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REGENERATING SINGULAR HYPERBOLIC STRUCTURES FROM Sol

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Abstract

Let M be a torus bundle over S^1 with an orientation preserving Anosov monodromy. The manifold M admits a geometric structure modeled on Sol. We prove that the Sol structure can be deformed into singular hyperbolic cone structures whose singular locus $\Sigma \subset M$ is the mapping torus of the fixed point of the monodromy.

The hyperbolic cone metrics are parametred by the cone angle α in the interval $(0, 2\pi)$. When $\alpha \to 2\pi$, the cone manifolds collapse to the basis of the fibration S^1, and they can be rescaled in the direction of the fibers to converge to the Sol manifold.

1. Introduction

Let M be the mapping torus of an orientation preserving Anosov homeomorphism of the 2-torus $\phi\colon T^2 \to T^2$. The compact oriented 3-manifold M fibers over S^1 with fiber a torus T^2,

$$T^2 \to M \to S^1.$$

The Anosov map $\phi\colon T^2 \to T^2$ lifts to a linear map of the universal covering $\mathbb{R}^2 \cong \widetilde{T}^2$. This linear map has two real eigenvalues different from ± 1. In particular, ϕ fixes $* \in T^2$ the projection of the origin in $\mathbb{R}^2$. Therefore there is a natural *zero-section* $\Sigma \subset M$ to $M \to S^1$ (i.e., $\Sigma \cap T^2 = \{*\}$ for each fibre T^2). Notice that the manifold M admits a geometric structure modeled on Sol. The main result of this paper is the following theorem:

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Theorem A. *Let M be a torus bundle with Anosov monodromy and let $\Sigma \subset M$ be as above. There exists a family of hyperbolic cone structures on M with singular set Σ parametrized by the cone angle $\alpha \in (0, 2\pi)$. When $\alpha \to 2\pi$ this family collapses to a circle, which is the basis of the fibration $M \to S^1$. In addition, the metrics can be rescaled in the direction of the fibers so that they converge to the* Sol *structure on M.*

It is well known that this family of cone manifolds converges to the complete hyperbolic structure on $M \smallsetminus \Sigma$ as $\alpha \to 0$.

When M is the manifold obtained by 0-surgery on the figure eight knot, this result is well illustrated in the literature. In [12] Jørgensen constructed the holonomy representations of these cone structures. This example was also developed in Thurston's notes [18], where, in the collapse, the developing maps are shown to converge to the developing map of a transversely hyperbolic foliation on M. In addition, Hilden, Lozano and Montesinos [9] construct an explicit family of Dirichlet polyhedra collapsing to a segment (whose ends are identified to give S^1). The third named author of the present paper shows in her thesis that this family of polyhedra can be rescaled to converge to a Sol structure [16], which motivated Theorem A.

The hyperbolic cone structure on M with singularity Σ induces a non-singular but non-complete hyperbolic metric on $M \smallsetminus \Sigma$. Hence $M \smallsetminus \Sigma$ has a developing map, that is usually called the developing map of the cone manifold. Notice that there is no unique choice of developing map, as is illustrated by the following proposition.

Proposition 1.1. *Let M be as in Theorem* A. *When $\alpha \to 2\pi$, the developing maps of the hyperbolic cone manifolds may be chosen to converge to the developing map of any of the three natural foliations, which are the foliation by tori and the two transversely hyperbolic foliations of codimension two.*

The developing map of the foliation by tori is the lift to $\mathbb{R}$ of the projection of the bundle $M \to S^1$. The two transversely hyperbolic foliations of M come from the one dimensional foliations of the fiber T^2 invariant by the Anosov monodromy $\phi : T^2 \to T^2$ (i.e., the eigenspaces of the linear map in $\mathbb{R}^2 = \widetilde{T}^2$).

The existence of the family of hyperbolic cone manifolds in Theorem A follows from Theorem B below and from a generalization of a result of Kojima [13].

The open manifold $N = M \setminus \Sigma$ is a fiber bundle

$$T^2 \setminus \{*\} \to N \to S^1$$

whose monodromy is the restriction of ϕ to the punctured torus $T^2 \setminus \{*\}$. In order to describe the structures on N with *generalized Dehn-filling coefficients*, we consider the compact manifold $M \setminus \mathcal{N}(\Sigma)$, where $\mathcal{N}(\Sigma)$ is an open tubular neighborhood of Σ. The manifold $M \setminus \mathcal{N}(\Sigma)$ is a compact core of N and its boundary is a torus. We choose $\{\mu, \lambda\}$ to be a basis for $H_1(\partial\overline{\mathcal{N}(\Sigma)}, \mathbb{Z})$ such that μ is the meridian of Σ. By using the basis $\{\mu, \lambda\}$ we can define the generalized Dehn-filling coefficients $(p, q) \in \mathbb{R}^2 \cup \{\infty\}$ for a given (incomplete) hyperbolic structure on N as follows. Let $\mathbb{C}^* \subset \mathrm{PSL}_2(\mathbb{C})$ be the group of isometries preserving an oriented line in $\mathbb{H}^3$. The holonomy map of the (incomplete) hyperbolic structure on $\mathcal{N}(\Sigma) \setminus \Sigma$ induces a homomorphism $h\colon H_1(\partial\overline{\mathcal{N}(\Sigma)}, \mathbb{Z}) \to \widetilde{\mathbb{C}^*} \cong \mathbb{C}$. Thurston defines the generalized Dehn-filling coefficients as the pair $(p, q) \in \mathbb{R}^2$ such that $p \cdot h(\mu) + q \cdot h(\lambda) = 2\pi i$.

The homomorphism h is not unique, but there is a natural choice such that Dehn-filling coefficients are related to the completion of the hyperbolic structure on N. If $(p, q) = k(p', q')$, where $k \in \mathbb{R}$ and p', q' are relatively prime integers, the hyperbolic structure on N can be completed to a cone hyperbolic structure on the manifold obtained by (p', q')-Dehn surgery on M along Σ with singular locus the core of the surgery and cone angle $2\pi/k$.

Generalized Dehn-filling coefficients can also be defined in an analogous way for other geometric structures on N and for foliations with a transverse geometric structure (see [10]). For instance, for the restriction to N of the Sol structure on M, the Dehn-filling coefficients are $(p, q) = (1, 0)$.

Theorem B. *Let N be a punctured torus bundle with hyperbolic monodromy. There exists $m \in \frac{1}{2}\mathbb{Z}$ and a neighborhood V of $(1, 0)$ in the half-plane $\{(x, y) \in \mathbb{R}^2 \mid x + my \geq 1\}$ such that each $(p, q) \in V$ is the generalized Dehn-filling coefficient of a geometric structure on N of the following kind:*

- *an incomplete hyperbolic structure when $p + mq > 1$;*
- *an transverse hyperbolic foliation when $p + mq = 1$.*

In addition $m \in \mathbb{Z}$ if and only if the monodromy has positive trace.

Notice that the meridian μ is uniquely determined, but λ is not. Thus, by replacing λ by $\lambda + r\mu$ for some $r \in \mathbb{Z}$, we could choose m to be either 0 or $\frac{1}{2}$, according to whether the trace of the monodromy is positive or negative.

A particular case of these hyperbolic structures on N are structures with coefficients of the form $(p, 0)$ (with $p > 1$), whose completion is a singular hyperbolic metric on M. The singular locus is the curve $\Sigma \subset M$ and the singularity is of cone type with cone angle $2\pi/p$.

Remark 1.2. The coefficients $(p, q) = (1, 0)$ correspond to the Sol structure and also to two transversely hyperbolic foliations induced by the foliations on T^2 invariant by ϕ (see Proposition 1.1).

Theorem B is a refinement of Theorem 4.18 of Hodgson's Thesis [10]. Hodgson shows that there is a complex curve of deformations corresponding to hyperbolic structures or to transversely hyperbolic foliations, but he does not compute explicitly the Dehn-filling parameter space. Instead of using Hodgson's general regeneration result, we shall give an explicit construction of the deformations of the Sol structure. This is possible by using Killing fields and a theorem about algebraic deformations of reducible representations proved in [8].

To prove Theorem B we first consider the case where the monodromy matrix of N has positive trace. In this case, the holonomy group of the complete Sol structure on M is contained in the component of the identity of $\mathrm{Isom}(\mathrm{Sol})$, i.e., in Sol itself. We are working with the space of representations of $\pi_1(N)$ in $\mathrm{SL}_2(\mathbb{C})$. Our starting point is the canonical exact sequence

$$1 \to \mathbb{R}^2 \to \mathrm{Sol} \xrightarrow{\mathrm{Lin}} \mathbb{R} \to 1 \tag{1}$$

(see Section 2). If $\mathrm{hol}\colon \pi_1(M) \to \mathrm{Sol}$ denotes the holonomy of the Sol structure on M, then we think of the composition $\mathrm{Lin} \circ \mathrm{hol}$ as a representation of $\pi_1(M)$ in the group of translations along a geodesic σ in hyperbolic 3-space $\mathbb{H}^3$, $\mathrm{Lin} \circ \mathrm{hol}\colon \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}_2(\mathbb{C})$. By composing this representation with the canonical surjection $\pi_1(N) \twoheadrightarrow \pi_1(M)$ we obtain a representation of $\pi_1(N)$ in $\mathrm{PSL}_2(\mathbb{C})$. We denote by $\rho_0\colon \pi_1(N) \to \mathrm{SL}_2(\mathbb{C})$ a lift of this representation in $\mathrm{SL}_2(\mathbb{C})$.

One of the elements of the proof of Theorem B is the construction of a *nice* deformation space

$$\mathcal{U} \subset \mathrm{Hom}(\pi_1(N), \mathrm{SL}_2(\mathbb{C}))$$

of ρ_0 which is homeomorphic to a neighborhood U of the origin in $\mathbb{C}^2$. In the sequel we denote by $\rho_{a,b} \in \mathcal{U}$ the representation corresponding to $(a,b) \in U$. The initial representation ρ_0 corresponds to the origin in $\mathbb{C}^2$.

The group $\mathrm{SL}_2(\mathbb{C})$ acts by conjugation on the representation space. The stabilizer of ρ_0 is a maximal torus $\mathbb{C}^* \subset \mathrm{SL}_2(\mathbb{C})$. The action of $\mathbb{C}^*$ restricted to $\mathcal{U}$ corresponds to the following action of $\mathbb{C}^*$ on $\mathbb{C}^2$:

$$\lambda \cdot (a,b) = (\lambda a, \lambda^{-1} b) \qquad \text{for } \lambda \in \mathbb{C}^* \text{ and } (a,b) \in \mathbb{C}^2.$$

The algebra of invariant functions on $\mathbb{C}^2$ is generated by ab. When $c \neq 0$, the level set $ab = c$ is precisely one orbit; moreover the level set $ab = 0$ is the union of three orbits, $\{(0,0)\}$, $\{0\} \times \mathbb{C}^*$ and $\mathbb{C}^* \times \{0\}$. Thus the quotient $\mathbb{C}^2/\mathbb{C}^*$ is the non-Hausdorff space $\mathbb{C}$ *with a triple point at the origin.*

The quotient $\mathcal{U}/\mathbb{C}^*$ of our deformation space is therefore a neighborhood of the triple point of $\mathbb{C}^2/\mathbb{C}^*$. This triple point corresponds precisely to the Sol structure and to the two transversely hyperbolic foliations induced by the invariant foliations of the torus. Real points in the quotient $\mathcal{U}/\mathbb{C}^*$ correspond to transversely hyperbolic foliations and the other points correspond to hyperbolic metrics. When we look at generalized Dehn-filling coefficients, the triple point of $\mathbb{C}^2/\mathbb{C}^*$ gives a single point, because generalized Dehn-filling coefficients are continuous functions. Moreover, complex conjugation in $\mathcal{U}/\mathbb{C}^*$ corresponds to change of orientation. The orientation conventions may be fixed so that Dehn-filling coefficients change sign, or not, under change of orientation. In any case, the neighborhood V of Theorem B is homeomorphic to the quotient of a neighborhood of the origin in $\mathbb{C}$ by conjugation.

To construct a family of developing maps $D_{a,b} \colon \widetilde{N} \to \mathbb{H}^3$ whose holonomy is the representation $\rho_{a,b} \in \mathcal{U}$, we proceed as follows. The geodesic $\sigma \subset \mathbb{H}^3$ gives the Cartan splitting of the Lie algebra

$$sl_2(\mathbb{C}) = \mathfrak{h}_0 \oplus \mathfrak{h}_+ \oplus \mathfrak{h}_-,$$

where $\mathfrak{h}_0$ is the subspace consisting of those Killing fields that preserve σ, while $\mathfrak{h}_+$ (resp. $\mathfrak{h}_-$) is the subspace of those parabolic Killing fields that fix one of the ends of σ (resp. the other end of σ). We remark that $\mathfrak{h}_0$ acts on $\mathfrak{h}_\pm$ and that we have an exact sequence of $\mathfrak{h}_0$-modules:

$$0 \to \mathfrak{h}_+ \oplus \mathfrak{h}_- \to sl_2(\mathbb{C}) \to \mathfrak{h}_0 \to 0. \tag{2}$$

This sequence is precisely the complexification of the sequence of real Lie algebras obtained from (1). Thus the Sol structure tells us how to deform ρ_0 at the first order.

We shall construct maps $\Delta_{a,b}\colon \mathrm{Sol} \to \mathbb{H}^3$ by taking the Riemannian exponential of Killing fields orthogonal to σ (evaluated at points of σ). The choice of those maps is motivated by the sequence (2). The construction of the developing map $D_{a,b}\colon \widetilde{N} \to \mathbb{H}^3$ starts with the map $\Delta_{a,b} \circ \mathcal{D}$ where $\mathcal{D}\colon \widetilde{N} \to \mathrm{Sol}$ denotes the developing map for the Sol structure on N induced by M. The construction of $D_{a,b}$ is then completed by using a well-known technique of Canary, Epstein and Green (see [6] and Section 6).

Finally, we prove Theorem B in the case where the monodromy matrix A of N has negative trace. It is clear that the two-fold cyclic covering $\widehat{N}$ of N has monodromy matrix A^2 with positive trace. Hence in order to prove the general result it is sufficient to show that the construction of $D_{a,b}$ is invariant under the covering transformation of $\widehat{N} \to N$.

The paper is organized as follows: In Section 2 we recall some basic facts about the group Sol, and in Section 3 we study representations of $\pi_1(M)$ in this group. Next in Section 4 we study the splitting of the Lie algebra and some applications of parabolic Killing fields. In Section 5 we construct the deformation space $\mathcal{U}$ for the representation ρ_0, leaving the proof of some results to the last section. Section 6 contains the construction the developing maps $D_{a,b}$. The Dehn-filling parameters are computed in Section 7 which concludes the proof of Theorem B. In Section 8 we prove Theorem A and Proposition 1.1. The case of torus bundles which are regular branched coverings of S^3 is an interesting example, and we discuss it in Section 9. The last section is devoted to the proofs of some technical results.

2. The group Sol

We recall that Sol can be defined as the component of the identity of the group of affine transformations in the Minkowski plane $\mathbb{R}^{1,1}$. That is, it is the group of transformations that preserve the Lorentz metric: $ds^2 = dx_1^2 - dy_1^2$. By making the change of coordinates: $(x_1, y_1) = (x_2 + y_2, x_2 - y_2)$, the Lorentz metric is written now as $ds^2 = 4dx_2dy_2$. In these coordinates, an orientation preserving affine transformation is of the form

$$(x_2, y_2) \to \left(k\, x_2 + a, \frac{1}{k}\, y_2 + b\right)$$

where $k \neq 0$ and $a, b \in \mathbb{R}$. The transformation belongs to the component of the identity if and only if $k > 0$.

Hence the group Sol is diffeomorphic to $\mathbb{R}^3$; we identify $(x, y, t) \in \mathbb{R}^3$ with the following affine transformation:

$$\begin{array}{rccl} (x,y,t)\colon & \mathbb{R}^{1,1} & \to & \mathbb{R}^{1,1} \\ & (x_2, y_2) & \mapsto & (e^t x_2 + x, e^{-t} y_2 + y). \end{array}$$

With this identification the product structure on Sol is given by:

$$(x, y, t)(x', y', t') = (x + e^t x', y + e^{-t} y', t + t').$$

It is clear from this that we have a split exact sequence

$$1 \to \mathbb{R}^2 \to \mathrm{Sol} \xrightarrow{\mathrm{Lin}} \mathbb{R} \to 1 \tag{3}$$

where $\mathrm{Lin}(x, y, t) = t$ and the kernel of Lin is the translation group. The action of $\mathbb{R}$ on $\mathbb{R}^2$ is

$$t \cdot (x, y) = (e^t x, e^{-t} y) \qquad \text{for } t \in \mathbb{R},\ (x, y) \in \mathbb{R}^2. \tag{4}$$

Definition 2.1. We define $\mathbb{R}_+$ and $\mathbb{R}_-$ to be the $\mathbb{R}$-modules

$$\begin{array}{rcl} \mathbb{R} \times \mathbb{R}_+ & \to & \mathbb{R}_+ \\ (t, x) & \mapsto & e^t x \end{array} \qquad \begin{array}{rcl} \mathbb{R} \times \mathbb{R}_- & \to & \mathbb{R}_- \\ (t, x) & \mapsto & e^{-t} x. \end{array}$$

The action of $\mathbb{R}$ on $\mathbb{R}^2$ in formula (4) decomposes into two actions:

$$\mathbb{R}^2 = \mathbb{R}_+ \oplus \mathbb{R}_- \qquad \text{as } \mathbb{R}\text{-modules.}$$

Definition 2.2. Given $p \in \mathbb{R}^{1,1}$, we take $\mathrm{trans}_p\colon \mathrm{Sol} \to \mathbb{R}^2$ to be the (set-theoretic) retraction for (3) defined by $\mathrm{trans}_p(\gamma) = \gamma(p) - p$.

Remark 2.3. This construction provides a natural family of retractions parametrized by $\mathbb{R}^{1,1}$. Such a retraction $\mathrm{trans}_p\colon \mathrm{Sol} \to \mathbb{R}^2$ satisfies the following cocycle condition:

$$\mathrm{trans}_p(\gamma_1\gamma_2) = \mathrm{trans}_p(\gamma_1) + \mathrm{Lin}(\gamma_1) \cdot \mathrm{trans}_p(\gamma_2)$$

for every $\gamma_1, \gamma_2 \in \mathrm{Sol}$, where $\mathbb{R}$ acts on $\mathbb{R}^2$ as in Equation (4) (i.e., $\mathbb{R}^2 \cong \mathbb{R}_+ \oplus \mathbb{R}_-$).

We fix a group Γ and a homomorphism $\rho\colon \Gamma \to \mathbb{R}$. We consider the action of Γ on $\mathbb{R}^2 \cong \mathbb{R}_+ \oplus \mathbb{R}_-$ induced by ρ and formula (4). Let $Z^1(\Gamma, (\mathbb{R}_+ \oplus \mathbb{R}_-)^\rho)$ and $B^1(\Gamma, (\mathbb{R}_+ \oplus \mathbb{R}_-)^\rho)$ denote respectively the space of cocycles and coboundaries twisted by ρ. That is:

$$Z^1(\Gamma, (\mathbb{R}_+ \oplus \mathbb{R}_-)^\rho) = \left\{ \theta\colon \Gamma \to \mathbb{R}^2 \;\middle|\; \begin{array}{c} \theta(\gamma_1\gamma_2) = \theta(\gamma_1) + \rho(\gamma_1)\cdot\theta(\gamma_2), \\ \forall \gamma_1, \gamma_2 \in \Gamma \end{array} \right\}$$

$$B^1(\Gamma, (\mathbb{R}_+ \oplus \mathbb{R}_-)^\rho) = \left\{ \theta\colon \Gamma \to \mathbb{R}^2 \;\middle|\; \begin{array}{c} \text{there exists } a \in \mathbb{R}^2 \text{ such that} \\ \theta(\gamma) = \rho(\gamma)\cdot a - a,\ \forall \gamma \in \Gamma \end{array} \right\}.$$

Let $H^1(\Gamma, (\mathbb{R}_+ \oplus \mathbb{R}_-)^\rho) = Z^1(\Gamma, (\mathbb{R}_+ \oplus \mathbb{R}_-)^\rho)/B^1(\Gamma, (\mathbb{R}_+ \oplus \mathbb{R}_-)^\rho)$ denote the first cohomology group.

Lemma 2.4. *Given a representation $\varphi\colon \Gamma \to \mathrm{Sol}$, $\mathrm{trans}_p \circ \varphi$ is a cocycle twisted by $\mathrm{Lin} \circ \varphi$.*

Given a fixed representation $\rho\colon \Gamma \to \mathbb{R}$, the map trans_p induces the following bijections, via the identification $\mathbb{R}_+ \oplus \mathbb{R}_- \cong \mathbb{R}^{1,1}$:

(i) $Z^1(\Gamma, (\mathbb{R}_+ \oplus \mathbb{R}_-)^\rho) \leftrightarrow \{\varphi \in \mathrm{Hom}(\Gamma, \mathrm{Sol}) \mid \mathrm{Lin} \circ \varphi = \rho\}$.

(ii) $B^1(\Gamma, (\mathbb{R}_+ \oplus \mathbb{R}_-)^\rho) \leftrightarrow \left\{ \varphi \in \mathrm{Hom}(\Gamma, \mathrm{Sol}) \;\middle|\; \begin{array}{l} \mathrm{Lin} \circ \varphi = \rho, \varphi(\Gamma) \textit{ has} \\ \textit{a fixed point in } \mathbb{R}^{1,1} \end{array} \right\}$.

(iii) $H^1(\Gamma, (\mathbb{R}_+ \oplus \mathbb{R}_-)^\rho) \leftrightarrow \{\varphi \in \mathrm{Hom}(\Gamma, \mathrm{Sol}) \mid \mathrm{Lin} \circ \varphi = \rho\}/\mathbb{R}^2$, *where $\mathbb{R}^2$ denotes the translation group acting by conjugation.*

Proof. Given a representation $\varphi\colon \Gamma \to \mathrm{Sol}$ such that $\mathrm{Lin} \circ \varphi = \rho$, by Remark 2.3 $\mathrm{trans}_p \circ \varphi \in Z^1(\Gamma, (\mathbb{R}_+ \oplus \mathbb{R}_-)^\rho)$. The bijectivity of (i) is a consequence of the isomorphism between Sol and the semidirect product $\mathbb{R}^2 \rtimes \mathbb{R}$ that maps $\gamma \in \mathrm{Sol}$ to $(\mathrm{trans}_p(\gamma), \mathrm{Lin}(\gamma)) \in \mathbb{R}^2 \rtimes \mathbb{R}$.

To prove (ii), we recall that $\mathrm{trans}_p \circ \varphi \in B^1(\Gamma, (\mathbb{R}_+ \oplus \mathbb{R}_-)^\rho)$ if there exist $a \in \mathbb{R}^2$ such that $\mathrm{trans}_p \circ \varphi(\gamma) = \rho(\gamma)(a) - a$ for every $\gamma \in \Gamma$. This equality can be rewritten as $\varphi(\gamma)(p) - p = \mathrm{Lin} \circ \varphi(\gamma)(a) - a$, or equivalently $\varphi(\gamma)(p - a) = p - a$.

Finally we prove (iii). Given $a \in \mathbb{R}^2$, let φ^a denote the conjugation of φ by the translation of vector a. An elementary computation shows that $\mathrm{trans}_p \circ \varphi^a(\gamma) = \mathrm{trans}_p \circ \varphi(\gamma) + \rho(\gamma)(a) - a$, and the bijection follows easily. q.e.d.

3. Representations of $\pi_1(N)$ in Sol

From now on we consider the case where the monodromy matrix of N has positive trace.

In this section we describe the representations of $\pi_1(N)$ in Sol by using the notation of Section 2. This approach is going to be useful for the study of deformations of representations. In particular we shall describe the holonomy representation of the Sol structure.

The fundamental group. As in the introduction, let M denote the mapping torus of an orientation preserving Anosov homeomorphism $\phi\colon T^2 \to T^2$. We denote by N the mapping torus of the restriction of ϕ to the punctured torus $T^2 \setminus \{*\}$ where $*$ is the fixed point of ϕ.

For the remaining of the paper, we fix $\Gamma = \pi_1(N)$, which has the following presentation:

$$\Gamma \cong \langle\, \lambda, \alpha, \beta \mid \lambda\alpha\lambda^{-1} = f(\alpha), \lambda\beta\lambda^{-1} = f(\beta)\,\rangle,$$

where α and β generate the free group $\pi_1(T^2 \setminus \{*\})$ and $f\colon \pi_1(T^2 \setminus \{*\}) \to \pi_1(T^2 \setminus \{*\})$ is the isomorphism induced by the restriction of ϕ. We assume that α and β satisfy $\mu = \alpha\beta\alpha^{-1}\beta^{-1}$ where μ is represented by the boundary of a regular neighborhood of the fixed point $* \in T^2$. Hence f preserves the commutator $\mu = \alpha\beta\alpha^{-1}\beta^{-1}$. The generators α and β of the free group provide a basis $\{[\alpha], [\beta]\}$ for $H_1(T^2; \mathbb{Z}) \cong \mathbb{Z}^2$ and the map f induces also an isomorphism $f_*\colon \mathbb{Z}^2 \to \mathbb{Z}^2$. Let $A \in \mathrm{SL}_2(\mathbb{Z})$ denote the matrix of f_* with respect to the basis $\{[\alpha], [\beta]\}$, i.e.,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad \text{where} \qquad \begin{aligned} f_*([\alpha]) = {} & a_{11}[\alpha] + a_{21}[\beta] \\ f_*([\beta]) = {} & a_{12}[\alpha] + a_{22}[\beta]\,. \end{aligned}$$

The holonomy representation. By Lemma 2.4 above, to describe a representation of Γ in Sol it is sufficient to give a representation in $\mathbb{R}$ and a twisted cocycle. Since ϕ is Anosov and we assume that $\operatorname{trace}(A) > 0$, the eigenvalues of A are of the form $e^{\pm l}$, with $l \in \mathbb{R},\ l > 0$. We fix the representation $\rho_0\colon \Gamma \to \mathbb{R}$ defined by

$$\rho_0(\alpha) = \rho_0(\beta) = 0 \quad \text{and} \quad \rho_0(\lambda) = l.$$

Let $\left(\begin{smallmatrix} b_{11} \\ b_{21} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} b_{12} \\ b_{22} \end{smallmatrix}\right)$ be eigenvectors of the transpose matrix A^{t} with respective eigenvalues e^l and e^{-l}. We set $B = \left(\begin{smallmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{smallmatrix}\right)$ and we assume that

$$\det(B) = b_{11}b_{22} - b_{12}b_{21} = 1. \tag{5}$$

Remark 3.1. Observe that all $b_{ij} \neq 0$ because $A \in \mathrm{SL}_2(\mathbb{Z})$.

A direct computation shows that there are exactly two cocycles $d_+\colon\ \Gamma \to \mathbb{R}_+$ and $d_-\colon\ \Gamma \to \mathbb{R}_-$ such that $d_\pm(\lambda) = 0$ and

$$\begin{array}{ll} d_+(\alpha) = b_{11} & d_-(\alpha) = b_{12} \\ d_+(\beta) = b_{21} & d_-(\beta) = b_{22}. \end{array}$$

We recall that each cocycle $d\colon\ \Gamma \to \mathbb{R}_\pm$ satisfies:

$$d(\gamma_1\gamma_2) = d(\gamma_1) + \gamma_1 \cdot d(\gamma_2)$$

where $\gamma \cdot \alpha = e^{\pm\rho_0(\gamma)}\alpha$ for every $\gamma \in \Gamma$ and $\alpha \in \mathbb{R}_\pm$.

The proof of the following lemma is a straightforward computation.

Lemma 3.2. *The cohomology classes of the cocycles d_+ and d_- form a basis for $H^1(\Gamma, (\mathbb{R}_+ \oplus \mathbb{R}_-)^{\rho_0}) \cong \mathbb{R}^2$. Equivalently,*

$$Z^1(\Gamma, (\mathbb{R}_+ \oplus \mathbb{R}_-)^{\rho_0}) = B^1(\Gamma, (\mathbb{R}_+ \oplus \mathbb{R}_-)^{\rho_0}) \oplus \langle d_+ \rangle_{\mathbb{R}} \oplus \langle d_- \rangle_{\mathbb{R}}.$$

In addition $B^1(\Gamma, (\mathbb{R}_+ \oplus \mathbb{R}_-)^{\rho_0}) = \{\theta\colon\ \Gamma \to \mathbb{R}_+ \oplus \mathbb{R}_- \mid \theta(\alpha) = \theta(\beta) = (0,0)\} \cong \mathbb{R}^2$.

Definition 3.3. The holonomy representation $\mathrm{hol}\colon\ \Gamma \to \mathrm{Sol}$ is the representation such that $\mathrm{Lin} \circ \mathrm{hol} = \rho_0$ and $\mathrm{trans} \circ \mathrm{hol} = d_+ \oplus d_-$.

With the coordinates $\mathrm{Sol} \cong \mathbb{R}^3$ from Section 2, this representation is given by $\mathrm{hol}(\gamma) = (d_+(\gamma), d_-(\gamma), \rho_0(\gamma)) \in \mathrm{Sol}$, for every $\gamma \in \Gamma$.

Remark 3.4. This representation is discrete and it induces a faithful representation of $\pi_1(M)$ the group of the compact manifold. Thus it is the holonomy of the complete Sol structure on M. It can be checked that this is the unique representation with these properties, up to conjugation by automorphisms of Sol [17], [19].

Since $N = M \setminus \Sigma$ has the rational homology of a circle, it has a well defined Alexander polynomial. The Alexander polynomial is precisely the characteristic polynomial of the monodromy A, hence of degree two. The following lemma allows us to apply the results of [8].

Lemma 3.5. *The eigenvalues $e^{\pm l}$ of A are simple roots of the Alexander polynomial of $N = M \setminus \Sigma$.*

4. Parabolic Killing fields

From now on we fix an *oriented* geodesic $\sigma \subset \mathbb{H}^3$. Parabolic Killing fields associated to the ends of σ give the connection between Sol and hyperbolic space $\mathbb{H}^3$. In this section we use them to construct a family of maps $\Delta_{a,b} : \mathrm{Sol} \to \mathbb{H}^3$, for $(a,b) \in \mathbb{C}^2$.

Definition 4.1. An orientation preserving isometry that fixes a geodesic σ is said to be *a hyperbolic translation of complex length $z \in \mathbb{C}$ along σ* when it is a pure translation of length $\mathrm{Re}(z)$ composed with a rotation of angle $\mathrm{Im}(z)$ along σ. This isometry is denoted by T_z.

Notice that this definition includes rotations, which are elliptic elements. A rotation of angle $\alpha \in \mathbb{R}$ along σ is a hyperbolic translation $T_{i\alpha}$ of complex length $i\alpha$.

The group of hyperbolic translations along σ is isomorphic to $\mathbb{C}^*$, the isomorphism being induced by taking the exponential of the complex length. The identity is also viewed as a hyperbolic translation of complex length valued in $2\pi i\mathbb{Z}$. Let $\sigma(+\infty)$ and $\sigma(-\infty)$ be the ends of σ. We also consider parabolic elements that fix one of the ends (i.e., isometries whose unique fixed point is one of the ends of σ). Thus, we consider the following subgroups of orientation preserving isometries:

$$\begin{aligned}
H_0 &= \{\gamma \in \mathrm{Isom}^+(\mathbb{H}^3) \mid \gamma \text{ is a hyperbolic translation along } \sigma\};\\
H_+ &= \{\gamma \in \mathrm{Isom}^+(\mathbb{H}^3) \mid \gamma \text{ is parabolic and fixes } \sigma(+\infty)\};\\
H_- &= \{\gamma \in \mathrm{Isom}^+(\mathbb{H}^3) \mid \gamma \text{ is parabolic and fixes } \sigma(-\infty)\}.
\end{aligned}$$

We remark that H_0 is also the subgroup of orientation preserving isometries that fix both ends of σ.

For instance, if we work in the upper half-space model of $\mathbb{H}^3$ and assume that σ is the geodesic with end-points 0 and ∞, then as subgroups of $\mathrm{PSL}_2(\mathbb{C})$,

$$\begin{aligned}
H_0 &= \left\{\pm \begin{pmatrix} e^{z/2} & 0 \\ 0 & e^{-z/2} \end{pmatrix} \mid z \in \mathbb{C}\right\},\\
H_+ &= \left\{\pm \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \mid z \in \mathbb{C}\right\}\\
H_- &= \left\{\pm \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \mid z \in \mathbb{C}\right\}.
\end{aligned}$$

The Lie algebra. The algebra of Killing fields of $\mathbb{H}^3$ is also the Lie algebra of the isometry group, which is $sl_2(\mathbb{C})$.

Remark 4.2. If $\mathfrak{h}_0$, $\mathfrak{h}_+$ and $\mathfrak{h}_-$ are the respective tangent spaces to H_0, H_+ and H_-, then

$$sl_2(\mathbb{C}) = \mathfrak{h}_0 \oplus \mathfrak{h}_+ \oplus \mathfrak{h}_-,$$

is a Cartan splitting.

The group $\mathrm{Isom}(\mathbb{H}^3)$ acts on $sl_2(\mathbb{C})$ by the adjoint action, and we consider the induced action of $H_0 \subset \mathrm{Isom}(\mathbb{H}^3)$. The Cartan splitting of Remark 4.2 is preserved by H_0, the group of hyperbolic translations along σ. A hyperbolic translation T_z of complex length $z \in \mathbb{C}$ acts by multiplication by $e^{\pm z}$ on $\mathfrak{h}_\pm$ and as the identity on $\mathfrak{h}_0$.

In this way, if we restrict the action of H_0 to the subgroup of translations T_s with real length $s \in \mathbb{R}$, $\mathfrak{h}_\pm$ is the complexification of the $\mathbb{R}$-module $\mathbb{R}_\pm$:

$$\mathfrak{h}_\pm \cong \mathbb{R}_\pm \otimes_{\mathbb{R}} \mathbb{C}\,.$$

The maps from Sol **to** $\mathbb{H}^3$**.** We fix now a parametrization $\sigma(t)$ of the geodesic by arc-length and two non-zero Killing fields $V^+ \in \mathfrak{h}_+$ and $V^- \in \mathfrak{h}_-$ such that

$$V^+_{\sigma(0)} = V^-_{\sigma(0)} \neq 0.$$

Notation 4.3. Given $p \in \mathbb{H}^3$ and a vector field X on $\mathbb{H}^3$, $\exp_p(X)$ denotes the Riemannian exponential of the vector $X_p \in T_p(\mathbb{H}^3)$ at the point p.

Definition 4.4. Let σ and $V^\pm$ be as above. For $(a, b) \in \mathbb{C}^2$ we define

$$\begin{array}{rcl} \Delta_{a,b}\colon \mathrm{Sol} & \to & \mathbb{H}^3 \\ (x, y, t) & \mapsto & \exp_{\sigma(t)}(a\,x\,V^+ + b\,y\,V^-). \end{array}$$

Lemma 4.5. *Denote by T_z the hyperbolic translation along σ of complex length $z \in \mathbb{C}$. For any $(a, b) \in \mathbb{C}^2$ and $g \in$ Sol:*

$$T_z(\Delta_{a,b}(g)) = \Delta_{e^z a, e^{-z} b}(g \cdot (0, 0, \mathrm{Re}(z))).$$

Proof. Writing $g = (x, y, t) \in$ Sol,

$$\begin{aligned} T_z(\Delta_{a,b}(g)) &= T_z(\exp_{\sigma(t)}(a\,x\,V^t + b\,y\,V^-)) \\ &= \exp_{\sigma(t+\mathrm{Re}(z))}\{Ad_{T_z}(a\,x\,V^+ + b\,y\,V^-)\} \\ &= \exp_{\sigma(t+\mathrm{Re}(z))}(a\,x\,e^z\,V^+ + b\,y\,e^{-z}\,V^-) \\ &= \Delta_{e^z a, e^{-z} b}(x, y, t + \mathrm{Re}(z)). \end{aligned}$$

q.e.d.

Coordinates in $\mathbb{C}\times\mathbb{R}$. Let W be the parallel vector field on σ such that $W_{\sigma(0)} = V^+_{\sigma(0)} = V^-_{\sigma(0)}$. Using the notation $e^{i\theta}W = (T_{i\theta})_* W$ for any $e^{i\theta}\in S^1\subset\mathbb{C}$, we define the diffeomorphism

$$\begin{aligned}\Psi\colon\ \mathbb{C}\times\mathbb{R} &\to \mathbb{H}^3 \\ (z,t) &\mapsto \exp_{\sigma(t)}(zW).\end{aligned} \tag{6}$$

For each $t\in\mathbb{R}$, $\Psi(\mathbb{C}\times\{t\})$ is the image by the exponential map of the tangent plane orthogonal to σ at $\sigma(t)$. In addition,

$$T_\omega(\Psi(z,t)) = \Psi(z\,e^{i\,\mathrm{Im}(\omega)}, t+\mathrm{Re}(\omega)) \qquad \forall z,\omega\in\mathbb{C}, t\in\mathbb{R}.$$

Lemma 4.6. *For every $(x,y,t)\in\mathrm{Sol}$ and $a,b\in\mathbb{C}$:*

$$\Delta_{a,b}(x,y,t) = \Psi(a\,x\,e^{-t}+\bar b\,y\,e^{t},t)$$

Proof. By Lemma 4.5 we have:

$$T_{-t}\Delta_{a,b}(x,y,t) = \Delta_{e^{-t}a,e^t b}(x,y,0) = \exp_{\sigma(0)}(a\,x\,e^{-t}V^+ + b\,y\,e^tV^-).$$

In addition $(e^{i\theta}V^\pm)_{\sigma(0)} = (Ad_{T_{\pm i\theta}}V^\pm)_{\sigma(0)} = (T_{\pm i\theta})_*V^\pm_{\sigma(0)} = e^{\pm i\theta}W_{\sigma(0)}$. Hence $T_{-t}\Delta_{a,b}(x,y,t) = \Psi(a\,x\,e^{-t}+\bar b\,y\,e^t,0)$ and since T_t preserves parallel vectors along σ the lemma follows. q.e.d.

Corollary 4.7. *The map $\Delta_{a,b}$ is a diffeomorphism if $ab\notin\mathbb{R}$. It preserves the orientation iff $\mathrm{Im}(ab)<0$ and it reverses it iff $\mathrm{Im}(ab)>0$. The map $\Delta_{a,b}$ is a submersion onto some hyperbolic plane $\mathbb{H}^2$ containing σ if $ab\in\mathbb{R}$ but $(a,b)\neq(0,0)$.*

5. Deforming representations

We view $\rho_0\colon\ \Gamma\to\mathbb{R}$ as a representation of Γ in the translation group along a geodesic $\sigma\subset\mathbb{H}^3$. It is possible to lift it to a representation in

$$\widetilde{\mathrm{Isom}^+}(\mathbb{H}^3)\cong\mathrm{SL}_2(\mathbb{C}),$$

which is the double covering of the orientation preserving isometry group $\mathrm{Isom}^+(\mathbb{H}^3)\cong\mathrm{PSL}_2(\mathbb{C})$. This representation is still denoted by $\rho_0\colon\ \Gamma\to\mathrm{SL}_2(\mathbb{C})$.

To get the holonomy of hyperbolic structures we shall deform $\rho_0\colon \Gamma \to \mathrm{SL}_2(\mathbb{C})$. The *nice* space of deformations $\mathcal{U}$ of ρ_0 is going to be a two dimensional set in an irreducible component of $\mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$:

$$\rho_0 \in \mathcal{U} \subset \mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{C})).$$

The group $\mathrm{SL}_2(\mathbb{C})$ acts by conjugation on $\mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ and the set $\mathcal{U}$ is going to be invariant under the restricted action of H_0.

We present at the beginning of this section some notation and facts which are needed in the sequel. The proofs of some technical results are postponed (see Section 10).

The variety of representations. The variety of representations

$$R(\Gamma, \mathrm{SL}_2(\mathbb{C})) = \mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$$

is an affine algebraic subset of $\mathbb{C}^{12}$, because $\mathrm{SL}_2(\mathbb{C}) \subset \mathbb{C}^4$ and Γ has three generators. The group $\mathrm{SL}_2(\mathbb{C})$ acts on $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ by conjugation and we may consider two quotients, the topological and the algebraic one.

The topological quotient is denoted by

$$R(\Gamma, \mathrm{SL}_2(\mathbb{C}))/\mathrm{SL}_2(\mathbb{C})$$

and it is not always Hausdorff. The quotient in the algebraic category denoted by

$$R(\Gamma, \mathrm{SL}_2(\mathbb{C}))/\!/\mathrm{SL}_2(\mathbb{C})$$

is called the *variety* of characters and it is a complex affine algebraic set (see for instance [5]). Since our representation ρ_0 is abelian, we have to be careful when studying the quotient. In fact a neighborhood of ρ_0 in $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))/\mathrm{SL}_2(\mathbb{C})$ is not Hausdorff.

The Zariski tangent space $T_{\rho_0} R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ is isomorphic to a subspace of $Z^1(\Gamma, sl_2(\mathbb{C})^{\rho_0})$, and the tangent space to the orbit by conjugation is isomorphic to the coboundary space $B^1(\Gamma, sl_2(\mathbb{C})^{\rho_0})$. The splitting

$$sl_2(\mathbb{C}) = \mathfrak{h}_0 \oplus \mathfrak{h}_+ \oplus \mathfrak{h}_- \tag{7}$$

is preserved by the action of ρ_0. This splitting is useful for the computation of the tangent space at the reducible representation ρ_0 (see Proposition 5.1).

We fix $V^\pm \in \mathfrak{h}_\pm$ and $V^0 \in \mathfrak{h}_0$ non-zero elements, such that $V^\pm$ are the same as in the previous section. Thus $\{V^+, V^-, V^0\}$ is a $\mathbb{C}$-basis for $sl_2(\mathbb{C})$.

Irreducible and abelian representations. Recall that a representation $\rho\colon \Gamma \to \mathrm{SL}_2(\mathbb{C})$ is called irreducible if it has no proper invariant subspaces of $\mathbb{C}^2$. The set of irreducible representations is not closed but we consider its closure:

$$R^{\mathrm{ir}}(\Gamma, \mathrm{SL}_2(\mathbb{C})) = \overline{\{\rho \in R(\Gamma, \mathrm{SL}_2(\mathbb{C})) \mid \rho \text{ is irreducible}\}}$$

Notice that $\rho_0 \in R^{\mathrm{ir}}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$, by the main theorem in [8]. The results of [8] are relevant because e^l is a simple root of its Alexander polynomial (see Lemma 3.5).

A representation is called abelian if its image is an abelian group. The set of abelian representations is closed and we consider:

$$R^{\mathrm{ab}}(\Gamma, \mathrm{SL}_2(\mathbb{C})) = \{\rho \in R(\Gamma, \mathrm{SL}_2(\mathbb{C})) \mid \rho \text{ is abelian}\}.$$

Notice that $\rho_0 \in R^{\mathrm{ir}}(\Gamma, \mathrm{SL}_2(\mathbb{C})) \cap R^{\mathrm{ab}}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$.

Proposition 5.1.

(a) *The analytic germ of $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ at ρ_0 has two irreducible components, which are precisely the germ of $R^{\mathrm{ir}}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ and of $R^{\mathrm{ab}}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$.*

(b) *In addition*

$$\begin{aligned} T_{\rho_0} R^{\mathrm{ir}}(\Gamma, \mathrm{SL}_2(\mathbb{C})) &= Z^1(\Gamma, (\mathfrak{h}_+ \oplus \mathfrak{h}_-)^{\rho_0}) \\ T_{\rho_0} R^{\mathrm{ab}}(\Gamma, \mathrm{SL}_2(\mathbb{C})) &= Z^1(\Gamma, \mathfrak{h}_0^{\rho_0}) \oplus B^1(\Gamma, (\mathfrak{h}_+ \oplus \mathfrak{h}_-)^{\rho_0}). \end{aligned}$$

Point (a) is proved in [8] and the proof of point (b) is given in Section 10.

Remark 5.2. As Γ-modules, $\mathfrak{h}_\pm \cong \mathbb{R}_\pm \otimes \mathbb{C}$ (i.e., $\mathfrak{h}_\pm$ is the complexification of $\mathbb{R}_\pm$). Therefore, the computations of cocycles and coboundaries of Section 3 apply here, just by complexifying. As in Lemma 3.2, we have:

$$Z^1(\Gamma, (\mathfrak{h}_+ \oplus \mathfrak{h}_-)^{\rho_0}) = B^1(\Gamma, (\mathfrak{h}_+ \oplus \mathfrak{h}_-)^{\rho_0}) \oplus \langle d_+ V^+ \rangle_{\mathbb{C}} \oplus \langle d_- V^- \rangle_{\mathbb{C}},$$

where $d_\pm V^\pm$ is the cocycle $\Gamma \to \mathfrak{h}_\pm$ that maps $\gamma \in \Gamma$ to $d_\pm(\gamma)\, V^\pm$.

The slice. We are interested in representations up to conjugation. Therefore we consider a slice with respect to the projection of $R^{\mathrm{ir}}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ onto the variety of characters. Since our representation is abelian, the slice does not give a parametrization of the space of orbits, but it is going to be a useful tool to study it.

Definition 5.3. Following [1], we define the slice:

$$\mathcal{S} = \{\rho \in R^{\mathrm{ir}}(\Gamma, \mathrm{SL}_2(\mathbb{C})) \mid \rho(\lambda) \text{ is a hyperbolic translation along } \sigma\}.$$

Notice that $\rho_0(\lambda)$ is a hyperbolic translation along σ of length l, and therefore $\rho_0 \in \mathcal{S}$.

The local parametrization for $\mathcal{S}$. Now we want to construct a parametrization for the slice $\mathcal{S}$ in a neighborhood of ρ_0. Given $\rho \in \mathcal{S}$ we consider $\rho(\alpha) \in \mathrm{SL}_2(\mathbb{C})$. Since $\rho_0(\alpha) = \mathrm{Id}$, we may choose ρ sufficiently close to ρ_0, so that

$$\rho(\alpha) \in \exp(\mathcal{W}),$$

where $\mathcal{W} \subset sl_2(\mathbb{C})$ is a neighborhood of the origin such that exp restricted to $\mathcal{W}$ is injective. Now we look at $\mathrm{pr}(\exp^{-1}(\rho(\alpha))) \in \mathfrak{h}_+ \oplus \mathfrak{h}_-$, where

$$\mathrm{pr}\colon\ sl_2(\mathbb{C}) \to \mathfrak{h}_+ \oplus \mathfrak{h}_-$$

denotes the projection with kernel $\mathfrak{h}_0$.

Proposition 5.4. *There is a neighborhood $\mathcal{U} \subset \mathcal{S}$ of ρ_0 so that the map*

$$\begin{aligned} \mathcal{U} &\to \mathfrak{h}_+ \oplus \mathfrak{h}_- \\ \rho &\mapsto \mathrm{pr}(\exp^{-1}(\rho(\alpha))) \end{aligned}$$

is a biholomorphism between $\mathcal{U}$ and a neighborhood U of the origin in $\mathfrak{h}_+ \oplus \mathfrak{h}_-$ which maps ρ_0 to the origin.

The proof of Proposition 5.4 is given in Section 10 (see Corollary 10.3).

Since $d_\pm(\alpha) \neq 0$ (see Remark 3.1), we can identify $\mathfrak{h}_+ \oplus \mathfrak{h}_-$ with $\mathbb{C}^2$ by sending $(a, b) \in \mathbb{C}^2$ to $a\, d_+(\alpha)\, V^+ + b\, d_-(\alpha) V^- \in \mathfrak{h}_+ \oplus \mathfrak{h}_-$. With this identification, denote by $U \subset \mathbb{C}^2$ the neighborhood of the origin in $\mathbb{C}^2 \cong \mathfrak{h}_+ \oplus \mathfrak{h}_-$ of Proposition 5.4.

Convention 5.5. Given $(a, b) \in U$, we shall denote by $\rho_{a,b} \in \mathcal{U} \subset \mathcal{S}$ the representation which is mapped to $a\, d_+(\alpha)\, V^+ + b\, d_-(\alpha) V^- \in \mathfrak{h}_+ \oplus \mathfrak{h}_-$ under the biholomorphism of Proposition 5.4.

In this way, we have parametrized a neighborhood $\mathcal{U}$ of ρ_0 in $\mathcal{S}$ by a neighborhood U of the origin in $\mathbb{C}^2$.

The action by conjugation. The action of $\mathrm{PSL}_2(\mathbb{C})$ by conjugation does not preserve the slice $\mathcal{S}$. The subgroup of isometries that preserve it is H_0, which is the group of hyperbolic translations along σ (and also the subgroup of orientation preserving isometries that fix both ends of σ).

Lemma 5.6. *For every $\gamma \in \Gamma$ and every $(a, b) \in U \in \mathbb{C}^2$,*

$$\rho_{e^z a, e^{-z} b}(\gamma) = T_z \, \rho_{a,b}(\gamma) \, T_{-z}$$

for every $z \in \mathbb{C}$ where it makes sense (i.e., for every $z \in \mathbb{C}$ such that $(e^z a, e^{-z} b) \in U$).

Proof. By construction, conjugation by T_z preserves the slice, and the lemma follows straightforward from Proposition 5.4 and the action of T_z on $\mathfrak{h}_\pm$. q.e.d.

By Lemma 5.6, the action of $H_0 \cong \mathbb{C}^*$ restricted to $\mathcal{U}$ transforms in the parameter space $U \subset \mathbb{C}^2$ into the action of $\mathbb{C}^*$ on $\mathbb{C}^2$ defined by

$$\lambda(a, b) = (\lambda a, \lambda^{-1} b).$$

Let $U/\sim$ denote the quotient of U by the action of $\mathbb{C}^*$ of Lemma 5.6. This quotient is not Hausdorff and has a triple point, which corresponds to the orbits $\{0\}$, $\{(a, 0) \mid a \neq 0\}$ and $\{(0, b) \mid b \neq 0\}$. If we identify these three points to a single one, then we obtain an open set of $\mathbb{C}$.

Corollary 5.7. *The quotient of a neighborhood of ρ_0 in the slice $\mathcal{S}$ by conjugation is isomorphic to $U/\sim$, which is a neighborhood of a triple point in $\mathbb{C}$. In addition, the algebra of invariant functions on U is generated by*

$$\begin{array}{ccc} U & \to & \mathbb{C} \\ (a, b) & \mapsto & ab. \end{array}$$

Representations in $\mathrm{SL}_2(\mathbb{R})$. By identifying $\mathrm{PSL}_2(\mathbb{C}) \cong \mathrm{Isom}^+(\mathbb{H}^3)$, we view $\mathrm{PSL}_2(\mathbb{R})$ as the subgroup of isometries that preserve some *oriented* hyperbolic plane $\mathbb{H}^2 \subset \mathbb{H}^3$. Thus $\mathrm{SL}_2(\mathbb{R})$ is the group of lifts of such isometries to $\mathrm{SL}_2(\mathbb{C})$. We assume also that the geodesic σ is contained in the plane $\mathbb{H}^2$. This implies that the image of ρ_0 belongs to $\mathrm{SL}_2(\mathbb{R})$ and we have:

Remark 5.8. All the results of this section hold true by replacing $\mathbb{C}$ by $\mathbb{R}$.

For instance the local parametrization of $\mathcal{S}$ in Proposition 5.4 induces a local parametrization of $\mathcal{S} \cap \mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{R}))$ with the neighborhood of the origin $U \cap \mathbb{R}^2$. In addition, when $a, b \in \mathbb{R}$ and $z \in \mathbb{R} + \pi i \mathbb{Z}$, the formula of Lemma 5.6 has a meaning in $\mathrm{SL}_2(\mathbb{R})$, because conjugation by $T_{\pi i}$ preserves $\mathrm{SL}_2(\mathbb{R})$. The proofs for the real case are similar to the proofs for the complex one, by using some results of [8] and the fact that $sl_2(\mathbb{C})$ is the complexification of $sl_2(\mathbb{R})$, i.e., $sl_2(\mathbb{C}) = sl_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$. For instance

$$Z^1(\Gamma, sl_2(\mathbb{C})^{\rho_0}) = Z^1(\Gamma, sl_2(\mathbb{R})^{\rho_0}) \otimes_{\mathbb{R}} \mathbb{C}$$

and so on for B^1 and H^1.

6. Deforming developing maps

The aim of this section is to construct a family of developing maps $D_{a,b} \colon \widetilde{N} \to \mathbb{H}^3$.

Recall that the manifold M admits a Sol structure and that Σ denotes the natural zero-section of the bundle $M \to S^1$. Moreover, we denote by $\pi \colon \widetilde{N} \to N$ the universal covering of the complement $N = M \setminus \Sigma$.

Proposition 6.1. *There exists a neighborhood $U \subset \mathbb{C}^2$ of the origin and a family of maps $D_{a,b} \colon \widetilde{N} \to \mathbb{H}^3$ parameterized by $(a, b) \in U$ which fulfills the following conditions:*

(i) *$D_{a,b}$ is $\rho_{a,b}$-equivariant.*

(ii) *$D_{a,b}$ is a submersion onto a hyperbolic plane $\mathbb{H}^2 \subset \mathbb{H}^3$ if $ab \in \mathbb{R}$ and $(a, b) \neq (0, 0)$.*

(iii) *$D_{a,b}$ is locally a diffeomorphism if $ab \notin \mathbb{R}$.*

(iv) *The structure on the end of N is of* Dehn type*.*

We explain now the meaning of assertion (iv) in the proposition. The end of N is of the form $(0, 1] \times T^2$, so that $\{1\} \times T^2$ is the boundary of a tubular neighborhood of Σ. Given a lift $(0, 1] \times \widetilde{T}^2$ to the universal cover $\widetilde{N}$, there corresponds a subgroup $\pi_1(T^2) \subset \pi_1(N)$. We shall show in next section that, if $(a, b) \neq 0$, then $\rho_{a,b}(\pi_1(T^2))$ is a loxodromic group and in particular it fixes a geodesic ς. Assertion (iv) means that given $x \in \widetilde{T}^2$, $D_{a,b}(s, x)$ is contained in the segment that minimizes the distance between $D_{a,b}(1,x)$ and the geodesic ς, and that $D_{a,b}(s,x) \to \varsigma$ as $s \to 0$.

Proof. Along the proof we use the coordinate system $\mathbb{C} \times \mathbb{R}$ of $\mathbb{H}^3$ provided by the diffeomorphism (6), which is:

$$\begin{array}{rccl} \Psi\colon & \mathbb{C} \times \mathbb{R} & \to & \mathbb{H}^3 \\ & (z,t) & \mapsto & \exp_{\sigma(t)}(zW), \end{array}$$

where W is the parallel vector field on σ such that $W_{\sigma(0)} = V^+_{\sigma(0)} = V^-_{\sigma(0)}$.

We are not going to use the real vector space structure of $\mathbb{C} \times \mathbb{R}$. Instead, we are going to use its $\mathbb{R}$- affine structure.

Construction of $D_{a,b}$. Let $\mathcal{D}\colon \widetilde{N} \to \mathrm{Sol}$ denote the developing map for the Sol structure on N induced by M. Observe that $\mathcal{D}$ is a covering map onto its image. The idea is to use the map $\Delta_{a,b} \circ \mathcal{D}$ and to make it $\rho_{a,b}$-equivariant. We follow the construction of Canary, Epstein and Green, in [6, Lemma 1.7.2]. We choose $U_0, U_1, \ldots, U_n$ a covering of N such that:

- U_0 is a punctured tubular neighborhood of Σ (i.e., $U_0 \cong (D^2 - \{*\}) \times S^1$).
- U_i is simply connected, for $i \geq 1$.

Let $\pi\colon \widetilde{N} \to N$ denote the projection of the universal covering, we construct the map $D^1_{a,b}\colon \pi^{-1}(U_1) \to \mathbb{H}^3$ as follows. We take

$$\pi^{-1}(U_1) = \bigcup_{\gamma \in \Gamma} \gamma V_1 \qquad \text{where } V_1 \text{ is simply connected.}$$

We define $D^1_{a,b}|_{V_1} = \Delta_{a,b} \circ \mathcal{D}|_{V_1}$, and we take $D^1_{a,b}$ to be the $\rho_{a,b}$-equivariant extension of $D^1_{a,b}|_{V_1}$.

Next we take a refinement $U^2_0, U^2_1, \ldots, U^2_n$ (i.e., a covering of N such that $\overline{U^2_i} \subset U_i$), and we shall construct $D^2_{a,b}\colon \pi^{-1}(U^2_1 \cup U^2_2) \to \mathbb{H}^3$. Let

$$\pi^{-1}(U_2) = \bigcup_{\gamma \in \Gamma} \gamma V_2 \qquad \text{and} \qquad \pi^{-1}(U^2_2) = \bigcup_{\gamma \in \Gamma} \gamma V^2_2,$$

where $\overline{V^2_2} \subset V_2$ and V_2 is simply connected. We define $f\colon V_2 \to \mathbb{H}^3$ as:

$$f = \phi\, D^1_{a,b} + (1-\phi)(\Delta_{a,b} \circ \mathcal{D}|_{V_2}), \tag{8}$$

where $\phi\colon V_2 \to [0,1]$ is a $\mathcal{C}^\infty$-function satisfying $\overline{\operatorname{supp}\phi} \subset V_2 \cap \pi^{-1}U_1$ and $\phi|_{\overline{V_2^2} \cap \pi^{-1}(U_1^2)} \equiv 1$. Notice that in equality (8) we use the $\mathbb{R}$-affine structure of the coordinate system $\mathbb{C} \times \mathbb{R} \cong \mathbb{H}^3$. By construction:

$$f|_{V_2^2 \cap \pi^{-1}(U_1^2)} = D^1_{a,b}|_{V_2^2 \cap \pi^{-1}(U_1^2)}.$$

Thus we extend $f|_{V_2^2}$ equivariantly to $\pi^{-1}(U_2^2)$ and we glue it to $D^1_{a,b}$ restricted to $\pi^{-1}(U_1^2)$. In this way we obtain $D^2_{a,b}\colon \pi^{-1}(U_1^2 \cup U_2^2) \to \mathbb{H}^3$.

We proceed inductively to construct

$$D^j_{a,b}\colon \pi^{-1}(U_1^j \cup U_2^j \cup \cdots \cup U_j^j) \to \mathbb{H}^3,$$

where $U_0^j, U_1^j, U_2^j, \ldots, U_n^j$ is a refinement of $U_0, U_1, U_2, \ldots, U_n$, until $j = n$

It remains to define the map on $\widetilde{U}_0$. We write $U_0 \cong (0,1) \times T^2$, where 0 points to the end and 1 to the interior of the manifold. We fix a lift $(0,1) \times \widetilde{T}^2$ to the universal covering $\widetilde{N}$ so that $\lambda, \mu \in \pi_1(T^2)$, hence $\rho_{a,b}(\pi_1(T^2))$ preserves the initial geodesic σ.

We take the closure $\overline{U}_0 \cong (0,1] \times T^2$ and we define a map

$$D^0_{a,b} : \widetilde{\overline{U}_0} \cong (0,1] \times \widetilde{T}^2 \to \mathbb{H}^3$$

as follows. For $x \in \widetilde{T}^2$, define $D^0_{a,b}(0,x)$ to be the point of σ that minimizes the distance to $D^n_{a,b}(1,x)$. Hence we take

$$D^0_{a,b}(s,x) = (1-s)D^0_{a,b}(0,x) + s\, D^n_{a,b}(1,x),$$

for $x \in \widetilde{T}^2$ and $s \in (0,1]$. We extend $D^0_{a,b}$ to the other lifts of $(0,1) \times T^2$ by taking the equivariant extension.

This can be done because of the following lemma:

Lemma 6.2. *Let $\varepsilon > 0$ be such that $(1-\varepsilon, 1] \times T^2 \subset U_1^n \cup \cdots \cup U_n^n$. Then:*

$$D^n_{a,b}(s,x) = (1-s)D^0_{a,b}(0,x) + sD^n_{a,b}(1,x)$$

for every $s \in (1-\varepsilon, 1]$ and $x \in \widetilde{T}^2$.

Proof. Working with the coordinate system $\mathbb{C}\times\mathbb{R}$, we write $D^n_{a,b}(1,x) = (\alpha(a,b,x), \beta(a,b,x))$, for $x \in \widetilde{T}^2$. In particular, $D^0_{a,b}(0,x) = (0, \beta(a, b,x))$. Thus the lemma will follow if we prove that there exist maps α_j and β_j independent of s, such that

$$D^j_{a,b}(s,x) = (s\,\alpha_j(a,b,x), \beta_j(a,b,x)) \tag{9}$$

for every $(s,x) \in (1-\varepsilon, 1] \times \widetilde{T}^2$ in the domain of $D^j_{a,b}$. We prove (9) for each step in the construction of $D^j_{a,b}$. We start with $D^1_{a,b}|_{V_1}$. We write $x \in \widetilde{T}^2$ as $x = (\theta_1, \theta_2) \in \mathbb{R}^2$, so that $\mathcal{D}(s,x) = (s\cos\theta_1, s\cos\theta_1, \theta_2)$. By Lemma 4.6, for $(x,s) \in V_1$,

$$D^1_{a,b}(s,x) = \Delta_{a,b} \circ \mathcal{D}(x,s) = (a\, s\cos\theta_1\, e^{-\theta_2} + \bar{b}\, s\sin\theta_1\, e^{\theta_2}, \theta_2),$$

which proves (9) for $D^1_{a,b}|_{V_1}$. Notice that this proof also applies to any building block $\Delta_{a,b} \circ \mathcal{D}|_{V_i^j}$.

Next we deal with the action of $\pi_1(T^2)$. Given $\gamma \in \pi_1(T^2)$, $\rho_{a,b}(\gamma)$ acts as a translation of complex length $h(\gamma) \in \mathbb{C}$ along σ. Thus, in coordinates, for $(z,t) \in \mathbb{C} \times \mathbb{R}$:

$$\rho_{a,b}(\gamma)(z,t) = T_{h(\gamma)}(z,t) = (e^{i\,\mathrm{Im}(h(\gamma))} z, t + \mathrm{Re}(h(\gamma)))$$

which is compatible with (9). Finally, the affine construction used for gluing in (8) is also compatible with (9), and therefore $D^j_{a,b}$ satisfies (9). q.e.d.

To check that $D_{a,b}$ satisfies properties (ii) and (iii) we need first the following two lemmas.

Lemma 6.3. *There exists a one-parameter group of diffeomorphisms $\{h_s\colon \widetilde{N} \to \widetilde{N}\}_{s\in\mathbb{R}}$ such that*

$$D_{e^z a, e^{-z} b} = T_z \circ D_{a,b} \circ h_{-\mathrm{Re}(z)} \qquad \textit{for every } z \in \mathbb{C},$$

where T_z is the hyperbolic translation of complex length z along σ.

To be a one-parameter group means that $h_{s+s'} = h_s \circ h_{s'}$ for every $s, s' \in \mathbb{R}$.

Proof. We take h_s to be the pullback under the covering map $\mathcal{D}$ of right-multiplication by $(0,0,s)$ in Sol (which preserves the image of $\mathcal{D}$). By Lemma 4.5

$$\Delta_{e^z a, e^{-z} b} = T_z \circ \Delta_{a,b} \circ h_{-\mathrm{Re}(z)} \qquad \text{for every } z \in \mathbb{C}.$$

Thus the lemma follows from Lemma 5.6, and linearity of constructions. q.e.d.

Lemma 6.4. *For $\gamma \in \Gamma$ and $K \subset \mathrm{Sol}$ a compact subset,*

$$d\big(\Delta_{a,b}(\mathrm{hol}(\gamma)g), \rho_{a,b}(\gamma)(\Delta_{a,b}(g))\big) = O(|(a,b)|^2) \qquad \forall g \in K,$$

where the estimation only depends on K and γ.

Here $O(|(a,b)|^2)$ stands for a smooth function $F\colon U \to \mathbb{R}$ such that $|F(a,b)|/|(a,b)|^2$ is bounded in a neighborhood of the origin.

We postpone the proof of Lemma 6.4 to Section 10.

Corollary 6.5. *For $K \subset \widetilde{N}$ a compact subset, we have*

$$d\big(D_{a,b}(x), \Delta_{a,b}(\mathcal{D}(x))\big) = O(|(a,b)|^2) \qquad \forall x \in K,$$

where the estimation only depends on K.

Proof. We recall that $D_{a,b}$ is constructed by using formula (8). Hence the corollary follows from Lemma 6.4 and the following two assertions. Here $\mathcal{N}_R(\sigma)$ denotes the tubular neighborhood of σ of radius R.

(i) There exists $R(a,b) = O(|(a,b)|^2)$ so that

$$\Delta_{a,b}(\mathcal{D}(K)) \subset \mathcal{N}_{R(a,b)}(\sigma) \quad \text{and}$$
$$\rho_{a,b}(\gamma)(\Delta_{a,b}(\mathcal{D}(\gamma^{-1}K))) \subset \mathcal{N}_{R(a,b)}(\sigma),$$

for every $\gamma \in \pi_1(N)$ with $K \cap \gamma K \neq \emptyset$.

(ii) The parametrization Ψ^{-1} restricted to $\mathcal{N}_{R(a,b)}(\sigma)$ is $(1+O(|(a,b)|^2))$-bi-Lipschitz.

Assertion (i) follows from construction and from Lemma 6.4, because there is a finite set of $\gamma \in \pi_1(N)$ such that $K \cap \gamma K \neq \emptyset$. Assertion (ii) is a consequence of the fact that the tangent map of the exponential at zero is the identity. q.e.d.

Lemma 6.6. *If $ab \in \mathbb{R} \smallsetminus \{0\}$, then the image of $D_{a,b}$ is contained in $(T\sigma)^{\perp} \cap T\mathbb{H}^2$ for some hyperbolic plane $\mathbb{H}^2$. In addition the tangent map of $D_{a,b}\colon \widetilde{N} \to \mathbb{H}^2$ is surjective.*

Proof. By Lemma 6.3, we can assume that $a, b \in \mathbb{R} \smallsetminus \{0\}$. In this case, by Remark 5.8 $\rho_{a,b}$ is a representation in $\mathrm{SL}_2(\mathbb{R})$, and by Corollary 4.7 the image of $D_{a,b}$ is contained in $\Psi(\mathbb{R} \times \mathbb{R}) = \mathbb{H}^2 \subset \mathbb{H}^3$, where $\Psi\colon \mathbb{C} \times \mathbb{R} \to \mathbb{H}^3$ is the diffeomorphism (6). To study the tangent map, it suffices to consider a compact subset $K \subset \widetilde{N}$, because of equivariance and Lemma 6.2. By Lemma 4.6:

$$\Psi^{-1}(\Delta_{a,b})(x,y,t) = (axe^{-t} + bye^{t}, t) \in \mathbb{R} \times \mathbb{R} \subset \mathbb{C} \times \mathbb{R}.$$

If $(\Psi^{-1}\Delta_{a,b})_*$ denotes the tangent map of $\Psi^{-1}\Delta_{a,b}$, then

$$(\Psi^{-1}\Delta_{a,b})_*(\partial_t) = \partial_t + (-\,a\,x\,e^{-t} + \,b\,y\,e^{t})\,\partial_x,$$
$$(\Psi^{-1}\Delta_{a,b})_*(\partial_x) = a\,e^{-t}\,\partial_x,$$
$$(\Psi^{-1}\Delta_{a,b})_*(\partial_y) = b\,e^{t}\,\partial_x.$$

By Corollary 6.5, $\Delta_{a,b} \circ \mathcal{D}$ and $D_{a,b}$ coincide up to some order two terms on a and b, thus the tangent map of $D_{a,b}$ is a surjection for (a,b) close the origin but non-zero. q.e.d.

Lemma 6.7. *If $ab \in \mathbb{C} \setminus \mathbb{R}$, then the tangent map of $D_{a,b} \colon \widetilde{N} \to \mathbb{H}^3$ is an isomorphism.*

Proof. By Lemma 6.3 we may assume that $a \in \mathbb{R}$ and $\mathrm{Im}(b) \neq 0$. A similar computation as in the proof of previous lemma works in this case:

$$\begin{aligned}
(\Psi^{-1}\Delta_{a,b})_*(\partial_t) &= \partial_t \,+\, (-a\,x\,e^{-t} \,+\, \mathrm{Re}(b)\,y\,e^{t})\,\partial_x + \mathrm{Im}(b)\,y\,e^{t}\,\partial_y,\\
(\Psi^{-1}\Delta_{a,b})_*(\partial_x) &= a\,e^{-t}\,\partial_x,\\
(\Psi^{-1}\Delta_{a,b})_*(\partial_y) &= \mathrm{Re}(b)\,e^{t}\,\partial_x \,-\, \mathrm{Im}(b)\,e^{t}\,\partial_y.
\end{aligned}$$

Again by Corollary 6.5, the tangent map of $D_{a,b}$ is an isomorphism. q.e.d.

Lemmas 6.6 and 6.7 imply assertions (ii) and (iii) of the proposition. The remaining assertions follow from construction. q.e.d.

7. Generalized Dehn Filling coefficients

In this section we compute the generalized Dehn-filling coefficient of a structure with developing map $D_{a,b}$ and holonomy $\rho_{a,b}$.

The Poincaré model. We work with the upper half-space model

$$\mathbb{H}^3 = \{z + e^t j \mid z \in \mathbb{C}, t \in \mathbb{R}\}, \tag{10}$$

with the warped product metric $ds^2 = e^{-2t} d|z|^2 + dt^2$. The orientation preserving isometry group $\mathrm{Isom}^+(\mathbb{H}^3)$ is isomorphic to $\mathrm{PSL}_2(\mathbb{C})$. We assume in the sequel that the geodesic σ is the real axis, i.e., $\sigma(t) = e^t j$, with $t \in \mathbb{R}$. The hyperbolic translation of complex length $z \in \mathbb{C}$ along σ is therefore

$$T_z = \pm \begin{pmatrix} e^{z/2} & 0 \\ 0 & e^{-z/2} \end{pmatrix}.$$

The two non-zero Killing fields $V^{\pm}$ are then given by $V^+ = \left(\begin{smallmatrix} 0 & c \\ 0 & 0 \end{smallmatrix}\right)$ and $V^- = \left(\begin{smallmatrix} 0 & 0 \\ c & 0 \end{smallmatrix}\right)$, for some non-vanishing constant $c \neq 0$. We assume that $c = 1$ (see Section 4).

The hyperbolic plane $\mathbb{H}^2 = \{x + e^t j \mid x \in \mathbb{R}, t \in \mathbb{R}\}$ contains the geodesic σ. The subgroup of isometries that preserve $\mathbb{H}^2$ is the subgroup generated by $\mathrm{PSL}_2(\mathbb{R})$ and the involution $T_{i\pi} = \pm\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$, which reverses the orientation of $\mathbb{H}^2$.

Recall that $\Gamma = \pi_1(N)$ has presentation

$$\Gamma \cong \langle\, \lambda, \alpha, \beta \mid \lambda\alpha\lambda^{-1} = f(\alpha), \lambda\beta\lambda^{-1} = f(\beta)\,\rangle,$$

where α and β generate the free group $\pi_1(T^2 - \{*\})$ and $f\colon \pi_1(T^2 \setminus \{*\}) \to \pi_1(T^2 \setminus \{*\})$ is an isomorphism that preserves the commutator $\mu = \alpha\beta\alpha^{-1}\beta^{-1}$. In particular, μ commutes with λ, so λ and μ generate a peripheral torus group.

Lemma 7.1. *For $(a, b) \in U$,*

$$\rho_{a,b}(\lambda) = \begin{pmatrix} e^{l/2} + O(|ab|) & 0 \\ 0 & e^{-l/2} + O(|ab|) \end{pmatrix}$$

and $$\rho_{a,b}(\mu) = \begin{pmatrix} 1 + ab + O(|ab|^2) & 0 \\ 0 & 1 - ab + O(|ab|^2) \end{pmatrix}.$$

Proof. The fact that $\rho_{a,b}(\lambda)$ and $\rho_{a,b}(\mu)$ are diagonal matrices comes from the choice $\sigma(t) = e^t j$, because $\rho_{a,b}(\lambda)$ is a translation along σ by construction of the slice, and μ commutes with λ. The formula for $\rho_{a,b}(\lambda)$ is easily deduced from the facts that $\rho_0(\lambda) = \begin{pmatrix} e^{l/2} & 0 \\ 0 & e^{-l/2} \end{pmatrix}$ and the coefficients are analytic functions on the trace of $\rho_{a,b}(\lambda)$ (hence analytic on ab by Corollary 5.7). To compute the coefficient of $\rho_{a,b}(\mu)$, we start with the following formulas, which follow from the construction of the parametrization (see Corollary 10.4 in Section 10).

$$\rho_{a,b}(\alpha) = \begin{pmatrix} 1 & a\, d_+(\alpha) \\ b\, d_-(\alpha) & 1 \end{pmatrix} + O(|(a,b)|^2)$$

and $$\rho_{a,b}(\beta) = \begin{pmatrix} 1 & a\, d_+(\beta) \\ b\, d_-(\beta) & 1 \end{pmatrix} + O(|(a,b)|^2).$$

Since $\mu = \alpha\beta\alpha^{-1}\beta^{-1}$, a long but easy computation shows that

$$\rho_{a,b}(\mu) = \begin{pmatrix} 1 + ab + O(|(a,b)|^3) & 0 \\ 0 & 1 - ab + O(|(a,b)|^3) \end{pmatrix}. \tag{11}$$

This computation uses the normalization $d_+(\alpha)\, d_-(\beta) - d_-(\alpha)\, d_+(\beta) = 1$, as fixed in (5), and analyzes the first and second order terms on (a, b)

(the first order term is zero and the second order term at the diagonal entries is $\pm ab$). Since the coefficients of $\rho_{a,b}(\mu)$ are functions on the product ab, the estimate $O(|(a,b)|^3)$ in (11) becomes $O(|ab|^2)$. q.e.d.

Definition 7.2. We define u and v as the complex functions on ab such that

$$\rho_{a,b}(\mu) = \pm \begin{pmatrix} e^{u(ab)/2} & 0 \\ 0 & e^{-u(ab)/2} \end{pmatrix} \text{ and } \rho_{a,b}(\lambda) = \pm \begin{pmatrix} e^{v(ab)/2} & 0 \\ 0 & e^{-v(ab)/2} \end{pmatrix}$$

with $u(0) = \pm 2\pi i$ and $v(0) = l + 2\pi i\, k$, $k \in \mathbb{Z}$.

For the choice of $u(0)$ and $v(0)$, it is necessary to determine the branch of the logarithm. The sign in $u(0) = \pm 2\pi i$ is determined by the developing map: we choose the sign $\pm$ according to whether $D_{a,b}$ preserves the orientation ($\mathrm{Im}(ab) < 0$) or reverses the orientation ($\mathrm{Im}(ab) > 0$). Up to sign, the integer $k \in \mathbb{Z}$ is the number of complete turns of λ around Σ the core of the solid torus, counted when we develop the Sol structure of M. The sign of k depends on the sign of $u(0) = \pm 2\pi i$. To simplify, we assume that $k = 0$ by replacing λ by $\lambda\mu^{\pm k}$. This simplification implies that the boundary of the Dehn-filling coefficients space is $p = 1$ but in general it is the line $p \pm k\, q = 1$.

Definition 7.3. Following Thurston, we define the *generalized Dehn-filling coefficients* to be the pair $(p, q) \in \mathbb{R}^2$ such that

$$pu + qv = 2\pi i$$

or equivalently:

$$\left.\begin{array}{rcl} p\ \mathrm{Re}(u) + q\ \mathrm{Re}(v) & = & 0 \\ p\ \mathrm{Im}(u) + q\ \mathrm{Im}(v) & = & 2\pi. \end{array}\right\}$$

Proposition 7.4. *The generalized Dehn-filling coefficients define a homeomorphism between a neighborhood of* 0 *in* $\{ab \in \mathbb{C} \mid \mathrm{Im}(ab) \leq 0\}$ *and a neighborhood of* $(1,0)$ *in* $\{(p,q) \in \mathbb{R}^2 \mid p \geq 1\}$.

Proof. Since we assume $\mathrm{Im}(ab) \leq 0$ we take $u(0) = 2\pi i$, so that when $ab = 0$, $(p,q) = (1,0)$. It follows from Lemma 7.1, that

$$(12) \quad u(ab) = 2\pi i + 2ab + O(|ab|^2) \qquad \text{and} \qquad v(ab) = l + O(|ab|).$$

In particular, $\mathrm{Re}(u) = O(|ab|)$ and $\mathrm{Im}(v) = O(|ab|)$.

When $ab \in \mathbb{R}$, we have that $\operatorname{Im}(u) = 2\pi$ and $\operatorname{Im}(v) = 0$, by Remark 5.8. In particular u and v are analytic on ab with real coefficients (except for the zero order term of u). Thus

$$p = 1 \quad \text{and} \quad q = -\operatorname{Re}(u)/\operatorname{Re}(v) = -(2ab/l) + O(|ab|^2),$$

which defines a homeomorphism between a neighborhood of 0 in the line $\{ab \in \mathbb{C} \mid \operatorname{Im}(ab) = 0\}$ and a neighborhood of $(1,0)$ in the line $\{(p,q) \in \mathbb{R}^2 \mid p = 1\}$.

For (a,b) in general we have:

$$p = \frac{-2\pi \operatorname{Re}(v)}{\operatorname{Re}(u)\operatorname{Im}(v) - \operatorname{Im}(u)\operatorname{Re}(v)} = \frac{-2\pi \operatorname{Re}(v)}{-\operatorname{Im}(u)\operatorname{Re}(v) + O(|ab|^2)}.$$

Since $\operatorname{Re}(v) = l + O(|ab|)$ and $\operatorname{Im}(u) = 2\pi + 2\operatorname{Im}(ab) + O(|ab|^2)$, it follows that

$$\begin{aligned} p = \frac{2\pi}{\operatorname{Im}(u) + O(|ab|^2)} &= \frac{2\pi}{2\pi + 2\operatorname{Im}(ab) + O(|ab|^2)} \\ &= 1 - \frac{1}{\pi}\operatorname{Im}(ab) + O(|ab|^2). \end{aligned}$$

In addition

$$\begin{aligned} q = \frac{2\pi \operatorname{Re}(u)}{\operatorname{Re}(u)\operatorname{Im}(v) - \operatorname{Im}(u)\operatorname{Re}(v)} &= \frac{2\pi \operatorname{Re}(u)}{-2\pi l + O(|ab|)} \\ &= -\frac{2\operatorname{Re}(ab)}{l} + O(|ab|^2). \end{aligned}$$

These computations show that the Dehn-filling coefficients defined in a neighborhood of 0 in the half plane $\{ab \in \mathbb{C} \mid \operatorname{Im}(ab) \leq 0\}$ are restriction of a homeomorphism defined in a neighborhood of 0 in the complex plane $\mathbb{C}$. Combining this with the fact that it maps the line $\operatorname{Im}(ab) = 0$ to the line $p = 1$, we obtain the proposition. q.e.d.

When $\operatorname{Im}(ab) > 0$, then D_{ab} reverses the orientation and the coefficients which we obtain are the same up to sign, according to the orientation convention.

Remark 7.5. In the proof of this proposition, we have shown that the coefficients of u and v as an analytic function on ab are real (except for the zero order term of u, which is $2\pi i$) by using Remark 5.8.

This finishes the proof of Theorem B in the case where the matrix of monodromy $\phi : T^2 \to T^2$ has positive trace.

Proof of Theorem B (case with negative trace)**.** When the monodromy matrix of M has negative trace, the same proof as in the case with positive trace applies, with some changes that we explain now. In this case the eigenvalues of the monodromy are $-e^{\pm l}$, with $l \in \mathbb{R}$, $l > 0$. Thus the holonomy of the Sol structure is not contained in Sol itself but in $\mathrm{Sol} \rtimes \mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Z}/2\mathbb{Z}$ is generated by the involution $(x, y, t) \mapsto (-x, -y, t)$. More precisely, the holonomy of λ is the transformation

$$\begin{array}{rccl} \mathrm{hol}(\lambda)\colon & \mathrm{Sol} & \to & \mathrm{Sol} \\ & (x, y, t) & \mapsto & (-e^{l}x, -e^{-l}, t + l). \end{array}$$

The exact sequence (1) generalizes to

$$1 \to \mathbb{R}^2 \to \mathrm{Sol} \rtimes \mathbb{Z}/2\mathbb{Z} \xrightarrow{\mathrm{Lin}} \mathbb{R} \times \mathbb{Z}/2\mathbb{Z} \to 0$$

The representation $\rho_0 = \mathrm{Lin} \circ \mathrm{hol}$ has cyclic image generated by $\rho_0(\lambda) = (l, -1) \in \mathbb{R} \times \mathbb{Z}/2\mathbb{Z}$. We view it as the abelian representation in $\mathrm{SL}_2(\mathbb{C})$ of translations and rotations around the geodesic σ, such that $\rho_0(\lambda) = T_{l+\pi i}$ (i.e., $\rho_0(\lambda)$ is a rotation of length l plus a rotation of angle π around σ). In the Poincaré model of $\mathbb{H}^3$, if σ has end-points 0 and ∞, then

$$\rho_0(\lambda) = \pm \begin{pmatrix} e^{(l+\pi i)/2} & 0 \\ 0 & e^{-(l+\pi i)/2} \end{pmatrix}.$$

All the constructions of Sections 2 to 6 apply to this situation. For instance $e^{\pm(l+\pi i)}$ is a simple root of the Alexander polynomial, hence the results of [8] allow to construct a deformation space as in Section 5. In Remark 5.8 the group $\mathrm{SL}_2(\mathbb{R})$ should be replaced by $\mathrm{SL}_2(\mathbb{R}) \cup \mathrm{SL}_2(\mathbb{R}i)$, which is the group that preserves a plane $\mathbb{H}^2$.

There are also some differences in the computation of Dehn-filling coefficients. The function $u(ab) = 2\pi i + 2ab + O(|ab|^2)$ does not change, but v becomes $v(ab) = l + \pi i + O(|ab|)$. This explains the change of line $p = 1$ to a line $p \pm \frac{1}{2} q = 1$, but the proof applies with no other change. q.e.d.

8. Proof of Theorem A.

Theorem B provides a family of hyperbolic cone structures C_α with cone angle $\alpha \in (2\pi - \varepsilon, 2\pi)$: it is sufficient to take $q = 0$ and $p = 2\pi/\alpha$.

In addition, Kojima's theorem [13] says that if we can reach angle $\alpha = \pi$, then the cone angle can decrease until zero. In order to make the cone angle decrease from $2\pi - \varepsilon$ until π, we work in a quotient of C_α.

From $2\pi - \varepsilon$ to π. Let τ denote the involution of the torus T^2 that has four fixed points. That is, if we view $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$, then τ lifts to

$$\begin{aligned} \widetilde{\tau}\colon \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ x &\mapsto -x. \end{aligned}$$

Notice that τ is central in the mapping class group of T^2, and in particular it commutes with the monodromy $\phi\colon T^2 \to T^2$. Thus τ extends to an involution of M and also of N, because the fixed point of ϕ is also fixed by τ. We also denote by τ the involution on N.

The quotient N/τ is a three-orbifold with fiber a punctured sphere with three singular points that have ramification index two. The ramification set of N/τ is a link that may have one, two or three components (according to the matrix of ϕ in $\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$). The orbifold M/τ is compact and has one singular component more than N/τ, which is the projection of Σ (that is fixed by τ).

Lemma 8.1. *The involution τ is an isometry of the hyperbolic cone manifold C_α, for $\alpha \in (2\pi - \varepsilon, 2\pi)$.*

Proof. Let $\widetilde{\tau}$ be a lift of τ to the universal covering of $\widetilde{N}$. We want to show that $\widetilde{\tau}$ may be chosen so that $D_{a,b} \circ \widetilde{\tau} = T_{\pi i} \circ D_{a,b}$, where $T_{\pi i}$ is a rotation around σ of angle π.

Let $\mathcal{D}\colon \widetilde{N} \to \mathrm{Sol}$ denote the developing map. We may choose $\widetilde{\tau}$ so that

$$\mathcal{D} \circ \widetilde{\tau} = \theta \circ \mathcal{D}$$

where θ is the involution of Sol that maps (x, y, t) to $(-x, -y, t)$. Then it follows that

$$\Delta_{a,b} \circ \mathcal{D} \circ \widetilde{\tau} = \Delta_{-a,-b} \circ \mathcal{D} = T_{\pi i} \circ \Delta_{a,b} \circ \mathcal{D}.$$

In addition, by choosing $\tau_*(\alpha) = \alpha^{-1}$, then $\rho_{a,b} \circ \tau_* = \rho_{-a,-b}$. (Here we have used the equality $d_\pm(\alpha^{-1}) = -e^{\pm\rho_0(\alpha)} d_\pm(\alpha) = -d_\pm(\alpha)$, because $\rho_0(\alpha) = 0$).

Hence it follows from the construction of $D_{a,b}$ that $D_{a,b} \circ \widetilde{\tau} = D_{-a,-b}$. By Lemma 6.3, we have that $D_{-a,-b} = T_{\pi i} \circ D_{a,b}$. Hence

$$D_{a,b} \circ \widetilde{\tau} = T_{\pi i} \circ D_{a,b}$$

and the lemma follows. q.e.d.

This lemma provides a family of cone manifolds C_α/τ with the same topological type $(|C_\alpha/\tau|, \Sigma_{C_\alpha/\tau})$ as the orbifold M/τ, cone angle $\alpha/2 \in (\pi-\varepsilon, \pi)$ in one singular component and cone angle π in the other singular components.

Now we would like to apply Kojima's Theorem to C_α/τ, but we have to be careful: in Kojima's deformation all cone angles decrease, but in C_α/τ only one cone angle decreases and the other ones stay constant equal π. With respect to Kojima's paper [13], the only thing we need to control is that, when a sequence (α_n) decreases, then C_{α_n}/τ does not collapse. The arguments in [13] go through once we establish next lemma, and allow to conclude that C_α is hyperbolic for $\alpha \in [\pi, 2\pi)$.

Lemma 8.2. *If (α_n) decreases, then C_{α_n}/τ does not collapse.*

Proof. The proof is by contradiction. We assume that C_{α_n}/τ collapses, namely the sequence $\sup\{\text{cone-inj}(x) \mid x \in C_{\alpha_n}/\tau\}$ converges to 0 as $n \to \infty$, where cone-inj denotes the cone injectivity radius (see [2] or [3] for the definition).

First we notice that the diameter of C_{α_n} (and of C_{α_n}/τ) does not converge to zero, because the volume of C_{α_n} increases with n. This is a consequence of Schläfli's variation formula (see Proposition 7.3.1 in [13]).

Next we want to show that the length of the singular component of C_{α_n} converges to zero. We choose a singular point $\bar{x}_n \in C_{\alpha_n}$ and we project it to $x_n \in C_{\alpha_n}/\tau$. The pointed rescaled sequence

$$\left(\frac{1}{\text{cone-inj}(x_n)} C_{\alpha_n}/\tau, x_n\right)$$

has a partial subsequence converging to a pointed non-compact Euclidean 3-manifold (E, x_∞), by the compactness theorem of [2] and because the diameter of C_{α_n} is bounded away from zero. The cone manifold E has at least one singular point with cone angle $\alpha_\infty/2 < \pi$. By the classification of non-compact Euclidean cone manifolds with cone angle $\leq \pi$, either $E = S^2(\alpha,\beta,\gamma) \times \mathbb{R}$ (where $S^2(\alpha,\beta,\gamma)$ is a sphere with three cone points such that $\alpha+\beta+\gamma = 2\pi$) or $E = S^1 \times$(open cone disk) is the infinite tubular neighborhood of a closed singular geodesic.

The case $E = S^2(\alpha,\beta,\gamma) \times \mathbb{R}$ can not occur by topological reasons. Here we use pointed bi-Lipschitz convergence that preserves the singular locus (denoted geometric convergence in [2, 3, 4]). Hence, if this case occurs, then $S^2(\alpha,\beta,\gamma)\times\{0\}$ is an embedded sphere of M/τ that intersects the singular locus in three points. In particular it does not separate in M/τ, and it lifts to a sphere in the torus bundle M which does not separate. Hence we have a contradiction and the Euclidean limit is $S^1\times$(open cone disk). It follows that the length of the singular set of C_{α_n} converges to zero at the same speed as cone-inj(x_n).

Now we can conclude in several ways. For instance, by Proposition 5.2.3 in [2], if C_{α_n}/τ collapses, then the fact that $E = S^1 \times$ (open cone disk) implies that the simplicial volume of $N = M \setminus \Sigma$ vanishes, which contradicts the hyperbolicity of N.

We can also use Proposition 5.3 of [4], which implies that the singular set of C_{α_n} has a tubular neighborhood of radius bounded below by some positive constant. With this lower bound, we can deduce that C_{α_n} does not collapse by applying either Proposition 5.1.1 of [13], or Proposition 2.6 of [4]. q.e.d.

Rescaling the cone manifolds. To finish the proof of Theorem A, it remains to check that the metrics on the cone manifolds C_α can be rescaled in the direction of the fibers, so that they to converge to the Sol-manifold M as $\alpha\to 2\pi$. This is the aim of the following lemma.

Lemma 8.3. *If the metric in the fiber direction of the cone manifolds C_α is rescaled by $1/\sqrt{2\pi-\alpha}$, then, as $\alpha\to 2\pi$, the rescaled cone manifolds converge to the* Sol-*manifold M.*

Proof. Let $(a(\alpha), b(\alpha))$ denote the parameters corresponding to C_α for α approaching 2π. By the computation of Dehn-filling coefficients in (12) and setting $u=\alpha i$, we have:

$$a(\alpha)b(\alpha) = -\frac{i}{2}(2\pi-\alpha) + O(|2\pi-\alpha|^2).$$

Hence we can choose $a(\alpha)=b(\alpha)=\frac{1-i}{2}\sqrt{2\pi-\alpha}+O(|2\pi-\alpha|)$.

We also choose $P\subset\widetilde{N}$ to be a fundamental domain for the action of $\pi_1(N)$. In particular $\mathcal{D}(P)\subset$ Sol is a fundamental domain for the action of $\pi_1(M)$ on Sol, where $\mathcal{D}$ is the developing map of the Sol-structure.

We consider $P_\alpha = D_{a(\alpha),a(\alpha)}(P)\subset\mathbb{H}^3$, so that C_α is obtained from $\overline{P_\alpha}$ by metric identifications on its boundary. Let P'_α and P''_α be P_α with the metric rescaled by $\frac{1}{\sqrt{2\pi-\alpha}}$ in the following directions:

- the directions tangent to the planes orthogonal to $\sigma\subset\mathbb{H}^3$, for P'_α;
- the directions tangent to the fibers, for P''_α.

It follows from the construction of $D_{a,a}$ that the angle between the plane tangent to the fibers and the plane orthogonal to σ is $O(|a(\alpha)|^2)$. Hence the metrics on P'_α and on P''_α differ by a term $O(|a(\alpha)|)$. In particular, since by construction $\overline{P'_\alpha} \to \overline{\mathcal{D}(P)}$, it follows that $\overline{P''_\alpha} \to \overline{\mathcal{D}(P)}$. This limit proves the lemma, because $\overline{\mathcal{D}(P)}/\!\sim = M$ and $\overline{P''_\alpha}/\!\sim$ is C_α with the metric rescaled on the fibers (here $\sim$ stands for metric identifications on the boundary).

This concludes the proof of Lemma 8.3 and also of Theorem A. q.e.d.

Proof of Proposition 1.1. The proof consists in constructing three different paths $\gamma_i : (2\pi - \varepsilon, 2\pi] \to U$, $i = 1, 2, 3$, with $\gamma_i(\alpha) = (a_i(\alpha), b_i(\alpha))$ such that:

(i) $D_{a_i(\alpha), b_i(\alpha)}$ is the developing map of the cone hyperbolic manifold C_α with cone angle $\alpha \in (2\pi - \varepsilon, 2\pi)$.

(ii) $D_{a_i(2\pi), b_i(2\pi))}$ is the developing map of the fibration by tori, when $i = 1$, and of a transversely hyperbolic foliation, when $i = 2, 3$.

Condition (i) implies that, for each α, the product $a_i(\alpha)b_i(\alpha)$ has a fixed value, corresponding to the geometric structure of C_α. In particular for the collapsed structures $a_i(2\pi)b_i(2\pi) = 0$. However there is some freedom when choosing $a_i(\alpha)$, and to satisfy condition (ii), we have to take $a_1(2\pi) = b_1(2\pi) = 0$, $a_2(2\pi) \neq 0$ but $b_2(2\pi) = 0$, and $a_3(2\pi) = 0$ but $b_3(2\pi) \neq 0$. q.e.d.

9. An example: torus bundles which are regular branched coverings of S^3

In this section we are interested in the case where the orientable torus bundle with hyperbolic monodromy M is a regular branched covering of S^3.

In [15], Sakuma shows that an orientable torus bundle M is a regular branched covering of S^3 if and only if M admits an orientation-preserving involution h which preserves the base circle Σ and acts on Σ by a reflection. In this case, M is in fact a branched covering of S^3 with deck-transformation group $\mathbb{Z}/2\mathbb{Z} + \mathbb{Z}/2\mathbb{Z}$, generated by any such involution h and the involution τ extending the standard involution of T^2 with four fixed points (see the proof of Theorem A), cf. Theorem VII in [15].

Let us first consider the quotient M/h. The involution h leaves two fibers T_1, T_2 invariant, whose quotient under h is either a Möbius strip or an annulus. Denote by Σ_i the boundary of T_i/h. Topologically, M/h is the union of two solid tori (tubular neighborhoods of T_1/h and T_2/h), i.e., a lens space. It is easy to check that the involution h is realized as a symmetry of order 2 of any complete Sol structure on M. In the universal cover, h lifts to a Sol isometry of the form

$$\widetilde{h}(x,y,t) = (e^s y, e^{-s}x, -t+s) \quad \text{or} \quad \widetilde{h}(x,y,t) = (-e^s y, -e^{-s}x, -t+s)$$

for some $s \in \mathbb{R}$. The quotient M/h is a Sol orbifold with underlying space a lens space and singularity the link $\Sigma_1 \cup \Sigma_2$, with all cone angles equal to π. The quotient of the base circle Σ under h is a segment $\overline{\Sigma}$ joining a point of Σ_1 to a point of Σ_2.

Now let us consider the quotient $M/<h,\tau>$. It is a Sol orbifold with underlying space S^3 and singularity a graph $\mathcal{G}$ that contains $\overline{\Sigma}$ as an edge. All cone angles are equal to π.

Remark 9.1. The pictures of the graphs $\mathcal{G}$ that appear in this way can be found in [7], but the distinguished edge $\overline{\Sigma}$ is not marked there.

Proposition 9.2. *Let $\mathcal{G}$ and $\overline{\Sigma}$ be as above. There exists a family of hyperbolic cone structures on S^3 with singular set the graph $\mathcal{G}$, cone angle α varying between 0 and π on the edge $\overline{\Sigma}$, and all other cone angles equal to π. When α tends to π, these structures collapse to a segment.*

Before proving this proposition, we give a corollary and an example.

Corollary 9.3. *The lens space M/h has hyperbolic cone structures with singularity the graph $\Sigma_1 \cup \Sigma_2 \cup \overline{\Sigma}$, cone angle $\alpha \in (0, 2\pi)$ on the edge $\overline{\Sigma}$ and all other cone angles equal to π. When $\alpha \to 2\pi$, these structures collapse to a segment.*

Example. As a particular example, let us consider the torus bundles which are two-fold branched cyclic covers of S^3, i.e., those for which the lens space M/h is already S^3. Sakuma shows in [14] that these are exactly the torus bundles $M_{m,n}$ with monodromy matrix of the form

$$A = \begin{pmatrix} -1 & -m \\ n & mn-1 \end{pmatrix}$$

for some pair of integers m, n. The branching locus is the link in S^3 in Figure 1.

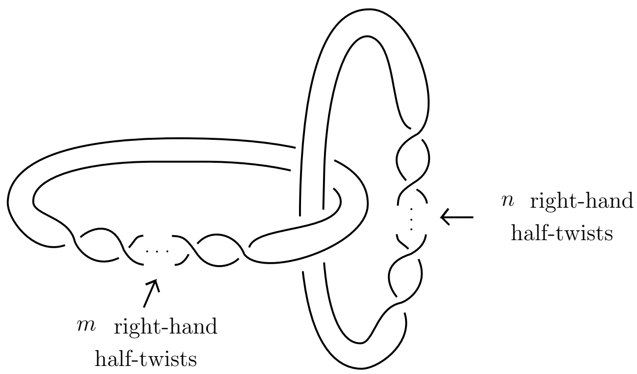

Figure 1: The singular set of M/h.

The torus bundle $M_{m,n}$ has Anosov monodromy when $|mn-2|>2$. Then Proposition 9.2 tells that S^3 has hyperbolic cone structures with singularity the graphs shown in Figures 2 and 3, cone angle $\alpha \in (0,2\pi)$ on the marked edge $\overline{\Sigma}$ in Figure 2 (resp. $\alpha \in (0,\pi)$ in Figure 3), and all other cone angles equal to π. When $\alpha \to 2\pi$ (resp. $\alpha \to \pi$) these structures collapse to the segment $\overline{\Sigma}$.

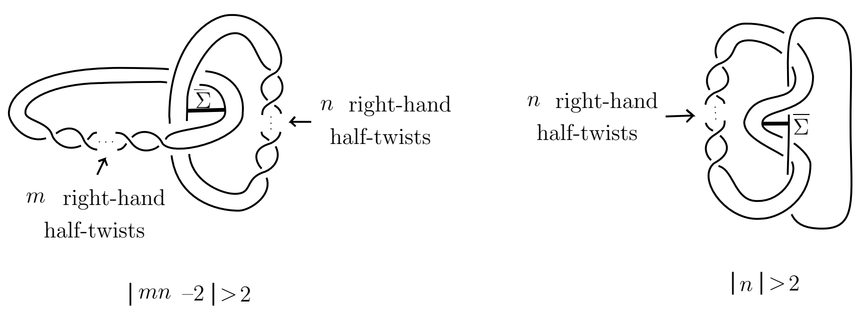

Figure 2: Cone angle $\alpha \in (0,2\pi)$ on the edge $\overline{\Sigma}$.

Proof of Proposition 9.2. Let $D_{a,b}$ the family of developing maps constructed in Proposition 6.1. We shall construct a path $(a(t),b(t))$ with $t\in[0,\varepsilon)$ such that the involutions can be lifted to $\widetilde{N}$, in such a way that, for every $t\in[0,\varepsilon)$, the following hold:

(i) $a(t)\,b(t)=\gamma(t)$, where $\gamma(t)$ corresponds to the singular cone structure with cone angle $2\pi-t$.

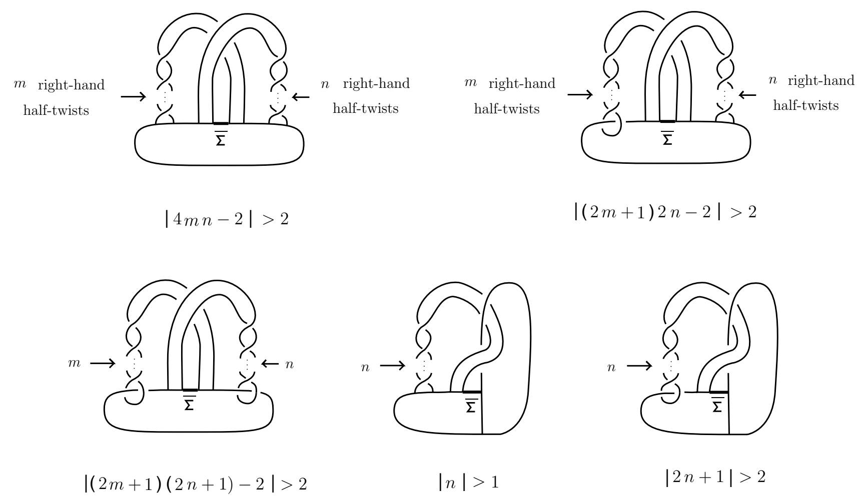

Figure 3: Cone angle $\alpha \in (0,\pi)$ on the edge $\overline{\Sigma}$.

(ii) $D_{a(t),b(t)} \circ \widetilde{\tau} = \Phi_1 \circ D_{a(t),b(t)}$ for some hyperbolic isometry Φ_1.

(iii) $D_{a(t),b(t)} \circ \widetilde{h} = \Phi_2 \circ D_{a(t),b(t)}$ for some hyperbolic isometry Φ_2.

Condition (i) implies that $D_{a(t),b(t)}$ is the developing map of the hyperbolic cone manifold with cone angle $2\pi - t$, $t \in (0,\varepsilon)$. Conditions (ii) and (iii) imply that those structures are compatible with τ and h and therefore they induce the structures of the proposition. Notice that condition (ii) is always satisfied, as proved in Lemma 8.1.

This only proves that the cone structures are invariant by τ and h when the cone angles are close to 2π. But this is sufficient, because the action of τ and h on the variety of representations is algebraic, and the local rigidity theorem of [11] shows that this space of representations is a path contained in a real curve (and the path consists of smooth points).

The value of $a(t)\, b(t) = \gamma(t)$ is fixed for each t. In Particular $\gamma(0) = 0$. However we have to choose specifically $a(t)$ and $b(t)$.

For simplicity, let us choose the Sol structure on M so that h lifts in the universal cover to one of the Sol isometries $\vartheta(x,y,t) = (y,x,-t)$ or $\vartheta(x,y,t) = (-y,-x,-t)$. Let us moreover assume that $\vartheta(x,y,t) = (y,x,-t)$.

Let $\widetilde{h}\colon \widetilde{N} \to \widetilde{N}$ be a lift of h such that $\mathcal{D} \circ \widetilde{h} = \vartheta \circ \mathcal{D}$.

Denote by $r_L\colon \mathbb{H}^3 \to \mathbb{H}^3$ the isometric involution consisting in a rotation of angle π around a line L perpendicular to σ at $\sigma(0)$. We want to prove that for every $t \in \mathbb{R} \setminus \{0\}$ close to 0, we can find a pair $(a,b) \in \mathbb{C}^2$ close to 0, such that $ab = \gamma(t)$ and

$$D_{a,b} \circ \widetilde{h} = r_L \circ D_{a,b}.$$

Let us choose the Killing vector fields $V^{\pm}$ so that $V^+_{\sigma(0)} = V^-_{\sigma(0)}$ is a vector tangent to the line L. Then $Ad_{r_L} V^+ = V^-$, $Ad_{r_L} V^- = V^+$, and we have:

$$\begin{aligned} r_L \circ \Delta_{a,b}(x,y,t) &= r_L(\exp_{\sigma(t)}(a\,x\,V^+ + b\,y\,V^-)) \\ &= \exp_{\sigma(-t)}(a\,x\,Ad_{r_L}V^+ + b\,y\,Ad_{r_L}V^-) \\ &= \exp_{\sigma(-t)}(a\,x\,V^- + b\,y\,V^+) \\ &= \Delta_{b,a}(y,x,-t) = \Delta_{b,a} \circ \vartheta(x,y,t) \end{aligned}$$

for all $(x,y,t) \in \mathrm{Sol}$. Hence:

$$\Delta_{a,b} \circ \mathcal{D} \circ \widetilde{h} = r_L \circ \Delta_{b,a} \circ \mathcal{D}.$$

Unfortunately $\rho_{a,b} \circ h_*$ does not necessarily equal $Ad_{r_L} \circ \rho_{b,a}$. If these two representations where equal, then we would have $D_{a,b} \circ \widetilde{h} = r_L \circ D_{b,a}$ and it would suffice to take $a(t) = b(t) = \sqrt{\gamma(t)}$. To solve this question, we have the following lemma:

Lemma 9.4. *There exists an analytic map $g\colon U \subset \mathcal{S} \to \mathcal{S}$ with real coefficients on (a,b), with $g(0) = 0$, and such that the following hold:*

(i) *The tangent map of g at 0 is the identity (in particular it is locally bi-analytic).*

(ii) $g(-x) = -g(x)$.

(iii) $\rho_{g(a,b)} \circ h_* = Ad_{r_L} \circ \rho_{g(b,a)}$.

Using this lemma, in the construction of $D_{a,b}$ we replace $\rho_{a,b}$ by $\rho_{g(a,b)}$. Then, by property (i) in the lemma, the partial derivatives of $\rho_{a,b}$ at the origin do not change, and the construction of $D_{a,b}$ applies without any change. In addition property (ii) implies that $D_{a,b}$ is still compatible with τ. It is also important to notice that the coefficients of g are real, so that Remark 5.8 can still be applied. Property (iii) implies the compatibility with h, hence taking $a(t) = b(t) = \sqrt{\gamma(t)}$ we obtain the required developing maps $D_{a,b}$.

Finally we prove Lemma 9.4.

Proof of Lemma 9.4. We consider the involution $\iota\colon \mathcal{S} \to \mathcal{S}$ defined by $\iota(\rho) = Ad_{r_L} \circ \rho \circ h_*$. In coordinates (a,b), it can be written as

$$\iota(a,b) = (\iota_1(a,b), \iota_2(a,b)).$$

We take $g^{-1}(a,b) = (a, \iota_1(a,b))$ and we check that g satisfies the properties. First at all, since ι is an involution, we have that $g^{-1} \circ \iota = \varsigma \circ g^{-1}$, where $\varsigma(a,b) = (b,a)$. Hence $\iota(g(a,b)) = g(b,a)$, which is property (iii).

Next we compute the tangent map of ι. We recall that hol denotes the Sol holonomy. By construction, $\mathrm{hol}(\alpha) = (d_+(\alpha), d_-(\alpha), 0)$ and $\mathrm{hol}(h_*(\alpha)) = (d_-(\alpha), d_+(\alpha), 0)$. Hence

$$\rho_{a,b}(h_*(\alpha)) = \exp(a d_-(\alpha) V^+ + b d_+(\alpha) V^- + O(|(a,b)|^2))$$

by Corollary 10.4. Thus

$$\begin{aligned}\iota(\rho_{a,b})(\alpha) &= Ad_{r_L}(\rho_{a,b}(h_*(\alpha)))\\ &= \exp(b d_+(\alpha) V^+ + a d_-(\alpha) V^- + O(|(a,b)|^2)),\end{aligned}$$

because $Ad_{r_L}(V^{\pm}) = V^{\mp}$. Hence the matrix of partial derivatives of ι at the origin with coordinates (a,b) is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and property (i) follows.

Since r_L and $T_{\pi i}$ commute, $\iota(-(a,b)) = -\iota(a,b)$ and (ii) also follows.

Finally, the coefficients of ι are real (it preserves real representations because r_L preserves the hyperbolic plane containing σ and r_L). This proves Lemma 9.4 and Proposition 9.2. q.e.d.

10. The tangent space to the variety of representations

In this section we prove some results stated in Sections 5 and 6 that involve the tangent space to the variety of representations and local parametrizations.

We start by computing $Z^1(\Gamma, sl_2(\mathbb{C})^{\rho_0})$. Since the image of ρ_0 is contained in the group of translation along σ, the action of ρ_0 preserves the Cartan splitting $sl_2(\mathbb{C}) = \mathfrak{h}_+ \oplus \mathfrak{h}_- \oplus \mathfrak{h}_0$. Thus:

$$Z^1(\Gamma, sl_2(\mathbb{C})^{\rho_0}) = Z^1(\Gamma, (\mathfrak{h}_+)^{\rho_0}) \oplus Z^1(\Gamma, (\mathfrak{h}_-)^{\rho_0}) \oplus Z^1(\Gamma, (\mathfrak{h}_0)^{\rho_0}).$$

Next we study each one of these spaces. Let $V^+ \in \mathfrak{h}_+$, $V^- \in \mathfrak{h}_-$ and $V^0 \in \mathfrak{h}_0$ be the elements defined in Section 5. Consider also the cocycles

$d_\pm\colon \Gamma \to \mathbb{R}_\pm$ defined in Section 3 and the morphism $d_0\colon \Gamma \to \mathbb{Z} \subset \mathbb{R}$ such that $d_0(\lambda) = 1$ and $d_0(\alpha) = d_0(\beta) = 0$. In the following lemma $d_0 V^+$ denotes the cocycle that maps $\gamma \in \Gamma$ to $d_0(\gamma)V^+ \in \mathfrak{h}_+$, and so on for other cocycles.

Lemma 10.1.

(i) $Z^1(\Gamma, (\mathfrak{h}_\pm)^{\rho_0}) \cong \mathbb{C}^2$ *is generated by* $d_0 V^\pm$ *and* $d_\pm V^\pm$.

(ii) $B^1(\Gamma, (\mathfrak{h}_\pm)^{\rho_0}) \cong \mathbb{C}$ *is generated by* $d_0 V^\pm$.

(iii) $Z^1(\Gamma, (\mathfrak{h}_0)^{\rho_0}) \cong \mathbb{C}$ *is generated by* $d_0 V^0$.

(iv) $B^1(\Gamma, (\mathfrak{h}_0)^{\rho_0}) \cong 0$.

Proof of Proposition 5.1 (b). We prove first that $T_{\rho_0} R^{\mathrm{ir}}(\Gamma, \mathrm{SL}_2(\mathbb{C})) = Z^1(\Gamma, (\mathfrak{h}_+ \oplus \mathfrak{h}_-)^{\rho_0})$. We know from [8] that $T_{\rho_0} R^{\mathrm{ir}}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ is four dimensional, hence it is sufficient to show that the cocycles $d_0 V^\pm$ and $d_\pm V^\pm$ belong to this tangent space. It is clear that $d_0 V^\pm$ belongs to $T_{\rho_0} R^{\mathrm{ir}}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$, because it is a coboundary. In addition the derivation $d_\pm V^\pm$ is tangent to a path of metabelian representations, that belong to $R^{\mathrm{ir}}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$, by [8]. Thus the equality holds true. A similar argument shows that $T_{\rho_0} R^{\mathrm{ab}}(\Gamma, \mathrm{SL}_2(\mathbb{C})) = Z^1(\Gamma, \mathfrak{h}_0^{\rho_0}) \oplus B^1(\Gamma, (\mathfrak{h}_+ \oplus \mathfrak{h}_-)^{\rho_0})$, because $d_0 V^0$ is tangent to a path of abelian representations. q.e.d.

We defined the slice as:

$$\mathcal{S} = \{\rho \in R^{\mathrm{ir}}(\Gamma, \mathrm{SL}_2(\mathbb{C})) \mid \rho(\lambda) \text{ is a hyperbolic translation along } \sigma\}.$$

Lemma 10.2. *$T_{\rho_0}\mathcal{S}$ is two dimensional generated by d_+V^+ and d_-V^-.*

Proof. It follows from Lemma 10.1 and Proposition 5.1 (b), that d_+V^+, d_-V^-, d_0V^+ and d_0V^- generate $T_{\rho_0} R^{\mathrm{ir}}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$. Moreover, $d_0(\lambda) \neq 0$ and $d_\pm(\lambda) = 0$. We consider the map

$$\begin{array}{rcl} F\colon R^{\mathrm{ir}}(\Gamma, \mathrm{SL}_2(\mathbb{C})) & \to & \hat{\mathbb{C}} \times \hat{\mathbb{C}} \\ \rho & \mapsto & (\rho(\lambda)(\sigma(+\infty)), \rho(\lambda)(\sigma(-\infty))) \end{array}$$

where $\hat{\mathbb{C}} = \partial \mathbb{H}^3$ is the Riemann sphere at infinity, and $\sigma(+\infty), \sigma(-\infty) \in \hat{\mathbb{C}}$ are the ends of σ. By construction,

$$\mathcal{S} = F^{-1}(\sigma(+\infty), \sigma(-\infty)).$$

The tangent map F_* at the point ρ_0 satisfies $F_*(d_0 V^+) = (0,1)$, $F_*(d_0 V^-) = (1,0)$ and $F_*(d_\pm V^\pm) = 0$, because $d_0(\lambda) = 1$ and $d_\pm(\lambda) = 0$. The lemma follows straightforward from these computations. q.e.d.

We obtain as a corollary Proposition 5.4.

Corollary 10.3 (Proposition 5.4). *There is a neighborhood $\mathcal{U} \subset \mathcal{S}$ of ρ_0 so that the map*

$$\begin{array}{rcl} \mathcal{U} & \to & \mathfrak{h}_+ \oplus \mathfrak{h}_- \\ \rho & \to & \mathrm{pr}(\exp^{-1}(\rho(\alpha))) \end{array}$$

is a biholomorphism between $\mathcal{U}$ and a neighborhood of the origin in $\mathfrak{h}_+ \oplus \mathfrak{h}_-$.

Proof. This follows now from Lemma 10.2 and the facts that $\rho_0(\alpha)$ is trivial and that $d_+(\alpha)$ and $d_-(\alpha)$ do not vanish. q.e.d.

Corollary 10.4. *For every $\gamma \in \Gamma$ there is a neighborhood of the origin $U_\gamma \subset U$ such that for every $(a,b) \in U_\gamma$, we have:*

$$\rho_{a,b}(\gamma) = \exp\left\{a\, d_+(\gamma) V^+ + b\, d_+(\gamma) V^- \ + O(|(a,b)|^2)\right\} \rho_0(\gamma).$$

Notice that the open set U_γ depends on γ, but we only need to apply it to finitely many elements.

Proof. Use Lemma 10.2, Corollary 10.3 and Convention 5.5. q.e.d.

It only remains to prove Lemma 6.4. Before we need the following one:

Lemma 10.5. *For a Killing field $Z \in sl_2(\mathbb{C})$ and $K \subset \mathbb{H}^3$ a compact subset,*

$$d\big(\exp_p(Z), \exp(Z)(p)\big) = O(|Z|^2) \qquad \forall p \in K,$$

where $\exp_p$ and $\exp$ denote the Riemannian exponential at p and the Lie group exponential respectively.

Proof. We decompose $Z = Z^r + Z^t$, where Z^r is a field of infinitesimal rotations around p and Z^t is a field of infinitesimal translations at p. Since $p \in K$, $|Z^r|, |Z^t| \leq O(|Z|)$. By construction:

$$\exp_p(Z) = \exp_p(Z_p) = \exp_p(Z^t_p) = \exp(Z^t)(p) = \exp(Z^t)\exp(Z^r)(p).$$

In addition, by Campell-Hausdorff's formula

$$\exp(Z^t)\exp(Z^r) = \exp(O(|Z|^2))\exp(Z),$$

and the lemma follows. q.e.d.

Proof of Lemma 6.4. We use the notation $O(n) = O(|(a,b)|^n)$. We want to prove that

$$d\big(\Delta_{a,b}(\mathrm{hol}(\gamma)g), \rho_{a,b}(\gamma)(\Delta_{a,b}(g))\big) = O(2).$$

We write $g = (x, y, t) \in K$ and $\mathrm{hol}(\gamma) = (d_+(\gamma), d_-(\gamma), s)$. In particular $\rho_0(\gamma) = T_s$. We use the following Killing fields:

$$X = a\,x\,V^+ + b\,y\,V^- \quad \text{and} \quad Y = a\,d_+(\gamma)\,V^+ + b\,d_-(\gamma)\,V^-,$$

so that $\exp_{\sigma(t)}(X) = \Delta_{a,b}(g)$ and $\exp_{\sigma(t+s)}(Y + Ad_{T_s}X) = \Delta_{a,b}(\mathrm{hol}(\gamma)g)$. In addition, by Corollary 10.4, $\rho_{a,b}(\gamma) = \exp(Y + O(2))T_s$, hence the claim is equivalent to:

$$d\big(\exp_{\sigma(t+s)}(Y + Ad_{T_s}X), \exp(Y + O(2))T_s(\exp_{\sigma(t)}(X))\big) = O(2),$$

which can be reformulated as:

$$d\big(\exp_{\sigma(t+s)}(Y + Ad_{T_s}X), \exp(Y)(\exp_{\sigma(t+s)}(Ad_{T_s}X))\big) = O(2).$$

And this estimate follows from Lemma 10.5 and

$$\exp(Y + Ad_{T_s}X) = \exp(Y)\exp(Ad_{T_s}X)\exp(O(2)),$$

because $|X|, |Y| \le O(1)$. q.e.d.

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