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ISOSPECTRAL METRICS ON FIVE-DIMENSIONAL SPHERES

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Abstract

We construct isospectral pairs of Riemannian metrics on S^5 and on B^6, thus lowering by three the minimal dimension of spheres and balls on which such metrics have been constructed previously ($S^{n\geq 8}$ and $B^{n\geq 9}$). We also construct continuous families of isospectral Riemannian metrics on S^7 and on B^8. In each of these examples, the metrics can be chosen equal to the standard metric outside certain subsets of arbitrarily small volume.

Introduction

During the past two decades, research on isospectral manifolds — that is, Riemannian manifolds sharing the same spectrum (including multiplicities) of the Laplace operator acting on functions — has been very active; see, for example, the survey article [5]. However, it was only recently that Zoltan I. Szabó and Carolyn Gordon independently discovered the first examples of isospectral metrics on spheres: Pairs of such metrics on $S^{n\geq 10}$ [20, 21], and continuous families on $S^{n\geq 8}$ [6].

In spite of the wealth of other isospectral manifolds obtained before, the construction of isospectral spheres had seemed beyond reach for a long time. The reason was that both of the main methods of construction which were known and used until 2000 had excluded spheres:

- The so-called Sunada method [18] and its various generalizations (see, e.g., [3]) produces isospectral quotients $(M/\Gamma_1\,, g)$, $(M/\Gamma_2\,, g)$ of a common Riemannian covering manifold (M, g); in particular, these isospectral manifolds were always nonsimply connected.

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Many interesting examples of locally isometric isospectral manifolds of the form $(M/\Gamma_1, g)$, $(M/\Gamma_2, g)$ — some of Sunada type, some using other special constructions — can be found, for example, in [24], [14], [10], [3], [12, 13], [15]; also the famous examples of isospectral plane domains [9] arise as orbifolds in the Sunada type setting.

- The method of principal torus bundles (see [4], [11], [7], [19], [8], [16, 17]) produces certain pairs of isospectral principal bundles for which the structural group is a torus of dimension at least two; the metrics are invariant under the torus action. Since no sphere is a principal $T^{k\geq 2}$-bundle, spheres cannot be obtained by using this method either (although products of spheres could [16]; these were the first examples of simply connected isospectral manifolds). See [17] for a detailed treatment of the method of principal torus bundles, a systematical approach for applying it, and many examples (among others, isospectral left invariant metrics on compact Lie groups).

The key to Gordon's construction of isospectral metrics on the spheres $S^{n\geq 8}$ (and on the balls $B^{n\geq 9}$) was a new approach which still involves $T^{k\geq 2}$-actions, but does not require them to be free anymore. On the other hand, Z.I. Szabó's pairs of isospectral metrics on $S^{n\geq 10}$ (and on $B^{n\geq 11}$) do not arise by such a construction and seem to be of a completely different type. A spectacular feature of his examples is that they include pairs of isospectral metrics on S^{11} in which one of the metrics is homogeneous while the other is not.

The present paper serves several purposes.

First, we reformulate Gordon's new theorem [6, Theorem 1.2] in a somewhat more elegant way (see Theorem 1.4 below), and in Theorem 1.6 we establish a special version of it which turns to be a useful tool for finding new applications. It also accounts for all applications of Theorem 1.4 which are known so far.

Second, we derive a general sufficient nonisometry condition for the type of Riemannian metrics occurring in Theorem 1.6; see Proposition 2.4 below. Although we will apply this criterion only to certain new examples constructed in this paper, we wish to point out that it could also be used to unify most of the various nonisometry proofs from [11], [7], [8], [6] (however, some of the locally homogeneous manifolds with boundary from [11] and [8] cannot be proven nonisometric using this approach because they violate a certain genericity condition;

see (G) in Proposition 2.3). Similarly, it would be possible to apply our method to the isospectral examples from [16, 17] where we had instead performed explicit curvature computations in order to show that objects like $\int \mathrm{scal}^2$, $\int \|R\|^2$, the critical values of the scalar curvature, or the dimension of their loci are not spectrally determined.

Our third (and main) purpose is to use Theorem 1.6 for constructing isospectral pairs of metrics on S^5 (and on B^6) and thus decreasing the minimal dimensions of Gordon's examples by three; see Example 3.3. We also obtain continuous isospectral families of metrics on S^7 (and on B^8); see Example 3.2. For the nonisometry proofs we use the general criterion mentioned above.

Fourth, we show that in these new examples — in particular, on S^5 and B^6 (pairs), and on S^7 and B^8 (continuous families) — it is possible to choose the isospectral metrics in such a way that they are equal to the round (resp. flat) metric outside certain subsets of arbitrarily small volume; see Theorem 5.3. This gives a nice contrast to the fact that in dimensions up to six, the round spheres themselves are completely determined by their spectra [22] (the corresponding problem in higher dimensions has not yet been solved), and to the fact that no round sphere in any dimension admits a continuous isospectral deformation [23].

The paper is organized as follows:

In Section 1 we present our reformulation (Theorem 1.4) of Gordon's theorem and the afore-mentioned specialization (Theorem 1.6). Section 2 contains our general sufficient nonisometry criterion for the metrics occurring in Theorem 1.6. In Section 3 we construct our new examples: Continuous families of isospectral metrics on S^7 and on B^8 (Example 3.2), and pairs of such metrics on S^5 and on B^6 (Example 3.3). Moreover, we give a survey of a number of related examples in 3.4; each of them can be obtained using Theorem 1.6. Section 4 gives the nonisometry proof for Examples 3.2 / 3.3. Finally, we show in Section 5 how to make the isospectral metrics round (resp. flat) on large subsets (Theorem 5.3).

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1. Isospectrality via effective torus actions

Definition 1.1. (i) The *spectrum* of a closed Riemannian manifold is the spectrum of eigenvalues, counted with multiplicities, of the associated Laplace operator acting on functions. The *Dirichlet spectrum* of a compact Riemannian manifold M with boundary is the spectrum of eigenvalues corresponding to eigenfunctions which satisfy the Dirichlet boundary condition $f_{|\partial M} = 0$. The *Neumann spectrum* of such a manifold is defined analogously with respect to the Neumann boundary condition $Nf = 0$, where N is the inward-pointing unit normal field on the boundary.

(ii) Two closed Riemannian manifolds are called *isospectral* if they have the same spectrum (including multiplicities). Two compact Riemannian manifolds with boundary are called *Dirichlet isospectral*, resp. *Neumann isospectral*, if they have the same Dirichlet spectrum, resp. the same Neumann spectrum.

Remark 1.2. Let (M, g) be a compact Riemannian manifold with or without boundary. Consider the Hilbert spaces $\mathcal{H}^N := H^{1,2}(M, g)$ and $\mathcal{H}^D := \overset{\circ}{\rightarrow} H^{1,2}(M, g) \subseteq \mathcal{H}^N$. Note that $\mathcal{H}^N$, resp. $\mathcal{H}^D$, is the completion of $C^\infty(M)$, resp. $\{f \in C^\infty(M) \mid f_{|\partial M} = 0\}$, with respect to the $H^{1,2}$-norm associated to (M, g). For each $f \in \mathcal{H}^N \setminus \{0\}$ the Rayleigh quotient is defined as

$$\mathcal{R}(f) := \int_M \|df\|_g^2 \, dvol_g \Big/ \int_M |f|^2 \, dvol_g = \left(\|f\|^2_{H^{1,2}(M,g)} \,/\, \|f\|^2_{L^2(M,g)}\right) - 1.$$

Let $0 \le \lambda_1^N \le \lambda_2^N \le \ldots \to \infty$, resp. $0 \le \lambda_1^D \le \lambda_2^D \le \ldots \to \infty$, denote the Neumann spectrum, resp. the Dirichlet spectrum, of (M, g); if $\partial M = \emptyset$ then both of these sequences coincide with the spectrum of (M, g). Finally, denote by L_k^N, resp. L_k^D, the set of all k-dimensional subspaces of $\mathcal{H}^N$, resp. $\mathcal{H}^D$. Then we have the following *variational characterization of eigenvalues* (see, e.g., [2]):

$$(1) \qquad \lambda_k^N = \inf_{U \in L_k^N} \sup_{f \in U \setminus \{0\}} \mathcal{R}(f) \quad \text{and} \quad \lambda_k^D = \inf_{U \in L_k^D} \sup_{f \in U \setminus \{0\}} \mathcal{R}(f).$$

Notation 1.3. By a *torus*, we always mean a nontrivial, compact, connected abelian Lie group. If a torus T acts smoothly and effectively by isometries on a compact connected Riemannian manifold (M, g) then we denote by $\hat{M}$ the union of those orbits on which T acts freely. Note that $\hat{M}$ is an open dense submanifold of M. The action of T gives $\hat{M}$ the structure of a principal T-bundle. By g^T we denote the unique Riemannian metric on the quotient manifold $\hat{M}/T$ such that the canonical projection $\pi : (\hat{M}, g) \to (\hat{M}/T, g^T)$ is a Riemannian submersion.

Theorem 1.4. *Let T be a torus which acts effectively on two compact connected Riemannian manifolds (M,g) and (M',g') by isometries. For each subtorus $W \subset T$ of codimension one, suppose that there exists a T-equivariant diffeomorphism $F_W : M \to M'$ which satisfies $F_W^* dvol_{g'} = dvol_g$ and induces an isometry $\overline{F}_W$ between the quotient manifolds $(\hat{M}/W, g^W)$ and $(\hat{M}'/W, g'^W)$. Then (M,g) and (M',g') are isospectral; if the manifolds have boundary then they are Dirichlet and Neumann isospectral.*

Remark. Theorem 1.4 above is a slight variation of Carolyn Gordon's Theorem 1.2 in [6]. Instead of our condition $F_W^* dvol_{g'} = dvol_g$, Gordon assumes the condition that $\overline{F}_{W*}$ maps the projected mean curvature vector field $\overline{H}_W$ of the submersion $(\hat{M}, g) \to (\hat{M}/W, g^W)$ to the corresponding vector field $\overline{H}'_W$. While the two conditions actually turn out to be equivalent in this context, our volume preserving condition is not only easier to formulate but also more convenient to check in applications. More than that, it is inherent and automatically satisfied in the specialization described below in 1.5 / 1.6 which covers all applications of the above theorem which are known so far. Our different formulation of the theorem has also led to a different proof (construction of a certain isometry between the $H^{1,2}$-spaces instead of intertwining the Laplacians). An advantage of our proof is that it does not require a certain additional condition which Gordon assumes in the case of manifolds with boundary (namely, that $\hat{M} \cap \partial M$ be dense in ∂M).

Proof of Theorem 1.4. Consider the Hilbert space $\mathcal{H} := H^{1,2}(M,g)$ in the case of manifolds without boundary or in case of Neumann boundary conditions, resp. $\mathcal{H} := \overset{\circ}{\to} H^{1,2}(M,g)$ in the case of Dirichlet boundary conditions. Let $\mathcal{H}'$ be defined analogously with respect to (M',g'). We claim that there is a Hilbert space isometry from $\mathcal{H}$ to $\mathcal{H}'$ which moreover preserves L^2-norms; the theorem will thus follow from the variational characterization of eigenvalues (1).

Consider the unitary representation of T on $\mathcal{H}$ defined by $(zf)(x) = f(zx)$ for all $f \in \mathcal{H}$, $z \in T$, $x \in M$. Write $T = \mathfrak{z}/\mathcal{L}$ and let $\mathcal{L}^*$ be the dual lattice. Since T is abelian, $\mathcal{H}$ decomposes as the orthogonal sum $\bigoplus_{\mu \in \mathcal{L}^*} \mathcal{H}_\mu$ with $\mathcal{H}_\mu = \{f \in \mathcal{H} \mid zf = e^{2\pi i \mu(Z)} f \text{ for all } z \in T\}$, where Z denotes any representative for z in $\mathfrak{z}$. In particular, this implies the coarser decomposition

$$\mathcal{H} = \mathcal{H}_0 \oplus \bigoplus\nolimits_W (\mathcal{H}_W \ominus \mathcal{H}_0), \tag{2}$$

where W runs though the set of all subtori of codimension 1 in T, and $\mathcal{H}_W$ is the sum of all $\mathcal{H}_\mu$ such that $\mu \in \mathcal{L}^*$ and $T_eW \subseteq \ker \mu$. In other words, $\mathcal{H}_W$ is just the space of W-invariant functions in $\mathcal{H}$. Let $\mathcal{H}'_W$ and $\mathcal{H}'_0$ be the analogously defined subspaces of $\mathcal{H}'$. Now let W be any subtorus of codimension 1 in T and choose a diffeomorphism F_W as in the assumption. Since F_W intertwines the T-actions, F_W^* sends $\mathcal{H}'_W$ to $\mathcal{H}_W$ and $\mathcal{H}'_0 \subset \mathcal{H}'_W$ to $\mathcal{H}_0 \subset \mathcal{H}_W$. We will now show that $F_W^* : \mathcal{H}'_W \to \mathcal{H}_W$ is a Hilbert space isometry which also preserves L^2-norms; in view of the decomposition (2) this will prove our above claim. Preservation of L^2-norms is trivial by the assumption $F_W^* \, dvol_{g'} = dvol_g$. Moreover, this assumption implies that it suffices to show that for each $\psi \in C^\infty(M')$ which is invariant under the W-action and for all $y \in M'$ we have $\|d\psi_{|y}\|_{g'} = \|d\varphi_{|x}\|_g$, where $\varphi := F_W^*\psi$ and $x := F_W^{-1}(y)$. We can assume $x \in \hat{M}$; let $\overline{\varphi}$ and $\overline{\psi}$ be the functions induced on $\hat{M}/W$ and $\hat{M}'/W$. Then $\overline{\varphi} = \overline{F}_W^*\overline{\psi}$; since g^W and ${g'}^W$ are the submersion metrics and $\overline{F}_W$ is an isometry we obtain indeed, at the appropriate points: $\|d\varphi\|_g = \|d\overline{\varphi}\|_{g^W} = \|d\overline{\psi}\|_{{g'}^W} = \|d\psi\|_{g'}$. q.e.d.

Notation and Remarks 1.5. In the following we fix a torus T with Lie algebra $\mathfrak{z} = T_eT$. Let $\mathcal{L}$ be the cocompact lattice in $\mathfrak{z}$ such that $\exp : \mathfrak{z} \to T$ induces an isomorphism from $\mathfrak{z}/\mathcal{L}$ to T, and denote by $\mathcal{L}^* \subset \mathfrak{z}^*$ the dual lattice. We also fix a compact connected Riemannian manifold (M, g_0), with or without boundary, and a smooth effective action of T on (M, g_0) by isometries.

(i) For $Z \in \mathfrak{z}$ we denote by Z^* the vector field $x \mapsto \frac{d}{dt}|_{t=0} \exp(tZ)x$ on M. For each $x \in M$ and each subspace $\mathfrak{w}$ of $\mathfrak{z}$ we let $\mathfrak{w}_x := \{Z_x^* \mid Z \in \mathfrak{w}\}$.

(ii) We call a smooth $\mathfrak{z}$-valued 1-form on M *admissible* if it is T-invariant and horizontal (i.e., vanishes on the vertical spaces $\mathfrak{z}_x$).

(iii) For any admissible $\mathfrak{z}$-valued 1-form λ on M we denote by g_λ the Riemannian metric on M given by

$$g_\lambda(X, Y) = g_0\big(X + \lambda(X)^*, Y + \lambda(Y)^*\big).$$

In other words, $g_\lambda = (\Phi_\lambda^{-1})^* g_0$, where Φ_λ is the smooth endomorphism field on M given by $X \mapsto X - \lambda(X)^*$ for all $X \in TM$. Note that Φ_λ is unipotent on each tangent space; in particular, $dvol_{g_\lambda} = dvol_{g_0}$.

(iv) Finally, note that g_λ is again invariant under the action of T, that g_λ restricts to the same metric as g_0 on the vertical subspaces $\mathfrak{z}_x$, and that the submersion metric g_λ^T on $\hat{M}/T$ is equal to g_0^T.

Theorem 1.6. *In the context of Notation* 1.5*, let λ, λ' be two admissible $\mathfrak{z}$-valued* 1*-forms on M. Assume:*

(∗) *For every $\mu \in \mathcal{L}^*$ there exists a T-equivariant $F_\mu \in \mathrm{Isom}(M, g_0)$ which satisfies $\mu \circ \lambda = F_\mu^*(\mu \circ \lambda')$.*

Then (M, g_λ) and $(M, g_{\lambda'})$ are isospectral; if M has boundary then the two manifolds are Dirichlet and Neumann isospectral.

Proof. We show that (M, g_λ) and $(M, g_{\lambda'})$ satisfy the hypotheses of Theorem 1.4. Let W be a subtorus of codimension 1 in T. Choose $\mu \in \mathcal{L}^*$ such that $\mathfrak{w} := T_eW$ equals $\ker\mu$, and choose a corresponding F_μ as in (∗). We claim that $F_W := F_\mu$ satisfies the conditions required in Theorem 1.4. First of all note that F_μ, being an isometry of g_0, trivially satisfies $F_\mu^* dvol_{g_{\lambda'}} = dvol_{g_\lambda}$ because of $dvol_{g_\lambda} = dvol_{g_0} = dvol_{g_{\lambda'}}$ (see 1.5(iii)). Thus it remains to prove that F_μ induces an isometry from $(\hat{M}/W, g_\lambda^W)$ to $(\hat{M}/W, g_{\lambda'}^W)$.

Let $V \in T_xM$ be any vector which is g_λ-orthogonal to $\mathfrak{w}_x$; then $V = \Phi_\lambda(X)$ for some X which is g_0-orthogonal to $\mathfrak{w}_x$. By condition (∗) we know that $\lambda(X)$ equals $\lambda'(F_{\mu*}X)$ modulo $\mathfrak{w}$. Keeping in mind that F_μ commutes with the T-action, we conclude that the vector $F_{\mu*}V = F_{\mu*}(\Phi_\lambda X)$ equals $Y := \Phi_{\lambda'}(F_{\mu*}X)$ up to an error in $\mathfrak{w}_{F_\mu(x)}$. But $F_{\mu*}X$ is g_0-orthogonal to $\mathfrak{w}_{F_\mu(x)}$; thus Y is the projection of $F_{\mu*}V$ to the $g_{\lambda'}$-orthogonal complement of $\mathfrak{w}_{F_\mu(x)}$. Our assertion now follows from $\|Y\|_{g_{\lambda'}} = \|F_{\mu*}X\|_{g_0} = \|X\|_{g_0} = \|V\|_{g_\lambda}$. q.e.d.

2. A sufficient condition for nonisometry

Throughout this section we let (M, g_0), T, $\mathfrak{z}$ be as in Notation 1.5, and we define the principal T-bundle $\pi : \hat{M} \to \hat{M}/T$ as in Notation 1.3. By λ, λ' we will always denote admissible $\mathfrak{z}$-valued 1-forms on M. In Proposition 2.4 below we will establish a sufficient condition for (M, g_λ), $(M, g_{\lambda'})$ to be nonisometric.

Notation and Remarks 2.1.

(i) We say that a diffeomorphism $F : M \to M$ is *T-preserving* if conjugation by F preserves $T \subset$ Diffeo(M). In that case, we denote by Ψ_F the automorphism of $\mathfrak{z} = T_eT$ induced by conjugation by F. Obviously, each T-preserving diffeomorphism F of M maps T-orbits to T-orbits and satisfies $F_*(Z^*) = \Psi_F(Z)^*$ for all $Z \in \mathfrak{z}$, where the vector fields Z^* on M are defined as in 1.5(iii).

(ii) We denote by $\mathrm{Aut}_{g_0}^T(M)$ the group of all T-preserving diffeomorphisms F of M which, in addition, preserve the g_0-norm of vectors tangent to the T-orbits and induce an isometry of $(\hat{M}/T, g_0^T)$. We denote the corresponding group of induced isometries by $\overline{\mathrm{Aut}}_{g_0}^T(M) \subset \mathrm{Isom}(\hat{M}/T, g_0^T)$.

(iii) We define $\mathcal{D} := \{\Psi_F \mid F \in \mathrm{Aut}_{g_0}^T(M)\} \subset \mathrm{Aut}(\mathfrak{z})$. Note that $\mathcal{D}$ is discrete because it is a subgroup of the discrete group $\{\Psi \in \mathrm{Aut}(\mathfrak{z}) \mid \Psi(\mathcal{L}) = \mathcal{L}\}$, where $\mathcal{L}$ is the lattice $\ker(\exp : \mathfrak{z} \to T)$.

(iv) Let ω_0 denote the connection form on $\hat{M}$ associated with g_0; i.e., for each $x \in \hat{M}$ the horizontal space $\ker(\omega_0|_{T_x\hat{M}})$ is the g_0-orthogonal complement of $\mathfrak{z}_x$ in $T_x\hat{M}$. Then the connection form on $\hat{M}$ associated with g_λ is obviously given by $\omega_\lambda := \omega_0 + \lambda$.

(v) Let Ω_λ denote the curvature form on $\hat{M}/T$ associated with the connection form ω_λ on $\hat{M}$. We have $\pi^*\Omega_\lambda = d\omega_\lambda$ because T is abelian.

(vi) Since λ is T-invariant and horizontal it induces some $\mathfrak{z}$-valued 1-form $\bar{\lambda}$ on $\hat{M}/T$. We conclude from $\pi^*\Omega_\lambda = d\omega_\lambda = d\omega_0 + d\lambda$ that $\Omega_\lambda = \Omega_0 + d\bar{\lambda}$. In particular, Ω_λ and Ω_0 differ by an exact $\mathfrak{z}$-valued 2-form.

Lemma 2.2. *Suppose that $F : (M, g_\lambda) \to (M, g_{\lambda'})$ is a T-preserving isometry.*

(i) *F preserves the g_0-norm of vectors tangent to the T-orbits, and it induces an isometry $\overline{F}$ of $(\hat{M}/T, g_0^T)$. In particular, $F \in \mathrm{Aut}_{g_0}^T(M)$ and $\Psi_F \in \mathcal{D}$.*

(ii) *$F^*\omega_{\lambda'} = \Psi_F \circ \omega_\lambda$; in particular, $F^*d\omega_{\lambda'} = \Psi_F \circ d\omega_\lambda$.*

(iii) *The isometry $\overline{F}$ of $(\hat{M}/T, g_0^T)$ satisfies*

$$\overline{F}^*\Omega_{\lambda'} = \Psi_F \circ \Omega_\lambda\,. \tag{3}$$

Proof. (i) This follows immediately from 1.5(iv).

(ii) We have $\omega_{\lambda'}(Z^*) = \omega_\lambda(Z^*) = Z$ for all $Z \in \mathfrak{z}$, hence $\omega_{\lambda'}(F_*(Z^*)) = \omega_{\lambda'}(\Psi_F(Z)^*) = \Psi_F(Z) = \Psi_F(\omega_\lambda(Z^*))$. The equation thus holds when applied to vectors tangent to the T-orbits. Since F is an isometry and maps orbits to orbits, it must map g_λ-horizontal vectors to $g_{\lambda'}$-horizontal vectors. Hence both sides of the asserted equation vanish when applied to a g_λ-horizontal vector.

(iii) This follows from (ii) and 2.1(v). q.e.d.

Proposition 2.3. *Let λ be an admissible $\mathfrak{z}$-valued 1-form on M such that the associated curvature form Ω_λ on $\hat{M}/T$ satisfies the following genericity condition:*

(G) *No nontrivial 1-parameter group in $\overline{\mathrm{Aut}}_{g_0}^T(M)$ preserves Ω_λ.*

Then T is a maximal torus in $\mathrm{Isom}(M, g_\lambda)$.

Proof. Let $F_t \in \mathrm{Isom}(M, g_\lambda)$ be a 1-parameter family of isometries commuting with T. In particular, the maps F_t are T-preserving and thus induce a 1-parameter family $\overline{F}_t \in \mathrm{Isom}(\hat{M}/T, g_0^T)$. The corresponding $\Psi_{F_t} \in \mathcal{D}$ satisfy $\Psi_{F_t} \equiv \mathrm{Id}$ because $\mathcal{D}$ is discrete and $\Psi_{F_0} = \mathrm{Id}$. Thus according to (3), each $\overline{F}_t$ preserves Ω_λ. The assumed property (G) of Ω_λ now implies $\overline{F}_t \equiv \mathrm{Id}$. We conclude that F_t restricts to a gauge transformation of the principal T-bundle $\hat{M}$. On the other hand, $F_t^*\omega_\lambda \equiv \omega_\lambda$ by 2.2(ii). But a gauge transformation which preserves the connection form of a principal bundle must act as an element of the structural group on each connected component of the bundle. Since an isometry of the connected Riemannian manifold (M, g_λ) is determined by its restriction to any nonempty open subset, it follows that the family F_t is contained in T. q.e.d.

Proposition 2.4. *Let λ, λ' be admissible 1-forms on M such that $\Omega_{\lambda'}$ has property* (G). *Furthermore, assume that*

$$\text{(N)} \qquad \Omega_\lambda \notin \mathcal{D} \circ \overline{\mathrm{Aut}}_{g_0}^T(M)^* \Omega_{\lambda'}\,.$$

Then (M, g_λ) and $(M, g_{\lambda'})$ are not isometric.

Proof. Suppose that there were an isometry $F : (M, g_\lambda) \to (M, g_{\lambda'})$. By Proposition 2.3, T is a maximal torus in $\mathrm{Isom}(M, g_{\lambda'})$. Since all maximal tori are conjugate, we can assume F — after possibly combining it with an isometry of $(M, g_{\lambda'})$ — to be T-preserving. But then Lemma 2.2 implies $\overline{F}^*\Omega_{\lambda'} = \Psi_F \circ \Omega_\lambda$ with $\overline{F} \in \overline{\mathrm{Aut}}_{g_0}^T(M)$ and $\Psi_F \in \mathcal{D}$, which contradicts our assumption (N). q.e.d.

Remark 2.5. Note that equally valid (but weaker) versions of Propositions 2.3 and 2.4 would be obtained by replacing $\overline{\mathrm{Aut}}_{g_0}^T(M)$ by the possibly larger group $\mathrm{Isom}(\hat{M}/T, g_0^T)$.

3. Examples

3.1 Notation

(i) Throughout the new examples given in 3.2 and 3.3 below we consider the two-dimensional torus $T := \mathbb{R}^2/\mathcal{L}$ with $\mathcal{L} := 2\pi\mathbb{Z} \times 2\pi\mathbb{Z}$. We denote the standard basis of its Lie algebra $\mathfrak{z} \cong \mathbb{R}^2$ by $\{Z_1\,, Z_2\}$.

(ii) We let $\mathcal{E}$ be the group of the four linear isomorphisms of $\mathfrak{z}$ which preserve each of the sets $\{\pm Z_1\}$ and $\{\pm Z_2\}$.

(iii) Let $m \in \mathbb{N}$. We identify the real vector spaces $\mathbb{C}^{m+1} = \mathbb{C}^m \oplus \mathbb{C}$ and $\mathbb{R}^{2m+2}$ via the linear isomorphism which sends $\{e_1\,, ie_1\,, \ldots\,, e_{m+1}\,, ie_{m+1}\}$ (in this order) to the standard basis of $\mathbb{R}^{2m+2}$, where $\{e_1\,, \ldots\,, e_{m+1}\}$ denotes the standard basis of $\mathbb{C}^{m+1}$. We let the torus T act on this space by

$$\exp(aZ_1 + bZ_2) : (p, q) \mapsto (e^{ia}p, e^{ib}q)$$

for all $a, b \in \mathbb{R}$, $p \in \mathbb{C}^m$, $q \in \mathbb{C}$. This action preserves the unit sphere $S^{2m+1} \subset \mathbb{R}^{2m+2}$ as well as the unit ball B^{2m+2}; by restriction we thus obtain an action ρ of T on S^{2m+1}, respectively on B^{2m+2}.

3.2 Example: Continuous isospectral families of metrics on $\mathbf{S^{2m+1\geq 7}}$ and on $\mathbf{B^{2m+2\geq 8}}$

Notation 3.2.1. For each linear map $j : \mathfrak{z} \cong \mathbb{R}^2 \to \mathfrak{su}(m)$ we define a $\mathfrak{z}$-valued 1-form $\lambda = (\lambda^1, \lambda^2)$ on $\mathbb{R}^{2m+2} \cong \mathbb{C}^m \oplus \mathbb{C}$ by letting

$$\lambda^k_{(p,q)}(X, U) = |p|^2 \langle j_{Z_k} p, X\rangle - \langle X, ip\rangle\langle j_{Z_k} p, ip\rangle \tag{4}$$

for $k = 1, 2$ and all $(X, U) \in T_p\mathbb{R}^{2m} \oplus T_q\mathbb{R}^2$. Here $\langle\,.\,,.\,\rangle$ denotes the standard euclidean inner product on $\mathbb{R}^{2m}$, and the skew-hermitian maps $j_{Z_k} := j(Z_k)$ act on $p \in \mathbb{R}^{2m}$ via the above identification $\mathbb{R}^{2m} \cong \mathbb{C}^m$.

By restriction we obtain a smooth $\mathfrak{z}$-valued 1-form λ on the unit sphere S^{2m+1}, respectively on the unit ball B^{2m+2}.

Remark 3.2.2. We observe that λ is admissible (see Notation 1.5(ii)) with respect to the action ρ of T defined above in 3.1(iii). In fact, invariance of λ under the action of T is immediate because multiplication with the complex scalar factor e^{ia} commutes with $j_{Z_k} \in \mathfrak{su}(m)$ and preserves the euclidean inner product. It remains to check that λ vanishes on the spaces $\mathfrak{z}_{(p,q)} = \operatorname{span}\{(ip,0),(0,iq)\} \subset T_p\mathbb{R}^{2m} \oplus T_q\mathbb{R}^2$. Indeed we have $\lambda^k_{(p,q)}(ip,0) = |p|^2\langle j_{Z_k}p, ip\rangle - \langle ip, ip\rangle\langle j_{Z_k}p, ip\rangle = 0$ for $k = 1, 2$, and $\lambda_{(p,q)}(0, iq) = 0$ by definition.

Definition 3.2.3. Let g_0 be the round standard metric on S^{2m+1}, respectively the standard metric on B^{2m+2}, and let g_λ be the metric associated with λ and g_0 as in Notation 1.5(iii).

Summarizing, for each linear map $j : \mathbb{R}^2 \to \mathfrak{su}(m)$ we have an associated Riemannian metric g_λ on S^{2m+1}, respectively on B^{2m+2}, via the corresponding 1-form λ as defined in (4).

Definition 3.2.4. Let $j, j' : \mathfrak{z} \cong \mathbb{R}^2 \to \mathfrak{su}(m)$ be two linear maps.

(i) We call j and j' *isospectral*, denoted $j \sim j'$, if for each $Z \in \mathfrak{z}$ there exists $A_Z \in \mathrm{SU}(m)$ such that $j'_Z = A_Z j_Z A_Z^{-1}$.

(ii) Let $Q : \mathbb{C}^m \to \mathbb{C}^m$ denote complex conjugation. We call j and j' *equivalent*, denoted $j \cong j'$, if there exists $A \in \mathrm{SU}(m) \cup \mathrm{SU}(m) \circ Q$ and $\Psi \in \mathcal{E}$ (see Notation 3.1(ii)) such that $j'_Z = A j_{\Psi(Z)} A^{-1}$ for all $Z \in \mathfrak{z}$.

(iii) We say j is *generic* if no nonzero element of $\mathfrak{su}(m)$ commutes with both j_{Z_1} and j_{Z_2}.

Proposition 3.2.5. *Let $j, j' : \mathfrak{z} \cong \mathbb{R}^2 \to \mathfrak{su}(m)$ be two linear maps, and let $g_\lambda, g_{\lambda'}$ be the associated pair of Riemannian metrics on S^{2m+1}, or on B^{2m+2}, as above. If $j \sim j'$ then the Riemannian manifolds (S^{2m+1}, g_λ) and $(S^{2m+1}, g_{\lambda'})$ are isospectral, and (B^{2m+2}, g_λ) and $(B^{2m+2}, g_{\lambda'})$ are Dirichlet and Neumann isospectral. If j and j' are not equivalent and if at least one of them is generic, then in both of these pairs the two manifolds are not isometric.*

Proof. We will prove the nonisometry statement in Section 4. To prove isospectrality we use Theorem 1.6. Fix an arbitrary $\mu \in \mathcal{L}^*$. We have to show that there exists an isometry F_μ of g_0 which commutes with the action of T and satisfies $\mu \circ \lambda = F_\mu^*(\mu \circ \lambda')$. Let $Z \in \mathfrak{z}$ be the vector corresponding to $\mu \in \mathcal{L}^* \subset \mathfrak{z}^*$ under the canonical identification of $\mathfrak{z}$ with $\mathfrak{z}^*$ associated with the basis $\{Z_1, Z_2\}$. Choose $A_Z \in \mathrm{SU}(m) \subset$

$\mathrm{SO}(2m)$ as in Definition 3.2.4(i), and let $F_\mu := (A_Z\,, \mathrm{Id}) \in \mathrm{SO}(2m+2)$. Then F_μ is an isometry of g_0 and satisfies

$$\begin{aligned}
&\big(F_\mu^*(\mu\circ\lambda')\big)_{(p,q)}(X,U)\\
&= (\mu\circ\lambda')_{(A_Zp,q)}(A_ZX,U)\\
&= |A_Zp|^2\langle j'_Z A_Zp, A_ZX\rangle - \langle A_ZX, iA_Zp\rangle\langle j'_Z A_Zp, iA_Zp\rangle\\
&= |p|^2\langle A_Z^{-1} j'_Z A_Zp, X\rangle - \langle X, ip\rangle\langle A_Z^{-1} j'_Z A_Zp, ip\rangle\\
&= |p|^2\langle j_Zp, X\rangle - \langle X, ip\rangle\langle j_Zp, ip\rangle = (\mu\circ\lambda)_{(p,q)}(X,U),
\end{aligned}$$

as desired. q.e.d.

The following result shows that there are in fact many examples to Proposition 3.2.5.

Proposition 3.2.6. *Let $m \geq 3$ and $\{Z_1\,, Z_2\}$ be the standard basis of $\mathbb{R}^2$.*

(i) *There exists a nonempty Zariski open subset $\mathcal{U}$ of the space $\mathcal{J}$ of linear maps $j : \mathbb{R}^2 \to \mathfrak{su}(m)$ such that for each $j \in \mathcal{U}$ there is a continuous family $j(t)$ in $\mathcal{J}$, defined on some open interval around $t = 0$, such that $j(0) = j$ and:*

  (1) *The maps $j(t)$ are pairwise isospectral in the sense of* 3.2.4(i).

  (2) *The function $t \mapsto \|j_{Z_1}(t)^2 + j_{Z_2}(t)^2\|^2 = \mathrm{tr}\big((j_{Z_1}(t)^2 + j_{Z_2}(t)^2)^2\big)$ is not constant in t in any interval around zero. In particular, the maps $j(t)$ are not pairwise equivalent in the sense of* 3.2.4(ii).

  (3) *The maps $j(t)$ are generic in the sense of* 3.2.4(iii).

(ii) *For $m = 3$, an explicit example of an isospectral family $j(t) : \mathbb{R}^2 \to \mathfrak{su}(3)$ with $\|j_{Z_1}(t)^2 + j_{Z_2}(t)^2\|^2 \neq \mathrm{const}$ is given by*

$$j_{Z_1}(t) := \begin{pmatrix} -i & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & i\end{pmatrix}, \qquad j_{Z_2}(t) := \begin{pmatrix} 0 & \cos t & \sqrt{2}\sin t\\ -\cos t & 0 & \cos t\\ -\sqrt{2}\sin t & -\cos t & 0\end{pmatrix}.$$

The $j(t)$ are pairwise isospectral since $\det\big(\lambda\mathrm{Id} - (sj_{Z_1}(t) + uj_{Z_2}(t))\big) = \lambda^3 + (s^2 + 2u^2)\lambda$ is independent of t. However, $\|j_{Z_1}(t)^2 + j_{Z_2}(t)^2\|^2 = 14 + 4\sin^2 t$ is nonconstant in t. The map $j(t)$ is generic in the sense of 3.2.4(iii) *if and only if $\cos t \neq 0$.*

Proof. (i) The same statement, but without requirement (3), was shown in [17, Proposition 3.6]. Since the section of two nonempty Zariski open sets is again such a set, it just remains to show that the subset $\mathcal{O} \subset \mathcal{J}$ of those elements which are generic in the sense of 3.2.4(iii) is nonempty and Zariski open. Note that the map $j \in \mathcal{J}$ given by

$$j_{Z_1} := \begin{pmatrix} i\alpha_1 & & & & \\ & i\alpha_2 & & & \\ & & i\alpha_3 & & \\ & & & \ddots & \\ & & & & i\alpha_m \end{pmatrix}, \qquad j_{Z_2} := \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & -1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & 0 & -1 \\ & & & 1 & 0 \end{pmatrix}$$

with pairwise different $\alpha_1, \ldots \alpha_m \in \mathbb{R}$ satisfying $\alpha_1 + \ldots + \alpha_m = 0$ is an element of $\mathcal{O}$; thus $\mathcal{O}$ is nonempty. To see that $\mathcal{O}$ is Zariski open, note that it is equal to the set of those $j \in \mathcal{J}$ for which the map $F_j : \mathfrak{su}(m) \ni \tau \mapsto ([j_{Z_1}, \tau], [j_{Z_2}, \tau]) \in \mathfrak{su}(m) \oplus \mathfrak{su}(m)$ has maximal rank $r_m := \dim(\mathfrak{su}(m))$. But the latter condition can be expressed as the nonvanishing of a certain polynomial in the coefficients of j, namely, the sum of the squared determinants of the $(r_m \times r_m)$-minors of a matrix representation of F_j.

(ii) This can be checked by straightforward calculation. q.e.d.

Corollary 3.2.7. *For every $m \geq 3$ there exist continuous families of isospectral metrics on S^{2m+1} and continuous families of Dirichlet and Neumann isospectral metrics on B^{2m+2}. In particular, there exist such families on S^7, resp. on B^8. An explicit example is given by the metrics $g_{\lambda(t)}$ associated with the family $j(t)$ from* 3.2.6(ii).

Remark. Carolyn Gordon [6] has previously given continuous families of isospectral metrics on each $S^{n \geq 8}$ and $B^{n \geq 9}$ using a related construction (see 3.4(i)). Those were the first examples of continuous isospectral families of metrics on balls and spheres.

3.3 Example: Isospectral pairs of metrics on $\mathbf{S^5}$ and on $\mathbf{B^6}$

Notation 3.3.1. In the context of Notation 3.1 we consider now the case $m = 2$.

(i) We fix a realization of the Hopf projection $P : S^3 \to S^2_{1/2} \subset \mathbb{R}^3$ in coordinates, say

$$P : (\alpha, \beta, \gamma, \delta) \mapsto \left(\frac{1}{2}(\alpha^2 + \beta^2 - \gamma^2 - \delta^2),\, \alpha\gamma + \beta\delta,\, \alpha\delta - \beta\gamma \right).$$

We extend P to a smooth map from $\mathbb{R}^4 \cong \mathbb{C}^2$ to $\mathbb{R}^3$, defined by the same formula (note that P will map S^3_a to $S^2_{a^2/2}$ for each radius $a \geq 0$).

(ii) Let $\mathrm{Sym}_0(\mathbb{R}^3)$ denote the space of symmetric traceless real (3×3)-matrices. For each linear map $c : \mathfrak{z} \cong \mathbb{R}^2 \to \mathrm{Sym}_0(\mathbb{R}^3)$ we define a $\mathfrak{z}$-valued 1-form $\lambda = (\lambda^1, \lambda^2)$ on $\mathbb{R}^6 \cong \mathbb{C}^2 \oplus \mathbb{C}$ by letting

$$\lambda^k_{(p,q)}(X, U) = \langle c_{Z_k} P(p) \times P(p), P_{*p}(X)\rangle \tag{5}$$

for $k = 1, 2$ and all $(X, U) \in T_p\mathbb{R}^{2m} \oplus T_q\mathbb{R}^2$, where $\langle\, .\, ,\, .\, \rangle$ denotes the standard euclidean inner product on $\mathbb{R}^3$, and $\times$ denotes the vector product in $\mathbb{R}^3$. By restriction we obtain a smooth $\mathfrak{z}$-valued 1-form λ on the unit sphere S^5, respectively on the unit ball B^6.

Remark 3.3.2. We observe that λ is admissible with respect to the action ρ of T on $S^5 \subset \mathbb{C}^2 \oplus \mathbb{C}$, resp. $B^6 \subset \mathbb{C}^2 \oplus \mathbb{C}$. Invariance of λ under the action of T is immediate since $P(e^{ia}p) = P(p)$ for all $a \in \mathbb{R}$, $p \in \mathbb{C}^2$. By the same reason we have $P_{*p}(ip) = 0$, which implies that λ vanishes on the spaces $\mathfrak{z}_{(p,q)}$.

Definition 3.3.3. Let g_λ be the metric associated with λ and the standard metric g_0 on S^5, resp. B^6. Thus for each linear map $c : \mathbb{R}^2 \to \mathrm{Sym}_0(\mathbb{R}^3)$ we have an associated Riemannian metric g_λ on S^5, respectively on B^6, via the corresponding 1-form λ as defined in (5).

Definition 3.3.4. Let $j, j' : \mathfrak{z} \cong \mathbb{R}^2 \to \mathrm{Sym}_0(\mathbb{R}^3)$ be two linear maps.

(i) We call c and c' *isospectral*, denoted $c \sim c'$, if for each $Z \in \mathfrak{z}$ there exists $E_Z \in \mathrm{SO}(3)$ such that $c'_Z = E_Z c_Z E_Z^{-1}$.

(ii) We call c and c' *equivalent*, denoted $c \cong c'$, if there exists $E \in \mathrm{O}(3)$ and $\Psi \in \mathcal{E}$ (see Notation 3.1(ii)) such that $c'_Z = E c_{\Psi(Z)} E^{-1}$ for all $Z \in \mathfrak{z}$.

(iii) We say c is *generic* if no nonzero element of $\mathfrak{so}(3)$ commutes with both c_{Z_1} and c_{Z_2}.

Proposition 3.3.5. *Let $c, c' : \mathfrak{z} \cong \mathbb{R}^2 \to \mathrm{Sym}_0(\mathbb{R}^3)$ be two linear maps, and let $g_\lambda, g_{\lambda'}$ be the associated pair of Riemannian metrics on S^5, or on B^6, as above. If $c \sim c'$ then the Riemannian manifolds (S^5, g_λ) and $(S^5, g_{\lambda'})$ are isospectral, and (B^6, g_λ) and $(B^6, g_{\lambda'})$ are Dirichlet and Neumann isospectral. If c and c' are not equivalent and if at least one of them is generic, then in both of these pairs the two manifolds are not isometric.*

Proof. Again, we postpone the proof of the nonisometry statement to Section 4, and we use Theorem 1.6 to prove isospectrality. Fix $\mu \in \mathcal{L}^*$ and let Z be the dual vector in $\mathfrak{z}$ as in the proof of 3.2.5. Choose $E_Z \in \mathrm{SO}(3)$ as in Definition 3.3.4(i), and choose $A_Z \in \mathrm{SU}(2) \subset \mathrm{SO}(4)$ such that for the Hopf projection $P : S^3 \to S^2_{1/2}$ from 3.3.1(i) we have $P \circ A_Z = E_Z \circ P$. Let $F_\mu := (A_Z\,, \mathrm{Id}) \in \mathrm{SO}(6)$. Then F_μ is an isometry of g_0 and satisfies

$$\begin{aligned}\big(F_\mu^*(\mu \circ \lambda')\big)_{(p,q)}(X,U) &= (\mu \circ \lambda')_{(A_Z p,q)}(A_Z X, U)\\ &= \langle c'_Z P(A_Z p) \times P(A_Z p), P_{*A_Z p}(A_Z X)\rangle\\ &= \langle c'_Z E_Z P(p) \times E_Z P(p), E_Z P_{*p}(X)\rangle\\ &= \langle E_Z^{-1} c'_Z E_Z P(p) \times P(p), P_{*p}(X)\rangle\\ &= \langle c_Z P(p) \times P(p), P_{*p}(X)\rangle = (\mu\circ\lambda)_{(p,q)}(X,U),\end{aligned}$$

as desired. q.e.d.

Proposition 3.3.6. *There exist pairs of linear maps $c, c' : \mathbb{R}^2 \to \mathrm{Sym}_0(\mathbb{R}^3)$ such that c and c' are isospectral in the sense of* 3.3.4(i)*, not equivalent in the sense of* 3.3.4(ii)*, and both generic in the sense of* 3.3.4(iii)*. An example of such a pair is given by*

$$c_{Z_1} = c'_{Z_1} = \begin{pmatrix} -1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}, \quad c_{Z_2} = \begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}, \quad c'_{Z_2} = \begin{pmatrix} 0 & 0 & \sqrt{2}\\ 0 & 0 & 0\\ \sqrt{2} & 0 & 0\end{pmatrix},$$

where $\{Z_1\,, Z_2\}$ is the standard basis of $\mathbb{R}^2$.

Proof. One easily checks that for each fixed pair $s, u \in \mathbb{R}$, the characteristic polynomials of $sc_{Z_1} + uc_{Z_2}$ and $sc'_{Z_1} + uc'_{Z_2}$ are equal (namely, to $\lambda^3 + (s^2 + 2u^2)\lambda$), which implies isospectrality. That c and c' are not equivalent can be seen from the fact that $\|c_{Z_1}^2 + c_{Z_2}^2\|^2 = 14$, while $\|c'^2_{Z_1} + c'^2_{Z_2}\|^2 = 18$. The maps are generic in the sense of 3.3.4(iii) because not even the map $c_{Z_1} = c'_{Z_1}$ alone commutes with any nonzero element of $\mathfrak{so}(3)$. q.e.d.

Corollary 3.3.7. *There exist nontrivial pairs of isospectral metrics on S^5, and there exist nontrivial pairs of Dirichlet and Neumann isospectral metrics on B^6.*

Remark. Our above pairs of isospectral 5-spheres constitute the lowest dimensional examples of isospectral spheres which have been constructed so far. The analogous statement holds for our isospectral 6-dimensional balls.

3.4 Survey of related examples

As a complement to our above new examples of isospectral spheres and balls we now give a short survey of related examples. Among them, (ii) and (v) are new; (i), (iii), and (iv) are already known, but were constructed first in slightly different settings.

Although we will not present any proofs here, we note that in each of the examples below the Riemannian manifolds can be described within the setting of 1.5 and proven to be isospectral by Theorem 1.6. Moreover, for each of them there is a nonisometry proof using the general results from Section 2 and following the lines of Section 4 below, where we will prove nonisometry for the above Examples 3.2 / 3.3. Since the formulas defining λ, λ' in the examples below (except (v)) are less complicated than those in 3.2 or 3.3, the corresponding computations are simpler than those in Section 4. Also note that in some of the examples the situation is simplified by the fact that $\hat{M} = M$ (in (iii) and (v)), or that $\Omega_0 = 0$ (in (i)–(iv)). For example, in (iii) below, the curvature form Ω_λ equals $d\bar{\lambda}$ on $M/T = S^{2m-1}$, where $d\bar{\lambda}^k$ is of the form $(X,Y) \mapsto 2\langle j_{Z_k}X, Y\rangle$.

The notation we use is similar to the one we used above. In particular, T again denotes the torus $\mathbb{R}^2/(2\pi\mathbb{Z} \times 2\pi\mathbb{Z})$, and $\{Z_1\,, Z_2\}$ is the standard basis of its Lie algebra $\mathfrak{z} \cong \mathbb{R}^2$. On all manifolds M which we consider below there is a canonical standard metric which will in each case play the role of g_0. When we call two manifolds with boundary isospectral we always mean Dirichlet and Neumann isospectral.

(i) *Continuous families of isospectral metrics on $S^{n\geq 8}$ and on $B^{n\geq 9}$* [6].
Let $M = S^{m+3} \subset \mathbb{R}^m \oplus \mathbb{C} \oplus \mathbb{C}$, resp. $M = B^{m+4} \subset \mathbb{R}^m \oplus \mathbb{C} \oplus \mathbb{C}$. Let the action of T on M be induced by its canonical action on the $\mathbb{C} \oplus \mathbb{C}$ component (generated by multiplication with i on each summand). For each linear map $j : \mathbb{R}^2 \to \mathfrak{so}(m)$ we define a $\mathfrak{z} \cong \mathbb{R}^2$-valued 1-form on $\mathbb{R}^m \oplus \mathbb{C} \oplus \mathbb{C} \cong \mathbb{R}^m \oplus \mathbb{R}^4$ (and hence on M) by letting

$$\lambda^k_{(p,q)}(X,U) = \langle j_{Z_k}p, X\rangle$$

for $k = 1,2$ and all $(X,U) \in T_p\mathbb{R}^m \oplus T_q\mathbb{R}^4$. The following conditions on a pair j, j' imply that the associated Riemannian manifolds (M, g_λ), $(M, g_{\lambda'})$ are isospectral and not isometric:

(1) j and j' are isospectral: For each $Z \in \mathfrak{z}$ there exists $A_Z \in \mathrm{O}(m)$ such that $j'_Z = A_Z j_Z A_Z^{-1}$.

(2) j and j' are nonequivalent: There is no $A \in \mathrm{O}(m)$ and no $\Psi \in \widetilde{\mathcal{E}}$ such that $j'_Z = A j_{\Psi(Z)} A^{-1}$ for all $Z \in \mathfrak{z}$; here $\widetilde{\mathcal{E}}$ denotes the set of the eight automorphisms of $\mathbb{R}^2$ which preserve the set $\{\pm Z_1, \pm Z_2\}$.

(3) At least one of j, j' is generic; j is called generic if no nonzero element of $\mathfrak{so}(m)$ commutes with both j_{Z_1} and j_{Z_2}.

It is known [11] that for $m \geq 5$ there are continuous families — even multiparameter families — $j(t)$ whose elements pairwise satisfy these conditions. The associated manifolds $(M, g_{\lambda(t)})$ are Carolyn Gordon's examples of isospectral deformations on spheres and balls.

(ii) *Isospectral pairs of metrics on S^6 and on B^7.*
This is just a slight modification of Example 3.3 which does not involve the Hopf projection P anymore; in turn, an additional dimension is needed for the construction. Let $M = S^6 \subset \mathbb{R}^3 \oplus \mathbb{C} \oplus \mathbb{C}$, resp. $M = B^7 \subset \mathbb{R}^3 \oplus \mathbb{C} \oplus \mathbb{C}$. Let T act canonically on the $\mathbb{C} \oplus \mathbb{C}$ component as in (i). For each linear map $c : \mathbb{R}^2 \to \mathrm{Sym}_0(\mathbb{R}^3)$ we define a $\mathfrak{z} \cong \mathbb{R}^2$-valued 1-form on $\mathbb{R}^3 \oplus \mathbb{C} \oplus \mathbb{C} \cong \mathbb{R}^3 \oplus \mathbb{R}^4$ (and hence on M) by letting

$$\lambda^k_{(p,q)}(X, U) = \langle c_{Z_k} p \times p, X \rangle$$

for $k = 1, 2$ and all $(X, U) \in T_p\mathbb{R}^3 \oplus T_q\mathbb{R}^4$. The conditions on a pair c, c' which cause (M, g_λ) and $(M, g_{\lambda'})$ to be isospectral and not isometric are the same as in 3.3, except that in the nonequivalence condition the group $\mathcal{E}$ has to be replaced by the larger group $\widetilde{\mathcal{E}}$ as in (i).

The specific pair c, c' given in 3.3.6 satisfies these conditions and thus yields a pair of isospectral metrics $g_\lambda, g_{\lambda'}$ on S^6 (resp. on B^7). Moreover, one can compute the loci N, N' of the maximal scalar curvature in (S^6, g_λ) and $(S^6, g_{\lambda'})$. It turns out that N' contains a 4-sphere, while N is a union of 2- and 3-spheres.

(iii) *Continuous families of isospectral metrics on $S^{m-1\geq 4} \times T^2$* [7] *and on $B^{m\geq 5} \times T^2$* [11]; *pairs of isospectral metrics on $S^2 \times T^2$* [17] *and on $B^3 \times T^2$.*
For $a, b, c > 0$ with $a^2 + b^2 + c^2 \leq 1$ we define $M_{a,b,c} := \{(p, u, v) \in \mathbb{R}^m \oplus \mathbb{C} \oplus \mathbb{C} \mid |p|^2 = a^2, |u|^2 = b^2, |v|^2 = c^2\}$. Then $M_{a,b,c}$ is diffeomorphic to $S^{m-1} \times T^2$ and is a submanifold of $B^{m+4} \subset \mathbb{R}^m \oplus \mathbb{C} \oplus \mathbb{C}$. The restrictions of two metrics $g_\lambda, g_{\lambda'}$ from (i) to one of these submanifolds are again isospectral and nonisometric under the same conditions as in (i) on the underlying pair of maps j, j'. In particular, one obtains

continuous isospectral families $g_{\lambda(t)}$ on $S^{m-1} \times T^2$ for $m \geq 5$. These are just the examples constructed in [7].

Similarly, for $m = 3$ the metrics $g_\lambda\,, g_{\lambda'}$ from (ii) restrict to isospectral nonisometric metrics on $M_{a,b,c}$ which is now diffeomorphic to $S^2 \times T^2$; these pairs were first constructed in [17].

In both cases we can replace the condition $|p|^2 = a^2$ by $|p|^2 \leq a^2$ in the definition of $M_{a,b,c}$ which then becomes diffeomorphic to $B^m \times T^2$. We obtain continuous families of isospectral metrics $g_{\lambda(t)}$ on $B^m \times T^2$ for $m \geq 5$ [11] and pairs of such metrics on $B^3 \times T^2$.

(iv) *Isospectral pairs of metrics on* $S^2 \times S^3$ [1].
Instead of considering the submanifolds $M_{a,b,c} \approx S^2 \times T^2$ of $B^7 \subset \mathbb{R}^3 \oplus \mathbb{C} \oplus \mathbb{C}$ as in the middle part of (iii) above, we now consider $M_{a,b} := \{(p,q) \in \mathbb{R}^3 \oplus \mathbb{R}^4 \mid |p|^2 = a^2, |q|^2 = b^2\}$ with $a, b > 0$. Then $M_{a,b}$ is diffeomorphic to $S^2 \times S^3$, and again the metrics $g_\lambda\,, g_{\lambda'}$ from (ii) restrict to isospectral nonisometric metrics on $M_{a,b}$.

(v) *Continuous isospectral families on* $S^{2m-1\geq 5} \times S^1$ *and on* $B^{2m\geq 6} \times S^1$*; pairs of isospectral metrics on* $S^3 \times S^1$ *and on* $B^4 \times S^1$*.*
We return to the context of our Examples 3.2 / 3.3 above and consider the submanifolds $M_{a,b} := \{(p,q) \in \mathbb{C}^m \oplus \mathbb{C} \mid |p|^2 = a^2, |q|^2 = b^2\}$ of B^{2m}. For $a, b > 0$ the manifold $M_{a,b}$ is diffeomorphic to $S^{2m-1} \times S^1$. If $j, j' : \mathbb{R}^2 \to \mathfrak{su}(m)$ satisfy the assumptions of Proposition 3.2.5 then the associated pair of metrics $g_\lambda\,, g_{\lambda'}$ on B^{2m} restricts to a pair of isospectral nonisometric metrics on $M_{a,b}$ as well. In particular, for $m \geq 3$ we obtain continuous families $g_{\lambda(t)}$ of such metrics as in 3.2.6 / 3.2.7.

Similarly, for $m = 2$ each pair $c, c' : \mathbb{R}^2 \to \mathrm{Sym}_0(\mathbb{R}^3)$ satisfying the assumptions of Proposition 3.3.5 also yields a pair of isospectral nonisometric metrics on the manifold $S^{2m-1} \times S^1 = S^3 \times S^1$.

In both cases we can again replace the condition $|p|^2 = a^2$ by $|p|^2 \leq a^2$ in the definition of $M_{a,b}$ which then becomes diffeomorphic to $B^{2m} \times S^1$. We obtain continuous families of isospectral metrics $g_{\lambda(t)}$ on $B^{2m} \times S^1$ for $m \geq 3$, and pairs of such metrics on $B^4 \times S^1$.

4. Nonisometry of the examples from 3.2 / 3.3

Our strategy for proving the nonisometry statements of Proposition 3.2.5 / 3.3.5 consists in showing that the nonequivalence condition on j, j' (resp. on c, c') implies condition (N) of Proposition 2.4, while the genericity condition on the pair of maps implies Property (G) for the associated curvature forms. See Proposition 4.3 below for these assertions. The desired nonisometry statements then follow immediately from Proposition 2.4.

Notation and Remarks 4.1. Throughout this section we denote by (M, g_0) either $S^{2m+1} \subset \mathbb{R}^{2m} \oplus \mathbb{R}^2$ or $B^{2m+2} \subset \mathbb{R}^{2m} \oplus \mathbb{R}^2$, endowed with the standard metric. We consider the action ρ of $T = \mathbb{R}^2/(2\pi\mathbb{Z} \times 2\pi\mathbb{Z})$ on M which was defined in 3.1(iii).

(i) For $a, b \geq 0$ let $M_{a,b} := \{(p, q) \in \mathbb{R}^{2m} \oplus \mathbb{R}^2 \mid |p|^2 = a^2, |q|^2 = b^2\}$. Thus S^{2m+1}, resp. B^{2m+2}, is the disjoint union of the submanifolds $M_{a,b}$ with $a^2 + b^2 = 1$, resp. $a^2 + b^2 \leq 1$. Note that in either case, $\hat{M}$ is the union of those $M_{a,b} \subset M$ with $a, b > 0$. Each $M_{a,b}$ is obviously T-invariant.

(ii) For $M_{a,b} \subset \hat{M}$ the manifold $(M_{a,b}/T, g_0^T)$ is isometric to $(\mathbb{C}P^{m-1}, a^2 g_{\mathrm{FS}})$, where g_{FS} denotes the Fubini-Study metric.

(iii) The first component of the $\mathfrak{z} \cong \mathbb{R}^2$-valued form which the curvature form Ω_0 induces on $(M_{a,b}/T, g_0^T) = (\mathbb{C}P^{m-1}, a^2 g_{\mathrm{FS}})$ is a scalar multiple of the standard Kähler form on $\mathbb{C}P^{m-1}$, and the second component is zero. In fact, for $(X, Z), (Y, W) \in T_{(p,q)} M_{a,b} = T_p S_a^{2m-1} \oplus T_q S_b^1$ we have $\omega_0^1(X, Z) = \langle X, ip\rangle/a^2$, $\omega_0^2(X, Z) = \langle Z, iq\rangle/b^2$, hence $d\omega_0^1((X, Z), (Y, W)) = 2\langle iX, Y\rangle/a^2$ and $d\omega_0^2((X, Z), (Y, W)) = 2\langle iZ, W\rangle/b^2 = 0$, where the last equation follows from the fact that $T_q S_b^1$ is one-dimensional. The statement now follows because Ω_0 is induced by $d\omega_0$.

(iv) Any isometry of $(\mathbb{C}P^{m-1}, a^2 g_{\mathrm{FS}}) = S_a^{2m-1}/S^1$ is induced by some $\mathbb{C}$-linear or $\mathbb{C}$-antilinear isometry $A \in \mathrm{SU}(m) \cup \mathrm{SU}(m) \circ Q$ (see 3.2.4) of $\mathbb{C}^m$.

Lemma 4.2. *If $F \in \mathrm{Aut}_{g_0}^T(M)$ then F preserves each of the submanifolds $M_{a,b}$ of $\hat{M}$. Moreover, $\Psi_F \in \mathcal{E}$ (see Notation* 3.1(ii)*). In particular, for the group $\mathcal{D}$ defined in* 2.1(ii) *we have $\mathcal{D} \subset \mathcal{E}$.*

Proof. Note that the T-orbit through $(p, q) \in M_{a,b} \subset \hat{M}$, endowed with the metric induced by g_0, is a rectangular torus with side lengths $2\pi a$ and $2\pi b$. It follows that F preserves the sets $M_{a,b} \cup M_{b,a} \subset \hat{M}$. We have to show that F cannot switch the components $M_{a,b}$ and $M_{b,a}$ if $a \neq b$. If it did then it would induce an isometry from $(M_{a,b}/T, g_0^T)$ to $(M_{b,a}/T, g_0^T)$. But by 4.1(ii) these manifolds have different volume. Thus F preserves $M_{a,b}$. Now choose $0 < a < b$ such that $M_{a,b} \subset \hat{M}$, and let $(p, q) \in M_{a,b}$. Since F preserves $M_{a,b}$, both of the orbits $T \cdot (p, q)$ and

$T \cdot F(p,q)$ are rectangular tori on which the shortest closed loops have length $2\pi a$ and are precisely the flow lines of Z_1^*. This, together with $Z_1^* \perp Z_2^*$ and $\|Z_2^*|_{(p,q)}\| = \|Z_2^*|_{F(p,q)}\| = b$, implies that $F_*(Z_1^*) \in \{\pm Z_1^*\}$ and $F_*(Z_2^*) \in \{\pm Z_2^*\}$; hence $\Psi_F \in \mathcal{E}$. Now $\mathcal{D} \subset \mathcal{E}$ follows from the definition of $\mathcal{D}$. q.e.d.

In view of Proposition 2.4, the following result implies immediately the nonisometry statements of Proposition 3.2.5 / 3.3.5 :

Proposition 4.3. *Let λ, λ' be of the type defined in* (4) (*resp. in* (5)), *and let $j, j' : \mathbb{R}^2 \to \mathfrak{su}(m)$* (*resp. $c, c' : \mathbb{R}^2 \to \mathrm{Sym}_0(\mathbb{R}^3)$*) *be the pair of linear maps with which λ, λ' are associated.*

(i) *If j and j'* (*resp. c and c'*) *are not equivalent in the sense of* 3.2.4(ii) (*resp.* 3.3.4(ii)), *then Ω_λ and $\Omega_{\lambda'}$ satisfy condition* (N) *of Proposition* 2.4.

(ii) *If j* (*resp. c*) *is generic in the sense of* 3.2.4(iii) (*resp.* 3.3.4(iii)), *then Ω_λ has Property* (G).

Proof. We choose one of the submanifolds $M_{a,b} =: L$ of $\hat{M}$ and denote by $\omega_\lambda^L, \Omega_\lambda^L$ the forms induced by $\omega_\lambda, \Omega_\lambda$ on $L \subset \hat{M}$ and $L/T \subset \hat{M}/T$, respectively.

(i) Suppose that condition (N) were not satisfied. Let $\Psi \in \mathcal{D} \subset \mathcal{E}$ (recall 4.2) and $\overline{F} \in \overline{\mathrm{Aut}}_{g_0}^T(M)$ such that $\Omega_\lambda = \Psi \circ \overline{F}^* \Omega_{\lambda'}$. Lemma 4.2 implies that $\overline{F}$ preserves L/T; hence $\Omega_\lambda^L = \Psi \circ \overline{F}^* \Omega_{\lambda'}^L$. Recall from 2.1(vi) that $\Omega_\lambda = \Omega_0 + d\bar{\lambda}$, $\Omega_{\lambda'} = \Omega_0 + d\overline{\lambda'}$. Each closed 2-form on $\mathbb{C}P^{m-1}$ is uniquely decomposable into an exact component and a multiple of the Kähler form. Therefore, the description of Ω_0^L given in 4.1(iii) now implies that

$$d\bar{\lambda}^L = \Psi \circ \overline{F}^* d\overline{\lambda'}^L \tag{6}$$

and hence

$$d\lambda^L = \Psi \circ A^* d\lambda'^L \tag{7}$$

for the map $A \in \mathrm{SU}(m) \cup \mathrm{SU}(m) \circ Q$ which induces the isometry $\overline{F}$ of $(L/T, g_0^T) = (\mathbb{C}P^{m-1}, a^2 g_{\mathrm{FS}})$ (recall 4.1(iv)). Here λ^L, λ'^L and $\bar{\lambda}^L, \overline{\lambda'}^L$ denote the forms which λ, λ' and $\bar{\lambda}, \overline{\lambda'}$ induce on L and L/T, respectively. We are going to show that

$$j'_{\Psi(Z)} = A j_Z A^{-1} \tag{8}$$

for all $Z \in \mathfrak{z}$ in the case of forms of type (4), and

$$c'_{\Phi(Z)} = Ec_Z E^{-1} \tag{9}$$

for all $Z \in \mathfrak{z}$ in the case of forms of type (5), where $E := \overline{F}|_{L/T} \in \mathrm{O}(3)$ and $\Phi := \det(E)\Psi \in \{\pm\Psi\} \subset \mathcal{E}$. This will contradict the assumed nonequivalence of j and j' (resp. of c and c').

In the first case, we have for all $(X, Z), (Y, W) \in T_{(p,q)}L = T_p S_a^{2m-1} \times T_q S_b^1$ and for $k = 1, 2$:

$$\begin{aligned}
\lambda^k(X, Z) &= a^2\langle j_k p, X\rangle - \langle j_k p, ip\rangle\langle X, ip\rangle, \quad \text{hence} \\
d\lambda^k((X, Z), (Y, W)) &= 2a^2\langle j_k X, Y\rangle - 2\langle j_k X, ip\rangle\langle Y, ip\rangle \\
&\quad + 2\langle j_k Y, ip\rangle\langle X, ip\rangle - 2\langle j_k p, ip\rangle\langle iX, Y\rangle \\
&= 2a^2\langle j_k X^h, Y^h\rangle - 2\langle j_k p, ip\rangle\langle iX, Y\rangle,
\end{aligned}$$

where $j_k := j_{Z_k}$ and X^h denotes the g_0-orthogonal projection of X to $(ip)^\perp$. Let $\varepsilon_k \in \{\pm 1\}$ be such that $\Psi(Z_k) = \varepsilon_k Z_k$. Then one derives from equation (7):

$$\begin{aligned}
&\varepsilon_k(a^2\langle j'_k(AX)^h, (AY)^h\rangle - \langle j'_k Ap, iAp\rangle\langle iAX, AY\rangle) \\
&\qquad\qquad = a^2\langle j_k X^h, Y^h\rangle - \langle j_k p, ip\rangle\langle iX, Y\rangle
\end{aligned}$$

for $k = 1, 2$ and all $X, Y \in T_p S_a^{2m-1}$. Since $(AX)^h = AX^h$ and since A either commutes or anticommutes with i, we get, letting $\tau_k := \varepsilon_k A^{-1} j'_k A - j_k \in \mathfrak{su}(m)$:

$$a^2\langle \tau_k X^h, Y^h\rangle - \langle \tau_k p, ip\rangle\langle iX, Y\rangle = 0.$$

In particular, we obtain for all $p \in S_a^{2m-1}$ and all nonvanishing $X \in \mathrm{span}\,\{p, ip\}^\perp$, by letting $Y := iX$:

$$\langle \tau_k X, iX\rangle/|X|^2 = \langle \tau_k p, ip\rangle/|p|^2.$$

The hermitian map $i\tau_k$ must therefore be a scalar multiple of the identity, and τ_k be a scalar multiple of $i\mathrm{Id}$. This implies $\tau_k = 0$ because of $\tau_k \in \mathfrak{su}(m)$; equation (8) now follows.

In the second case (namely, with λ, λ' of type (5)) we have

$$\bar{\lambda}^k(X) = a^3\langle c_k x \times x, X\rangle$$

for $k = 1, 2$ and all $X \in T_x(L/T) = T_x S^2_{a/2}$, where $c_k := c_{Z_k}$. The factor a^3 is due to the fact that if $\pi : S^3_a \to S^2_{a/2}$ denotes the Riemannian submersion, then for the projection P as defined in 3.3.1 we have $|P(p)| = a|\pi(p)|$. We obtain

$$\begin{aligned} &d\bar{\lambda}^k(X, Y)/a^3 \\ &\quad = \langle c_k X \times x + c_k x \times X, Y\rangle - \langle c_k Y \times x + c_k x \times Y, X\rangle \\ &\quad = \langle c_k X \times x, Y\rangle + 2\langle c_k x \times X, Y\rangle - \langle c_k Y \times x, X\rangle \\ &\quad = \langle c_k X \times x, Y\rangle + 2\langle c_k x \times X, Y\rangle + \langle Y \times c_k x, X\rangle + \langle Y \times x, c_k X\rangle \\ &\quad = 3\langle c_k x \times X, Y\rangle. \end{aligned}$$

Note that in the third equation we have used $\mathrm{tr}(c_k) = 0$. Equation (6) now implies for $\overline{F}|_{S^2_{a/2}} = E \in \mathrm{O}(3)$:

$$\varepsilon_k \langle c'_k Ex \times EX, EY\rangle = \langle c_k x \times X, Y\rangle$$

for $k = 1, 2$ and all $X, Y \in T_x S^2_{a/2}$, where ε_k is defined as above. Letting $\tau_k := \varepsilon_k \det(E) E^{-1} c'_k E - c_k \in \mathrm{Sym}_0(\mathbb{R}^3)$ we obtain

$$\langle \tau_k x \times X, Y\rangle = 0$$

for all $X, Y \perp x$, which implies $\tau_k x \perp x$ for all $x \in S^2_{a/2}$. Since τ_k is symmetric, it must be zero; equation (9) now follows.

(ii) Suppose to the contrary that λ does not have Property (G); i.e., there is a 1-parameter family $\overline{F}_t \in \overline{\mathrm{Aut}}^T_{g_0}(M)$ such that $\overline{F}^*_t \Omega_\lambda \equiv \Omega_\lambda$. Proceeding as in the proof of (i) (in the special situation $\Psi = \mathrm{Id}$ and $\lambda = \lambda'$) we now obtain 1-parameter families $A_t \in \mathrm{SU}(m)$ (resp. $E_t \in \mathrm{SO}(3)$) preserving $d\lambda^L$ (resp. $d\bar{\lambda}^L$), and we derive that $j_Z \equiv A_t j_Z A_t^{-1}$ (resp. $c_Z \equiv E_t c_Z E_t^{-1}$) for each $Z \in \mathfrak{z}$ (cf. (8), (9)). But this contradicts the genericity assumption made on j (resp. c). q.e.d.

5. Isospectral metrics on spheres and balls which are equal to the standard metric on large subsets

The main idea of this section is to simultaneously multiply λ, λ' by some smooth scalar function f on the sphere (resp. the ball) which has small support but is chosen such that both isospectrality and nonisometry of the associated metrics $g_{f\lambda}$ and $g_{f\lambda'}$ continue to hold.

The following observation is trivial.

Remark 5.1. In the context of Theorem 1.6, let λ, λ' be two admissible $\mathfrak{z}$-valued 1-forms on M which satisfy $(*)$. Let $f \in C^\infty(M)$ be any function which is invariant under T and under each of the isometries F_μ occurring in $(*)$. Then $f\lambda$ and $f\lambda'$ are again admissible and satisfy $(*)$. In particular, $(M, g_{f\lambda})$ and $(M, g_{f\lambda'})$ are isospectral.

Proposition 5.2. *Let* $(M, g_0) = S^{2m+1} \subset \mathbb{R}^{2m} \oplus \mathbb{R}^2$*, resp.* $(M, g_0) = B^{2m+2} \subset \mathbb{R}^{2m} \oplus \mathbb{R}^2$*. Choose any* $\varphi \in C^\infty([0,1]^2)$ *which does not vanish identically; in case* $M = S^{2m+1}$ *we assume that* φ *does not vanish identically on the set* $\{(s, 1-s) \mid s \in [0,1]\}$*. Let* $f \in C^\infty(M)$ *be defined by* $f(p,q) := \varphi(|p|^2, |q|^2)$ *for* $(p,q) \in M \subset \mathbb{R}^{2m} \oplus \mathbb{R}^2$*. Then under the assumptions of Proposition* 3.2.5 (resp. 3.3.5) *on the pair* j, j' (*resp.* c, c') *which defines* λ, λ' *as in* (4) (*resp.* (5))*, the manifolds* $(M, g_{f\lambda})$ *and* $(M, g_{f\lambda'})$ *are isospectral and not isometric.*

Proof. The function f is obviously invariant under the action of $T = \mathbb{R}^2/\mathcal{L}$ as defined in 3.1(iii). Moreover, f is invariant under those F_μ occurring in the isospectrality proofs of Proposition 3.2.5 / 3.3.5: Recall that the F_μ used there were of the form (A_Z, Id) with $A_Z \in \mathrm{SO}(2m)$. The isospectrality statement thus follows from Remark 5.1 above.

For the nonisometry proof we must only slightly modify Proposition 4.3 and its proof. In both statements of that proposition we replace each occurrence of Ω_λ by $\Omega_{f\lambda}$, and similarly $\Omega_{\lambda'}$ by $\Omega_{f\lambda'}$. In the proof, we now choose $L := M_{a,b} \subset \hat{M}$ not arbitrarily, but such that $C := \varphi(a^2, b^2) \neq 0$. This is possible by the assumptions made on φ. The forms induced by $f\lambda$, $f\lambda'$ on L are then equal to the ones induced by $C\lambda$ and $C\lambda'$, respectively. But replacing $\lambda^L, {\lambda'}^L$ by $C\lambda^L, C{\lambda'}^L$ does not afflict any of the subsequent arguments of the proof. Thus both statements of Proposition 4.3, modified as described, remain true. q.e.d.

Theorem 5.3. *Let* $\varepsilon > 0$ *be given, and let* $m \geq 2$*. Then there exist nonisometric pairs of isospectral metrics on* S^{2m+1} *whose volume element is the standard one and which are equal to the round standard metric outside a subset of volume smaller than* ε*. For* $m \geq 3$ *there are even continuous families of such metrics. The analogous statements hold for* B^{2m+2} (*with metrics which are both Dirichlet and Neumann isospectral*)*. The mentioned subset of small volume can be chosen as a tubular neighborhood around any of the submanifolds* $S_a^{2m-1} \times S_b^1$ *with* $0 \leq a, b \leq 1$ *and* $a^2 + b^2 = 1$ (*in the case of* S^{2m+1})*, resp.* $a^2 + b^2 \leq 1$ (*in the case of* B^{2m+2})*.*

Proof. We can choose the function $\varphi \in C^\infty([0,1]^2)$ in Proposition 5.2 such that its support is contained in a sufficiently small rectangle around (a^2, b^2). Concerning the coincidence of the volume elements of g_0 and $g_{f\lambda}$ recall 1.5(iii). q.e.d.

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