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THE MASS OF ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

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Abstract

Motivated by certain problems in general relativity and Riemannian geometry, we study manifolds which are asymptotic to the hyperbolic space in a certain sense. It is shown that an invariant, the so called total mass, can be defined unambiguously. A positive mass theorem is established by using the spinor method.

1. Introduction

In [13] Min-Oo studied manifolds asymptotic to the hyperbolic space in a strong sense and proved a scalar curvature rigidity theorem. The asymptotics he assumed are very restrictive. People are interested in relaxing his asymptotics and having a numerical measure like the mass in the asymptotically flat case. This is the first motivation behind this work. Min-Oo's method was later refined by Andersson and Dahl [2]. The techniques developed there are crucial to this work.

Another motivation comes from general relativity. Einstein's theory of general relativity asserts that spacetime structure and gravitation are described by a spacetime $(N^4, \bar{g})$ where N^4 is a 4-dimensional manifold and $\bar{g}$ is a Lorentz metric satisfying Einstein's equation

$$G = 8\pi T, \tag{1}$$

where T is the energy-momentum tensor, $G = \mathrm{Ric}\,(\bar{g}) - \frac{1}{2}R(\bar{g}) \cdot \bar{g}$ is the Einstein tensor, $\mathrm{Ric}\,(\bar{g})$ is the Ricci tensor, and $R(\bar{g})$ is the scalar curvature of $\bar{g}$.

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The notion of energy and the law of conservation of energy play a key role in all physical theories. In general relativity, the energy properties of matter are represented by the energy-momentum tensor T. Thus the local energy density of matter as measured by a given observer is well-defined. On physical grounds the total energy is the sum of energy content of matter and the gravitational field energy. However there is no known meaningful notion of the energy density of the gravitational field in general relativity.

Despite this difficulty, there does exist a useful and meaningful notion of the total energy of an isolated system, i.e., the total energy-momentum 4-vector present in an asymptotically flat spacetime. One of the fundamental problems in general relativity is to understand the relationship between the local energy density and the total energy-momentum vector. The positive mass theorem, proved by Schoen-Yau [16] and Witten [19], and the Penrose conjecture can both be thought of as basic attempts in this direction. The Penrose conjecture in its Riemannian version has recently been proved by H. Bray [6] and by Huisken and Ilmanen [10].

There have been attempts by physicists to generalize these results to Einstein's theory with a negative cosmological constant. In this theory the spacetime $(N^4, \bar{g})$ satisfies the equation

$$G + \Lambda \bar{g} = 8\pi T, \tag{2}$$

where Λ is a negative constant which we normalize to be -3. The Anti-de Sitter spacetime $(\mathbb{R}^4, g_0)$ with $g_0 = -(1+r^2)dt^2+(1+r^2)^{-1}dr^2+r^2d\omega^2$ replaces the Minkowski spacetime as the groundstate of the theory. It is easy to see that each time slice is the hyperbolic space $\mathbb{H}^3$. Let (M^3, g) be a space-like hypersurface of $(N^4, \bar{g})$ with second fundamental form h_{ij} in N^4. Equation (2) implies that the local energy density μ and the local current density J^i are given by

$$\mu = \frac{1}{16\pi}\left[R - \sum_{i,j} h^{ij}h_{ij} + \left(\sum_i h_i^i\right)^2 + 6\right],$$

$$J^i = \frac{1}{8\pi}\sum_j \nabla_j \left[h^{ij} - \left(\sum_k h_k^k\right) g^{ij}\right],$$

where R is the scalar curvature of the metric g. These two equations are called the constraint equations for M^3 in N^4. The assumption of nonnegative energy density everywhere in N^4 implies that

$$\mu \geq \left(\sum_i J^i J_i \right)^{1/2} .$$

Thus we see that if we restrict our attention to 3-manifolds which have zero mean curvature in N^4, the constraint equations and the energy condition imply that M^3 has scalar curvature $R \geq -6$.

In the physics literature there have been many papers trying to define mass for a spacetime asymptotic to the Anti-de Sitter spacetime in a certain sense and to prove positivity assuming the energy condition, see e.g., [1], [3], [7] and references therein. The problem is much more complicated than the asymptotically flat case. It seems that the picture is still far from clear. In this paper we try to study the problem from a purely Riemannian geometric point of view, i.e., we study the special case of a spacelike hypersurface with zero second fundamental form in the spacetime. But the method can be generalized to incorporate a nonzero second fundamental form with appropriate decay. Another application of our results is to generalize the definition of Bondi mass of a constant mean curvature hyperboloid in an asymptotically flat spacetime.

The paper is organized as follows. In Section 2 we give the definition of an asymptotically hyperbolic manifold. Roughly speaking a Riemannian n-manifold (X, g) is asymptotically hyperbolic if it is conformally compact with the standard sphere (S^{n-1}, g_0) as its conformal infinity such that near infinity we have the expansion

$$g = \sinh^{-2}(r) \left(dr^2 + g_0 + \frac{r^n}{n} h + O\left(r^{n+1}\right) \right).$$

If X is spin, asymptotically hyperbolic and the scalar curvature $R \geq -n(n-1)$ we prove:

Theorem 1.1.

$$\int_{S^{n-1}} \operatorname{tr}_{g_0}(h) d\mu_{g_0} \geq \left| \int_{S^{n-1}} \operatorname{tr}_{g_0}(h) x d\mu_{g_0} \right| .$$

Moreover equality holds if and only if (X, g) is isometric to the hyperbolic space $\mathbb{H}^n$.

The proof is based on ideas developed by Min-Oo [13] and Andersson and Dahl [2]. The choice of g_0 is not unique and plays a subtle role in the asymtotics. This is studied in Section 3. The change of coordinates near infinity is analyzed in detail and the following result is proved:

Theorem 1.2. *The following quantity*

$$\left(\int_{S^2} m_{g_0}(x)d\mu_{g_0}\right)^2 - \left|\int_{S^2} m_{g_0}(x)xd\mu_{g_0}\right|^2$$

where $m_{g_0} = \operatorname{tr}_{g_0}(h)$, *is an invariant for an asymptotically hyperbolic 3-manifold.*

Theorem 1.1 tells us that this invariant is nonnegative when $R \geq -6$. Its squre root M is our definiton for total mass. We have done the calculation in dimension 3, but the same result should be true in any dimension. In Section 4, we study the special case that the manifold is globally conformal to the hyperbolic space. One can see even in this very simple case the mass is complicated and reveals interesting phenomena.

In closing the introduction, we briefly describe the Penrose conjecture in the asymptotically hyperbolic case, formulated as follows:

Conjecture. *Let* (X, g) *be an asymptotically hyperbolic manifold with* $R \geq -6$. *Then*

$$M \geq \sqrt{\frac{|N_0|}{16\pi}},$$

where M *is the total mass and* N_0 *is the outmost surface of mean curvature* $H = 2$. *Moreover the identity holds if and only if* X *is isometric to the Schwartzschild-Anti-de Sitter space outside their respective horizons.*

In Huisken and Ilmanen's work [10] on Penrose conjecture in the asymtotically flat case, the quasi-local mass proposed by Hawking plays an important role. Geroch made the key observation that Hawking mass is non-decreasing under the inverse mean curvature flow. It is easy to see that Hawking's definition can be generalized to this new setting. Let (X, g) be a 3-dimensional asymptotically hyperbolic manifold. We define the Hawking mass of a compact surface Σ in X to be

$$m(\Sigma) = \left(\frac{|\Sigma|}{16\pi}\right)^{1/2}\left(1 + \frac{|\Sigma|}{4\pi} - \frac{1}{16\pi}\int_\Sigma H^2 d\sigma\right), \tag{3}$$

where $|\Sigma|$ is the area of Σ and H its mean curvature. Compared to Hawking's original definition we add a term $\frac{|\Sigma|}{4\pi}$. Geroch's argument can be applied without much change to show that if Σ_t is an inverse mean curvature flow $m(\Sigma_t)$ is non-decreasing provided the scalar curvature $R \geq -6$ and Σ_t is connected.

Let $N_0 \subset X$ be an outmost surface of mean curvature $H = 2$. One can prove then N_0 minimizes the functional $A(\Sigma) - 2V(\Sigma)$ among all surfaces Σ outside N_0, where $A(\Sigma)$ is the area of Σ and $V(\Sigma)$ is the volume enclosed by N_0 and Σ. In particular N_0 is a strictly minimizing hull in the sense of [10]. The inverse mean curvature flow of N_0 in the classical sense does not necessarily exist for all time. In [10] Huisken and Ilmanen propose a generalized solution for the inverse mean curvature flow in a 3-manifold and establish existence and uniqueness under certain conditions on the initial surface and the 3-manifold. In particular their theorem applies to our situation. Let N_t be this generalized flow with initial condition N_0. They also prove that Geroch monotonicity still holds for the generalized flow, i.e., $m(N_t)$ is nondecreasing in t. It follows

$$\lim_{t\to\infty} m(N_t) \geq \sqrt{\frac{|N_0|}{16\pi}}. \tag{4}$$

To identify the limit will be a very difficult problem. We pick a metric g_0 on the conformal infinity. In terms of the corresponding special defining function r we write $g = \sinh^{-2}(r)(dr^2 + g_r)$ with $g_r = g_0 + \frac{h}{3}r^3 + \cdots$. Note $m(x) = \operatorname{tr}_{g_0} h$. By elementary calculation one can show:

Proposition 1.3. *Let S_r be the coordinate sphere, then*

$$\lim_{r\to 0} m(S_r) = \frac{1}{16\pi}\int_{S^2} m(x)\,d\mu_{g_0}.$$

One possible approach to proving the conjecture is to use Huisken-Ilmanen's generalized inverse mean curvature flow. As we have shown, the key is to study the asymptotic behavior of the flow N_t and relate $\lim_{t\to\infty} m(N_t)$ to the total mass M of (X, g). We expect that there exists a unique choice of coordinates near infinity such that in these coordinates the flow N_t is asymptotic to the coordinate sphere defined by $r = t$. If this true by the above proposition we get

$$\lim_{t\to\infty} m(N_t) \geq= \frac{1}{16\pi}\int_{S^2} \operatorname{tr} h.$$

As $\frac{1}{16\pi}\int_{S^2} \operatorname{tr} h \geq M$ this would prove the conjecture. But this problem of studying the asymptotics of N_t seems much harder than the asymptotically flat case.

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The referee also informed the author that similar results on the invariant definition of mass have been subsequently obtained by Chrusciel and Nagy.

2. Definitions and the spinor argument

Let $\overline{X}$ be a compact n-dimensional manifold with boundary M and Let X be the interior. If r is a smooth function on $\overline{X}$ with a first order zero on the boundary of $\overline{X}$, positive on X, then r is called a defining function. Let g be a Riemannian metric on X.

Definition 2.1. The Riemannian manifold (X, g) is called conformally compact if for any defining function r, $\bar{g} = r^2 g$ extends as a C^3 metric on $\overline{X}$.

Example. The hyperbolic space

$$\left(B^n, g = \frac{4}{(1-|x|^2)^2} dx^2\right)$$

is apparently conformally compact.

The restriction of $\bar{g}$ to M gives a metric on M. This metric changes by a conformal factor if the defining function is changed, so M has a well-defined conformal structure. We call M with this induced conformal structure the conformal infinity of (X, g). A straightforward computation (see [12]) shows that the sectional curvatures of g approach $-|dr|^2_{\bar{g}}$ on M. Accordingly, one says g is weakly asymptotically hyperbolic if $|dr|^2_{\bar{g}} = 1$ on M. One can easily check this definition is independent of the choice of a defining function. The following lemma is proved by several authors (see [2]).

Lemma 2.2. *Let (X, g) be a conformally compact manifold and g_0 a metric on M which represents the induced conformal structure. If (X, g) is weakly asymptotically hyperbolic, then there is a unique defining function r in a collar neighborhood of $M = \partial\overline{X}$, such that $g = \sinh^{-2}(r)(dr^2 + g_r)$, with g_r an r-dependent family of metrics on M*

such that $g_r|_{r=0}$ is the given metric g_0. We call r the special defining function determined by g_0.

Definition 2.3. A weakly asymptotically hyperbolic manifold (X, g) is called asymptotically hyperbolic if it satisfies:

1. The conformal infinity is the standard sphere (S^{n-1}, g_0).

2. Let r be the corresponding special defining function so that we can write

$$g = \sinh^{-2}(r)(dr^2 + g_r) \tag{5}$$

in a collar neighborhood of the conformal infinity. Then

$$g_r = g_0 + \frac{r^n}{n} h + O(r^{n+1}), \tag{6}$$

where h is a symmetric 2-tensor on S^{n-1}. Moreover the asymptotic expansion can be differentiated twice.

Let $\pi : \overline{X}^\epsilon := \{p \in \overline{X} | r(p) < \epsilon\} \to M$ be the nearest-point projection, i.e., $\pi(p) \in M$ is the nearest point to p with respect to the metric $\bar{g} = (dr^2 + g_r)$. This is a well-defined smooth map if ϵ is very small. Let $\{x^i\}$ be local coordinates on S^{n-1}. We introduce local coordinates on X^ϵ such that the coordinates of $p \in X^\epsilon$ is (r, x^i), where (x^i) is the coordinates of $\pi(p)$. In terms of such local coordinates we write $\bar{g} = dr^2 + g_{ij}(r, x)dx^i dx^j$.

The tensor h in the definition of an asymptotically hyperbolic manifold X measures the deviation of the space from the hyperbolic space $\mathbb{H}^n$. If we assume the scalar curvature $R \geq -n(n-1)$ we expect certain restriction on h. Using ideas developed by Min-Oo [13] and Andersson and Dahl [2] we prove the following theorem.

Let $\mathcal{S}$ be a representation of the Clifford algebra Cl_n and $\mathcal{S}(X)$ the corresponding spinor bundle over the spin manifold X.

Theorem 2.4. *If X is spin and has scalar curvature $R \geq -n(n-1)$, we have*

$$\int_{S^{n-1}} \operatorname{tr}_{g_0}(h) \left(1 + \sqrt{-1}\langle x \cdot u, u\rangle\right) d\mu_{g_0} \geq 0,$$

for any unit vector $u \in \mathcal{S}$. Moreover the equality holds for some u if and only if X is isometric to the hyperbolic space $\mathbb{H}^n$.

Remark. In the theorem we identify the conformal infinity (S^{n-1}, g_0) as the unit sphere in $\mathbb{R}^n \subset Cl_n$.

We first introduce some basic constructions and notations. We will follow mostly notations in Andersson-Dahl [2]. We denote the Levi-Civita connection on the spinor bundle $\mathcal{S}(X)$ by ∇. We define a modified connection on $\mathcal{S}(X)$ by

$$\widehat{\nabla}_V = \nabla_V + \frac{\sqrt{-1}}{2} V\cdot,$$

where $V\cdot$ is the Clifford action of the tangent vector V on spinors. Spinors parallel with respect to this connection are called (imaginary) Killing spinors. Let

$$\widehat{D} = e_i \cdot \widehat{\nabla}_{e_i}$$

be the corresponding Dirac operator. We have the following Lichnerowicz formula

$$\widehat{D}^* \widehat{D} = \widehat{\nabla}^* \widehat{\nabla} + \frac{\widehat{R}}{4}, \tag{7}$$

where $\widehat{R} = R + n(n-1)$.

On the hyperbolic space $\mathbb{H}^n$ there is a full set of Killing spinors. We describe them using the disc model $B^n = \{x \in \mathbb{R}^n | |x| < 1\}$ with the metric $g = \frac{4}{(1-|x|^2)^2} dx^2$. Let $e_i = \frac{1-|x|^2}{2} \frac{\partial}{\partial x_i}$. With respect to the orthonormal frame $\{e_1, \dots, e_n\}$ we get a trivialization of the spinor bundle. In this trivialization the Killing spinors are

$$\phi_u(x) = \left(\frac{2}{1 - |x|^2} \right)^{1/2} (1 - \sqrt{-1} x\cdot) u, \tag{8}$$

where $u \in \mathcal{S}$. Now we are ready to prove the theorem.

Proof of Theorem 2.4. We introduce the background hyperbolic metric on a neighborhood $\mathcal{O}$ of the conformal infinity by setting

$$g' = \sinh^{-2}(r)(dr^2 + g_0).$$

Define the gauge transformation A by

$$g(AV, AW) = g'(V, W), \ g(AV, W) = g(V, AW).$$

Note that $A(\frac{\partial}{\partial r}) = \frac{\partial}{\partial r}$ and $g_0(V,W) = g_r(AV, AW)$ on S^{n-1}. By our assumption

$$B \triangleq A - I = O(r^n).$$

Let ∇' be the Levi-Civita connection of g'. Define the gauge transformed connection $\overline{\nabla}$ by

$$\overline{\nabla} V = A\nabla'(A^{-1}V).$$

The connection $\overline{\nabla}$ preserves the metric g and has torsion

$$\begin{aligned} \overline{T}(V,W) &= \overline{\nabla}_V W - \overline{\nabla}_V W - [V,W] \\ &= -(\nabla'_V A)A^{-1}W + (\nabla'_W A)A^{-1}V. \end{aligned} \tag{9}$$

The difference $\Lambda_V = \overline{\nabla}_V - \nabla_V$ is given by

$$2\langle \Lambda_U V, W\rangle = \langle \overline{T}(U,V), W\rangle - \langle \overline{T}(V,W), U\rangle + \langle \overline{T}(W,U), V\rangle. \tag{10}$$

Here and below $\langle\,,\rangle$ refers to the metric g. Let $\{e'_i\}$ be a local orthonormal frame with respect to g' with $e'_1 = -\sinh(r)\frac{\partial}{\partial r}$. Set $e_i = Ae'_i$. Then $\{e_i\}$ is an orthonormal frame for g. Let $\{\omega_{ij}\}$ and $\{\varpi_{ij}\}$ be the connection 1-forms for ∇ and $\overline{\nabla}$, respectively. On a spinor ϕ, the two connections are related by

$$\begin{aligned} \overline{\nabla}_V\phi - \nabla_V\phi &= \frac{1}{4}\sum_{ij}(\omega_{ij}(V) - \varpi_{ij}(V))e_ie_j\cdot\phi_0 \\ &= \frac{1}{4}\sum_{i<j}\langle\Lambda_V e_i, e_j\rangle e_ie_j\phi. \end{aligned}$$

By (9) and (10) we have the estimate

$$|(\Lambda_V e_i, e_j)| \le C|A^{-1}||\nabla' A||V|. \tag{11}$$

The gauge transformation A induces a map between the two spinor bundles $\mathcal{S}(X,g)$ and $\mathcal{S}(X,g')$ near infinity, also denoted by A (for detail see [2]). Let ϕ' be a Killing spinor. Let f be a smooth function with supp(df) compact, $f = 0$ outside $\mathcal{O}$ and $f = 1$ near infinity. Define $\phi_0 = fA\phi'$. We have

$$\widehat{\nabla}_V\phi_0 = (\nabla_V - \overline{\nabla}_V)\phi_0 - \frac{\sqrt{-1}}{2}(AV - V)\phi_0$$

near infinity (where $f = 1$). By (11)

$$|\widehat{\nabla}\phi_0|^2 \le C(|A^{-1}|^2|\nabla' A|^2 + |A - Id|^2)r^{-1} = O(r^{2n-3}).$$

Then it is easy to see that $\widehat{D}\phi_0 \in L^2(X, \mathcal{S}(X))$. By standard method (see [2]), there exists a spinor $\epsilon \in H^1(X, \mathcal{S}(X))$ such that

$$\widehat{D}(\epsilon + \phi_0) = 0.$$

Let $\phi = \phi_0 + \epsilon$. By the Lichnerocwicz formula (7) we have

$$\int_X \left(|\widehat{\nabla}\phi|^2 + \frac{\widehat{R}}{4}|\phi|^2 \right) = \lim_{\delta \to 0} \int_{X_\delta} \left\langle \left(\widehat{\nabla}_\nu + \nu \widehat{D} \right) \phi, \phi \right\rangle,$$

where $X_\delta = \{x \in X | r(x) = \delta\}$. One can show that the limit on the right hand side is equal to

$$\lim_{\delta \to 0} \int_{X_\delta} \left\langle \left(\widehat{\nabla}_\nu + \nu \widehat{D} \right) \phi_0, \phi_0 \right\rangle.$$

The problem is then to calculate this limit.

$$\begin{aligned}
\left\langle \left(\widehat{\nabla}_\nu + \nu \widehat{D} \right) \phi_0, \phi_0 \right\rangle &= \sum_i \left\langle (\delta_{1i} + e_1 e_i) \widehat{\nabla}_{e_i} \phi_0, \phi_0 \right\rangle \\
&= \sum_i \left\langle (\delta_{1i} + e_1 e_i) \overline{\nabla}_{e_i} \phi_0, \phi_0 \right\rangle \\
&\quad + \frac{\sqrt{-1}}{2} \sum_i \langle (\delta_{1i} + e_1 e_i) e_i \phi_0, \phi_0 \rangle \\
&\quad + \frac{1}{4} \sum_{ikl} (\omega_{kl}(e_i) - \overline{\omega}_{kl}(e_i)) \langle (\delta_{1i} + e_1 e_i) e_k e_l \phi_0, \phi_0 \rangle \\
&= \frac{1}{4} \sum_{ikl} (\omega_{kl}(e_i) - \overline{\omega}_{kl}(e_i)) \langle (\delta_{1i} + e_1 e_i) e_k e_l \phi_0, \phi_0 \rangle \\
&\quad - \frac{\sqrt{-1}}{2} \sum_i \langle e_1 e_i (A e_i - e_i) \phi_0, \phi_0 \rangle \\
&= I + II.
\end{aligned}$$

We can write the first term as

$$\begin{aligned}
&\frac{1}{4} \sum_{i \neq 1; k, l} (\omega_{kl}(e_i) - \overline{\omega}_{kl}(e_i)) \langle e_1 e_i e_k e_l \phi_0, \phi_0 \rangle \\
&\qquad\qquad = \frac{1}{2} \sum_{i, k \neq 1} (\omega_{1k}(e_i) - \overline{\omega}_{1k}(e_i)) \langle e_i e_k \phi_0, \phi_0 \rangle
\end{aligned}$$

$$- \frac{1}{2} \sum_{i,k \neq 1} (\omega_{ik}(e_i) - \bar{\omega}_{ik}(e_i)) \langle e_1 e_k \phi_0, \phi_0 \rangle$$
$$+ \frac{1}{4} \sum_{ikl} (\omega_{kl}(e_i) - \bar{\omega}_{kl}(e_i)) \langle \sigma_{1ikl} \phi_0, \phi_0 \rangle,$$

where $\sigma_{1ikl} = e_1 e_i e_k e_l$ if $1, i, k, l$ are different, otherwise it is zero. We only need to consider the real part. It is easy to see that

$$I \sim \frac{1}{2} \sum_i (\omega_{i1}(e_i) - \bar{\omega}_{i1}(e_i)) \langle \phi_0, \phi_0 \rangle + \frac{1}{4} \sum_{ikl} (\omega_{kl}(e_i) - \bar{\omega}_{kl}(e_i)) \langle \sigma_{1ikl} \phi_0, \phi_0 \rangle,$$

where by $\sim$ we mean the real parts of the two sides are equal. By (10) we have

$$2(\omega_{kl}(e_i) - \bar{\omega}_{kl}(e_i)) = -\langle \bar{T}(e_i, e_k), e_l \rangle + \langle \bar{T}(e_i, e_l), e_k \rangle + \langle \bar{T}(e_k, e_l), e_i \rangle.$$

The last two terms taken together are symmetric in ik and vanishes when summed against σ_{1ikl}. Hence

$$I \sim \left[\frac{1}{2} \langle (\nabla'_{e_1} A) A^{-1} e_i, e_i \rangle - \frac{1}{2} \langle (\nabla'_{e_i} A) A^{-1} e_1, e_i \rangle \right] \langle \phi_0, \phi_0 \rangle$$
$$+ \frac{1}{4} \sum_{ikl} \langle (\nabla'_{e_i} A) A^{-1} e_k, e_l \rangle \langle \sigma_{1ikl} \phi_0, \phi_0 \rangle.$$

We write $Ae_i = A_i^j e_j$ and similarly for B. To calculate the term II we use the fact that A is symmetric to get

$$\begin{aligned} \langle e_1 e_i (Ae_i - e_i) \phi_0, \phi_0 \rangle &= (A_i^j - \delta_i^j) \langle e_1 e_i e_j \phi_0, \phi_0 \rangle \\ &= -(A_i^i - \delta_i^i) \langle e_1 \phi_0, \phi_0 \rangle \\ &= -(\operatorname{tr} A - n) \langle e_1 \phi_0, \phi_0 \rangle. \end{aligned}$$

Therefore we get

$$\begin{aligned} &\langle (\widehat{\nabla}_\nu + \nu \widehat{D}) \phi_0, \phi_0 \rangle \\ &\quad \sim \left[\frac{1}{2} \langle (\nabla'_{e_1} A) A^{-1} e_i, e_i \rangle - \frac{1}{2} \langle (\nabla'_{e_i} A) A^{-1} e_1, e_i \rangle \right] \langle \phi_0, \phi_0 \rangle \\ &\qquad + \frac{1}{4} \sum_{ikl} \langle (\nabla'_{e_i} A) A^{-1} e_k, e_l \rangle \langle \sigma_{1ikl} \phi_0, \phi_0 \rangle + \frac{\sqrt{-1}}{2} (\operatorname{tr} A - n) \langle e_1 \phi_0, \phi_0 \rangle. \end{aligned}$$

To estimate the first term we have

$$\langle (\nabla'_{e_i} A) A^{-1} e_k, e_l \rangle \sigma_{1ikl}$$

$$
\begin{aligned}
&= g'(A^{-1}(\nabla'_{e_i}A)e'_k, e'_l)\sigma_{1ikl}\\
&= g'(A^{-1}\nabla'_{e_i}(Ae'_k) - \nabla'_{e_i}e'_k, e'_l)\sigma_{1ikl}\\
&= \Big[e_iA^j_k g'(A^{-1}e'_j, e'_l) + A^j_k g'(A^{-1}\nabla'_{e_i}e'_j, e'_l) - g'(\nabla'_{e_i}e'_k, e'_l)\Big]\,\sigma_{1ikl}\\
&= \Big[e_iA^l_k + e_iA^j_k g'((A^{-1}-I)e'_j, e'_l) + (A^j_k - \delta^j_k)g'(A^{-1}\nabla'_{e_i}e'_j, e'_l)\\
&\qquad\quad + g'((A^{-1}-I)\nabla'_{e_i}e'_k, e'_l)\Big]\,\sigma_{1ikl}\\
&= \Big[e_iA^l_k + e_iA^j_k g'((A^{-1}-I)e'_j, e'_l) + B^j_k g'(\nabla'_{e_i}e'_j, e'_l)\\
&\qquad + B^j_k g'((A^{-1}-I)\nabla'_{e_i}e'_j, e'_l) + g'(\nabla'_{e_i}e'_k, (A^{-1}-I)e'_l)\Big]\,\sigma_{1ikl}\\
&= \Big[e_iA^l_k + B^j_k g'(\nabla'_{e_i}e'_j, e'_l) - B^j_l g'(\nabla'_{e_i}e'_k, e'_j) + O(r^{2n})\Big]\,\sigma_{1ikl}\\
&= \Big[e_iA^l_k + B^j_k g'(\nabla'_{e_i}e'_j, e'_l) + B^j_l g'(\nabla'_{e_i}e'_j, e'_k) + O(r^{2n})\Big]\,\sigma_{1ikl}\\
&= O(r^{2n-2}).
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
&\langle(\nabla'_{e_1}A)A^{-1}e_i, e_i\rangle\\
&\qquad = g'(A^{-1}(\nabla'_{e_1}A)e'_i, e'_i)\\
&\qquad = g'(A^{-1}\nabla'_{e_1}(Ae'_i), e'_i) - g'(\nabla'_{e_1}e'_i, e'_i)\\
&\qquad = e_1A^j_i g'(A^{-1}e'_j, e'_i) + A^j_i g'(A^{-1}\nabla'_{e_1}e'_j, e'_i)\\
&\qquad\quad - g'(\nabla'_{e_1}e'_i, e'_i)\\
&\qquad = e_1A^i_i + B^j_i g'(\nabla'_{e_1}e'_j, e'_i) - g'(B\nabla'_{e_1}e'_i, e'_i) + O(r^{2n})\\
&\qquad = e_1\mathrm{tr}\,(A) + O(r^{2n-2}),
\end{aligned}
$$

and

$$
\begin{aligned}
\langle(\nabla'_{e_i}A)A^{-1}e_1, e_i\rangle &= g'(A^{-1}(\nabla'_{e_i}A)e'_1, e'_i)\\
&= g'((A^{-1}-I)\nabla'_{e_i}e'_1, e'_i)\\
&= -g'(B\nabla'_{e_i}e'_1, e'_i) + O(r^{2n})\\
&= -B^j_i g'(\nabla'_{e_i}e'_1, e'_j) + O(r^{2n})\\
&= -B^j_i g'(\nabla'_{e'_i}e'_1, e'_j) + O(r^{2n})\\
&= -B^j_i \cosh(r)\delta_{ij} + O(r^{2n})\\
&= -\mathrm{tr}\,(B) + O(r^{n+2}).
\end{aligned}
$$

Therefore we have

$$\left\langle \left(\widehat{\nabla}_\nu + \nu \widehat{D} \right) \phi_0, \phi_0 \right\rangle \sim (e_1 \mathrm{tr}\,(B) + \mathrm{tr}\,(B)) \langle \phi_0, \phi_0 \rangle + \frac{\sqrt{-1}}{2} \mathrm{tr}\,(B) \langle e_1 \phi_0, \phi_0 \rangle + O(r^{n+2}). \tag{12}$$

By our assumption we have

$$\mathrm{tr}\,(B) = -\frac{1}{2n} \mathrm{tr}\,(h) r^n + O(r^{n+1}).$$

By (8) we have

$$\lim_{r \to 0} r \langle \phi_0, \phi_0 \rangle = |(1 - \sqrt{-1} x \cdot) u|^2,$$

$$\lim_{r \to 0} r \langle e_1 \phi_0, \phi_0 \rangle = \sqrt{-1} |(1 - \sqrt{-1} x \cdot) u|^2.$$

The volume form $d\mu_r$ on X_r is asymptotically $r^{-(n-1)} d\mu_{g_0}$, where $d\mu_{g_0}$ is the volume form on S^{n-1}. Putting all these formulas together we get

$$\lim_{r \to 0} \int_{X_r} \left\langle \left(\widehat{\nabla}_\nu + \nu \widehat{D} \right) \phi_0, \phi_0 \right\rangle = \frac{1}{2} \int_{S^{n-1}} \mathrm{tr}\,(h) \left(1 + \sqrt{-1} \langle x \cdot u, u \rangle \right) d\mu_{g_0}.$$

Therefore

$$\frac{1}{2} \int_{S^{n-1}} \mathrm{tr}\,(h) \left(1 + \sqrt{-1} \langle x \cdot u, u \rangle \right) d\mu_{g_0} = \int_X \left(|\widehat{\nabla} \phi|^2 + \frac{\widehat{R}}{4} |\phi|^2 \right) \geq 0.$$

If the equality holds, then ϕ is a Killing spinor on X. By a theorem due to Baum [4], X is isometric to a warp product $P \times \mathbb{R}$ with metric $e^{2t} h + dt^2$. As X is asymptotically hyperbolic, this implies that X is isometric to the hyperbolic space $\mathbb{H}^n$. q.e.d.

It is better to reformulate the result without reference to spinors. We denote the unit sphere in $\mathcal{S}$ by $\mathcal{S}_1$. First we have

$$\sqrt{-1} \langle x \cdot u, u \rangle = \sum_{i=1}^n x_i \sqrt{-1} \langle e_i \cdot u, u \rangle = x \cdot \xi_u,$$

where $\xi_u = \sum_{i=1}^n \sqrt{-1} \langle e_i \cdot u, u \rangle e_i \in \mathbb{R}^n$. Obviously we can take $\{e_i\}$ to be any orthonormal basis of $\mathbb{R}^n$ in the expression for ξ_u. Because x acts on $\mathcal{S}$ as a skew-Hermitian operator and its square is $-|x|^2$, we have

$$\sup{}_{u \in \mathcal{S}_1} |x \cdot \xi_u| = \sup{}_{u \in \mathcal{S}_1} |\sqrt{-1} \langle x \cdot u, u \rangle| = |x|. \tag{13}$$

Let $\varrho : \mathrm{Spin(n)} \to \mathrm{SO(n)}$ be the canonical homomorphism. For any $\sigma \in \mathrm{Spin(n)}$ and any $u \in \mathcal{S}_1$ we have

$$\begin{aligned}
\xi_{\sigma u} &= \sum_i \sqrt{-1}\langle e_i \cdot \sigma u, \sigma u\rangle e_i \\
&= \sum_i \sqrt{-1}\langle \sigma^t e_i \sigma u, u\rangle e_i \\
&= \sum_i \sqrt{-1}\langle \varrho(\sigma) e_i u, u\rangle e_i \\
&= \varrho(\sigma)^{-1}\left(\sum_i \sqrt{-1}\langle \varrho(\sigma) e_i u, u\rangle \varrho(\sigma) e_i\right) \\
&= \varrho(\sigma)^{-1}\xi_u.
\end{aligned}$$

Therefore $\forall u \in \mathcal{S}_1$ the orbit $\{\xi_{\sigma u} \in \mathbb{R}^n | \sigma \in \mathrm{Spin(n)}\}$ is the sphere of radius $|\xi_u|$ in $\mathbb{R}^n$. By (13) we conclude that

$$|\xi_u| \le 1, \forall u \in \mathcal{S}_1; \exists u_0 \in \mathcal{S}_1, |\xi_{u_0}| = 1.$$

It follows that $\{\xi_{\sigma u_0} \in \mathbb{R}^n | \sigma \in \mathrm{Spin(n)}\}$ is the unit sphere in $\mathbb{R}^n$. Therefore the conclusion of Theorem 2.4 is equivalent to

$$\int_{S^{n-1}} \mathrm{tr}_{\,g_0}(h)(1 + x\cdot\xi)d\mu_{g_0} \ge 0, \forall \xi \in S^{n-1}.$$

We can restate Theorem 2.4 as follows.

Theorem 2.5. *Let (X, g) be an asymptotically hyperbolic manifold as defined in Definition 2.3. If X is spin and has scalar curvature $R \ge -n(n-1)$ then we have*

$$\int_{S^{n-1}} \mathrm{tr}_{\,g_0}(h)d\mu_{g_0} \ge \left|\int_{S^{n-1}} \mathrm{tr}_{\,g_0}(h)x d\mu_{g_0}\right|.$$

Moreover equality holds if and only if (X, g) is isometric to the hyperbolic space $\mathbb{H}^n$.

Remark. In the above formulation we still identify the conformal infinity (S^{n-1}, g_0) as the unit sphere in $\mathbb{R}^n$. If we want to avoid using this identification we should state the result as

$$\int_{S^{n-1}} \mathrm{tr}_{\,g_0}(h)d\mu_{g_0} \ge \left|\int_{S^{n-1}} \mathrm{tr}_{\,g_0}(h)\mathcal{F}_{g_0}(x)d\mu_{g_0}\right|.$$

where $\mathcal{F}_{g_0} : S^{n-1} \to \mathbb{R}^n$ is a map such that the coordinate functions form an orthogonal basis for the first eigenspace of $-\triangle_{g_0}$ and each satisfies $\int_{S^{n-1}} f^2 d\mu_{g_0} = \omega_{n-1}$(the volume of S^{n-1}).

3. Change of coordinates and the total mass

In the definition of an asymptotically hyperbolic manifold we use a round metric g_0 on S^{n-1}, so a natural problem is to understand its role in the definition. For simplicity we work in dimension three. Let $\widehat{g}_0 = e^{2\omega_0} g_0$ be another round metric on S^2. The function ω_0 is given by the following formula

$$\omega_0(x) = -\log(\cosh(t) + \sinh(t)\xi \cdot x), \tag{14}$$

for some $t \geq 0$ and $\xi \in S^2$. One can verify that ω_0 satisfies the following equation

$$\nabla^2 \omega_0 = \frac{1}{2}(1 - e^{2\omega_0} - |\nabla \omega_0|^2) g_0 + d\omega_0 \otimes d\omega_0. \tag{15}$$

If we replace the metric g_0 on M by the metric $\widehat{g}_0 = e^{2\omega_0} g_0$ in the same conformal class, we get a new special defining function $\widehat{r}$ such that $g = \sinh^{-2}(\widehat{r})(d\widehat{r}^2 + \widehat{g}_{\widehat{r}})$. Set $e^\omega = \frac{\sinh(\widehat{r})}{\sinh(r)}$. Then

$$\left.\frac{\partial \widehat{r}}{\partial r}\right|_{S^2} = e^{\omega_0}, \tag{16}$$

$$\widehat{\bar{g}} = \sinh^2(\widehat{r}) g = \left(\frac{\sinh(\widehat{r})}{\sinh(r)}\right)^2 \bar{g} = e^{2\omega} \bar{g}. \tag{17}$$

The function ω (hence $\widehat{r}$) is determined by

$$\omega|_M = \omega_0 \text{ and } |d\widehat{r}|_{\widehat{\bar{g}}} = 1 \text{ in a neighborhood of } M.$$

From $e^\omega = \frac{\sinh(\widehat{r})}{\sinh(r)}$ we get $\cosh(\widehat{r}) d\widehat{r} = e^\omega(\cosh(r) dr + \sinh(r) d\omega)$. Thus

$$\begin{aligned}
\cosh^2(\widehat{r}) |d\widehat{r}|^2_{\widehat{\bar{g}}} &= |\cosh(r) dr + \sinh(r) d\omega|^2_{\bar{g}} \\
&= \cosh^2(r) + 2\cosh(r)\sinh(r)(dr, d\omega)_{\bar{g}} + \sinh^2(r)|d\omega|^2_{\bar{g}} \\
&= \cosh^2(r) + 2\cosh(r)\sinh(r)\frac{\partial \omega}{\partial r}
\end{aligned}$$

$$+\sinh^2(r)\left(\left(\frac{\partial\omega}{\partial r}\right)^2+|d_M\omega|^2_{g_r}\right).$$

Therefore the condition $|d\widehat{r}|_{\widehat{\widehat{g}}}=1$ is equivalent to

$$\sinh(r)(e^{2\omega}-1)=2\cosh(r)\frac{\partial\omega}{\partial r}+\sinh(r)\left(\left(\frac{\partial\omega}{\partial r}\right)^2+|d_M\omega|^2_{g_r}\right). \tag{18}$$

From this equation one can easily prove

$$\left.\frac{\partial\omega}{\partial r}\right|_{r=0}=0, \tag{19}$$

$$\left.\frac{\partial^2\omega}{\partial r^2}\right|_{r=0}=\frac{1}{2}\left[e^{2\omega_0}-1-|d\omega_0|^2_{g_0}\right], \tag{20}$$

$$\left.\frac{\partial^3\omega}{\partial r^3}\right|_{r=0}=0. \tag{21}$$

To write the metric $\widehat{\widehat{g}}$ in the original coordinates (r,x) we calculate

$$\begin{aligned}
\widehat{\widehat{g}} &= d\widehat{r}^2+\widehat{g}_{ij}(\widehat{r},\widehat{x})d\widehat{x}^i d\widehat{x}^j\\
&= e^{2\omega}\cosh^{-2}(\widehat{r})(\cosh^2(r)dr^2+\sinh(2r)drd\omega+\sinh^2(r)d\omega^2)\\
&\quad +\widehat{g}_{ij}\left(\frac{\partial\widehat{x}^i}{\partial r}dr+\frac{\partial\widehat{x}^i}{\partial x^k}dx^k\right)\left(\frac{\partial\widehat{x}^j}{\partial r}dr+\frac{\partial\widehat{x}^j}{\partial x^l}dx^l\right)\\
&= \left[e^{2\omega}\left(\frac{\cosh^2(r)}{\cosh^2(\widehat{r})}+\frac{\sinh(2r)}{\cosh^2(\widehat{r})}\omega_r+\frac{\sinh^2(r)}{\cosh^2(\widehat{r})}{\omega_r}^2\right)\widehat{g}_{ij}\frac{\partial\widehat{x}^i}{\partial r}\frac{\partial\widehat{x}^j}{\partial r}\right]dr^2\\
&\quad +\left[e^{2\omega}\left(\frac{\sinh(2r)}{\cosh^2(\widehat{r})}\frac{\partial\omega}{\partial x^k}+2\frac{\sinh^2(r)}{\cosh^2(\widehat{r})}\omega_r\frac{\partial\omega}{\partial x^k}ig\right)\right.\\
&\qquad\quad \left.+2\widehat{g}_{ij}\frac{\partial\widehat{x}^i}{\partial r}\frac{\partial\widehat{x}^j}{\partial x^k}\right]drdx^k\\
&\quad +\left[e^{2\omega}\frac{\sinh^2(r)}{\cosh^2(\widehat{r})}\frac{\partial\omega}{\partial x^k}\frac{\partial\omega}{\partial x^l}+\widehat{g}_{ij}\frac{\partial\widehat{x}^i}{\partial x^k}\frac{\partial\widehat{x}^j}{\partial x^l}\right]dx^kdx^l.
\end{aligned}$$

By (17) we must have

(22)

$$e^{2\omega}\left(\frac{\cosh^2(r)}{\cosh^2(\widehat{r})}+\frac{\sinh(2r)}{\cosh^2(\widehat{r})}\omega_r+\frac{\sinh^2(r)}{\cosh^2(\widehat{r})}{\omega_r}^2\right)+\widehat{g}_{ij}\frac{\partial\widehat{x}^i}{\partial r}\frac{\partial\widehat{x}^j}{\partial r}=e^{2\omega},$$

$$e^{2\omega}\left(\frac{\sinh(2r)}{\cosh^2(\widehat{r})}\frac{\partial\omega}{\partial x^k}+2\frac{\sinh^2(r)}{\cosh^2(\widehat{r})}\omega_r\frac{\partial\omega}{\partial x^k}\right)+2\widehat{g}_{ij}\frac{\partial\widehat{x}^i}{\partial r}\frac{\partial\widehat{x}^j}{\partial x^k}=0, \tag{23}$$

$$e^{2\omega}\frac{\sinh^2(r)}{\cosh^2(\widehat{r})}\frac{\partial\omega}{\partial x^k}\frac{\partial\omega}{\partial x^l}+\widehat{g}_{ij}\frac{\partial\widehat{x}^i}{\partial x^k}\frac{\partial\widehat{x}^j}{\partial x^l}=e^{2\omega}g_{kl}(r,x). \tag{24}$$

By definition $\widehat{x}^i|_{r=0}=x^i$ and hence in particular $\frac{\partial\widehat{x}^i}{\partial x^j}|_{r=0}=\delta_{ij}$. Then from Equation (23) we get $\frac{\partial\widehat{x}^i}{\partial r}|_{r=0}=0$. By these identities differentiation of (23) gives

$$g_{ik}\frac{\partial^2\widehat{x}^i}{\partial r^2}\bigg|_{r=0}=-\frac{\partial\omega_0}{\partial x^k}. \tag{25}$$

Differentiation of (24) with respect to r gives

$$\begin{aligned}e^{2\omega}\left(\frac{\partial g_{kl}}{\partial r}+2\frac{\partial\omega}{\partial r}g_{kl}\right)=&\left(\frac{\partial\widehat{g}_{ij}}{\partial\widehat{r}}\frac{\partial\widehat{r}}{\partial r}+\frac{\partial\widehat{g}_{ij}}{\partial\widehat{x}^m}\frac{\partial\widehat{x}^m}{\partial r}\right)\frac{\partial\widehat{x}^i}{\partial x^k}\frac{\partial\widehat{x}^j}{\partial x^l}\\&+\widehat{g}_{ij}\frac{\partial}{\partial r}\left(\frac{\partial\widehat{x}^i}{\partial x^k}\frac{\partial\widehat{x}^j}{\partial x^l}\right)\\&+\frac{\partial}{\partial r}\left(e^{2\omega}\cosh^{-2}(\widehat{r})\sinh^2(r)\frac{\partial\omega}{\partial x^k}\frac{\partial\omega}{\partial x^l}\right).\end{aligned} \tag{26}$$

This easily implies

$$\frac{\partial\widehat{g}_{ij}}{\partial\widehat{r}}\bigg|_{S^2}=0. \tag{27}$$

Differentiation of (26) with respect to r gives

$$\begin{aligned}&e^{2\omega}\left[\frac{\partial^2 g_{kl}}{\partial r^2}+2\frac{\partial^2\omega}{\partial r^2}g_{kl}+4\omega_r\frac{\partial g_{kl}}{\partial r}+4(\omega_r)^2g_{kl}\right]\\&\quad=\Bigg[\frac{\partial^2\widehat{g}_{ij}}{\partial\widehat{r}^2}\left(\frac{\partial\widehat{r}}{\partial r}\right)^2+\frac{\partial\widehat{g}_{ij}}{\partial\widehat{r}}\frac{\partial^2\widehat{r}}{\partial r^2}+\left(2\frac{\partial^2\widehat{g}_{ij}}{\partial\widehat{x}^m\partial\widehat{r}}\frac{\partial\widehat{r}}{\partial r}+\frac{\partial^2\widehat{g}_{ij}}{\partial\widehat{x}^m\partial\widehat{x}^n}\frac{\partial\widehat{x}^n}{\partial r}\right)\frac{\partial\widehat{x}^m}{\partial r}\\&\qquad+\frac{\partial\widehat{g}_{ij}}{\partial\widehat{x}^m}\frac{\partial^2\widehat{x}^m}{\partial r^2}\Bigg]\frac{\partial\widehat{x}^i}{\partial x^k}\frac{\partial\widehat{x}^j}{\partial x^l}\\&\qquad+2\left(\frac{\partial\widehat{g}_{ij}}{\partial\widehat{r}}\frac{\partial\widehat{r}}{\partial r}+\frac{\partial\widehat{g}_{ij}}{\partial\widehat{x}^m}\frac{\partial\widehat{x}^m}{\partial r}\right)\frac{\partial}{\partial r}\left(\frac{\partial\widehat{x}^i}{\partial x^k}\frac{\partial\widehat{x}^j}{\partial x^l}\right)+\widehat{g}_{ij}\frac{\partial^2}{\partial r^2}\left(\frac{\partial\widehat{x}^i}{\partial x^k}\frac{\partial\widehat{x}^j}{\partial x^l}\right)\\&\qquad+\frac{\partial^2}{\partial r^2}\left[e^{2\omega}\cosh^{-2}(\widehat{r})\sinh^2(r)\frac{\partial\omega}{\partial x^k}\frac{\partial\omega}{\partial x^l}\right].\end{aligned} \tag{28}$$

Restricted on the boundary we have

$$
\begin{aligned}
(29) \qquad 2e^{2\omega_0} g_{kl} \frac{\partial^2 \omega}{\partial r^2}\bigg|_{r=0} =& \frac{\partial^2 \widehat{g}_{ij}}{\partial \widehat{r}^2} \left(\frac{\partial \widehat{r}}{\partial r}\right)^2 + \widehat{g}_{ij} \frac{\partial^2}{\partial r^2} \left(\frac{\partial \widehat{x}^i}{\partial x^k} \frac{\partial \widehat{x}^j}{\partial x^l}\right) \\
&+ \frac{\partial \widehat{g}_{kl}}{\partial \widehat{x}^m} \frac{\partial^2 \widehat{x}^m}{\partial r^2} + 2e^{2\omega_0} \frac{\partial \omega_0}{\partial x^k} \frac{\partial \omega_0}{\partial x^l}.
\end{aligned}
$$

On the other hand by (25) we have

$$
\begin{aligned}
\widehat{g}_{ij} \frac{\partial^2}{\partial r^2} \left(\frac{\partial \widehat{x}^i}{\partial x^k} \frac{\partial \widehat{x}^j}{\partial x^l}\right)\bigg|_{S^2} &= e^{2\omega_0} \left[g_{il} \frac{\partial}{\partial x^k} \left(\frac{\partial^2 \widehat{x}^i}{\partial r^2}\right) + g_{kj} \frac{\partial}{\partial x^l} \left(\frac{\partial^2 \widehat{x}^j}{\partial r^2}\right) \right] \\
&= -e^{2\omega_0} \left[g_{il} \frac{\partial}{\partial x^k} \left(g^{ij} \frac{\partial \omega_0}{\partial x^j}\right) + g_{kj} \frac{\partial}{\partial x^l} \left(g^{ij} \frac{\partial \omega_0}{\partial x^i}\right) \right] \\
&= -e^{2\omega_0} \left[2 \frac{\partial^2 \omega_0}{\partial x^k \partial x^l} - g^{ij} \left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^l}\right) \frac{\partial \omega_0}{\partial x^j} \right] \\
&= -e^{2\omega_0} \left[2\nabla^2_{k,l} \omega_0 - g^{ij} \frac{\partial g_{kl}}{\partial x^i} \frac{\partial \omega_0}{\partial x^j} \right],
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial \widehat{g}_{kl}}{\partial \widehat{x}^m} \frac{\partial^2 \widehat{x}^m}{\partial r^2}\bigg|_{S^2} &= -\frac{\partial}{\partial x^m} (e^{2\omega_0} g_{kl}) g^{mi} \frac{\partial \omega_0}{\partial x^i} \\
&= -e^{2\omega_0} \left[|\nabla \omega_0|^2 g_{kl} + g^{ij} \frac{\partial g_{kl}}{\partial x^i} \frac{\partial \omega_0}{\partial x^j} \right].
\end{aligned}
$$

Plugging these two identities and (20) in (29) we obtain

$$
\begin{aligned}
&\frac{1}{2} e^{-2\omega_0} \frac{\partial^2 \widehat{g}_{ij}}{\partial \widehat{r}^2} \left(\frac{\partial \widehat{r}}{\partial r}\right)^2 \bigg|_{S^2} \\
&\qquad = \frac{1}{2} (e^{2\omega_0} - 1 + |\nabla \omega_0|^2) g_{kl} + \nabla^2_{k,l} \omega_0 + \frac{\partial \omega_0}{\partial x^k} \frac{\partial \omega_0}{\partial x^l} = 0,
\end{aligned}
$$

where we have used (15) in the last step. So we proved

$$
(30) \qquad \frac{\partial^2 \widehat{g}_{ij}}{\partial \widehat{r}^2}\bigg|_{\widehat{r}=0} = 0.
$$

Therefore the definition is independent of the choice of a particular metric on S^2. We can write

$$
\widehat{g} = \widehat{g}_0 + \frac{\widehat{r}^3}{3} \widehat{h} + O(\widehat{r}^4).
$$

Differentiation of (23) twice with respect to r gives

$$\left.\frac{\partial^3 \widehat{x}^i}{\partial r^3}\right|_{S^2} = 0, \tag{31}$$

where we have used (19), (27), (30) etc. Then differentiate (28) with respect to r and restrict on the boundary M, using (31), (27) (30) etc, and we obtain

$$\widehat{h} = e^{-\omega_0} h. \tag{32}$$

Let $m_{g_0} : S^2 \to \mathbb{R}$ be the function given by $\operatorname{tr}_{g_0} h$. From (32) we also obtain

$$m_{\widehat{g}_0} = e^{-3\omega_0} m_{g_0}, \tag{33}$$

where $m_{\widehat{g}_0} = \operatorname{tr}_{\widehat{g}_0} \widehat{h}$.

From the above discussion we see that for an asymptotically hyperbolic manifold the asymptotics is very complicated and is measured by a tensor h on the conformal infinity S^2 while in asymptotically flat case the asymptotics is simply measured by a number. Another difficulty comes from the fact that the asymptotic model is not unique and as a result the tensor h and its trace depend on the metric g_0 on S^2 we choose. If we replace g_0 by $\widehat{g}_0 = e^{2\omega_0} g_0$, the quantities h and m change according to (32) and (33). If X has scalar curvature $R \geq -6$, by Theorem 2.5 (in dimension three the spin assumption is automatically true) and the remark that follows it we have

$$\int_{S^2} m(x) d\sigma_{g_0} - \left| \int_{S^2} m(x) \mathcal{F}_{g_0}(x) d\sigma_{g_0} \right| \geq 0, \tag{34}$$

where $\mathcal{F}_{g_0} : S^2 \to \mathbb{R}^3$ is a map such that the three coordinate functions form orthogonal basis for the first eigenspace of $-\triangle_{g_0}$ and each satisfies $\int_{S^2} f^2 d\mu_{g_0} = 4\pi$. The above discussion shows both the quantities $\int_{S^2} m_{g_0}(x) d\mu_{g_0}$ and $\left|\int_{S^2} m_{g_0}(x) \mathcal{F}_{g_0}(x) d\mu_{g_0}\right|$ appearing in the inequality (34) depend on coordinates used to define them. However the difference of their squares is an invariant.

Theorem 3.1. *The number*

$$\left(\int_{S^2} m_{g_0}(x) d\mu_{g_0} \right)^2 - \left| \int_{S^2} m_{g_0}(x) \mathcal{F}_{g_0}(x) d\mu_{g_0} \right|^2$$

is independent of the choice of g_0.

Proof. We identify (S^2, g_0) with the unit sphere in $\mathbb{R}^3$ and then we can choose $\mathcal{F}_{g_0}(x) = x$. Let $\widehat{g}_0 = e^{2\omega_0} g_0$ where ω_0 is given by (14). Without loss of generality we can assume that ξ is the north pole in (14). Therefore ω_0 is given by the formula

$$\omega_0(x) = -\log(\cosh(t) + \sinh(t)x_3). \tag{35}$$

Define $\psi : S^2 \to S^2$ by

$$\psi(x) = \left(\frac{x_1}{\cosh(t) + \sinh(t)x_3}, \frac{x_2}{\cosh(t) + \sinh(t)x_3}, \frac{\sinh(t) + \cosh(t)x_3}{\cosh(t) + \sinh(t)x_3} \right).$$

It is easy to verify that $\widehat{g}_0 = \psi^* g_0$. Hence we can take ψ as our map $\mathcal{F}_{\widehat{g}_0} : S^2 \to \mathbb{R}^3$. By (33) and (35) we have

$$\begin{aligned}
\int_{S^2} m_{\widehat{g}_0}(x) d\mu_{\widehat{g}_0} &= \int_{S^2} m_{g_0}(x) e^{-\omega_0} d\mu_{g_0} \\
&= \int_{S^2} m_{g_0}(x)(\cosh(t) + \sinh(t)x_3) d\mu_{g_0} \\
&= \cosh(t) \int_{S^2} m_{g_0}(x) d\mu_{g_0} \\
&\quad + \sinh(t) \int_{S^2} m_{g_0}(x) x_3 d\mu_{g_0}.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
&\int_{S^2} m_{\widehat{g}_0}(x) \mathcal{F}_{\widehat{g}_0}(x) d\mu_{\widehat{g}_0} \\
&\quad = \int_{S^2} m_{g_0}(x) \left(x_1, x_2, \sinh(t) + \cosh(t)x_3 \right) d\mu_{g_0} \\
&\quad = \left(\int_{S^2} m_{g_0}(x) x_1 d\mu_{g_0}, \int_{S^2} m_{g_0}(x) x_2 d\mu_{g_0}, \sinh(t) \int_{S^2} m_{g_0}(x) d\mu_{g_0} \right. \\
&\qquad\qquad \left. + \cosh(t) \int_{S^2} m_{g_0}(x) x_3 d\mu_{g_0} \right).
\end{aligned}$$

By simple calculation we get

$$\begin{aligned}
&\left(\int_{S^2} m_{\widehat{g}_0}(x) d\mu_{\widehat{g}_0} \right)^2 - \left| \int_{S^2} m_{\widehat{g}_0}(x) \mathcal{F}_{\widehat{g}_0}(x) d\mu_{\widehat{g}_0} \right|^2 \\
&\quad = \left(\int_{S^2} m_{g_0}(x) d\mu_{g_0} \right)^2 - \left| \int_{S^2} m_{g_0}(x) x d\mu_{g_0} \right|^2
\end{aligned}$$

$$= \left(\int_{S^2} m_{g_0}(x) d\mu_{g_0} \right)^2 - \left| \int_{S^2} m_{g_0}(x) \mathcal{F}_{g_0}(x) d\mu_{g_0} \right|^2 ,$$

i.e., the quantity is independent of the coordinates we use. q.e.d.

Definition 3.2. For an asymptotically hyperbolic 3-manifold (X, g) we denote the invariant

$$\left(\frac{1}{16\pi} \int_{S^2} m_{g_0}(x) d\mu_{g_0} \right)^2 - \left| \frac{1}{16\pi} \int_{S^2} m_{g_0}(x) \mathcal{F}_{g_0}(x) d\mu_{g_0} \right|^2$$

by $\mathcal{E}(X, g)$.

Remark. Consider the vector

$$\left(\frac{1}{16\pi} \int_{S^2} m_{g_0}(x) d\mu_{g_0}, \frac{1}{16\pi} \int_{S^2} m_{g_0}(x) \mathcal{F}_{g_0}(x) d\mu_{g_0} \right)$$

in Minkowski space $\mathbb{R}^{1,3}$. If we replace g_0 by $\widehat{g}_0 = \epsilon^{2\omega_0} g_0$ the proof of Theorem 3.1 actually shows that this vector is transformed by a proper Lorentz transformation. This vector can be interpreted as the total Energy-Momentum vector. The invariant $\mathcal{E}(X, g)$ is its Lorentz length. If $R \geq -6$ Theorem 2.5 says that the total Energy-Momentum vector is strictly timelike and future-directed unless X is isometric to the hyperbolic space $\mathbb{H}^3$.

We can restate Theorem 2.5 in dimension 3 as follows.

Theorem 3.3. *Let (X, g) be an asymptotically hyperbolic 3-manifold. If it has scalar curvature $R \geq -6$, then $\mathcal{E}(X, g) \geq 0$. Moreover it is zero if and only if X is isometric to the hyperbolic space $\mathbb{H}^3$.*

Definition 3.4. For an asymptotically hyperbolic 3-manifold (X, g) with $R \geq -6$, we define its total mass M to be $\sqrt{\mathcal{E}(X, g)}$.

To illustrate the various definitions, let us consider the Anti de Sitter-Schwarzschild space $]r_0, \infty[\times S^2$ with the metric

$$g = \frac{dr^2}{1 + r^2 - M/r} + r^2 d\omega^2, \tag{36}$$

where $M > 0$ is a constant and r_0 is the zero of the function $1 + r^2 - M/r$. This space has two ends with the same asymptotic behavior, so we only analyze the end $r \to \infty$. We change coordinates by solving the following ODE

$$\begin{cases} \dot{r}(t) = -\sinh^{-1}(t)\sqrt{1+r^2-M/r} \\ r(0) = \infty. \end{cases}$$

Let $r(t) = \sinh^{-1}(t)u(t)$. We get

$$\begin{cases} \cosh(t)u - \sinh(t)\dot{u} = \sqrt{\sinh^2(t) + u^2 - M\sinh(t)/u} \\ u(0) = 1. \end{cases}$$

In the new coordinates $g = \sinh^{-2}(t)(dt^2 + u(t)^2 d\omega^2)$. It is easy to prove that u has the asymptotic expansion

$$u(t) = 1 + \frac{M}{3!}t^3 + O(t^4).$$

Therefore ADS-Schwarzschild space is asymptotically hyperbolic in the sense of Definition 2.3. Its total mass is obviously the parameter M in (36).

4. A nonlinear PDE on the hyperbolic space $\mathbb{H}^n$

In this section we study a special class of asymptotically hyperbolic manifolds. Let $(\mathbb{H}^n, g_0)$ be the hyperbolic n-space. We will consider Riemannian metric $g = u^{4/(n-2)}g_0$ such that the scalar curvature $R_g \geq -n(n-1) = R_{g_0}$ and u is asymptotic to 1 in certain sense. First we have the following equation

$$-\triangle u - \frac{n(n-2)}{4}u = \frac{n-2}{4(n-1)}R_g u^{(n+2)/(n-2)}. \tag{37}$$

If $R_g = -n(n-1)$, we have the trivial solution $u_0 \equiv 1$. The linearized equation of (37) at u_0 is

$$-\triangle\phi + n\phi = 0. \tag{38}$$

Let $G(x,y)$ be its fundamental solution. In dimension three, we have the explicit formula $G(x,y) = \frac{1}{4\pi^2}e^{-2d(x,y)}/\sinh d(x,y)$. In the following we will work in dimension three to simplify the presentation though the results are true in any dimension. We solve the inhomogeneous equation on $\mathbb{H}^3$

$$-\triangle\phi + 3\phi = f \tag{39}$$

by the formula

$$\phi(x) = \int_{\mathbb{H}^n} G(x,y) f(y) dV_y.$$

Theorem 4.1. *If $f(x)e^{3d(o,x)}$ is bounded and integrable on $\mathbb{H}^3$, then the above formula solves Equation (39). Moreover $\phi(x)e^{3d(o,x)}$ extends continuously to the compactification of $\mathbb{H}^3$ with boundary value $\int_{\mathbb{H}^3} f(y)e^{3B_\theta(y)}dV_y$ on the sphere S^2 at ∞, where $B_\theta(y)$ is the Buseman function.*

Proof. Let $\{x_n\} \subset \mathbb{H}^3$ be a sequence converging to $\theta \in \mathbb{S}^2_\infty$. Given any $\epsilon > 0$, we can write the integral as a sum of two parts

$$\begin{aligned} \phi(x)e^{3d(0,x)} &= \int_{H^3} \frac{e^{3d(o,x)-2d(x,y)}}{4\pi^2 \sinh d(x,y)} f(y) dV_y \\ &= \int_{\{y|d(y,x)>\epsilon\}} + \int_{\{y|d(y,x)\leq\epsilon\}} \frac{e^{3d(o,x)-2d(x,y)}}{4\pi^2 \sinh d(x,y)} f(y) dV_y \\ &= I + II. \end{aligned}$$

We have the estimate

$$\begin{aligned} |II| &\leq \frac{1}{4\pi^2} \int_{\{y\in\mathbb{H}^3|d(y,x)\leq\epsilon\}} \frac{e^{d(x,y)}}{\sinh d(x,y)} |f(y)| e^{3d(o,y)} dV_y \\ &\leq C \int_0^\epsilon e^r \sinh(r) dr \\ &\leq C'\epsilon. \end{aligned}$$

Let $E_{x,\epsilon} = \{y \in \mathbb{H}^3 | d(y,x) > \epsilon\}$. We write the first part as

$$I = \int_{\mathbb{H}^3} \frac{1}{4\pi^2} \frac{e^{3d(o,x)-2d(x,y)}}{\sinh d(x,y)} f(y) \chi_{E_{x,\epsilon}}(y) dV_y.$$

The integrand, dominated by the integrable function

$$\frac{e^\epsilon}{4\pi^2 \sinh \epsilon} |f(y)| e^{3d(o,y)},$$

converges pointwise to the function $\frac{1}{4\pi^2} e^{3B_{\theta(y)}} f(y)$ if $x = x_n, n \to \infty$. Therefore by Lebesgue dominated convergence theorem, we obtain

$$\overline{\lim_{n\to\infty}} \left| \phi(x_n) e^{3d(o,x_n)} - \int_{\mathbb{H}^3} f(y) e^{3B_\theta(y)} dV_y \right| \leq C\epsilon.$$

As ϵ is arbitrary we conclude

$$\lim_{n\to\infty} \phi(x_n)e^{3d(o,x_n)} = \int_{\mathbb{H}^3} f(y)e^{3B_\theta(y)}dV_y.$$

Therefore $\phi(x)e^{3d(o,x)}$ has continuous extension to $\overline{\mathbb{H}}^3$ with the claimed boundary value. q.e.d.

Next we consider the nonlinear equation on $\mathbb{H}^3$

$$\triangle u + \frac{3}{4}u = \frac{3}{4}(1-f)u^5, u \geq 0. \tag{40}$$

We assume that $f \geq 0$, which means the scalar curvature of the metric $g = u^4 g_0$ is greater or equal to -6, the scalar curvature of g_0.

Proposition 4.2. *If* $\sup f < 1$ *and* $f(x)e^{3d(o,x)}$ *are bounded and integrable, then Equation* (40) *has a solution* u *which is asymptotic to* 1 *at infinity. Moreover* $(u-1)e^{3d(o,x)}$ *has a continuous extension on* $\overline{\mathbb{H}}^3$.

Proof. It is obvious that $u_0 \equiv 1$ is a subsolution for Equation (40). Let $u = 1 + v$ then we have

$$\begin{aligned} &-\triangle u - \frac{3}{4}u + \frac{3}{4}(1-f)u^5 \\ &\quad = -\triangle v + 3v - \frac{3}{4}f(1+5v) + \frac{3}{4}(1-f)(10v^2 + \cdots + v^5) \\ &\quad \geq -\triangle v + 3v - \frac{3}{4}f - C_\epsilon f^2 + \frac{3}{4}(1-f-\epsilon)(10v^2 + \cdots + v^5). \end{aligned}$$

If $\sup f < 1$, we can choose $\epsilon > 0$ such that $1-f-\epsilon \geq 0$. By Theorem 4.1, we can solve the following linear equation

$$-\triangle v + 3v = \frac{3}{4}f + C_\epsilon f^2. \tag{41}$$

Then $\bar{u} = 1+v$ is a supersolution of Equation (40). By standard theory in PDE, there exists a solution u for Equation (40) such that $1 \leq u \leq \bar{u}$. The continuous extension of $(u-1)e^{3d(o,x)}$ to $\overline{\mathbb{H}}^3$ follows easily from Theorem 4.1. q.e.d.

To proceed further we use the disk model B^3. We assume that

$$\phi(x) = (u(x)-1)(1-r^2)^{-3}$$

extends to a C^2 function on $\overline{B}^3$. Denote its restriction on the boundary by $m : S^2 \to \mathbb{R}$. By the maximum principle we have $u \geq 1$ and hence $m \geq 0$.

Theorem 4.3. *Either $m \equiv 0$ on S^2 or $m(\theta) > 0, \forall\theta \in S^2$.*

Proof. On the disk model the hyperbolic Laplacian $\triangle$ is related to the Euclidean Laplacian $\triangle_0$ by the following formula

$$\triangle = \frac{(1-r^2)^2}{4}\left(\triangle_0 + \frac{2r}{1-r^2}\frac{\partial}{\partial r}\right). \tag{42}$$

From this formula and (40) we obtain

$$\begin{aligned}
\frac{(1-r^2)^2}{4}&\left(\triangle_0 u + \frac{2r}{1-r^2}\frac{\partial u}{\partial r}\right)\\
&\leq \frac{3}{4}u^5 - \frac{3}{4}u\\
&= \frac{3}{4}\left(1+\phi\left(1-r^2\right)^3\right)^5 - \frac{3}{4}\left(1+\phi\left(1-r^2\right)^3\right)\\
&= 3\phi\left(1-r^2\right)^3 + \frac{3}{4}\left(10\phi^2\left(1-r^2\right)^6 + \cdots + \phi^5\left(1-r^2\right)^{15}\right).
\end{aligned}$$

It follows then

$$(1-r^2)\triangle_0\phi - 10\nabla\phi\cdot x - 30\phi \leq 3\left(10\phi^2(1-r^2)^2 + \cdots + \phi^5(1-r^2)^{11}\right).$$

We can rewrite the above inequality as

$$(1-r^2)\triangle_0\phi - 10\nabla\phi\cdot x + c\phi \leq 0, \tag{43}$$

where $c = -30 - 3(10\phi(1-r^2)^2 + \cdots + \phi^4(1-r^2)^{11}) < 0$. By the strong maximum principle, either $u \equiv 1$ or $u > 1$ in B^3. Assuming $u > 1$ in B^3, we are to prove $\phi > 0$ on $S^2 = \partial B^3$. Suppose $\phi(\xi) = 0$ for some $\xi \in S^2$. Let $y = a\xi$ where $a \in (0,1)$ is very close to 1. Consider the function $v = e^{-\alpha|x-y|^2} - e^{-\alpha R^2}$ on the annulus $B(y,R) - B(y,\rho)$, where $R = 1-a$ and $\rho < R$. Easy calculation shows

$$v_i = -2\alpha e^{-\alpha|x-y|^2}(x_i - y_i),$$

$$v_{ii} = e^{-\alpha|x-y|^2}(4\alpha^2(x_i-y_i)^2 - 2\alpha),$$

$$\triangle_0 v = e^{-\alpha|x-y|^2}(4\alpha^2|x-y|^2 - 6\alpha).$$

It follows then

$$\begin{aligned}&(1-r^2)\triangle_0 v - 10\nabla v\cdot x + cv\\ &\quad\geq e^{-\alpha|x-y|^2}[(4\alpha^2(1-|x|^2)|x-y|^2 - 6\alpha(1-|x|^2)\\ &\quad\quad +20\alpha(x-y)\cdot x + c].\end{aligned}$$

If $x_3 \geq \frac{1+a}{2}$, we have the estimate

$$\begin{aligned}&(1-r^2)\triangle_0 v - 10\nabla v\cdot x + cv\\ &\quad\geq e^{-\alpha|x-y|^2}\left[(4\alpha^2\rho^2 - 6\alpha)(1-|x|^2) + 5\alpha(1-a^2) + c)\right]\\ &\quad\geq 0,\ \text{ if } \alpha \text{ is big enough.}\end{aligned}$$

If $x_3 \leq \frac{1+a}{2}$, we have the estimate

$$\begin{aligned}&(1-r^2)\triangle_0 v - 10\nabla v\cdot x + cv\\ &\quad\geq e^{-\alpha|x-y|^2}\left[\alpha^2(3+a)(1-a)\rho^2 - 26\alpha + c)\right]\\ &\quad\geq 0, \qquad \text{if } \alpha \text{ is big enough.}\end{aligned}$$

Consider the function $f = -\phi + \epsilon v$ on $B(y,R) - B(y,\rho)$. We have

$$(1-r^2)\triangle_0 f - 10\nabla f\cdot x + cf \geq 0.$$

Choose ϵ small enough such that $f \leq 0$ on $\partial B(y,\rho)$. By the maximum principle, $f \leq 0$ on $B(y,R) - B(y,\rho)$. Taking normal derivative at ξ, we get

$$\frac{\partial\phi}{\partial\nu}(\xi) \leq \epsilon\frac{\partial v}{\partial\nu}(\xi) = -\epsilon\alpha R e^{-\alpha R^2} < 0.$$

On the other hand from (43) we see that $\frac{\partial\phi}{\partial\nu}(\xi) \geq 0$, a contradiction. Therefore $\phi > 0$ and hence $m > 0$ on S^2. q.e.d.

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