega$. Set $Q \doteqdot e^{\bar{t}} \widetilde{\mathrm{Rm}}$. Then by Proposition 35,

$$
\begin{aligned}
\frac{\partial}{\partial t} Q &= \frac{d\bar{t}}{dt} \frac{\partial}{\partial \bar{t}} \left(e^{\bar{t}} \widetilde{\mathrm{Rm}} \right) \\
&= \frac{\partial}{\partial \bar{t}} \widetilde{\mathrm{Rm}} + \widetilde{\mathrm{Rm}} \\
&= \widetilde{\Delta} \widetilde{\mathrm{Rm}} + \widetilde{\mathrm{Rm}}^2 + \widetilde{\mathrm{Rm}}^{\#} + 2 \widetilde{\mathrm{Rm}} + \left(\widetilde{\Gamma}_0 + \widetilde{A} \right) \vee \widetilde{\mathrm{Rm}}.
\end{aligned}
\tag{3.11}
$$

Since $Q \ge 0$ at $t = 0$, it will suffice to apply the space-time maximum principle (Proposition 25) to Q for $0 \le t < \Omega$. Notice that for $j, k \ge 1$,

$$
\begin{aligned}
\left(\widetilde{\Gamma}_0 + \widetilde{A} \right)_j^k &= -t R_j^k - \frac{1}{2} \delta_j^k \\
\left(\widetilde{\Gamma}_0 + \widetilde{A} \right)_0^k &= -\frac{t^2}{2} \nabla^k R + t \left(\delta A \right)^k \\
\left(\widetilde{\Gamma}_0 + \widetilde{A} \right)_j^0 &= 0 \\
\left(\widetilde{\Gamma}_0 + \widetilde{A} \right)_0^0 &= -\frac{1}{2},
\end{aligned}
$$

and define a generalized symmetric connection $\hat{\nabla}$ on $\mathcal{M} \times [0, \Omega)$ for $i, j, k \ge 1$ by

$$
\begin{aligned}
\hat{\Gamma}_{ij}^k &\doteqdot \Gamma_{ij}^k \\
\hat{\Gamma}_{i0}^k &\doteqdot -R_i^k \\
\hat{\Gamma}_{00}^k &\doteqdot -\left(\frac{1}{2} \nabla^k R + F^k \right) \\
\hat{\Gamma}_{00}^0 &= \hat{\Gamma}_{i0}^0 = \hat{\Gamma}_{ij}^0 = 0,
\end{aligned}
$$

where the Γ_{ij}^k are determined by the Levi-Civita connection of g, and

$$F^k \doteqdot \frac{1}{2}(t-1)\nabla^k R - (\delta A)^k .$$

Let $\hat{g}$ be the space-time metric on $\mathcal{M} \times [0, \Omega)$ induced by g: namely, $\hat{g}^{ij} = g^{ij}$ and $\hat{g}^{i0} = \hat{g}^{00} = 0$ for $i, j \geq 1$. By Lemma 26, $\hat{g}$ is parallel with respect to $\hat{\nabla}$. One verifies by straightforward calculation that

$$\widetilde{\Delta}\widetilde{\mathrm{Rm}} \doteqdot \widetilde{g}^{pq}\widetilde{\nabla}_p\widetilde{\nabla}_q\widetilde{\mathrm{Rm}} = \hat{g}^{pq}\hat{\nabla}_p\hat{\nabla}_q Q \doteqdot \hat{\Delta} Q$$

and

$$\widetilde{R}^2_{ijk\ell} = \widetilde{g}^{ad}\widetilde{g}^{bc}\widetilde{R}_{ijab}\widetilde{R}_{cdk\ell} = g^{ad}g^{bc}Q_{ijab}Q_{cdk\ell} \doteqdot Q^2_{ijk\ell},$$

and

$$\widetilde{R}^{\#}_{ijk\ell} = \widetilde{R}_{abcd}\widetilde{R}_{pqrs}C_{ij}^{ab,pq}C_{\ell k}^{cd,rs} = Q_{abcd}Q_{pqrs}c_{ij}^{ab,pq}c_{\ell k}^{cd,rs} \doteqdot Q^{\#}_{ijk\ell},$$

where $c_{ij}^{ab,cd} \doteqdot \delta_i^a\delta_j^d g^{bc} - \delta_i^c\delta_j^b g^{ad} = e^{-\bar{t}}C_{ij}^{ab,cd}$. Thus Equation (3.11) can be written in the form

$$\hat{\nabla}_0 Q = \hat{\Delta} Q + Q^2 + Q^{\#},$$

whence the theorem follows by applying Proposition 25 to Q for $0 \leq t < \Omega$. q.e.d.

If $\widetilde{\mathrm{Rm}}$ is symmetric, we define another symmetric bilinear form $\widetilde{H}$ on $\Lambda^2 T\widetilde{\mathcal{M}}$ by stipulating that $\widetilde{H}$ obey the symmetries

$$\widetilde{H}_{abcd} = -\widetilde{H}_{bacd} = -\widetilde{H}_{abdc} = \widetilde{H}_{cdab}$$

for all $a, b, c, d \geq 0$ and satisfy

$$\begin{aligned} \widetilde{H}_{ijk\ell} &\doteqdot \overline{R}_{ijk\ell} \\ \widetilde{H}_{0jk\ell} &\doteqdot \overline{\nabla}_j\overline{A}_{k\ell} \\ \widetilde{H}_{0j0\ell} &\doteqdot A_j^p\overline{A}_{p\ell} + \overline{\nabla}_j\overline{E}_\ell \end{aligned}$$

for all $i, j, k, \ell \geq 1$. Then we define and expand

$$\begin{aligned} \Psi(A, E, U, W) &\doteqdot e^{\bar{t}}\widetilde{H}\left(\widetilde{X}, \widetilde{X}\right) \\ &= R_{ijk\ell}U^{ij}U^{\ell k} + 2W^j\nabla_j A_{k\ell}U^{\ell k} \\ &\quad - \left(A_j^p A_{p\ell} + \nabla_j E_\ell\right)W^j W^\ell. \end{aligned}$$

Theorem 39. *Let $(\mathcal{M}^n, g(t))$ be a solution of the Ricci flow on a closed manifold and a time interval $[0, \Omega)$. Let A_0 be a 2-form which is closed at $t = 0$, and let E_0 be a 1-form which is closed at $t = 0$. Then there is a solution $A(t)$ of*

$$\frac{\partial}{\partial t} A = \Delta_d A, \qquad A(0) = A_0$$

and a solution $E(t)$ of

$$\frac{\partial}{\partial t} E = \Delta_d E - d\,|A|_g^2, \qquad E(0) = E_0$$

which exist for all $t \in [0, \Omega)$. Suppose that

$$\begin{aligned} 0 &\leq \Psi(A, E, U, W)|_{t=0} \\ &= \mathrm{Rm}(U, U) - 2\langle \nabla_W A, U\rangle + |A(W)|^2 - \langle \nabla_W E, W\rangle \end{aligned}$$

for any 2-form U and 1-form W on $\mathcal{M}$. Then $\Psi(A, E, U, W)$ remains nonnegative for all $t \in [0, \Omega)$.

Proof. Suppose $(\mathcal{M}, g(t))$ exists for $t \in [0, \Omega)$, and define

$$\Phi(A, E, U, W) \doteqdot e^t \widetilde{\mathrm{Rm}}\left(\widetilde{X}, \widetilde{X}\right),$$

where $\widetilde{\mathrm{Rm}}$ is the curvature of the generalized connection on $(\widetilde{\mathcal{M}}, \widetilde{g})$ with $B \equiv E - 2t\delta A$ and $C = 1/2$. By hypothesis,

$$0 \leq \Psi(A, E, U, W) \leq \Phi(A, E, U, W)$$

at $t = 0$ for all U and W. So $\Phi(A, E, U, W) \geq 0$ for all $t \in [0, \Omega)$ and all U, W by Theorem 38. Now since $\Psi\left(\lambda A, \lambda^2 E, U, W\right) = \Psi(A, E, U, \lambda W)$, it follows from Remark 32 that $\Phi\left(\lambda A, \lambda^2 E, U, W\right) \geq 0$ for all $t \in [0, \Omega)$, all U, W, and all $\lambda > 0$. In particular, at each fixed $t \in [0, \Omega)$, we have $0 \leq \Phi\left(\lambda A, \lambda^2 E, U, \lambda^{-1} W\right)$ for all U, W and $\lambda > 0$, and hence

$$0 \leq \lim_{\lambda \to \infty} \Phi\left(\lambda A, \lambda^2 E, U, \lambda^{-1} W\right) = \Psi(A, E, U, W).$$

q.e.d.

Corollary 40. *Let the hypotheses of Theorem 39 hold. Then for any 1-form V, we have*

$$\begin{aligned} 0 \leq \psi(A, E, V) &\doteqdot \mathrm{Rc}(V, V) - 2(\delta A)(V) + |A|^2 + \delta E \\ &= \mathrm{Rc}(V, V) - 2(\delta A)(V) + |A|^2 + \delta B \end{aligned}$$

for as long as the solution $(\mathcal{M}, g(t))$ exists.

Proof. For an orthonormal frame $\{e_k\}$, take $U_{ij} = \frac{1}{2}(V_i W_j - V_j W_i)$ and trace over $W \in \{e_1, \ldots, e_n\}$. q.e.d.

4. Examples

We shall now develop some examples, which may be regarded as further corollaries of Theorem 39. Our intent is to explore the utility of the new LYH quadratics by comparing their implications with known results in a few special cases. Although our principle examples come from Kähler geometry, we emphasize that the main results of this paper are more general, and in no way require a Kähler structure.

4.1 Kähler examples

By definition, a Riemannian manifold $(\mathcal{M}^n, g)$ is Kähler if there is an almost-complex structure $J : T\mathcal{M} \to T\mathcal{M}$ such that g is Hermitian $g(JX, JX) = g(X, X)$ and J is parallel $\nabla_X (JY) = J(\nabla_X Y)$. The latter condition immediately implies the symmetry $R(X,Y)JZ = J(R(X,Y)Z)$ of the Riemannian curvature.

There is a one-to-one correspondence between J-invariant symmetric 2-tensors and J-invariant 2-forms on a Kähler manifold. Indeed, given a J-invariant symmetric 2-tensor $\phi(JX, JY) = \phi(X,Y) = \phi(Y,X)$, let $F(X,Y) \doteqdot \phi(JX, Y)$. Then F is a 2-form $F(Y,X) = -F(X,Y)$ which is J-invariant $F(JX, JY) = F(X,Y)$. Conversely, given a J-invariant 2-form $H(JX, JY) = H(X,Y) = -H(Y,X)$, let $\eta(X,Y) \doteqdot -H(JX, Y)$. Then η is symmetric $\eta(Y,X) = \eta(X,Y)$ and J-invariant $\eta(JX, JY) = \eta(X,Y)$. The *Kähler form* ω is defined to be the 2-form induced by a Hermitian metric g, and the *Ricci form* ρ is defined to be the 2-form induced by the Ricci tensor Rc of g. These definitions are justified by the following standard facts:

Lemma 41. *Let J be an almost-complex structure on $(\mathcal{M}^n, g)$.*

1. *If $(\mathcal{M}, g)$ is Kähler, then* Rc *is J-invariant.*
2. *If $(\mathcal{M}, g)$ is Kähler, then ρ is closed.*
3. *If g is Hermitian, then J is parallel if and only if ω is closed.*

It follows easily that the Ricci flow preserves a Kähler structure. Indeed, (1) implies that g remains Hermitian, whence (2) and (3) imply that J remains parallel, because

$$\frac{\partial}{\partial t}(d\omega) = d\left(\frac{\partial}{\partial t}\omega\right) = d(-2\rho) = 0.$$

A key observation for the constructions in this section is the following:

Lemma 42. *Let ϕ be a J-invariant symmetric 2-tensor and let F be the corresponding J-invariant 2-form. Then ϕ satisfies the Lichnerowicz-Laplacian heat equation*

$$\frac{\partial}{\partial t}\phi = \Delta_L \phi,$$

if and only if F satisfies the Hodge-Laplacian heat equation

$$\frac{\partial}{\partial t}F = \Delta_d F.$$

Proof. If $\frac{\partial}{\partial t}\phi = \Delta_L \phi$, then direct computation gives

$$\begin{aligned}\frac{\partial}{\partial t}F_{ij} = \frac{\partial}{\partial t}\left(J_i^k \phi_{kj}\right) &= J_i^k \left(\Delta \phi_{kj} + 2R_{kpqj}\phi^{pq} - R_j^\ell \phi_{k\ell} - R_k^\ell \phi_{j\ell}\right) \\ &= \Delta F_{ij} + 2J_i^k R_{kpqj}\phi^{pq} - R_j^\ell F_{i\ell} - J_i^k R_k^\ell \phi_{j\ell}.\end{aligned}$$

The Hodge–de Rham Laplacian of F is

$$\begin{aligned}\Delta_d F_{ij} &= -(d\delta F)_{ij} - (\delta dF)_{ij} \\ &= \left(\nabla_i \nabla^k F_{kj} - \nabla_j \nabla^k F_{ki}\right) + \nabla^k\left(\nabla_k F_{ij} - \nabla_i F_{kj} + \nabla_j F_{ki}\right) \\ &= \Delta F_{ij} - \left(\nabla^k \nabla_i - \nabla_i \nabla^k\right) F_{kj} + \left(\nabla^k \nabla_j - \nabla_j \nabla^k\right) F_{ki} \\ &= \Delta F_{ij} + 2g^{k\ell} R_{kij}^m F_{\ell m} - R_i^k F_{kj} - R_j^k F_{ik}.\end{aligned}$$

The identity

$$\begin{aligned}\langle R(Z,W)X, Y\rangle &= \langle R(X,Y)Z, W\rangle \\ &= \langle R(X,Y)JZ, JW\rangle = \langle R(JZ,JW)X, Y\rangle\end{aligned}$$

implies in particular that $R(JZ,W)X = -R(Z,JW)X$, and hence

$$J_i^k R_{kpqj}\phi^{pq} = -R_{ikqj} J_p^k \phi^{pq} = g^{k\ell} R_{kijq} J_\ell^p \phi_p^q = g^{k\ell} R_{kij}^q F_{\ell q}.$$

Thus

$$\frac{\partial}{\partial t}F_{ij} = \Delta_d F_{ij} - J_i^k R_k^\ell \phi_{j\ell} + R_i^k F_{kj}.$$

But since the Ricci tensor is J-invariant, we have

$$J_k^\ell R_i^k = J_i^k R_k^\ell,$$

so that

$$-J_i^k R_k^\ell \phi_{j\ell} = -J_k^\ell R_i^k \phi_{j\ell} = -R_i^k F_{kj}.$$

Hence $\frac{\partial}{\partial t} F = \Delta_d F$. The converse is proved similarly. q.e.d.

Proposition 43. *If $(\mathcal{M}^n, g(t))$ is a Kähler solution of the Ricci flow with nonnegative curvature operator on a closed manifold, the choices*

$$A = t\rho + \frac{1}{2}\omega$$

and

$$E = -\frac{t^2}{2}\, dR$$

yield the estimate

$$\begin{aligned} 0 \leq \operatorname{Rm}(U,U) - 2\langle \nabla_W \rho, U\rangle + \frac{1}{4t^2}|W|^2 + \frac{1}{t}\operatorname{Rc}(W,W) \\ + \operatorname{Rc}^2(W,W) + \frac{1}{2}(\nabla\nabla R)(W,W) \end{aligned}$$

for all $t > 0$ such that the solution exists.

Proof. The choice $A = t\rho + \frac{1}{2}\omega$ satisfies

$$\frac{\partial}{\partial t} A = \rho + t\frac{\partial}{\partial t}\rho + \frac{1}{2}(-2\rho) = t\Delta_d \rho = \Delta_d A$$

by the preceding lemma, and

$$|A|^2 = t^2 |\operatorname{Rc}|^2 + tR + \frac{n}{4}.$$

If $E = df$ for some smooth function f, then E will satisfy

$$\frac{\partial}{\partial t} E = \Delta_d E - d|A|^2$$

if and only if

$$d\left(\frac{\partial}{\partial t} f\right) = \frac{\partial}{\partial t} E = d\left(\Delta f - t^2|\operatorname{Rc}|^2 - tR - \frac{n}{4}\right).$$

The choice $f = -t^2 R/2$ satisfies

$$\frac{\partial}{\partial t} f = \Delta f - t^2 |\mathrm{Rc}|^2 - tR.$$

So to apply Theorem 39, we need only check that $\Psi(A, E, U, W) \geq 0$ at $t = 0$ for any 2-form U and 1-form W. Noting that $\mathrm{Rc}^2(JW, JW) = \mathrm{Rc}^2(W, W)$, we compute

$$\begin{aligned}\Psi(A, E, U, W) = {} & \mathrm{Rm}(U, U) - 2t \langle \nabla_W \rho, U\rangle + \frac{1}{4}|W|^2 + t\,\mathrm{Rc}(W, W) \\ & + t^2 \mathrm{Rc}^2(W, W) + \frac{t^2}{2}(\nabla\nabla R)(W, W).\end{aligned}$$

This is clearly nonnegative at $t = 0$ whenever the curvature operator is. Hence Theorem 39 implies in particular that $0 \leq \Psi\left(A, E, U, \frac{1}{t}W\right)$ for all U and W at any $t > 0$ such that the solution exists. q.e.d.

To apply Corollary 40, we note that $\delta E = \delta df = \frac{t^2}{2}\Delta R$. Because

$$(\delta A)_i = t(\delta\rho)_i + \frac{1}{2}(\delta\omega)_i = t\nabla^j \rho_{ij} = tJ_i^k \nabla^j R_{kj} = \frac{t}{2} J_i^k \nabla_k R,$$

we have

$$(\delta A)(V) = \frac{t}{2} J_i^k \nabla_k R V^i = \frac{t}{2}\langle \nabla R, JV\rangle.$$

Then setting $V = tJX$, we compute

$$\begin{aligned}\psi(A, E, V) &= \mathrm{Rc}(V, V) - t\langle \nabla R, JV\rangle \\ &\quad + \left(t^2 |\mathrm{Rc}|^2 + tR + \frac{n}{4}\right) + \frac{t^2}{2}\Delta R \\ &= \frac{t^2}{2}\left[\begin{array}{c} \Delta R + 2|\mathrm{Rc}|^2 + \frac{R}{t} \\ +2\langle \nabla R, X\rangle + 2\,\mathrm{Rc}(JX, JX) \end{array}\right] \\ &\quad + \frac{1}{2}\left(tR + \frac{n}{2}\right),\end{aligned}$$

obtaining:

Example 44. If $(\mathcal{M}, g(t))$ is a Kähler solution of the Ricci flow with nonnegative curvature operator on a closed manifold, then for any vector field X and all $t > 0$ such that a solution exists, we have the estimate

$$0 \leq \left[\frac{\partial}{\partial t} R + \frac{R}{t} + 2\langle \nabla R, X\rangle + 2\,\mathrm{Rc}(X, X)\right] + \left(\frac{R}{t} + \frac{n}{2t^2}\right). \tag{4.1}$$

The terms in square brackets compose Hamilton's trace quadratic. So this special case of our general inequality is weaker than that estimate. To gauge the potential usefulness of our inequality, however, we can make qualitative comparisons. Taking $X = 0$, we obtain

$$0 \leq t^2 \frac{\partial R}{\partial t} + 2tR + \frac{n}{2} = \frac{\partial}{\partial t}\left[t\left(tR + \frac{n}{2}\right)\right].$$

Hence at any $x \in \mathcal{M}$ and times $t_2 \geq t_1$ with $t_2 \neq 0$, we have

$$R(x, t_2) \geq \left(\frac{t_1}{t_2}\right)^2 R(x, t_1) + \frac{n}{2t_2}\left(\frac{t_1}{t_2} - 1\right).$$

If we have an ancient solution, so that the interval of existence is $t \in (-\infty, \Omega)$, we can translate in time $t \mapsto t - \tau$ and take the limit as $\tau \to \infty$ to conclude that R is a pointwise nondecreasing function of t. Thus this particular example (4.1) of our general result is strong enough to recover that important fact. (Compare [12] and [13].)

On the other hand, suppose $t_1 > 0$ and there is some constant $C > 0$ such that $R(x, t_1) \geq nC/t_1$. Then for all $t_2 \in [t_1, Ct_1]$ we have

$$R(x, t_2) \geq \frac{R(x, t_1)}{2C^2} + \left(\frac{t_1}{t_2}\right)^2 \frac{nC}{2t_1} + \frac{n}{2t_2}\left(\frac{1}{C} - 1\right) \geq \frac{R(x, t_1)}{2C^2}.$$

In all known applications of a trace LYH inequality, one has $t_1 \geq c > 0$. Thus in this sense also, the special case (4.1) of Corollary 40 is qualitatively equivalent to Hamilton's estimate.

Remark 45. If $(\mathcal{M}^n, g(t))$ is a Kähler solution of the Ricci flow on a closed manifold, we can choose $A = \rho$ and $E = -dR/2$ (thereby dropping the explicit time dependency from Example 44) and thus obtain the quadratic

$$\Psi\left(\rho, -\frac{1}{2}dR, U, W\right) = \mathrm{Rm}(U, U) - 2\langle \nabla_W \rho, U\rangle + \mathrm{Rc}^2(W, W) + \frac{1}{2}(\nabla\nabla R)(W, W).$$

One would not, however, expect this to be positive for general initial data.

4.2 Other examples

Proposition 46. *Let $(\mathcal{M}^2, g(t))$ be a solution of the Ricci flow on a closed surface, and let dS denote the area element of g. Then for any pair (ϕ, f) solving the system*

$$\frac{\partial}{\partial t}\phi = \Delta\phi + R\phi \tag{4.2a}$$

$$\frac{\partial}{\partial t}f = \Delta f + \phi^2, \tag{4.2b}$$

the choices $A = \phi\, dS$ and $E = -2df$ yield the trace quadratic

$$\psi(A, E, X) = R|X|^2 + 2\langle\nabla\phi, X\rangle + \frac{\partial}{\partial t}f$$

and the matrix quadratic

$$\begin{aligned}&\Psi(A, E, U, W)\\ &= R|U|^2 - 2\langle\nabla\phi, W\rangle\langle\omega, U\rangle + \phi^2|W|^2 + 2(\nabla\nabla f)(W, W).\end{aligned}$$

If $\psi \geq 0$ $(\Psi \geq 0)$ at $t = 0$, then $\psi \geq 0$ $(\Psi \geq 0)$ for as long as the solution exists.

Example 47. On any closed surface $(\mathcal{M}^2, g)$ of nonnegative curvature evolving by the Ricci flow, the pair (ϕ, f) given by

$$\begin{aligned}\phi &= tR + 1\\ f &= t\phi = t^2R + t\end{aligned}$$

yield the estimate

$$0 \leq \frac{\partial}{\partial t}R + 2\langle\nabla R, X\rangle + R|X|^2 + \frac{2tR + 1}{t^2}$$

for any vector field X and all $t > 0$ such that a solution exists.

It is remarkable that this special case of Proposition 46 recovers the result in Example 44. Even though Proposition 46 is *a priori* more general than Proposition 43 in the sense that it depends only on the dimension, the construction in Proposition 46 is specific to surfaces, and does not generalize readily to higher-dimension manifolds (whether Kähler or not).

In analogy with Remark 45, one can drop the explicit time dependency in Example 47 and notice that the choices $\phi = f = R$ solve system (4.2a)-(4.2b), obtaining:

Remark 48. On any closed surface $\left(\mathcal{M}^2, g(t)\right)$ evolving by the Ricci flow, one has the trace LYH quadratic

$$\psi(A, E, X) = \frac{\partial}{\partial t} R + 2\langle \nabla R, X \rangle + R |X|^2 .$$

Proof of Proposition 46. Write the area element of g as $(dS)_{ij} = J_i^k g_{kj}$, and suppose that $A = \phi \, dS$ for some smooth function ϕ. Then using the standard fact that $\frac{\partial}{\partial t} dS = -R \, dS$, we get

$$\frac{\partial}{\partial t} A = \left(\frac{\partial}{\partial t} \phi - R\phi \right) dS.$$

Because $\Delta_d A = (\Delta \phi) \, dS$, it follows that A satisfies the Hodge heat equation $\frac{\partial}{\partial t} A = \Delta_d A$ if and only if ϕ evolve by Equation (4.2a). Now suppose $E = -2df$ for some smooth function f. Then since

$$|A|^2 = \phi^2 g^{ik} g^{j\ell} J_i^p g_{pj} J_k^q g_{q\ell} = 2\phi^2,$$

it follows that E satisfies $\frac{\partial}{\partial t} E = \Delta_d E - d |A|^2$ if and only if

$$-2d\left(\frac{\partial}{\partial t} f \right) = 2 (d\delta + \delta d)(df) - 4\phi \, d\phi = -2d\left(\Delta f + \phi^2 \right).$$

Hence we can apply Theorem 39 with any solution f of (4.2b).

To apply Corollary 40, we first calculate

$$(\delta A)_i = (\delta (\phi \, dS)_i) = \left(\nabla^j \phi \right) (dS)_{ij} = \nabla^j \phi J_i^k g_{kj} = J_i^k \nabla_k \phi = (J \, d\phi)_i .$$

Then we need only note that $\delta E = 2\Delta f$ and set $V = 2JX$ in the resulting expression:

$$\psi(A, E, V) = \frac{1}{2} R |V|^2 - 2\langle \nabla \phi, JV \rangle + 2\phi^2 + 2\Delta f.$$

q.e.d.

Proof of Example 47. The choice $\phi = tR + 1$ satisfies

$$\frac{\partial}{\partial t} \phi = t\left(\Delta R + R^2 \right) + R = \Delta \phi + R\phi.$$

Then if $f = t\phi = t^2 R + t$, we get

$$\frac{\partial}{\partial t} f = t(\Delta \phi + R\phi) + \phi = \Delta f + (tR + 1)\phi = \Delta f + \phi^2.$$

Hence the trace LYH quadratic takes the form

$$\psi (A, E, tX) = t^2 \left(R |X|^2 + 2 \langle \nabla R, X \rangle + \Delta R + R^2 \right) + 2tR + 1,$$

which is certainly positive at $t = 0$. q.e.d.

Remark 49. On any surface $\left(\mathcal{M}^2, g(t)\right)$ of strictly positive curvature evolving by the Ricci flow, the trace inequality

$$0 \leq R |X|^2 + 2 \langle \nabla \phi, X \rangle + \frac{\partial}{\partial t} f$$

in Proposition 46 can be deduced from the following calculations. Let $F \doteqdot \Delta f + \phi^2 - R^{-1} |\nabla \phi|^2$ and observe that

$$\begin{aligned} \frac{\partial}{\partial t} F = {} & \Delta \left(\Delta f + \phi^2 \right) + RF + R\phi^2 + 2\phi \Delta \phi - |\nabla \phi|^2 \\ & - 2R^{-1} \left(\langle \nabla \phi, \nabla \Delta \phi \rangle + \phi \langle \nabla \phi, \nabla R \rangle \right) + R^{-2} |\nabla \phi|^2 \Delta R. \end{aligned}$$

Since by Bochner,

$$\begin{aligned} \Delta F = {} & \Delta \left(\Delta f + \phi^2 \right) - 2R^{-1} \left(\langle \nabla \Delta \phi, \nabla \phi \rangle + |\nabla \nabla \phi|^2 + \frac{R}{2} |\nabla \phi|^2 \right) \\ & + R^{-2} \left(|\nabla \phi|^2 \Delta R + 2 \left\langle \nabla R, \nabla |\nabla \phi|^2 \right\rangle \right) - 2R^{-3} |\nabla R|^2 |\nabla \phi|^2 . \end{aligned}$$

we have

$$\begin{aligned} \frac{\partial}{\partial t} F = {} & \Delta F + RF + R\phi^2 + 2\phi \Delta \phi \\ & + 2R^{-1} \left(\left| \nabla \nabla \phi - R^{-1} \nabla R \otimes \nabla \phi \right|^2 - \phi \langle \nabla \phi, \nabla R \rangle \right) \\ \geq {} & \Delta F + RF + R^{-1} \left(\Delta \phi - \langle \nabla \phi, \nabla \ln R \rangle + \phi R \right)^2 . \end{aligned}$$

The inequality on the last line is equivalent to the estimate

$$\begin{aligned} N \doteqdot {} & R \left(R\phi^2 + 2\phi \Delta \phi \right) + 2 \left(\left| \nabla \nabla \phi - R^{-1} \nabla R \otimes \nabla \phi \right|^2 - \phi \langle \nabla \phi, \nabla R \rangle \right) \\ & - \left(\Delta \phi - \langle \nabla \phi, \nabla \ln R \rangle + \phi R \right)^2 \\ = {} & 2 \left(|\nabla \nabla \phi|^2 - \left\langle \nabla |\nabla \phi|^2 , \nabla \ln R \right\rangle + |\nabla \phi|^2 |\nabla \ln R|^2 \right) \\ & - \left((\Delta \phi)^2 - 2\Delta \phi \langle \nabla \phi, \nabla \ln R \rangle + \langle \nabla \phi, \nabla \ln R \rangle^2 \right) \\ = {} & 2 |\nabla \nabla \phi - \nabla \phi \otimes \nabla \ln R|^2 - \left(\Delta \phi - \langle \nabla \phi, \nabla \ln R \rangle \right)^2 \geq 0. \end{aligned}$$

Our final example makes no use of a Kähler structure:

Example 50. If $(\mathcal{M}^n, g(t))$ is any solution of the Ricci flow on a closed manifold, the choice $A \equiv 0$ lets us take $E = -df$ for any solution f of the ordinary heat equation $\frac{\partial}{\partial t} f = \Delta f$, and leads to the LYH quadratic

$$\Psi(0, -df, U, W) = \mathrm{Rm}(U, U) + (\nabla\nabla f)(W, W).$$

In case the curvature operator of $(\mathcal{M}, g)$ is nonnegative initially (hence for all time), Theorem 39 applies whenever $\nabla\nabla f \geq 0$ initially and is equivalent to the statement that the weak convexity of f is preserved. (Compare Remark 2 in §6 of [6].)

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University of California, San Diego

University of Wisconsin