J. DIFFERENTIAL GEOMETRY
59 (2001) 269-300

LOCAL FORMULA FOR THE INDEX OF A FOURIER INTEGRAL OPERATOR

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Abstract

Let X and Y be two closed connected Riemannian manifolds of the same dimension and $\phi : S^*X \mapsto S^*Y$ a contact diffeomorphism. We show that the index of an elliptic Fourier operator Φ associated with ϕ is given by $\int_{B^*(X)} e^{\theta_0} \hat{A}(T^*X) - \int_{B^*(Y)} e^{\theta_0} \hat{A}(T^*Y)$ where θ_0 is a certain characteristic class depending on the principal symbol of Φ and, $B^*(X)$ and $B^*(Y)$ are the unit ball bundles of the manifolds X and Y. The proof uses the algebraic index theorem of Nest-Tsygan for symplectic Lie Algebroids and an idea of Paul Bressler to express the index of Φ as a trace of 1 in an appropriate deformed algebra.

In the special case when $X = Y$ we obtain a different proof of a theorem of Epstein-Melrose conjectured by Atiyah and Weinstein.

1. Introduction

Let X and Y be two smooth closed connected Riemannian manifolds of the same dimension such that there exists a contact diffeomorphism $\phi : S^*X \mapsto S^*Y$ between the two unit cotangent bundles which induces a homogeneous symplectomorphism, still denoted by ϕ, from $T^*X \setminus X$ onto $T^*Y \setminus Y$.

We first recall the definition of the index of ϕ, *which is defined only when* $\dim X \geq 3$, following [15]. We will denote by $\Omega_{\frac{1}{2}}$ the half-density bundle over X or Y. Let C_ϕ be the graph of ϕ^{-1} in $(T^*Y \setminus Y) \times (T^*X \setminus X)$ and L_{C_ϕ} be the associated Maslov bundle. Let $A : L^2\left(X, \Omega_{\frac{1}{2}}\right) \to L^2\left(Y, \Omega_{\frac{1}{2}}\right)$ be an elliptic Fourier integral operator of order zero whose canonical relation is C_ϕ and whose principal symbol is an invertible

Received April 18, 2000

section of the bundle $\Omega_{\frac{1}{2}} \otimes L_{C_\phi} \to C_\phi$ (see [15], [7]) for details). Suppose that $B : L^2\left(Y, \Omega_{\frac{1}{2}}\right) \to L^2\left(X, \Omega_{\frac{1}{2}}\right)$ is an elliptic Fourier integral operator of order zero whose canonical relation is $C_{\phi^{-1}}$. Then $B \circ A : L^2\left(X, \Omega_{\frac{1}{2}}\right) \to L^2\left(X, \Omega_{\frac{1}{2}}\right)$ is an elliptic *scalar* pseudo-differential operator of order zero. Since $\dim X \geq 3$ there exists a smooth non vanishing function $x \in X \to a(x) \in \mathbb{C}^*$ such that the principal symbol of $B \circ A$ is homotopic to $(x, \xi) \in T^*X \to a(x) \in \mathbb{C}^*$. In particular the index of $B \circ A$ is zero. Thus $\mathrm{Ind}B = -\mathrm{Ind}A$ for any Fourier integral operators A and B as above, and, as the corollary of this fact, $\mathrm{Ind}A$ does not depend on the choice of A. Since it only depends on the transformation ϕ, it is called the index of ϕ and denoted by $\mathrm{Ind}\,\phi$. A. Weinstein has proved (see [15]) that the integer $\mathrm{Ind}\,\phi$ naturally appears if one wants to compare the spectrum $(\,\lambda_k(X)\,)_{k\in\mathbb{N}}$ of the Laplace Beltrami operator Δ_X of X (associated with the metric g_X) with the one of Δ_Y (associated with the metric g_Y); for instance if $T^*X \setminus X$ is simply connected then the sequence $(\lambda_k(X) - \lambda_{k-\mathrm{Ind}\phi}(Y))_{k\in\mathbb{N}}$ is bounded.

The goal of this paper is to provide a geometric formula for the index of an elliptic Fourier integral operator Φ of order zero whose canonical relation is C_ϕ (we do not assume $\dim X \geq 3$).

Let us first fix some notation. Given a smooth manifold X, we will use T^*X to denote the cotangent bundle of X and $\overline{B}^*X$ to denote the projective compactification of T^*X. We will use M to denote the smooth manifold obtained by glueing at infinity $\overline{B}^*X$ and $\overline{B}^*Y$ with the help of the map

$$\phi' : (x, \xi) \to \phi(x, -\xi).$$

Let $S^0(T^*X)$ and $S^0(T^*Y)$ denote the algebras of asymptotic symbols of pseudodifferential operators of order at most zero on X and Y. Given an element $a \in S^0(T^*X)$, we denote by $a_\hbar$ the symbol a scaled by $\hbar$ in the cotangent direction and by $\mathrm{Op}(a)$ the pseudodifferential operator associated to a. Given a pseudodifferential operator A, we denote by $\sigma(A)$ its full symbol (for the precise definition see the next section).

The general strategy is as follows. We interpret conjugation by Φ as an isomorphism of the algebras of pseudodifferential operators (modulo the smoothing ones) on X and Y. Translated into terms of formal deformations of the cotangent bundles, this allows us to construct a formal deformation $\mathbb{A}^\hbar(M)$ of $C^\infty(M)$ which represents on T^*X and T^*Y the calculus of differential operators, while on the common cosphere at infinity it represents the calculus of pseudodifferential operators. While the symplectic structures on T^*X and T^*Y do not glue together (so there is in general no almost complex structure on $B^*X \cup_{\phi'} B^*Y$), there is a (noncanonical) symplectic Lie algebroid structure $(\mathcal{E}, [\cdot,\cdot], \omega)$ over M and $\mathbb{A}^\hbar(M)$ is a deformation associated to it in the sense of [10]. The usual traces on the algebras of smoothing operators on X and Y give rise to a trace τ_{can} on $\mathbb{A}^\hbar(M)$ such that $\operatorname{ind}\Phi = \tau_{\mathrm{can}}(1)$. An application of the general algebraic index theorem from [10] gives the local formula for the index.

The content of the paper is given below.

• In the first section we recall the relation between the calculus of *smoothing* operators on X and a formal deformation of T^*X which is basically given by the full symbol of a pseudodifferential operator.

• The manifold $\overline{B}^*X$ carries a structure of symplectic Lie algebroid $(\mathcal{E}_X, [,], \omega)$ described in Section 2. The symbolic calculus of pseudodifferential operators gives rise to a formal deformation of the sphere at infinity of $\overline{B}^*X$ which, together with the formal deformation of T^*X given above, gives rise to a formal deformation $\mathbb{A}^\hbar(X)$ of $\overline{B}^*X$ associated to $(\mathcal{E}_X, [,], \omega)$.

• Let us fix an almost unitary elliptic Fourier integral operator Φ whose canonical relation is given by the graph of ϕ^{-1}. In Section 4 we show how to glue together the deformations $\mathbb{A}^\hbar(X)$ and $\mathbb{A}^\hbar(Y)$ into a formal deformation $\mathbb{A}^\hbar(M)$ of M associated to a symplectic Lie algebroid structure $(\mathcal{E}, [,], \omega)$ on M. The construction is based on the following strengthening of the Egorov theorem (see Theorem 2).

1. *The map which to any $a \in S^0(T^*X)$ associates the asymptotic expansion at $\hbar = 0$ of $(\sigma(\Phi \mathrm{Op}(a_\hbar)\Phi^*))_{\hbar^{-1}}$ induces an algebra isomorphism*

$$\widetilde{\Phi} : S^0(T^*X) \to S^0(T^*Y).$$

2. *For each $k \in \mathbb{N}^*$, there exists an $\mathcal{E}_X$−differential operator D_k on $\overline{B}^*X$ such that, for any $a \in S^0(T^*X)$, the following identity holds:*

$$\widetilde{\Phi}(a) = \left(a + \sum_{k \geq 1} \hbar^k D_k(a) \right) \circ \phi^{-1}.$$

Egorov's theorem corresponds to the leading term in the above expansion.

The real symplectic vector bundle $\mathcal{E}$ is isomorphic to TM (as a vector bundle over M) and hence TM is the realification of a complex vector bundle on M which will be denoted by $\mathcal{E}_{\mathbb{C}}$.

• In Section 5 we identify the space traces on $\mathbb{A}^{\hbar}(M)$ and relate it to the traces on the algebras of smoothing and of pseudodifferential operators.

• In Section 6 we identify the index of the Fourier integral operator with the value of the trace τ_{can} on the unit 1 in the formal deformation algebra $\mathbb{A}^{\hbar}(M)$.

The local index formula for $\operatorname{Ind}\Phi$ follows from the algebraic index theorem of [10], the class θ_0 being the coefficient of $\hbar^0$ in the characteristic class θ (cf. [5], [10]) of the deformation. Notice that the first algebraic type index theorem was proved in [5] where θ appears as the curvature of a suitable connection.

The main result can be formulated as follows:

3. *Let Φ be a Fourier integral operator and $\mathbb{A}^{\hbar}(M)$ the formal deformation of M associated to it as in Definition* 3. *Then*

$$\operatorname{ind}\Phi = \int_M \mathrm{e}^{\theta_0}\hat{A}(M),$$

where θ_0 denotes the characteristic class of the deformation of the Lie Algebroid $(\mathcal{E},[\,,\,],\omega)$ given by $\mathbb{A}^{\hbar}(M)$.

4. *Let ∇_X be a connection on the tangent bundle $T(\overline{B}^*X)$ and $\hat{A}(T^*X)$ an associated representative form of the $\hat{A}$-class of ∇_X. The symplectomorphism ϕ induces a connection $\phi_*(\nabla_X)$ on the tangent space of $\overline{B}^*Y\setminus B^*(Y)$. Let ∇_Y denote its extension to a connection of $T(\overline{B}^*Y)$ and $\hat{A}(T^*Y)$ an associated representative differential form of the $\hat{A}$-class of ∇_Y. Then*

$$\operatorname{ind}\Phi = \int_{B^*(X)} \mathrm{e}^{\theta_0}\hat{A}(T^*X) - \int_{B^*(Y)} \mathrm{e}^{\theta_0}\hat{A}(T^*Y). \tag{1}$$

Remark 1. It is easy to see that our local formula implies the following fact:

If ϕ extends as a symplectomorphism : $T^*X \to T^*Y$ up to the zero section, then $\operatorname{Ind}\Phi = 0$.

• The computation of the characteristic class of the deformation is given in the last section, where we simultaneously construct a deformation of M and the Fourier integral operator whose index is given by the trace of the 1 in the deformed algebra. As the starting point we give a somewhat nonstandard definition of the characteristic class of a formal deformation which is more amenable to computations in the case of deformations associated to (twisted) differential or pseudodifferential operators.

As a corollary we get the following result:

5. *There exists an almost unitary Fourier integral operator Φ_0 whose canonical relation is C_ϕ and such that:*

$$\operatorname{ind}\Phi_0 = \int_M \hat{A}(M)e^{\frac{1}{2}c_1(\mathcal{E}_{\mathbb{C}})}.$$

6. *If the dimension of M is at least three, then*

$$\operatorname{ind}(\phi) = \int_M \hat{A}(M)e^{\frac{1}{2}c_1(\mathcal{E}_{\mathbb{C}})}$$

(*cf.* [4], [14]).

In the case when $X = Y$ Epstein and Melrose ([4]) had shown that $\operatorname{ind}\Phi_0$ is equal to the index of the Dirac operator associated to the Spin^c-structure of the mapping torus of (X,ϕ) then, using the Atiyah-Singer index formula they proved the following Theorem which provides a geometric formula for $\operatorname{ind}\Phi_0$. We show how to derive this Theorem from our previous result.

Theorem 1 ([4], Section 7)**.** *Let ϕ be a contact automorphism of S^*X, denote by Z_ϕ the mapping torus of (S^*X,ϕ) which, as explained in Section* 3 *of* [4], *is endowed with a "positive" almost complex structure. Then:*

$$\operatorname{ind}\Phi_0 = \int_{Z_\phi} \exp\left(\frac{c_1(T^{1,0}Z_\phi)}{2}\right)\hat{A}(Z_\phi). \tag{2}$$

Proof. Recall that the mapping torus of ϕ is given by

$$Z_\phi = \{S^*X \times [0,1]\}/\{(\xi,0) \sim (\phi(\xi),1)\}.$$

We will write it as the manifold obtained from glueing together two copies of $\{\xi \in \overline{B}^*X \,|\, 1 \le ||\xi|| \le \infty\}$, where at the sphere $||\xi|| = 1$ we use the identity map, while at $||\xi|| = \infty$ we use ϕ. We can choose two connections ∇_1 and ∇_2 on T^*X which coincide at $||\xi|| \leq \frac{1}{2}$ and are related by ϕ at $||\xi|| \geq 2$ They glue together to a connection ∇ on the mapping torus Z_ϕ. One can use the Mayer-Vietoris exact sequence for the cohomology of Z_ϕ associated to the open covering by two open sets — a neighbourhood of $||\xi|| = 1$ and a neighbourhood of $||\xi|| = \infty$ — to evaluate the right hand side of (2). But, provided that the above ∇ is used to compute the characteristic classes appearing, the result becomes the same as the left hand side of (1) expressed with connections ∇_1 and ∇_2. q.e.d.

Remark 2. The methods of this paper extend in a fairly straightforward manner to the case of a Fourier integral operator Φ between L^2 sections of vector bundles E and F of the same dimension on X and Y. In the case when both X and Y possess a metalinear structure, the corresponding index formula is given by the expression

$$\operatorname{ind}\Phi = \int_{B^*(X)} \operatorname{ch}(\mathcal{L})\hat{A}(T^*X) - \int_{B^*(Y)} \operatorname{ch}(\mathcal{L})\hat{A}(T^*Y).$$

Here $\mathcal{L}$ is the vector bundle over M obtained by glueing together pullbacks (by the canonical projections π_X^* and π_Y^*) to the cotangent bundles of X (resp. Y) of the bundles $\Lambda^{\frac{n}{2}}(X) \otimes E$ and $\Lambda^{\frac{n}{2}}(Y) \otimes F$ with the help of the symbol of Φ.

Note that existence of an isomorphism of $\pi_X^*(\Lambda^{\frac{n}{2}}(X) \otimes E)$ and $\pi_Y^*(\Lambda^{\frac{n}{2}}(Y) \otimes F)$ over ϕ' is equivalent to existence of an elliptic Fourier integral operator from $L^2(X, E)$ to $L^2(Y, F)$.

Acknowledgements. The authors are very grateful to the referees, whose detailed observations and comments have allowed us to improve the exposition of this paper.

2. Symbolic calculus for ΨDO's and formal deformations

2.1 Deformation of T^*X

We will recall the pertinent facts from [12].

Let χ be a smooth, nonnegative function on $X \times X$ satisfying the following conditions:

(1) $\chi(x, y) = \chi(y, x)$.

(2) $\chi \equiv 1$ on an open set containing the diagonal in $X \times X$.

(3) For each $x \in X$, the set $D_x = \{y \in X / (x, y) \in \operatorname{supp} \chi\}$ is geodesically convex.

We denote by $\operatorname{Exp}_x^{-1}$ the unique smooth inverse to the exponential map:

$$\operatorname{Exp}_x : T_x X \to X$$

defined on D_x and such that $\operatorname{Exp}_x^{-1}(x) = 0$.

Given $x \in X$, $y \in D_x$, let z denote the midpoint of the unique geodesic joining x and y within D_x, and let $v \in T_z X$ be given by

$$v/2 \,=\, \operatorname{Exp}_x^{-1}(y) \,=\, -\operatorname{Exp}_z^{-1}(x). \tag{3}$$

Now, denote by $S^m(T^*X)$ the space of classical symbols of order m on X, i.e., smooth functions θ on T^*X satisfying estimates of the form:

$$\sup_{(x,\xi)} |\partial_x^\alpha \partial_\xi^\beta \theta(x, \xi)| \,\leq\, C_{\alpha,\beta} (1 + |\xi|^2)^{\frac{m-|\beta|}{2}}$$

$S^m(T^*X)$ is given the topology of (Frechet) topological vector space by the "best" $C_{\alpha,\beta}$.

We will denote by $S^\infty(T^*X) = \cup_{m\in\mathbb{R}} S^m(T^*X)$ the set of all classical symbols on T^*X.

With the above notation (3), the map:

$$\operatorname{Op} : S^m(T^*X) \to \operatorname{End}(C^\infty(X))$$

given by

$$\operatorname{Op}(\theta)(u)(x) \,=\, \int_{T_z^*X} d\xi \int dy\, \chi(x, y)\, \mathrm{e}^{i\,\xi.v}\, \theta(z, \xi)\, u(y)$$

defines a pseudo-differential operator. Conversely, if P is a pseudo-differential operator on X we define its complete symbol to be:

$$\sigma(P)(z, \xi) = P_y(\, \chi(x, y)\, \mathrm{e}^{i\,\xi.v}\,)|_{x=y=z} \tag{4}$$

where z is the midpoint of the geodesic joining x and y and v satisfies (3). We observe that $P - \operatorname{Op}(\sigma(P))$ is a smoothing operator whose Schwartz kernel vanishes to infinite order on the diagonal. Now, for a given $\theta \in C^\infty(T^*X)$, we set: $\theta_\hbar(x, \xi) \,=\, \theta(x, \hbar\xi)$.

Following [12], we endow the algebra

$$\mathbb{A}^\hbar(T^*X) = C^\infty(T^*X) \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]]$$

with a star product $\star_X$ by defining, for any symbols $\theta^1, \theta^2, \in S^m(T^*X)$, $\theta^1 \star_X \theta^2$ to be the asymptotic expansion at $\hbar = 0$ of:

$$\sigma(\operatorname{Op}(\theta^1_\hbar) \circ \operatorname{Op}(\theta^2_\hbar))_{\hbar^{-1}}(x,\xi) = \sigma(\operatorname{Op}(\theta^1_\hbar) \circ \operatorname{Op}(\theta^2_\hbar))(x, \hbar^{-1}\xi). \tag{5}$$

One sees immediately that $\star_X$ extends to $\mathbb{A}^\hbar(T^*X)$.

Recall that there exists, unique up to normalization, a canonical trace on $(\mathbb{A}^\hbar(T^*X), \star_X)$, $\operatorname{Tr}^X_{\text{can}}$, given by:

$$\forall a \in S^{-\infty}(T^*X),$$
$$\operatorname{Tr}^X_{\text{can}}(a) = \operatorname{Tr}(\operatorname{Op}(a_\hbar)) = \frac{1}{n!\,\hbar^n} \int_{T^*X} a\,(\omega^X)^n \in \mathbb{C}[\hbar^{-1}, \hbar]].$$

(See Proposition 2.5 (3) of [12], the uniqueness is addressed in [5, p. 172] and in [11]).

2.2 Lie algebroid structure and deformation quantization of the projective completion $\overline{B}^* X$

For any $x \in X$, we set $B^*_x X = \frac{\mathbb{R}_+ \oplus T^*_x X}{\mathbb{R}^*_+} \setminus \{(0,0)\}$, and embed $T^*_x X$ in $\overline{B}^*{}_x X$ by sending ξ to the class of $1 \oplus \xi$. We view $B^*_x X$ as a compactification of $T^*_x X$. Then we consider the fiber bundle $\overline{B}^* X$ over X defined by $\overline{B}^* X = \cup_{x \in X} B^*_x X$. Therefore $\overline{B}^* X$ is a compactification of T^*X and a smooth compact manifold with boundary: $\partial \overline{B}^* X = \overline{B}^* X \setminus T^*X$. Similarly one defines the bundle $\overline{B}^* Y$ over Y. We observe that the map from S^*X into $\overline{B}^* X$ given by $\xi \to 0 \oplus \xi$ defines an isomorphism between S^*X and $\overline{B}^* X \setminus T^*X$. For any $\xi = (x, \xi_x) \in T^*_x X$ we will define $-\xi$ to be $(x, -\xi_x) \in T^*_x X$. Clearly, ϕ induces a natural smooth isomorphism of manifolds with boundary:

$$\phi' : \overline{B}^* X \setminus X \mapsto \overline{B}^* Y \setminus Y$$

defined by

$$\phi'(\lambda \oplus \xi) = (\lambda \oplus \phi(-\xi)) \text{ if } \xi \in T^*X \setminus X \text{ and } \lambda \geq 0,$$

$$\forall \lambda \in \mathbb{R}^{+*},\ \phi'(\lambda \oplus 0) = (\lambda \oplus 0).$$

By glueing $\overline{B}^*X$ and $\overline{B}^*Y$ along the boundary $\overline{B}^*X \setminus T^*X$ with the help of ϕ', we define the following smooth compact manifold M:

$$M = \overline{B}^*X \cup_{\phi'} \overline{B}^*Y. \tag{6}$$

Let $\Pi_X : \overline{B}^*X \to X$ be the projection map. We denote by Ξ^X the set of smooth vectors fields of $\overline{B}^*X$ which are tangent to all the submanifolds $\Pi_X^{-1}(x) \cap (\overline{B}^*X \setminus T^*X)$, $x \in X$. Let $(x,\xi) = (x_1, \ldots, x_n; \xi_1, \ldots, \xi_n)$ be a local chart of T^*X and (ρ, θ) be the polar coordinates: $\rho = ||\xi||$, $\theta = \frac{\xi}{||\xi||}$, where $||\cdot||$ denotes the Euclidean norm of T^*X. Then a local chart of $\overline{B}^*X$ near $\overline{B}^*X \setminus T^*X$ is given by

$$\left(x_1, \ldots, x_n; t = \frac{1}{\rho}, \theta = (\theta_1, \ldots, \theta_{n-1})\right) \; t \geq 0, \; \theta \in S^{n-1}. \tag{7}$$

In this local chart, Ξ^X is generated by the vector fields $t\frac{\partial}{\partial x_j}$, $t\frac{\partial}{\partial t}$, $\frac{\partial}{\partial \theta_l}$, where $1 \leq j \leq n$, $1 \leq l \leq n-1$. We will make several applications of the following lemma. It is easily proved using polar coordinates.

Lemma 1. *The vector fields* $\frac{\partial}{\partial \xi_j}$ $(1 \leq j \leq n)$ *belong to the* $C^\infty(\overline{B}^*X)$*-module* $t\Xi^X$ *generated by* $t^2\frac{\partial}{\partial t}$, $t\frac{\partial}{\partial \theta_l}$ $(1 \leq l \leq n-1)$.

Moreover we observe that the set of classical symbols of order zero on T^*X is nothing else but $C^\infty(\overline{B}^*X)$.

Before we continue, let us recall the definition of a symplectic Lie algebroid (see for instance [8], [10]).

Definition 1.

1) A symplectic Lie algebroid on M is a quadruple $(\mathcal{E}, \rho, [\,,\,], \omega)$ on M, where $\mathcal{E}$ is a smooth vector bundle on M, $[\,,\,]$ is a Lie algebra structure on the sheaf of sections of $\mathcal{E}$, ρ is a smooth map of vector bundles,

$$\rho : \mathcal{E} \to TM,$$

such that the induced map

$$\Gamma(\rho) : C^\infty(M, \mathcal{E}) \to C^\infty(M, TM)$$

is a Lie algebra homomorphism and, for any sections σ and τ of $\mathcal{E}$ and any smooth function f on M, the following identity holds:

$$[\,\sigma, f\tau\,] = \rho(\sigma)(f).\tau + f[\sigma, \tau].$$

Lastly, ω is a closed $\mathcal{E}$-two form on M such that the associated linear map:

$$C^\infty(M, \mathcal{E}) \times C^\infty(M, \mathcal{E}) \ni (U, V) \mapsto \omega(U, V) \in C^\infty(M)$$

defines a symplectic structure on $\mathcal{E}$.

2) The ring of $\mathcal{E}$-differential operators is by definition the ring generated by smooth functions on M and smooth sections of $\mathcal{E}$.

3) We denote by ${}^{\mathcal{E}}\Omega^\bullet = C^\infty(M, \Lambda^\bullet \mathcal{E}^*)$ the set of smooth sections on M of the bundle of alternating multilinear forms on $\mathcal{E}$.

The following proposition will allow us to construct the relevant Lie algebroid $\mathcal{E}$ over M.

Proposition 1. *For any $p \in \overline{B}^* X$, we set*

$$\mathcal{E}_p^X = \frac{\Xi^X}{I_p \Xi^X}$$

where I_p is the set of smooth real-valued functions on $\overline{B}^* X$ which vanish at p. Then:

1) *$(\mathcal{E}_p^X)_{p \in \overline{B}^* X}$ form a smooth vector bundle, denoted $\mathcal{E}^X$, over $\overline{B}^* X$ such that the set of smooth sections over $\overline{B}^* X$ of $\mathcal{E}^X$ coincides with Ξ^X. If $U, V \in \Xi^X$ then the Lie bracket $[U, V]$ also belongs to Ξ^X.*

2) *The fundamental two-form $\omega^X (= \sum_{j=1}^n d\xi_j \wedge dx_j)$ of T^*X induces a smooth form, still denoted ω^X, in $C^\infty(\overline{B}^* X; {}^{\mathcal{E}^X}\Omega^2)$. Moreover, $(\mathcal{E}^X, [,], \omega^X)$ defines a symplectic Lie algebroid over $\overline{B}^* X$.*

Proof. We shall use the local coordinates (7).

1) Near a point $p^0 = (x^0; 0 = t, \theta^0)$, each vector field U of Ξ^X is of the form:

$$\sum_{j=1}^n A_j(x; t, \theta) t \frac{\partial}{\partial x_j} + A_{n+1}(x; t, \theta) t \frac{\partial}{\partial t} + \sum_{j=n+2}^{2n} A_j(x; t, \theta) t \frac{\partial}{\partial \theta_{j-n-1}}.$$

Then by considering the Taylor formulas:

$$\begin{aligned} A_j(x; t, \theta) = A_j(x^0; 0, \theta^0) &+ \sum_{k=1}^n (x_k - x_k^0) B_{j,k}(x; t, \theta) \\ &+ t C_{j,0}(x; t, \theta) + \sum_{l=1}^{n-1} (\theta_l - \theta_l^0) C_{j,l}(x; t, \theta) \end{aligned}$$

one sees immediately that $\mathcal{E}_p^X$ defines a smooth vector bundle over $\overline{B}^* X$ whose set of smooth sections coincides with Ξ^X. Next, working with the vector fields

$$t\frac{\partial}{\partial x_j},\, t\frac{\partial}{\partial t},\, \frac{\partial}{\partial \theta_l}$$

one checks immediately that for any $U, V \in \Xi^X$, $[U,V]$ is still in Ξ^X.

2) The fundamental form ω^X is not defined on the boundary of $T\overline{B}^* X$ but, using the polar coordinates (7), one checks easily for any $U, V \in \Xi^X$, $\omega^X(U,V)$ is well defined up to the boundary of $\overline{B}^* X$ ($= \{t = 0\}$). One then immediately gets the proposition. q.e.d.

Proposition 2. *The star product $\star_X$ on T^*X extends to a star product, still denoted $\star_X$, on $\overline{B}^* X$ such that for any $f, g \in C^\infty(\overline{B}^* X)$ we have:*

$$f \star_X g = fg + \sum_{n \geq 1} \hbar^n A^{(n)}(f\,,\, g)$$

where the $A^{(n)}$ are $\mathcal{E}^X$-bidifferential operators.

Proof. Let $(x, \xi) = (x_1, \ldots, x_n; \xi_1, \ldots, \xi_n)$ be a local chart of T^*X. Then for any $f, g \in C^\infty(\overline{B}^* X)$ and (x, ξ) in the domain of this local chart we have:

$$f \star_X g(x,\xi) = \sum_{\alpha,\beta \in \mathbb{N}^n,\, |\beta| \leq |\alpha|} \frac{\hbar^{|\alpha|}}{\alpha!}\, c_{\alpha,\beta}(x) D_\xi^\alpha f(x,\xi) \frac{\partial^\beta}{\partial^\beta x} g(x,\xi)$$

then, using the local coordinates (7) and Lemma 1, one gets easily all the results of the proposition. q.e.d.

Proposition 1 allows us to formulate the following definition:

Definition 2.

1) A smooth real-vector bundle $\mathcal{E}^Y$ over $\overline{B}^* Y$ is defined by setting $\mathcal{E}^Y_{|T^*Y} = T(T^*Y)$ and $\mathcal{E}^Y_{|\overline{B}^* Y \setminus Y} = \phi_* \left(\mathcal{E}^X_{|\overline{B}^* X \setminus X} \right)$.

2) By glueing $\mathcal{E}^X$ and $\mathcal{E}^Y$ along $\overline{B}^* X \setminus T^*X$ with the help of ϕ', one defines a smooth vector bundle $\mathcal{E}$ over M which is isomorphic to TM. A smooth exact differential form $\omega \in C^\infty(M; {}^{\mathcal{E}}\Omega^2)$ is defined by setting $\omega_{|\overline{B}^* X} = \omega^X$, $\omega_{|\overline{B}^* Y \setminus Y} = (\phi^{-1})^* \left(\omega^X_{|\overline{B}^* X \setminus X} \right)$ and $\omega_{|T^*Y} = \omega^Y$ (where ω^Y is the canonical two form of T^*Y).

3) The natural injection

$$C^\infty(M,\mathcal{E}) \to C^\infty(M,TM)$$

is induced by a bundle map $\rho : \mathcal{E} \to TM$ as in Proposition 1 and $(\mathcal{E},\rho,[,],\omega)$ defines a symplectic Lie algebroid which will be denoted $(\mathcal{E},[,],\omega)$ in the sequel.

3. Regularized index formula for a Fourier integral operator

Let C_ϕ be the graph of ϕ^{-1} in $(T^*Y\setminus Y)\times(T^*X\setminus X)$ and L_{C_ϕ} be the associated Maslov bundle over C_ϕ. We fix $\Phi : L^2\Big(X,\Omega_{\frac12}\Big) \to L^2\Big(Y,\Omega_{\frac12}\Big)$ an elliptic Fourier integral operator of order zero whose canonical relation is C_ϕ and whose principal symbol a is a unitary section of the bundle $\Omega_{\frac12}\otimes L_{C_\phi} \to C_\phi$: this means that a is homogeneous of degree zero (i.e., constant on each ray) and that $a\overline{a}\equiv 1$: see [15]. *We can, and will, assume in the sequel that* $\Phi\Phi^* - \mathrm{Id}$ *and* $\Phi^*\Phi - \mathrm{Id}$ *are smoothing*. As observed in [15] Φ is Fredholm, with index defined by $\operatorname{ind}\Phi = \dim\ker\Phi - \dim\operatorname{coker}\Phi$. In order to give a formula "via regularization" for $\operatorname{ind}\Phi$ we introduce the following algebra $\mathcal{A}$ which will have a "regularized" trace:

$$\mathcal{A} = \{(A,B)\in\Psi^0(X,\Omega_{\frac12})\times\Psi^0(Y,\Omega_{\frac12}) \mid A-\Phi^*B\Phi \text{ is smoothing}\}.$$

Proposition 3.

1) *The map* $\tau : \mathcal{A}\to\mathbb{C}$ *given by*

$$(8)\quad \forall (A,B)\in\mathcal{A},\ \tau(A,B) = \mathrm{Tr}(A-\Phi^*B\Phi) - \mathrm{Tr}(B(\mathrm{Id}-\Phi\Phi^*))$$

is a trace.

2) $\operatorname{ind}\Phi = \tau(\mathrm{Id},\mathrm{Id})$.

Proof. 1). If (A,B), $(A',B')\in\mathcal{A}$ are smoothing operators then one immediately checks that

$$(9)\quad \tau(\,(A,B)\cdot(A',B')\,) = \mathrm{Tr}(AA') - \mathrm{Tr}(BB') = \tau(\,(A',B')\cdot(A,B)\,).$$

Now if the operators (A,B), (A',B') are not smoothing one considers operators of the form $\left(A\circ\chi\left(\frac1N\sqrt{\Delta_X}\right), B\circ\chi\left(\frac1N\sqrt{\Delta_Y}\right)\right)$ where $\chi\in C^\infty(\mathbb{R},[0,1])$ is equal to 1 on $[-1,1]$ and to zero on $[-2,2]$. By applying

Equation (9) and letting N go to ∞ one immediately gets part 1). Part 2) means that

$$\operatorname{ind}\Phi = \operatorname{Tr}(\operatorname{Id} - \Phi^*\Phi) - \operatorname{Tr}(\operatorname{Id} - \Phi\Phi^*),$$

which is well-known. q.e.d.

Remark 3. $\tau(A, B)$ is a "regularization" of $\operatorname{Tr}A - \operatorname{Tr}B$.

4. Algebraization of a Fourier integral operator

We are going to use the following (deformed quantized algebra), where the manifold Z is equal to X or Y:

$$\mathbb{B}^\hbar(\overline{B}^*Z) = \frac{C^\infty(\overline{B}^*Z)}{C_0^\infty(\overline{B}^*Z)} \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]]$$

where $C_0^\infty(\overline{B}^*Z)$ denotes the set of smooth functions which vanish to infinite order to $\overline{B}^*Z \setminus T^*Z$. We observe that $\star_Z$ induces a star-product, still denoted $\star_Z$, on $\mathbb{B}^\hbar(\overline{B}^*Z)$.

Theorem 2.

1) *The map which to any $a \in S^0(T^*X)$ associates the asymptotic expansion at $\hbar = 0$ of $(\sigma(\Phi \operatorname{Op}(a_\hbar)\Phi^*))_{\hbar^{-1}}$ induces an algebra isomorphism $\widetilde{\Phi}$ from $(\mathbb{B}^\hbar(\overline{B}^*X), \star_X)$ onto $(\mathbb{B}^\hbar(\overline{B}^*Y), \star_Y)$.*

2) *For each $k \in \mathbb{N}^*$, there exists an $\mathcal{E}$-differential operator D_k on $\overline{B}^*X$ such that for any $a \in C^\infty(\overline{B}^*X)$ which is identically zero in a neighborhood of the zero section, we have the following identity:*

$$\widetilde{\Phi}(a) = (a + \sum_{k \geq 1} \hbar^k D_k(a)) \circ \phi^{-1}$$

in the vector space $\mathbb{B}^\hbar(\overline{B}^*Y)$.

Before proving this theorem we state the next proposition which is an easy consequence of Proposition 2 and Theorem 2:

Proposition 4. *The star product $\star_Y$ on T^*Y extends to a star product, still denoted $\star_Y$, on $\overline{B}^*Y$ such that for any $f, g \in C^\infty(\overline{B}^*Y)$ we have:*

$$f \star_Y g = fg + \sum_{n \geq 1} \hbar^n B^{(n)}(f\,,\,g)$$

where the $B^{(n)}$ are $\mathcal{E}^Y$-bidifferential operators.

Proof of Theorem 2. Let us first assume part 2). Then, using the results of Section 2.1 and the fact that $\Phi\Phi^* - \mathrm{Id}$ and $\Phi^*\Phi - \mathrm{Id}$ are smoothing, one proves easily that $\widetilde{\Phi}$ is an isomorphism whose inverse is given by:

$$b \in S^0(T^*Y) \to (\sigma(\Phi^*\mathrm{Op}(b_\hbar)\Phi))_{\hbar^{-1}}.$$

Now let us prove part 2). Following [7] page 26, we recall that the Schwartz kernel of Φ is the finite sum of a smooth function and of oscillatory integrals (supported in small coordinates charts) of the following type:

$$K(y,x) = \int_{\mathbb{R}^n} e^{i(\varphi(y,\eta)-x\cdot\eta)}\, b(y,\eta)d\eta \tag{10}$$

where $b(y,\eta) \in S^0(T^*Y)$ vanishes for $||\eta|| \leq 1$, $\varphi(y,\eta)$ is an homogeneous phase function parametrizing locally the graph C_ϕ of ϕ^{-1} which satisfies $\det \frac{\partial^2\varphi}{\partial y\partial\eta} \neq 0$ so that locally we have:

$$\{(y, \varphi'_y(y,\eta);\, \varphi'_\eta(y,\eta),\eta)\} = C_\phi$$

and $\phi^{-1}(y,\varphi'_y(y,\eta)) = (\varphi'_\eta(y,\eta),\eta)$. Notice moreover that $(y,\eta) \to (y,\varphi'_y(y,\eta))$ and $(y,\eta) \to (\varphi'_\eta(y,\eta),\eta)$ are local diffeomorphisms.

With these notations, the Schwartz kernel of Φ^* is the finite sum of a smooth function and of oscillatory integrals (supported in small coordinates charts) of the following type:

$$K^*(x,y) = \int_{\mathbb{R}^n} e^{-i(\varphi(y,\eta)-x\cdot\eta)}\, b_1(y,\eta)d\eta. \tag{11}$$

Let $a \in C^\infty(\overline{B}^*X)$ which is identically zero in a neighborhood of the zero section. Recall that our goal is to study the operator $\Phi\circ\mathrm{Op}(a_\hbar)\circ\Phi^*$ and its complete symbol. We shall first analyze $\Phi\circ\mathrm{Op}(a_\hbar)$, to this aim it is enough to study the operator $K\circ\mathrm{Op}(a_\hbar)$ where K denotes the operator whose Schwartz kernel is given by (10).

The Schwartz kernel of $K\circ\mathrm{Op}(a_\hbar)$ is given by

$$T(y,z) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(\varphi(y,\eta)-x\cdot\eta)}\, b(y,\eta)\, a(x,\hbar\xi) e^{i(x-z)\cdot\xi} d\xi dx d\eta.$$

In this integral we replace $a(x,\hbar\xi)$ by its Taylor expansion:

$$\sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\partial_x^\alpha a(z,\hbar\xi)(x-z)^\alpha.$$

Using the following two identities

$$(x-z)^\alpha e^{i(x-z)\cdot\xi} = D_\xi^\alpha(e^{i(x-z)\cdot\xi})$$
$$\int_{\mathbb{R}^n} e^{ix\cdot(\xi-\eta)}dx = (2\pi)^n\delta_{\xi=\eta}$$

and integrating by parts we see that $T(y,z)$ is the sum of a smooth function and of $H(y,z) =$

$$\int\int\int e^{i(\varphi(y,\eta)-x\cdot\eta)}\, b(y,\eta)\sum_{\alpha\in\mathbb{N}^n}\frac{1}{\alpha!}(-\hbar)^{|\alpha|}\partial_x^\alpha D_\xi^\alpha a(z,\hbar\xi)e^{i(x-z)\cdot\xi}d\xi dx d\eta$$
$$= (2\pi)^n\int_{\mathbb{R}^n} e^{i(\varphi(y,\eta)-z\cdot\eta)}\, b(y,\eta)\sum_{\alpha\in\mathbb{N}^n}\frac{1}{\alpha!}(-\hbar)^{|\alpha|}\partial_x^\alpha D_\xi^\alpha a(z,\hbar\eta)d\eta.$$

Now for $\alpha\in\mathbb{N}^n$ we set

$$c_\alpha(z;\hbar\eta) = \partial_x^\alpha D_\xi^\alpha a(z,\hbar\eta)$$

and we consider

$$H_\alpha(y,z) = \int_{\mathbb{R}^n} e^{i(\varphi(y,\eta)-z\cdot\eta)}\, b(y,\eta)\, c_\alpha(z,\hbar\eta)d\eta.$$

If we replace $c_\alpha(z,\hbar\eta)$ by its Taylor expansion

$$\sum_{\beta\in\mathbb{N}^n}\frac{1}{\beta!}\partial_z^\beta c_\alpha(\varphi'_\eta;\hbar\eta)(z-\varphi'_\eta)^\beta$$

then, using integrations by parts as above and the formula

$$\partial_\eta^\beta\left(e^{i(\varphi(y,\eta)-z\cdot\eta)}\right) = (-1)^{|\beta|}(z-\varphi'_\eta)^\beta e^{i(\varphi(y,\eta)-z\cdot\eta)},$$

it follows easily that $H_\alpha(y,z)$ is the sum of a smooth function and of

$$\int_{\mathbb{R}^n} e^{i(\varphi(y,\eta)-z\cdot\eta)}\sum_{\beta\in\mathbb{N}^n}\frac{1}{\beta!}D_\eta^\beta\Big(\, b(y,\eta)\,\partial_z^\beta c_\alpha(\varphi'_\eta;\hbar\eta)\,\Big)d\eta.$$

We observe that if we apply the Leibniz rule for the term $D_\eta^\beta\Big(....\Big)$ in the previous integral then the following differential operators will appear

$$D_\eta^{\beta-\gamma}b(y,\eta)\, D_\eta^{\gamma-\gamma'}(\varphi'_\eta)D_\eta^{\gamma'}\partial_z^\beta. \tag{12}$$

It is clear from Lemma 1 that, expressed in the coordinates $(\varphi'_\eta(y,\eta),\, \eta)$, these differential operators (12) are $\mathcal{E}$-differential operators. Therefore we have just proved that $T(y,z)$ is the sum of a smooth function and of:

$$\int_{\mathbb{R}^n} e^{i(\varphi(y,\eta)-z\cdot\eta)} \sum_{k\in\mathbb{N}} \hbar^k P_k(a)(\varphi'_\eta(y,\eta),\hbar\eta)d\eta$$

where the P_k are $\mathcal{E}$-differential operators.

Now we recall that the Schwartz kernel of Φ^* is the finite sum of a smooth function and of terms of the type (11). So in order to study the complete symbol of $\Phi \circ \mathrm{Op}(a_\hbar) \circ \Phi^*$ it is enough to study the operator $K \circ \mathrm{Op}(a_\hbar) \circ K^*$ whose Schwartz kernel is the finite sum of a smooth function and of integrals of the type

$$\begin{aligned}&\int\int\int e^{i(\varphi(y,\eta)-x\cdot\eta)} e^{-i(\varphi(y',\eta')-x\cdot\eta')} P_k(a)(\varphi'_\eta(y,\eta),\hbar\eta)b_1(y',\eta')dxd\eta' d\eta \\ &= (2\pi)^n \int_{\mathbb{R}^n} e^{i(\varphi(y,\eta)-\varphi(y',\eta)} P_k(a)(\varphi'_\eta(y,\eta),\hbar\eta)b_1(y',\eta)d\eta.\end{aligned} \tag{13}$$

Moreover we can write $\varphi(y,\eta) - \varphi(y',\eta) = (y-y').\hat{\eta}(y,y',\eta)$ where $\hat{\eta}(y,y,\eta) = \varphi'_y(y,\eta)$ and we can assume (at the expense of shrinking the local coordinates charts) that $\eta \to \hat{\eta}(y,y',\eta)$ is a local diffeomorphism whose inverse is denoted $\hat{\eta} \to \eta(y,y',\hat{\eta})$. With these notations, we set:

$$A_k(y,y',\hbar,\hat{\eta}) = P_k(a)(\varphi'_\eta(y,\eta)\,,\,\hbar\eta)b_1(y',\eta).$$

Then a change of variable formula allows us to see that the oscillatory integral (13) is equal to

$$(2\pi)^n \int_{\mathbb{R}^n} e^{i(y-y')\cdot\hat{\eta}}\, A_k(y,y',\hbar,\hat{\eta}) \left|\frac{D\eta}{D\hat{\eta}}\right| d\hat{\eta}.$$

We observe that, expressed in the coordinates $(\varphi'_\eta(y,\eta),\, \eta)$, the vector fields $\partial_\eta(\varphi'_\eta)\partial_y$ are $\mathcal{E}$-differential operators. Therefore one proves easily the assertion of part 2) of the Theorem by replacing $A_k(y,y',\hbar,\hat{\eta})$ by its Taylor expansion

$$\sum_{\beta\in\mathbb{N}^n} \frac{1}{\beta!}\partial^\beta_{y'} A_k(y,y,\hbar,\hat{\eta})_{|y=y'}\,(y'-y)^\beta$$

and using, as before, integration by parts. q.e.d.

5. The formal deformation and traces on $\overline{B}^* X \cup_\phi \overline{B}^* Y$ and regularized traces on ΨDO's

Recall first that $C^\infty(M)$ is exactly the set of functions $(f, g) \in C^\infty(\overline{B}^* X) \times C^\infty(\overline{B}^* Y)$ such that $f - g \circ \phi'$ vanish of infinite order to the boundary of $\overline{B}^* X$.

We are going to use the $\star$-products denoted $\star_X, \star_Y$ on $\overline{B}^* X$ and $\overline{B}^* Y$ defined in Propositions 2 and 4. We set $\mathbb{A}^\hbar(\overline{B}^* X) = C^\infty(\overline{B}^* X) \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]]$ and $\mathbb{A}^\hbar(\overline{B}^* Y) = C^\infty(\overline{B}^* Y) \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]]$.

Let $\mathbb{A}^\hbar(M)$ be the vector space given by

$$\{(a, b) \in \mathbb{A}^\hbar(\overline{B}^* X) \times \mathbb{A}^\hbar(\overline{B}^* Y) \mid \widetilde{\Phi}(\overline{a}) = \bar{b}\}$$

where $\overline{a}$ (resp. $\bar{b}$) denotes an element of $\mathbb{B}^\hbar(\overline{B}^* X)$ (resp. $\mathbb{B}^\hbar(\overline{B}^* Y)$) induced by a (resp. b). Theorem 2 shows that $\mathbb{A}^\hbar(M)$ is an algebra with respect to the diagonal product: $(\star_X, \star_Y)$. In particular, pairs of the form $(\sigma(\Phi^*\Phi), \sigma(\Phi\Phi^*))$ belong to $\mathbb{A}^\hbar(M)$.

In the statement of the next proposition we will use the notations of Theorem 2.

Proposition 5.

1) *Let $\chi \in C^\infty(T^* X, [0,1])$ be such that $\chi(x, \xi) = 0$ for $||\xi|| \leq 1/2$ and $\chi(x, \xi) = 1$ for $||\xi|| \geq 1$. For any $f \in C^\infty(\overline{B}^* X)$ we have*

$$\widetilde{\Phi}(\chi f) - (\chi f) \circ \phi^{-1} \in \hbar \mathbb{B}^\hbar(\overline{B}^* Y).$$

2) *For each $b \in C^\infty(\overline{B}^* Y)$ one defines $b^- \in C^\infty(\overline{B}^* Y)$ by setting $b^-(\eta) = b(-\eta)$ for any $\eta \in \overline{B}^* Y$. Then the formula*

$$\forall (a, b) \in \mathbb{A}^\hbar(M),\ \mathcal{U}(a, b) = \left(a + \sum_{k \geq 1} \hbar^k D_k(a), b^- \right) \in C^\infty(M) \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]]$$

defines a $\mathbb{C}[[\hbar]]$-linear isomorphism $\mathcal{U}$ from $\mathbb{A}^\hbar(M)$ to $C^\infty(M) \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]]$.

3) *The product $\mathcal{U}(*_X, *_Y)$ defines an $\mathcal{E}$-deformation of M (or a star product) associated to the symplectic Lie algebroid $(\mathcal{E}, [,], \omega)$ (see [10] Section 3.3).*

Proof. Part 1) is a consequence of the theorem of Egoroff (see [7]). Part 2) is an easy consequence of Theorem 2. Part 3) is an easy consequence of 2) and of Theorem 2 (2). q.e.d.

Definition 3.

- $\mathbb{A}^\hbar(M)$ denotes the formal deformation of M associated to the symplectic Lie algebroid $(\mathcal{E}, [,], \omega)$ constructed in Proposition 5.

- The linear functional

$$\tau_{\mathrm{can}} : \mathbb{A}^\hbar(M) \to \mathbb{C}[\hbar^{-1}, \hbar]]$$

is given by

$$\forall (a,b) \in \mathbb{A}^\hbar(M), \tag{14}$$
$$\tau_{\mathrm{can}}(a,b) = \left\{ \begin{array}{c} \text{asymptotic expansion at } \hbar = 0 \text{ of} \\ \hbar \mapsto \tau(\mathrm{Op}(a_\hbar),\, \mathrm{Op}(b_\hbar)) \end{array} \right\}$$

where τ is the trace defined in Proposition 3. It follows immediately from the definition that τ_{can} is a trace.

Computation of τ.

Since the space of traces on $\mathbb{A}^\hbar(M)$ may be very big we introduce the following algebra:

$$\mathbb{D}^\hbar(M) = \mathbb{A}^\hbar(M)\left[\left(\chi\,||\xi||,\, (\chi\,||\xi|| + \sum_{k\geq 1} \hbar^k D_k(\chi\,||\xi||)) \circ \phi^{-1} \right)\right].$$

Another way of describing $\mathbb{D}^\hbar(M)$ is given by glueing from

$$\widetilde{\phi} : \mathbb{A}^\hbar(\overline{B}^* X)[\chi||\xi||] \to \mathbb{A}^\hbar(\overline{B}^* Y)[\chi||\xi||]$$

where χ is as in Proposition 5. It is easily seen that τ_{can} defines, by the same formula as (14), a trace on $\mathbb{D}^\hbar(M)$.

Next proposition describes the space of traces on $\mathbb{D}^\hbar(M)$.

Proposition 6 *The space of traces with values in $\mathbb{C}[\hbar^{-1}, \hbar]]$ on the algebra $\mathbb{D}^\hbar(M)$ is two dimensional over $\mathbb{C}[\hbar^{-1}, \hbar]]$. A basis is given by $(\tau_{\mathrm{can}}, \tau_1)$ where for any $(a,b) \in \mathbb{D}^\hbar(M)$ $\tau_1(a,b) = \mathrm{Res}_W(a)$. Here Res_W denotes Wodzicki's noncommutative residue.*

Proof. For $Z = X$ or Y we set

$$C_0^\infty(\overline{B}^* Z) \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]] \;=\; \mathbb{A}_0^\hbar(T^* Z)$$

where $C_0^\infty(\overline{B}^* Z)$ denotes the set of smooth functions which vanish of infinite order at $\overline{B}^* Z \setminus T^* Z$. Then we have the following exact sequence of $\mathbb{C}[[\hbar]]$-algebras:

$$0 \to \mathbb{A}_0^\hbar(T^* X) \oplus \mathbb{A}_0^\hbar(T^* Y) \to \mathbb{D}^\hbar(M) \to \mathcal{T}(M) \to 0.$$

Here $\mathcal{T}(M)$ denotes the induced formal $\mathcal{E}$-deformation of the algebra of transversal Laurent series with coefficients given by smooth functions on the sphere at infinity. A more direct construction of this deformation may be described as follows. Let $\mathcal{P}^i$ denote the space of pseudodifferential operators on, say X, of order $\leq i$ modulo the smoothing operators. The space $\mathcal{S}P$ of doubly infinite sequences

$$\{P_i\}_{i \in \mathbb{Z}}, \;\; P_i \in \mathcal{P}^i, \;\text{ and there exists } i_0 \text{ such that } P_i \in \mathcal{P}^{i_0} \text{ for } i \text{ large}$$

is a flat module over $\mathbb{C}[[\hbar]]$, where the multiplication by $\hbar$ acts as the right translation. We endow $\mathcal{S}P$ with the product

$$\{P_i\}_{i \in \mathbb{Z}} \{Q_i\}_{i \in \mathbb{Z}} = \left\{ \sum_{i+j=n} P_i Q_j \right\}_{n \in \mathbb{Z}}.$$

Note that, given the particular sequences $\{P_i\}_i$ and $\{Q_i\}_i$, there exists N such that all the pseudodifferential operators involved have order bounded by N, hence the doubly infinite sums $\sum_{i+j=n} P_i Q_j$ make sense. In fact, given n, the top order of operators appearing as products $P_i Q_{n-i}$ are at most 2N, and given any integer $k \leq 2N$, there are at most finitely many terms in the sum of the order k.

The space $\mathcal{S}P$ is immediately seen to be isomorphic to $\mathcal{T}(M)$. Any trace τ on $\mathcal{T}(M)$ is given by a sequence of $\mathbb{C}$-linear, $\mathbb{C}[[\hbar]]$-valued functionals τ_n on $\mathcal{P}^n$ such that

$$\tau(\{P_i\}) = \sum \tau_i(P_i).$$

The $\hbar$-linearity of τ implies that $\tau_{i+1} = \hbar \tau_i$ and the trace condition on τ implies that each τ_i is a trace on the algebra of pseudodifferential operators modulo the smoothing operators. Recall that, on this latter algebra, the Wodzicki residue *res* is the unique trace up to multiplicative constant. Thus τ is, up to multiplicative constant, uniquely determined by $\tau_{-n} = res$, and hence the space of traces on $\mathcal{T}(M)$ is one-dimensional.

We recall that $\mathbb{A}_0^\hbar(T^*X)$ is H-unital (in the sense of Wodzicki, see [17]), so we have the following long exact sequence in cyclic cohomology:

$$0 \to HC^0(\mathcal{T}(M)) \to HC^0(\mathbb{D}^\hbar(M)) \to HC^0(\, \mathbb{A}_0^\hbar(T^*X) \oplus \mathbb{A}_0^\hbar(T^*Y)\,) \to HC^1(\mathcal{T}(M)) \to \cdots .$$

From Section 2.1 we recall that the space of $\mathbb{C}[[\hbar]]$-linear traces (with values in $\mathbb{C}[\hbar^{-1}, \hbar]]$) on $\mathbb{A}_0^\hbar(T^*X)$ is one dimensional and generated by $\mathrm{Tr}_{\mathrm{can}}^X$. By above, $HC^0(\mathcal{T}(M))$ is one-dimensional. The connecting map

$$\delta :\, HC^0(\, \mathbb{A}_0^\hbar(T^*X) \oplus \mathbb{A}_0^\hbar(T^*Y)\,) \to HC^1(\mathcal{T}(M))$$

is given by taking a trace on $\mathbb{A}_0^\hbar(T^*X) \oplus \mathbb{A}_0^\hbar(T^*Y)$, extending it to a linear functional on $\mathbb{D}^\hbar(M)$ and taking its Hochschild boundary. In particular, it is not zero because V. Nistor has shown in [13] that the nonvanishing of this Hochschild boundary is equivalent to the existence of a pseudodifferential operator with nonzero index!. This implies that $HC^0(\mathbb{D}^\hbar(M))$ is either one or two dimensional. Since, with the notations of the Proposition, $\tau_{\mathrm{can}}, \tau_1$ are two linearly independent elements of the vector space of traces on $\mathbb{D}^\hbar(M)$, the rest of the statement of Proposition 6 follows. q.e.d.

6. The algebraic index theorem for the Lie algebroid $\mathcal{E}$

The following algebraic index theorem is proved in [10] (see Theorem 6.1) and is an extension to the symplectic Lie algebroid $(\mathcal{E}, [,], \omega)$ of the Riemann Roch theorem (on symplectic manifolds) for periodic cochains of [1], [9]. We recall that the set of $\mathcal{E}$-differential forms ${}^{\mathcal{E}}\Omega^*$ is endowed with a de Rham differential ${}^{\mathcal{E}}d$ (see Definition 2.2 in [10]) so that there is an associated de Rham cohomology ${}^{\mathcal{E}}H^*(M)$.

Theorem 3. *The following diagram is commutative:*

$$\begin{tikzcd}
CC_*^{\mathrm{per}}(\mathbb{A}^\hbar(M)) \arrow[r, "\sigma"] \arrow[dr, "\mu^\hbar"'] & CC_*^{\mathrm{per}}(C^\infty(M)) \arrow[d, "\mu\cup\hat{A}(M)\cup e^\theta"] \\
 & (\,{}^{\mathcal{E}}\Omega^{2n-*}(M)[\hbar^{-1}, \hbar]], d\,)
\end{tikzcd}$$

where σ is the specialization map at $\hbar = 0$, μ is the transposed of the map constructed by A. Connes (*see Lemma* 45 *of* [3]) *in order to identify* *the continuous Hochschild cohomology groups of $C^\infty(M)$ with the space of de Rham currents, $\mu^\hbar$ is the trace density map defined in Section 6 of [10] and $\theta = \frac{1}{\sqrt{-1}\hbar}\omega + \sum_{k\geq 0}\hbar^k\theta_k \in {}^{\mathcal{E}}H^2(M, \mathbb{C}[[\hbar]])$ is the characteristic class of the deformation of the symplectic Lie algebroid $(\mathcal{E}, [,], \omega)$ ([10]).*

Remark 4.

1) The first index theorem of algebraic index type has been proved by Fedosov (see [5] page 189) for a symplectic manifold N (and Lie algebroid TN), it computes the index as an element of $\mathbb{C}[\hbar^{-1}, \hbar]]$. Theorem 3 is a far reaching generalization of Fedosov's theorem since it involves higher cyclic cocycles. In fact, the next proposition and theorem implicitly show how Theorem 3 implies Fedosov's index theorem.

2) In the case of a regular affine algebra A, the analogue of the above map μ had been constructed by Hochschild-Kostant-Rosenberg (see [6]) in order to identify the Hochschild homology groups of A with spaces of differential forms.

The natural injection $\mathbb{A}^\hbar(M) \to \mathbb{D}^\hbar(M)$ induces a natural map

$$CC_*^{\mathrm{per}}(\mathbb{A}^\hbar(M)) \to CC_*^{\mathrm{per}}(\mathbb{D}^\hbar(M)).$$

Since the traces τ_{can} and τ_1 of Proposition 6, and the trace density map $\mu^\hbar$ extend to $CC_*^{\mathrm{per}}(\mathbb{D}^\hbar(M))$, they can be identified using the following result.

Proposition 7.

1) *The $\mathbb{C}$-vector space ${}^{\mathcal{E}}H^{2n}(M, \mathbb{C})$ is two-dimensional. The vector space of $\mathbb{C}$-linear forms on ${}^{\mathcal{E}}\Omega^{2n}(M)$ which vanish on the range of the $\mathcal{E}$-exterior derivative ${}^{\mathcal{E}}d$ admits a unique linear basis $({}^{\mathrm{reg}}\!\int, \int_1)$, characterized by the following properties:*

   For any $(\alpha, \beta) \in {}^{\mathcal{E}}\Omega^{2n}(M)$ such that α (resp. β) is zero in a neighborhood of the boundary of $\overline{B}^*X$ (resp. $\overline{B}^*Y$)

$$^{\mathrm{reg}}\!\!\int(\alpha, \beta) = \int_{T^*X}\alpha - \int_{T^*Y}\beta, \quad \int_1(\alpha, \beta) = 0.$$

   Moreover $\int_1 \circ \mu^\hbar = \mathrm{Res}_W$.

2) *There exists a constant C such that*

$$\tau_{\mathrm{can}} = {}^{\mathrm{reg}}\!\!\int \circ\, \mu^\hbar + C\int_1 \circ \mu^\hbar.$$

Moreover $\int_1 \circ \mu^\hbar(1,1) = 0$ *and for any* $(a,b) \in \mathbb{D}^\hbar(M)$ *such that* a *is zero in a neighborhood of the boundary of* $\overline{B}^* X$, $\int_1 \circ \mu^\hbar(a,b) = 0$.

Proof.

1) A standard Mayer-Vietoris sequence argument shows that ${}^{\mathcal{E}}H^{2n}(M, \mathbb{C})$ is indeed two dimensional. The fact that $({}^{\mathrm{reg}}\!\int, \int_1)$ defines a basis is left to the reader.

2) This is an easy consequence from part 1) and of the properties (see [9], [10]) of the trace density map $\mu^\hbar$. To begin with, ${}^{\mathrm{reg}}\!\int \circ \mu^\hbar$ and τ_{can} coincide on $\mathbb{A}_0^\hbar(T^*X)$, while res_W vanishes there. In particular ${}^{\mathrm{reg}}\!\int \circ \mu^\hbar$ is not proportional to res_W and, since the space of traces on $\mathbb{D}^\hbar(M)$ is two dimensional,

$$\tau_{\mathrm{can}} = {}^{\mathrm{reg}}\!\int \circ \mu^\hbar + C \int_1 \circ \mu^\hbar$$

holds. Since $res_W(f)$ depends only on the jet of f at the cosphere at infinity, and here it is given by the Wodzicki residue of the corresponding pseudodifferential operator, the rest of the statement follows immediately. q.e.d.

7. Local formula for the index of a Fourier integral operator

Theorem 4.

- *Let* Φ *be a Fourier integral operator and* $\mathbb{A}^\hbar(M)$ *the formal deformation of* M *associated to it as in Definition* 3. *Then*

$$\operatorname{ind} \Phi = \int_M \mathrm{e}^{\theta_0} \hat{A}(M),$$

  where θ_0 *denotes the characteristic class of the deformation of the Lie Algebroid* $(\mathcal{E}, [\,,\,], \omega)$ *given by* $\mathbb{A}^\hbar(M)$.

- *Let* ∇_X *be a connection* ∇_X *on the tangent bundle* $T(\overline{B}^* X)$ *and* $\hat{A}(T^*X)$ *an associated representative form of the* $\hat{\mathrm{A}}$*-class of* ∇_X. *The symplectomorphism* ϕ *induces a connection* $\phi_*(\nabla_X)$ *on the tangent space of* $\overline{B}^* Y \setminus B^*(Y)$. *Let* ∇_Y *denote its extension to a connection of* $T(\overline{B}^* Y)$ *and* $\hat{A}(T^*Y)$ *an associated representative differential form of the* $\hat{\mathrm{A}}$*-class of* ∇_Y. *Then*

$$\operatorname{ind} \Phi = \int_{B^*(X)} \mathrm{e}^{\theta_0} \hat{A}(T^*X) - \int_{B^*(Y)} \mathrm{e}^{\theta_0} \hat{A}(T^*Y).$$

Proof. Proposition 3 shows that

$$\operatorname{ind}\Phi \;=\; \tau(\mathrm{Id},\mathrm{Id}) \;=\; \tau_{\mathrm{can}}(1,1).$$

Proposition 7 (2) shows that

$$\tau_{\mathrm{can}}(1,1) = {}^{\mathrm{reg}}\!\!\int \circ\, \mu^{\hbar}(1,1).$$

The result follows then from Theorem 3 by letting $\hbar \to 0^+$, provided that the involved characteristic classes of vector bundles on M are in fact standard de Rham cohomology classes. We will prove this below and see that the regularized integral coincides with the orientation class of M. q.e.d.

The previous formula shows that if ϕ extends as a symplectomorphism $T^*X \to T^*Y$ up to the zero section then $\operatorname{ind}\Phi = 0$.

For a deformation associated with a Fourier integral operator (as in Proposition 5) the characteristic class θ of Theorem 3 is in fact of the form

$$\theta = \frac{1}{\sqrt{-1}\hbar}\omega + \theta_0$$

where $\theta_0 \in H^2(M,\mathbb{C})$ is a closed differential form (not only an $\mathcal{E}$-differential form). In order to do this and to identify the relevant characteristic class we will give below a slightly nonstandard description of a formal deformation.

7.1 General construction of the characteristic class of a formal deformation

Let us start with some notation.

Let $\mathbb{A}^{\hbar}$ denote the Weyl algebra of the symplectic vector space $\mathbb{R}^{2n}$ with the standard symplectic structure, i.e., the algebra generated by the vectors $\hat{x}_l, \hat{\xi}_l$ $(1 \le l \le n)$ satisfying the relations $[\hat{\xi}_k, \hat{x}_l] = \sqrt{-1}\hbar\delta_{k,l}$. The algebra $\mathbb{A}^{\hbar}$ is completed in the topology associated to the ideal generated by $\{\hat{x}_l, \hat{\xi}_l, \hbar;\ 1 \le l \le n\}$ and has the grading induced by

$$\deg\,\hat{x}_l = \deg\,\hat{\xi}_l = 1, \quad \deg\,\hbar = 2.$$

The corresponding Lie algebra $\frac{1}{\hbar}\mathbb{A}^{\hbar}$ will be denoted by $\widetilde{\mathfrak{g}}$. We set

$$\mathfrak{g} = \mathrm{Der}\,(\mathbb{A}^{\hbar}) = \widetilde{\mathfrak{g}}/\mathrm{center}$$

and

$$G = \mathrm{Aut}(\mathbb{A}^{\hbar}) = \exp(\mathfrak{g}_{\geq 0}).$$

We set

$$\widetilde{G} = \left\{ g \in \frac{1}{\hbar}\mathbb{A}^{\hbar} \mid g \in \mathfrak{sp}(2n, \mathbb{R}) \bmod \mathfrak{g}_{\geq 1} \right\}$$

and will endow it with the group structure coming from the exponential map. Note that $\widetilde{G}$ is an extension of G associated to the (Lie algebra) central extension $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$.

We endow the bundle $\mathbb{R}^{2n} \times \mathbb{A}^{\hbar}$ with the obvious fiber-wise action of $\widetilde{G}$ and with the $\widetilde{g}$-valued (Fedosov) connection

$$\widetilde{\nabla}^0 = \sum_{l=1}^{n} \left(d\xi_l \left(\partial_{\xi_l} - \frac{1}{\sqrt{-1}\hbar} \hat{x}_l \right) + dx_l \left(\partial_{x_l} + \frac{1}{\sqrt{-1}\hbar} \hat{\xi}_l \right) \right)$$

where $(dx_1, \ldots, dx_n; d\xi_1, \ldots, d\xi_n)$ denote the local (usual) dual basis of $T^*\mathbb{R}^n$. Let us recall (see Section 2.2) that a local chart of $\overline{B}^*\mathbb{R}^n$ near $\overline{B}^*\mathbb{R}^n \setminus T^*\mathbb{R}^n$ is given by:

$$\left(x_1, \ldots, x_n; t = \frac{1}{||\xi||}, \theta = (\theta_1, \ldots \theta_{n-1}) \right) \; t \geq 0, \; \theta \in S^{n-1}. \tag{15}$$

By using the local coordinates (15) one checks easily that $\widetilde{\nabla}^0$ extends as an $\mathcal{E}^{\mathbb{R}^n}$-connection, still denoted $\widetilde{\nabla}^0$, of $\overline{B}^*\mathbb{R}^n \times \mathbb{A}^{\hbar}$.

The description given below of a formal deformation of a symplectic Lie algebroid structure on M is just the representation of the Fedosov construction in terms of the bundle of jets on M with the fiber-wise product structure induced by the *-product (which is isomorphic to Weyl bundle) .

Local description of the characteristic class θ of a formal deformation.

The deformation is described by a local (Darboux) cover $\{U_i\}_{i\in I}$ of (M, ω), a collection of functions $\{g_{i,j} : U_i \cap U_j \to \widetilde{G}\}$ and a collection of $\widetilde{\mathfrak{g}}$-valued $\mathcal{E}$-connections $\widetilde{\nabla}_i$ on $U_i \times \mathbb{A}^{\hbar}$ which, when expressed in terms of local Darboux coordinates $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ (resp (15)) if U_i does not meet (resp. meets) the boundary at infinity, are equal to $\widetilde{\nabla}^0$ modulo $\widetilde{\mathfrak{g}}_{\geq 1}$ and so that the three following conditions hold:

1) *The cocycle condition holds:*

$$g_{i,j}g_{j,i} = 1 \quad \text{and} \quad g_{i,j}g_{j,k} = g_{i,k} \quad \text{on} \quad U_i \cap U_j \cap U_k.$$

In particular $\{g_{i,j} : U_i \cap U_j \to \widetilde{G}\}$ *define a smooth bundle* $\mathcal{W}$ *of algebras over* M *with fiber isomorphic to* $\mathbb{A}^\hbar$ *and the structure group* $\widetilde{G}$.

2) *The local connections* $\widetilde{\nabla}_i$ *define a* $\widetilde{\mathfrak{g}}$*-valued connection* $\widetilde{\nabla}$ *on the bundle* $\mathcal{W}$*, i.e.,*

$$g_{i,j}\widetilde{\nabla}_j = \widetilde{\nabla}_i g_{i,j}.$$

3) *The induced* $\mathfrak{g}$*-valued connection* ∇ *on the bundle* $\mathcal{W}$ *is flat, i.e.,* $\theta = \widetilde{\nabla}^2$ *is a globally defined differential form on* M *with values in the center of* $\widetilde{\mathfrak{g}}$*, necessarily of the form*

$$\frac{1}{\sqrt{-1}\hbar}\omega + \theta_0 \quad \textit{where } \theta_0 \in \Omega^2(M, \mathbb{C}[[\hbar]]).$$

The algebra of ∇*-flat sections of* $\mathcal{W}$ *is a formal deformation of* (M, ω) *whose characteristic class is* θ.

7.2 Local canonical liftings

We endow $\mathbb{R}^{2n}$ with its canonical symplectic structure $\omega = \sum_{l=1}^n d\xi_l \wedge dx_l$. Given any smooth, $\mathbb{C}[[\hbar]]$-valued function H on $\mathbb{R}^{2n}$, we define the following sections of the bundle $\mathbb{R}^{2n} \times \mathbb{A}^\hbar$ (over $\mathbb{R}^{2n}$):

(1) $H_0 = H(x, \xi)|_{\hbar=0}$, $H_1 = \sum_{l=1}^n (\hat{x}_l \partial_{x_l} H_0 + \hat{\xi}_l \partial_{\xi_l} H_0)$.

(2) $\widetilde{H} = \sum \frac{\hat{x}^\alpha \hat{\xi}^\beta}{\alpha! \beta!} \partial_x^\alpha \partial_\xi^\beta H$.

Using those, we can associate to H the $\widetilde{\mathfrak{g}}$-lift of the Lie derivative $\mathcal{L}_{\{H,\ \}}$ given by

$$\mathcal{D}_H = \mathcal{L}_{\{H,\ \}} + \frac{1}{\hbar}\left(\widetilde{H} - H_0 - H_1 + \frac{1}{2}\hbar \sum_{l=1}^n \partial^2_{x_l, \xi_l} H_0 \right).$$

We can think of it as an element of the Lie algebra of the semidirect product of $C^\infty(\mathbb{R}^{2n}, \widetilde{G})$ by the pseudogroup of local diffeomorphisms of $\mathbb{R}^{2n}$. The $\mathcal{D}_H$'s form a Lie algebra, in fact

$$[\mathcal{D}_H, \mathcal{D}_K] = \mathcal{D}_{\frac{1}{\hbar}(H*K - K*H)}, \tag{16}$$

and they satisfy

$$[\mathcal{D}_H, \widetilde{\nabla}^0] = -\frac{1}{2} d \left(\sum_{l=1}^{n} \partial^2_{x_l,\xi_l} H_0 \right). \tag{17}$$

We will also have an occasion to use

$$\mathcal{D}^0_H = \mathcal{L}_{\{H,\ \}} + \frac{1}{\hbar}(\,\widetilde{H} - H_0 - H_1\,), \tag{18}$$

which satisfies

$$[\widetilde{\nabla}^0, \mathcal{D}^0_H] = 0. \tag{19}$$

The identities (16), (17) and (19) follow from the definition of the quantities involved (and a completely straighforward computation).

7.3 The cotangent bundle case

The deformation of T^*X associated to the sheaf of differential operators on X can be now described as follows.

Locally on a coordinate domain $U \subset X$ we use coordinates on U to give an explicit symplectomorphism

$$T^*U \to \mathbb{R}^{2n}$$

and use Weyl deformation of $\mathbb{R}^{2n}$ to construct the deformation of T^*U. This amounts to the choice of a ($\widetilde{\mathfrak{g}}$-valued) connection given in our local coordinates $(x_1, \ldots, x_n)$ on U and the induced local coordinates $(x_i, \xi_i)_{i=1,\ldots n}$ on $T^*U \simeq U \times \mathbb{R}^n$ by

$$\widetilde{\nabla}^0 = d + \frac{1}{\hbar} \sum_{l=1}^{n} (dx_l \hat{\xi}_l - d\xi_l \hat{x}_l).$$

The infinitesimal change of coordinates on U is given by a vector field of the form $\sum_{l=1}^n X_l \partial_{x_l}$ and the associated infinitesimal symplectomorphism of T^*U is given by the Hamiltonian vector field $\{\sum_{l=1}^n X_l \xi_l, \cdot\}$.

It is immediate to see that the map

$$\sum_l X_l \partial_{x_l} \mapsto \mathcal{D}_{\sum_l X_l \xi_l}$$

is the Lie algebra homomorphism.

The associated local diffeomorphisms (coordinate changes) $\exp \sum_l X_l \partial_{x_l}$ lift to a local isomorphisms of the bundle $T^*U \times \mathbb{A}^\hbar$ given by $\exp \mathcal{D}_{\sum_l X_l \xi_l}$.

Given a local coordinate cover $\{U_i\}_{i \in I}$ of X it is now immediate to construct the associated $\widetilde{G}$-valued cocycle $\{g_{ij}\}$ glueing the bundles together. Note that, since $\mathcal{D}$'s do not commute with the connection $\widetilde{\widetilde{\nabla}}^0$, the corresponding collection of connections

$$\widetilde{\widetilde{\nabla}}_i = \widetilde{\widetilde{\nabla}}^0 \text{ in i'th coordinate system on } T^*U_i$$

do not glue together. But it is not difficult to check that (see for instance [5] page 260)

$$g_{ij} \widetilde{\widetilde{\nabla}}_i g_{ji} = \widetilde{\widetilde{\nabla}}_j + \frac{1}{2} d \log \det D g_{ij},$$

where Dg_{ij} is the induced action of g_{ij} on the tangent bundle.

The cocycle

$$\frac{1}{2} d \log \det D g_{ij} \tag{20}$$

is in $\check{C}^1(T^*X, \Omega^1(T^*X))$ and represents the pull back to T^*X of one half of the first Chern class of the complexified tangent bundle of X over X. Since this class vanishes (with coefficients in $\mathbb{R}$), we can find a collection α_i of closed one-forms on T^*X satisfying

$$\frac{1}{2} d \log \det D g_{ij} = \alpha_i - \alpha_j.$$

Setting

$$\nabla_i = \widetilde{\widetilde{\nabla}}_i + \alpha_i,$$

we get a globally defined $\widetilde{\mathfrak{g}}$-connection ∇ whose curvature, and hence also the characteristic class of the associated deformation, is represented in the complex $\check{C}^*(T^*X, \Omega^1(T^*X))$ by the cocycle $\frac{1}{2} d \log \det D g_{ij}$.

It is immediate that the deformation constructed in this way coincides with the one associated to the calculus of differential operators on X, while its jet at $\xi = \infty$ gives the deformation associated to the calculus of pseudodifferential operators on X.

If we recall that the real symplectic vector bundle $\mathcal{E}|_{T^*X}$ (see Definition 2) is the realification of a restriction of a complex vector bundle $\mathcal{E}_{\mathbb{C}}$ to T^*X, the characteristic class of this deformation can be described as the jet of $\frac{1}{2} c_1(\mathcal{E}_{\mathbb{C}}|_{T^*X})$ at $\xi = \infty$ and represented by the explicit cochain given by the formula (20).

7.4 The Lie algebroid case

Recall now that the Lie algebroid (on M) $(\mathcal{E}, [\,,\,], \omega)$ is given by glueing (at infinity) the two cotangent bundles (T^*X, ω^X) and (T^*Y, ω^Y) by the symplectomorphism ϕ'. To construct the deformation in this case, we will use the following data, whose existence follows immediately from the compactness of the co-sphere bundles of X and Y:

(1) A local coordinate cover $\{U_i\}_{i\in I}$ of X and an open relatively compact neighborhood U_X of the zero section in T^*X.

(2) A local coordinate cover $\{V_i\}_{i\in I}$ of Y and an open relatively compact neighborhood U_Y of the zero section in T^*Y.

(3) For each $i \in I$ we can choose local coordinates on U_i and V_i, so that both T^*U_i and T^*V_i become identified with open subsets of (the same) copy of $\mathbb{R}^{2n} = T^*\mathbb{R}^n$ with its standard symplectic structure and $\phi_i = \phi|_{T^*U_i \setminus U_X}$ is the induced symplectomorphism between two open subsets of $\mathbb{R}^{2n}$.

(4) For each $i \in I$ there exists a smooth real-valued function H_i on $\mathbb{R}^{2n} \times [0,1]$, which is one-homogeneous in the cotangent variables and such that ϕ_i can be obtained by integrating the (time dependent) hamiltonian flow $\mathcal{L}_{H_i}$.

The existence of such a function H_i follows from the fact that any symplectomorphism between two open subsets of a connected symplectic manifold is locally given as ϕ_1 for a 1-parameter Hamiltonian flow $\phi_t,\ t \in [0,1]$.

Using the above data, we can construct cocycles

$$T^*U_i \cap T^*U_j \ni p \to g_{ij}(p) \in C^\infty(T^*U_i \cap T^*U_j, \widetilde{G})$$

and

$$T^*V_i \cap T^*V_j \ni p \to h_{ij}(p) \in C^\infty(T^*V_i \cap T^*V_j, \widetilde{G})$$

intertwining the flat connections $\widetilde{\nabla}_i^0$ up to the term $\frac{1}{2} d \log \det Dg_{ij}$ ($\frac{1}{2} d \log \det Dh_{ij}$ respectively) as in the cotangent bundle case.

Let Ψ_i denote the lifting of ϕ_i given by:

$$\exp \mathcal{D}^0_{H_i} : \Gamma(T^*U_i \setminus U_X, \mathbb{A}^\hbar) \to \Gamma(T^*V_i \setminus U_Y, \mathbb{A}^\hbar).$$

We can and will view Ψ_i as local isomorphisms of jets at infinity of the $\widetilde{G}$-bundles on compactified cotangent bundles of X and Y constructed from the cocycles $g_{i,j}$ and $h_{i,j}$.

While both $g_{i,j}$'s and $h_{i,j}$'s do satisfy the cocycle conditions on $T^*(X)$ ($T^*(Y)$ respectively), however

$$\lambda_{ij} = \Psi_j^{-1} h_{ji} \Psi_i g_{ij} \neq 1$$

and hence we do not yet have the data necessary to construct the bundle $\mathcal{W}$ over M.

The following facts are easy corollaries of the construction:

(1) $\lambda_{ij} = 1 \mod \widetilde{G}_{\geq 1}$.

(2) $\lambda_{ij} \widetilde{\nabla}_j^0 \lambda_{j,i} = \widetilde{\nabla}_i^0 - \frac{1}{2} d \log \det Dg_{ij} - \frac{1}{2} d \log \det Dh_{ij}$.

(3) λ_{ij} form a two-cocycle with values in $\widetilde{G}$.

To begin with, recall that both $\frac{1}{2} d \log \det Dg_{ij}$ and $\frac{1}{2} d \log \det Dh_{ij}$ as cohomology classes on $T^*X \setminus X$ and $T^*Y \setminus Y$ represent (under our symplectomorphism) the same cohomology class, to wit half of the first Chern class of the tangent bundle with the complex structure induced by the symplectic form. Since these vanish, we can find a zero-Čech cochain τ_i with the coefficient in the sheaf of smooth nowhere vanishing functions and such that $\tau_i \lambda_{ij} \tau_j^{-1}$ intertwines the (local) flat connections $\widetilde{\nabla}_i^0$ and $\widetilde{\nabla}_j^0$.

In particular, $\tau_i \lambda_{ij} \tau_j^{-1}$ are given by exponentials of jets of $\widetilde{\nabla}_i^0$-flat sections of the bundle $T^*U_i \times \mathbb{A}^\hbar$ and, using a partition of unity, they can be written in the form $\tau_i \lambda_{ij} \tau_j^{-1} = \lambda_i \lambda_j^{-1}$, where λ_i is a jet of a flat section of the Weyl bundle supported on $T^*U_i \setminus U_X$. We now define an operator Ψ, acting on the set of sections of the Weyl bundle $\mathcal{W}$, by setting for each $i \in I$:

$$\Psi_{|T^*U_i \setminus U_X} = \Psi_i \tau_i \mathcal{M}_{\lambda_i}.$$

Here $\mathcal{M}_{\lambda_i}$ stands for the operator of multiplication with the flat section λ_i. It is easy to see that Ψ descends to an isomorphism of jets at the sphere at infinity of the deformations of cotangent bundles constructed above so that

$$\forall a \in C^\infty(\overline{B}^* X),\ \Psi(a) = (\, a + \sum_{k \geq 1} \hbar^k \hat{D}_k(a) \,) \circ \phi^{-1}$$

holds in $\mathbb{B}^\hbar(\overline{B}^* Y)$, where the $\hat{D}_k$ are $\mathcal{E}$-differential operators. Hence, as in Proposition 5, Ψ induces a deformation of the Lie algebroid $(\mathcal{E}, [\,,\,], \omega)$.

7.5 The characteristic class of $\mathbb{A}^{\hbar}(M)$

The characteristic class of the deformation constructed above can now be easily obtained as follows. The collection g_{ij}, h_{ij}, and the jet at infinity of $\Psi_i \mathcal{M}_{\lambda_i}$ give a cocycle with values in $\widetilde{G}$, and it commutes with local flat connections up to the Cech cocycle given by the collection of differential forms

$$\frac{1}{2} d \log \det Dg_{ij}, d \log \tau_i, \frac{1}{2} d \log \det Dh_{ij}. \tag{21}$$

As in the case of cotangent bundle, we can correct local connections by a scalar term and the characteristic class θ_0 of the deformation is given by (21) as a cochain in $\check{C}^1(M, \Omega^1(T^*M))$. Moreover, in the case that both X and Y admit metalinear structures, the collection $\{\tau_i\}_{i\in I}$ can be thought of as glueing of the pulled back)of the half-top form bundles of X and Y along the graph of the symplectomorphism into a line bundle $\mathcal{L}$ over M and, in this case,

$$\theta_0 = c_1(\mathcal{L}).$$

7.6 The Fourier integral operator

To get the Fourier integral operator we will work locally. We will dispense with the half-density bundles (trivial in any case) for the sake of simplicity of notation. We will begin by constructing, for each i, an operator on $L^2(\mathbb{R}^n)$ as follows. Choosing local coordinates on U_i and V_i, we can assume that H_i (introduced at the end of Section 7.4) is a smooth function on $T^*\mathbb{R}^{2n}$ which is 1-homogeneous in the cotangent direction. The differential equation

$$\frac{d}{dt} T_i(t) = \mathrm{Op}(\sqrt{-1} H_i) \circ T_i(t),\ T_i(0) = 1$$

has a solution given by a smooth family of bounded operators. Using the fact that $\{\tau_i\}_{i\in I}$ is a Čech zero - cochain with coefficients in the sheaf of everywhere nonzero functions and proceeding as in Section 7.3.2 of [5] one checks that $T_i(1)$ satisfies

$$\mathrm{Ad}\ (T_i(1)\mathrm{Op}(\lambda_i))\mathrm{Op}(f_\hbar) \sim\ \mathrm{Op}(\Psi(f)_\hbar)$$

mod $\hbar^\infty$ as $\hbar \to 0$ whenever supp$f \subset T^*U_i \setminus X$ (recall that λ_i is introduced in Section 7.4). In other words, the deformation constructed above is associated (in the sense of Proposition 5) to the almost unitary Fourier integral operator $\Phi = \sum_{i\in I} T_i(1)\mathrm{Op}(\lambda_i)$ whose canonical relation is C_ϕ. Moreover, the index of this operator Φ_0 is given by

$$\int_M \hat{A}(M)e^{\theta_0}.$$

Remark 5. The result above depends on the choice of the τ_i's which in turn determine the homotopy class of the symbol of the Fourier integral operator. Note however that in the case of scalar coefficients and the dimension of the underlying manifold greater then two this class is unique, since any invertible function on the cosphere bundle is homotopic to the constant one.

Moreover, since the characteristic classes involved are given by differential forms associated to connections on a vector bundle over M and $\Omega \subset {}^{\mathcal{E}}\Omega$, the $\mathcal{E}$-classes involved in the index formulas are in fact identical with corresponding standard characteristic classes.

Let us recall that the real vector bundle $\mathcal{E} \simeq TM$ is given by realification of a complex vector bundle $\mathcal{E}_{\mathbb{C}}$ on M (the almost complex structure coming from the symplectic vector bundle structure on $\mathcal{E}$). Moreover, it is easy to see, that there exists a choice of the τ_i's such that the associated characteristic class of the deformation coincides with $\frac{1}{2}c_1(\mathcal{E}_{\mathbb{C}})$. This gives the following result (compare with [4] and [15]):

Theorem 5. *There exists an almost unitary Fourier integral operator* Φ_0 (*as in Section* 3) *whose canonical relation is* C_ϕ *and such that:*

$$\operatorname{ind}\Phi_0 = \int_M \hat{A}(M)e^{\frac{1}{2}c_1(\mathcal{E}_{\mathbb{C}})}.$$

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