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COLLAPSING THREE-MANIFOLDS UNDER A LOWER CURVATURE BOUND

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Abstract

The purpose of this paper is to completely characterize the topology of three-dimensional Riemannian manifolds with a uniform lower bound of sectional curvature which converges to a metric space of lower dimension.

0. Introduction

We study the topology of three-dimensional Riemannian manifolds with a uniform lower bound of sectional curvature converging to a metric space of lower dimension.

Given a positive integer n and $D > 0$, let us consider the set $\mathcal{M}(n, D)$ of isometry classes of n-dimensional closed Riemannian manifolds M with sectional curvature $K \geq -1$ and diameter $\operatorname{diam}(M) \leq D$. By the Gromov Precompactness Theorem [16], the closure of $\mathcal{M}(n, D)$ is compact with respect to the Gromov-Hausdorff distance. Thus any sequence M_i, $i = 1, 2, \ldots,$ in $\mathcal{M}(n, D)$ has a convergent subsequence whose limit is a compact Alexandrov space X of dimension $\leq n$ and curvature ≥ -1. We now assume that M_i itself converges to X, i is sufficiently large, and consider the following:

Problem 0.1. Describe the topological structure of M_i by using the geometry and topology of X.

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Some answers are known in the extremal cases: If $\dim X = 0$, the fundamental group of M_i is almost nilpotent (Fukaya and Yamaguchi [13]) and if $\dim X = n$, M_i is homeomorphic to X (Perelman [26, 27], cf. Grove, Petersen and Wu [19]). In particular, for the above problem it suffices to consider only the case of $\dim X \le n-1$, the so called *collapsing case*.

If X has no singular points, then X is a C^0-Riemannian manifold (Otsu and Shioya [25]), and the Fibration Theorem (Yamaguchi [37]) implies that M_i is a fibre bundle over X with almost nonnegatively curved fibre. Actually the Fibration Theorem still holds if X has only 'weak' singularities ([38]) in some sense. According to Perelman ([28]), it is also known that if X has no 'bad' singularities (precisely called extremal subsets), there is an isomorphism $\pi_k(M_i, F_i) \simeq \pi_k(X)$ for homotopy groups, where F_i is a 'general fibre' and i is large enough compared with k.

When $\dim X \le n-1$ and X may have 'bad' singularities, no solution to Problem 0.1 is known as of now. In this paper we completely solve it in the case when $n = 3$ and $\dim X = 1$ or 2. Note that if $n = 3$ and $\dim X = 0$ (i.e., X is a single point), it has already been obtained in [13] that some finite cover of M_i is homeomorphic to either a homotopy sphere, $S^1 \times S^2$, T^3, or a nilmanifold.

From now on, we assume that each element $M_i \in \mathcal{M}(3, D)$ in the sequence is *orientable* and i is *sufficiently large*. We first state our main results in the case of $\dim X = 2$. Recall that X is a topological manifold possibly with boundary in this case.

Theorem 0.2. *If $\dim X = 2$ and X has no boundary, then M_i is homeomorphic to a Seifert fibred space over X, for which the orbit type (μ, ν) of the singular fibre over a point $p \in X$ satisfies $\mu \le 2\pi/L(\Sigma_p)$.*

Here, $L(\Sigma_p)$ is the length of the space of directions, Σ_p, at p. Observe that every fibre in M_i shrinks to a point.

Theorem 0.3. *If $\dim X = 2$ and X has nonempty boundary, then there is a Seifert fibred space $\mathrm{Seif}_i(X)$ over X such that:*

(1) *M_i is homeomorphic to the union $\mathrm{Seif}_i(X) \cup (\partial X \times D^2)$ glued along their boundaries, where the fibres of $\mathrm{Seif}_i(X)$ over boundary points $x \in \partial X$ are identified with $\{x\} \times \partial D^2$.*

(2) *the orbit type (μ, ν) of the singular fibre of $\mathrm{Seif}_i(X)$ over a point $x \in \mathrm{int}\, X$ satisfies $\mu \le 2\pi/L(\Sigma_x)$.*

It should be noted that the Euler characteristic of X and the number of singular fibres of M_i in Theorems 0.2 and 0.3 are estimated by a constant depending only on the upper diameter bound D (see Remark 4.6). Observe that $\partial X \times D^2 \subset M_i$ collapses to ∂X.

We have the following corollary of Theorem 0.3.

Corollary 0.4. *Let M_i, X, and $\mathrm{Seif}_i(X)$ be as in Theorem 0.3, and let g and k denote the genus of X and the number of components of ∂X respectively. Then we have the following factorization:*

$$M_i \simeq S^3 \# \underbrace{S^2 \times S^1 \# \cdots \# S^2 \times S^1}_{f(g,k)} \# L(\mu_1, \nu_1) \# \cdots \# L(\mu_\ell, \nu_\ell),$$

where

$$f(g,k) = \begin{cases} 2g + k - 1 & \text{if } X \text{ is orientable,} \\ g + k - 1 & \text{if } X \text{ is non-orientable,} \end{cases}$$

and (μ_j, ν_j), $1 \le j \le \ell$, denote all the orbit types of singular fibres of $\mathrm{Seif}_i(X)$, and $L(\mu_j, \nu_j)$ the lens space of type (μ_j, ν_j).

Notice here that $f(g,k)$ and ℓ are both bounded by some constant depending only on D. It also follows from Theorem 0.3 and Corollary 0.4 that *the set of homeomorphism types of $\{M_i\}$ collapsing to a two-dimensional Alexandrov space with boundary is finite.*

Let us next consider the case when $\dim X = 1$, i.e., X is isometric to either a circle or a closed interval. When X is isometric to a circle, we readily observe from the Fibration Theorem [37] that M_i is a fibre bundle over S^1 whose fibre is either S^2 or T^2. The rest is to investigate the case when X is isometric to a closed interval. We denote by Mö $\tilde{\times} S^1$ a twisted S^1-bundle over a Möbius band (see Section 5 for the definition of twisted bundles).

Theorem 0.5. *Assume that X is isometric to a closed interval. Then M_i is homeomorphic to a gluing of B and C along their boundaries, where B and C are respectively either D^3, $P^3 - \mathrm{int}\, D^3$, $S^1 \times D^2$ or Mö $\tilde{\times} S^1$.*

Note that there are six combinations for the choice of B and C with a number of gluing $B \cup C$. If we express M_i in Theorem 0.5 as $M_i = B \cup A \cup C$, where $A = \partial B \times [0, \ell]$, so that as $i \to \infty$, A, B and C collapse to $[0, \ell]$, $\{0\}$ and $\{\ell\}$ respectively, we call (B, C) the *collapsing data* of the collapsing $M_i \to [0, \ell]$. For more concrete topological information of M_i, see Table 1 in Section 8.

Every prism manifolds can be written as a gluing $S^1 \times D^2 \cup \text{Mö} \tilde{\times} S^1$, and there is an infinite sequence of pairwise non-homeomorphic prism manifolds with constant curvature $K = 1$ which collapses to a closed interval with the collapsing data $(S^1 \times D^2, \text{Mö} \tilde{\times} S^1)$ (Example 7.1). It is unclear if a fixed prism manifold admits a sequence of metrics collapsing to a closed interval under $K \geq -1$.

Theorem 0.6. *Let (M, X) be one of the following:*

(1) *X is a compact surface without boundary, and M a Seifert fibred space over X.*

(2) *X is a compact surface with boundary, and M the union of a Seifert fibred space over X and $\partial X \times D^2$ glued as in Theorem* 0.3.

(3) *X is a closed interval, and M any gluing of B and C, where (B, C) is any of the six possible choices as in Theorem* 0.5. *Suppose that M is not a prism manifold in this case.*

Then there exist a sequence of Riemannian metrics g_i on M and a smooth orbifold metric g on X such that (M, g_i) collapses to (X, g) under $K \geq -1$.

Combining Theorems 0.2, 0.3 and 0.5 improves the result of [13] in the case $\dim X = 0$ previously stated in the following way.

Corollary 0.7. *Suppose that X is a point. Then a finite cover of M_i is homeomorphic to $S^1 \times S^2$, T^3, a nilmanifold or a simply connected Alexandrov space with nonnegative curvature.*

Conjecture 0.8. *Any three-dimensional compact, simply connected, nonnegatively curved Alexandrov space without boundary which is a topological manifold is homeomorphic to a sphere.*

If Conjecture 0.8 is solved, everything will be clear about Problem 0.1 for $n = 3$. The above conjecture is certainly true in the Riemannian case ([20]).

It is known by Thurston (cf. [35]) that there are eight geometric structures of three-manifolds modelled on S^3, $\mathbb{R}^3$, H^3, $S^2 \times \mathbb{R}^1$, $H^2 \times \mathbb{R}$, $\widetilde{SL}_2(\mathbb{R})$, Nil and Sol.

Corollary 0.9. *For any $D > 0$, there exists a constant $\epsilon = \epsilon(D) > 0$ such that if a closed, prime three-manifold with infinite fundamental group admits a Riemannian metric contained in $\mathcal{M}(3, D)$ with volume $< \epsilon$, then it admits a geometric structure modelled on one of the seven geometries except H^3.*

Observe that a hyperbolic manifold M does not collapse under $K \geq -1$ because of the non-vanishing property $\|M\| \neq 0$ for simplicial volume (Gromov [17], Thurston [34]).

Combining our results above, we obtain the following corollary on the existence of geometric structures for the elements of $\mathcal{M}(3, D)$.

Corollary 0.10. *All elements but finitely many homeomorphism classes in $\mathcal{M}(3, D)$ admit geometric structures.*

In the proofs of our results, we essentially use a critical point-rescaling argument to understand the topology of a small neighborhood of M_i converging to a small neighborhood of a singular point of X. When $\dim X = 2$, as the limit space of the rescaled M_i, we have a three-dimensional complete open nonnegatively curved Alexandrov space Y which is a topological manifold. Here an Alexandrov space is called open if it is noncompact and without boundary. It is significant to determine the topology of such a space Y by using its soul S.

Theorem 0.11. *Let Y be a three-dimensional complete open Alexandrov space of nonnegative curvature. Suppose that Y is a topological manifold. Then Y is homeomorphic to the normal bundle $N(S)$ of the soul S of Y.*

This extends the Cheeger-Gromoll Soul Theorem [9] in dimension three. Actually we classify all the three-dimensional complete open Alexandrov spaces with nonnegative curvature which are not necessarily topological manifolds (Theorem 9.6). This seems to be of independent interest.

The organization of this paper is as follows: In Section 1, we sketch the essential idea of the proofs of Theorems 0.2, 0.3 and 0.5.

In Section 2, we present some basic notions and results on Alexandrov spaces needed in the subsequent sections.

The main body of this paper consists of two parts. In Part 1, we discuss the collapsing of three-dimensional Riemannian manifolds by assuming the Generalized Soul Theorem, which is proved in Part 2.

In Section 3, we prove a key lemma, which is the important first step to understand the topology of a small neighborhood of a point of M_i converging to a singular point of X.

The proofs of Theorems 0.2, 0.3 and 0.5 are given in Sections 4, 5 and 6 respectively. We also prove Corollaries 0.7, 0.9 and 0.10 in Section 6.

In Section 7, we construct collapsing metrics with a lower curvature bound on three-manifolds to prove Theorem 0.6.

In Section 8, we discuss the bounded curvature collapsing of three-manifolds with some construction of collapsing metrics, and compare our main results with them.

In Part 2, we classify all the three-dimensional complete open Alexandrov spaces Y with nonnegaive curvature (the Generalized Soul Theorem). First in Section 9, we state the main results in Part 2 with some examples, and prove the rigidity part of the Generalized Soul Theorem.

We essentially use the topological Morse theory for distance functions in the proof of non-rigidity part. We divide the proof into the two cases, depending on the dimension of the minimum set C of a Busemann function on Y. After some preliminary arguments in Sections 10 and 11, we prove the non-rigidity part of the Generalized Soul Theorem in the case of $\dim C = 2$ in Section 12. The case of $\dim C = 1$ is proved in Section 13.

In Appendix, we discuss the total curvature, the Gauss-Bonnet Theorem and the Cohn-Vossen Theorem for Alexandrov surfaces, and give a classification of nonnegaitvely curved Alexandrov surfaces. Those are needed in the proof of Theorem 0.5.

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1. Idea of proofs

Let a sequence of three-dimensional closed orientable Riemannian manifolds M_i with $K \geq -1$ and $\mathrm{diam}(M_i) \leq D$ collapse to a compact Alexandrov space X with $\dim X = 1$ or 2. Assume for simplicity that i is always large enough.

Let us first consider the most basic case when $\dim X = 2$ and X has no boundary. For a sufficiently small fixed $\epsilon > 0$, take the singular points x_j of X with $L(\Sigma_{x_j}) \leq 2\pi - \epsilon$. There are only finitely many such points x_j, say $1 \leq j \leq k$. For a sufficiently small $r > 0$, we consider $X' = X - (B(x_1, r) \cup \cdots \cup B(x_k, r))$, where $B(x, r)$ denotes the closed r-ball around x. Applying the Fibration Theorem (Theorem 2.2) to X', we have a domain $M_i' \subset M_i$ converging to X' which is a circle bundle over X'. We here need some new idea to determine the topology of the components B_{ij} of $M_i - M_i'$ converging to $B(x_j, r)$.

For simplicity we fix a j and set $B_i = B_{ij} \subset M_i$, $x = x_j$ and $B = B(x, r) \subset X$. To investigate the topology of B_i, we want to measure the diameter, say δ_i, of the 'fibres' of the convergence $B_i \to B$ and consider the convergence of the rescaled pointed manifolds $(\frac{1}{\delta_i} M_i, p_i)$ as $i \to \infty$, where p_i is the center of B_i. If this convergence does not collapse, then $\frac{1}{\delta_i} B_i$ should be homeomorphic to its limit which is a nonnegatively curved complete open Alexandrov space, which could be characterized by generalizing the Soul Theorem.

However, the main problem here is the difficulty to measure the diameter δ_i of the 'fibres'. To find δ_i, we shift the point p_i slightly to a 'peak' $\hat{p}_i$ of B_i, precisely a point where the average of the distance to all points on ∂B_i takes a local maximum. By shifting p_i, the ball B_i isotopically moves, so that the topology does not change. Because of the boundary condition $\partial B_i \simeq T^2$, there are critical points in B_i of the distance function from $\hat{p}_i$. We define δ_i to be the furthest distance from $\hat{p}_i$ to the critical points in B_i. It then follows that $\delta_i \to 0$.

By passing to a subsequence, the sequence of the rescaled pointed manifolds $(\frac{1}{\delta_i} M_i, \hat{p}_i)$ converges to a pointed noncompact nonnegatively curved Alexandrov space (Y, y_0). As a crucial lemma (Key Lemma 3.6), we prove that $\dim Y = \dim X + 1 = 3$. Since the convergence $(\frac{1}{\delta_i} M_i, \hat{p}_i) \to (Y, y_0)$ does not collapse, a discussion using Perelman's Stability Theorem (Theorem 2.4) shows that $\operatorname{int} B_i \simeq Y$.

The next step is to establish the Generalized Soul Theorem for three-dimensional Alexandrov spaces to determine the topology of Y. For general Alexandrov space, the Soul Theorem as in the Riemannian case does not hold. This happens essentially because of the appearance of topological singular points. We prove the Soul Theorem as in the Riemannian case for three-dimensional complete open nonnegaitvely curved Alexandrov space which is a topological manifold, by generalizing the notion of gradient flows of distance functions with the use of the topological Morse theory.

Applying the Soul Theorem to Y together with the boundary condition $\partial B_i \simeq T^2$, we conclude that the soul of Y is isometric to a circle, and hence $B_i \simeq S^1 \times D^2$. Finally, we put a structure of fibred solid torus on B_i which is compatible to the circle bundle structure on M_i'. Thus we obtain the Seifert bundle structure on M_i over X, and conclude Theorem 0.2.

We summarize the above discussion as follows:

(1) For the almost regular part of X, we use Fibration Theorem 2.2 to obtain the circle bundle structure on $M_i' \subset M_i$.

(2) For singular points of X, we use the flow chart:

$$\begin{array}{c}\text{the rescaling argument (Key Lemma 3.6)}\\ \downarrow \\ \text{Stability Theorem 2.4} \\ \downarrow \\ \text{the Generalized Soul Theorem}\end{array}$$

to obtain the topology of a small neighborhood $B_i \subset M_i$ near a singular point of X.

(3) We put a fibred solid torus structure on B_i and finally check the compatibility.

Next we consider the case when $\dim X = 2$ and X has non-empty boundary. For a sufficiently small fixed $\epsilon > 0$, take the boundary points $x_j \in \partial X$ with $L(\Sigma_{x_j}) \leq \pi - \epsilon$. There are only finitely many such points x_j, say $1 \leq j \leq \ell$. For a sufficiently small $r > 0$ and $\delta \ll r$, we decompose X into three kinds of parts: $B(x_j, r)$, $1 \leq j \leq \ell$, $H = B(\partial X, \delta) - \mathrm{int}(B(x_1, r) \cup \cdots \cup B(x_\ell, r))$, and X', the complement of $\mathrm{int}\, H$ and $\mathrm{int}\, B(x_j, r)$. Corresponding to these parts, we decompose M_i into three kinds of parts B_{ij}, H_i, and M_i' which are respectively Gromov-Hausdorff close to $B(x_j, r)$, H, and X'. Applying Theorem 0.2 to X', we obtain a Seifert bundle structure on M_i' over X'. The generalized Margulis lemma ([13]) implies that H_i is homeomorphic to $D^2 \times (\partial X \cap H)$. Finally by using the critical point-rescaling argument (2) above with the boundary condition $\partial B_{ij} \simeq S^2$, we conclude that $B_{ij} \simeq D^3$. Combining those topological information, we obtain Theorem 0.3.

When X is isometric to a closed interval I, we decompose M_i into three parts A_i, B_i, and C_i, where B_i and C_i are two metric balls close to the two endpoints of I respectively. The Fibration Theorem implies that $A_i \simeq F_i \times I$, where F_i is either S^2 or T^2. To investigate the topologies of B_i and C_i, we use the same discussion as (2) above. However, we only have $3 \geq \dim Y \geq \dim X + 1 = 2$ in this case. If $\dim Y = 2$, we apply the critical point-rescaling argument (2) to the convergence $\frac{1}{\delta_i} B(p_i, R\delta_i) \to B(y_0, R)$ with large $R > 0$ instead of $B_i \to B$. This determines the topology of B_i and C_i and proves Theorem 0.5.

2. Preliminaries

In this section, we present some results on Alexandrov spaces and the Gromov-Hausdorff convergence related with Alexandrov spaces, which will be needed in the subsequent sections. We refer to [4] for the basic materials and the details of the results on Alexandrov spaces mentioned below.

First we give some basic definitions and notations. Let X be a geodesic space in the sense that every two points can be joined by a minimal geodesic. We assume that all geodesic have unit speed unless otherwise stated. For a fixed real number κ and a geodesic triangle Δxyz in X with vertices x, y and z, we denote by $\tilde{\Delta} xyz$ a *comparison triangle* in the simply connected complete surface M_κ with constant curvature κ. This means that each side length of $\tilde{\Delta} xyz$ is equal to the corresponding one of Δxyz. Here we suppose that the perimeter of Δxyz is less than $2\pi/\sqrt{\kappa}$ if $\kappa > 0$. We say that an open set U in X satisfies the *Alexandrov convexity* if for any geodesic triangle in U with vertices x, y and z and for any point w on the geodesic segment yz joining y to z, we have $d(x, w) \geq d(\tilde{x}, \tilde{w})$, where $\Delta \tilde{x}\tilde{y}\tilde{z} = \tilde{\Delta} xyz$, $\tilde{w}$ is the point on $\tilde{y}\tilde{z}$ corresponding to w. The space X is called an Alexandrov space with curvature $\geq \kappa$ if each point of X has a neighborhood satisfying the Alexandrov convexity. Actually it is known that the whole space X satisfies the Alexandrov convexity. From now on we assume that X is of finite dimension.

The angle between the geodesics xy and yz in X is denoted by $\angle xyz$, and the corresponding angle of $\tilde{\Delta} xyz$ by $\tilde{\angle} xyz$. It holds that $\angle xyz \geq \tilde{\angle} xyz$. We denote by $\Sigma_p = \Sigma_p(X)$ the space of directions at $p \in X$, and by $K_p = K_p(X)$ the tangent cone at p with vertex o_p, the Euclidean cone $K(\Sigma_p)$ over Σ_p. It is known that Σ_p is an Alexandrov space with curvature ≥ 1.

For a compact set $A \subset X$ and $p \in X - A$, we denote by $A' = A'_p$ the closed set of Σ_p consisting of all the directions of minimal geodesics from p to A. Let us now consider the distance function $d_x(\cdot) = d(x, \cdot)$ from x. A point p is called a *critical* point of d_x if $\tilde{\angle} xpy \leq \pi/2$, or equivalently $\angle(x'_p, y'_p) \leq \pi/2$, for every $y \in X - \{p\}$. Otherwise p is a *regular* point of d_x.

A (not necessarily continuous) map $\varphi : Y \to Z$ between metric spaces is called an ϵ-*approximation* if:

(1) $|d(x, y) - d(\varphi(x), \varphi(y))| < \epsilon \quad$ for all $x, y \in Y$.

(2) $\varphi(Y)$ is ϵ-dense in Z.

The Gromov-Hausdorff distance $d_{GH}(Y, Z)$ between Y and Z is defined to be the infimum of such ϵ that there exist ϵ-approximations $Y \to Z$ and $Z \to Y$. We say that pointed spaces (X_i, x_i) converge to (X, x) with respect to the pointed Gromov-Hausdorff topology if the metric balls $B(x_i, R_i; X_i)$ converge to $B(x, R; X)$ with respect to the Gromov-Hausdorff distance for any $R > 0$ and some monotone nonincreasing sequence $R_i \to R$. We recall that K_p is isometric to the pointed Gromov-Hausdorff limit of $(\frac{1}{\epsilon}X, p)$ as $\epsilon \to 0$.

Let X have dimension n, and $\delta > 0$. A system of n pairs of points, $(a_i, b_i)_{i=1}^n$ is called an (n, δ)-*strainer* at $p \in X$ if it satisfies

$$\tilde{\angle} a_i p b_i > \pi - \delta, \quad \tilde{\angle} a_i p a_j > \pi/2 - \delta,$$
$$\tilde{\angle} b_i p b_j > \pi/2 - \delta, \quad \tilde{\angle} a_i p b_j > \pi/2 - \delta,$$

for every $i \neq j$. The number $\min\{d(a_i, p), d(b_i, p) \,|\, 1 \leq i \leq n\}$ is called the *length* of the strainer.

Let X_δ denote the set of (n, δ)-strained points of X. This has the structure of a Lipschitz n-manifold. Note that every point in X_δ has a small neighborhood almost isometric to an open subset of $\mathbb{R}^n$ for small δ.

The boundary ∂X of X is inductively defined as the set of points p such that Σ_p has non-empty boundary.

In dimension two, we have the following result ([4], [1]).

Theorem 2.1. *Any two-dimensional Alexandrov space X with curvature bounded below is a topological manifold possibly with boundary. Furthermore for every positive number δ, the set of interior points* (*resp. boundary points*) *of X at which the length of the space of directions is smaller than $2\pi - \delta$* (*resp.* $\pi - \delta$) *is discrete.*

Next we recall the Fibration Theorem from [37], [38]. Let X be an Alexandrov space. The δ-*strain radius* at a point $p \in X_\delta$ is defined as the supremum of those $r > 0$ that there exists an (n, δ)-strainer at p of length r. The δ-strain radius of a closed domain $Y \subset X_\delta$ is, by definition,

$$\delta\text{-str.rad}(Y) = \inf_{p \in Y} \delta\text{-strain radius at } p.$$

The δ-strain radius plays a role similar to the injectivity radius of a Riemannian manifold. We denote by $\tau(\epsilon_1, \dots, \epsilon_k)$ a function depending on a priori constants and ϵ_i satisfying $\lim_{\epsilon_i \to 0} \tau(\epsilon_1, \dots, \epsilon_k) = 0$.

Theorem 2.2 (Fibration Theorem [38]). *Given n and $\mu > 0$ there exist positive numbers $\delta = \delta_n$ and $\epsilon = \epsilon_n(\mu)$ satisfying the following : Let $Y \subset X_\delta \subset X$ be as above such that δ-str.rad$(Y) > \mu$. Let M be a complete Riemannian manifold with $K \geq -1$ and suppose that the Gromov-Hausdorff distance $d_{GH}(M, X) < \epsilon$. Then there exists a closed domain $N \subset M$ and a locally trivial fibre bundle $f : N \to Y$ such that:*

(1) *It is a $\tau(\delta, \epsilon)$-Lipschitz submersion.*

(2) *It is a $\tau(\epsilon)$-approximation.*

For the definition of τ-Lipschitz submersion, see [38].

Remark 2.3. It is essentially proved in [37] that the first Betti number of the fibre of $f : N \to Y$ is less than or equal to its dimension, and in [13] that the fundamental group of the fibre is almost nilpotent.

When X is compact, two dimensional and without boundary, one can take as Y the complement of a small neighborhood of the finite set $X - X_\delta$. This explains a reason that our methods work in dimension three.

In [38], the general convergence when M is also an Alexandrov space was discussed. Although the theorem above is not stated explicitly in [38], it follows directly from the proof there. However for reader's convenience, we give a sketch of the proof of the above theorem. For the details, see [38].

Let $f_X : X \to L^2(X)$ be the embedding of X into the Hilbert space $L^2(X)$ defined by using the distance functions from the points of X. Since a small neighborhood of each point of $f_X(X_\delta)$ can be approximated by an n-plane in $L^2(X)$, $f_X(X_\delta)$ has a normal bundle ν in a generalized sense. Namely, ν is a map of $f_X(X_\delta)$ into the Grassmann manifold consisting of all subspaces of $L^2(X)$ of codimension n. This map ν, called a normal bundle of $f_X(X_\delta)$, is Lipschitz and $f_X(X_\delta)$ has a tubular neighborhood U with respect to ν. The C^1-map $f_M : M \to L^2(X)$ is constructed in a similar way (Remark 4.20 in [38]). Note that $f_M(N) \subset U$ for a closed domain N of M with small $d_{GH}(N, Y)$. Thus the map $f = f_X^{-1} \circ \pi \circ f_M : N \to Y$ is well defined, where $\pi : U \to f_X(X_\delta)$ is the projection along ν. For $p \in N$, let $T = T_p$ be an n-plane in $L^2(X)$ approximating a small neighborhood of $\pi(f_M(p))$ in $f_X(X_\delta)$. It follows from the proof of Lemma 4.6 in [38] that for every unit vector $\bar{\xi} \in T$ there exists $\xi \in T_p(M)$ such that

$$|df_M(\xi) - \bar{\xi}|$$

is small when the given constant δ and ϵ are small. This implies that $\pi_T \circ f_M$ gives a locally trivial fibre bundle on a neighborhood of p over a neighborhood of $\pi_T \circ f_M(p)$ in T, where π_T denotes the nearest point projection to T. Since $\pi : T_p \to f_X(X_\delta)$ is homeomorphic on a small neighborhood of $\pi \circ f_M(p)$ ([38, Lemma 3.7]), it follows that $\pi \circ f_M$ and hence f provides a fibre bundle structure on N over Y.

A point p of an Alexandrov space X is called an *essential* singular point if $\mathrm{rad}(\Sigma_p) \le \pi/2$, where

$$\mathrm{rad}(\Sigma_p) = \min_{\xi \in \Sigma_p} \max_{\eta \in \Sigma_p} \angle(\xi, \eta)$$

is the radius of Σ_p. Notice that if a point $p \in X$ is not an essential singular point, then Σ_p is homeomorphic to a sphere ([18]) and a small metric ball around p is homeomorphic to $\mathbb{R}^n$ ([26, 27]), where n is the dimension of X. We also say that p is a *topological* singular point if Σ_p is not homeomorphic to a sphere.

When no collapsing occurs, we have the following stability result.

Theorem 2.4 (Stability Theorem [26]). *Let a sequence of compact n-dimensional Alexandrov spaces X_i with curvature ≥ -1 converge to a compact Alexandrov space X of dimension n. Then X_i is homeomorphic to X for sufficiently large i.*

Part 1. Analyzing collapsed three-manifolds

3. Key lemma

Let a sequence of pointed complete n-dimensional Riemannian manifolds (M_i, p_i) with $K \ge -1$ converge to a pointed k-dimensional Alexandrov space (X, p), where $k \le n$. In this section, we investigate the topology of the metric ball $B(p_i, r)$ for i large enough compared to a fixed small $r > 0$ under some assumption for p.

Let A be a metric space and $\epsilon > 0$ a number. A discrete subset N of A is called an *ϵ-discrete net of A* if $d(x, y) \ge \epsilon$ for any $x \ne y \in N$. Set

$$\beta_A(\epsilon) = \max\{\#N \mid N \text{ is an } \epsilon\text{-discrete net of } A\}.$$

An ϵ-discrete net N of A is said to be *maximal* if $\#N = \beta_A(\epsilon)$. If A is a relatively compact open subset of an n-dimensional Alexandrov space, there are two constants c_1 and c_2 depending on A such that $0 < c_1 \leq \epsilon^n \beta_A(\epsilon) \leq c_2 < \infty$ for any $\epsilon > 0$ (see [4]).

Let $\phi_i \colon X \to M_i$ be a μ_i-approximation, where $\mu_i \to 0$ as $i \to \infty$. For any $\epsilon > 0$, we take a maximal ϵ-discrete net $\{\xi^j\}_{j=1,\ldots,\beta_{\Sigma_p}(\epsilon)}$ of Σ_p. For a small enough $r > 0$ compared to p and ϵ, there are $x^j \in \partial B(p, r)$, $j = 1, \ldots, \beta_{\Sigma_p}(\epsilon)$, such that the direction η^j at p of a minimal segment from p to x^j satisfies $\angle(\xi^j, \eta^j) < \epsilon^2$. Set $x_i^j = \phi_i(x^j)$ and

$$f_i = \frac{1}{\beta_{\Sigma_p}(\epsilon)} \sum_j d(x_i^j, \cdot) \colon M_i \to \mathbb{R}.$$

By letting δ_x be Dirac's δ-measure, there exists a sequence $\epsilon_\ell \to 0$ such that the measure $\frac{1}{\beta_{\Sigma_p}(\epsilon_\ell)} \sum_j \delta_{\xi^j}$ for $\epsilon = \epsilon_\ell$ converges to some Borel measure m_p on Σ_p as $\ell \to 0$ in the weak* topology. Remark that the measure m_p is (possibly) not unique because of the variety of the choices of ξ^j and ϵ_ℓ. However, we observe that m_p coincides with the normalized Hausdorff measure over Σ_p if $k \leq 2$. Let $\psi_r \colon K_p \to \frac{1}{r} B(p, r)$ be an ν_r-approximation, $\lim_{r \to 0} \nu_r = 0$, and let

$$\bar{f} = \int_{\xi \in \Sigma_p} d(\iota(\xi), \cdot)\, dm_p : K_p \to \mathbb{R},$$

where $\iota : \Sigma_p \to K_p$ is the natural embedding.

Lemma 3.1. *We have*

$$\lim_{\ell \to \infty} \lim_{r \to 0} \lim_{i \to \infty} f_i \circ \phi_i \circ \psi_r = \bar{f},$$

where the convergence is uniform on any compact set.

Proof. Since, as $i \to \infty$, $d(x_i^j, \phi_i(\cdot))$ converges to $d(x^j, \cdot)$, the function $f_i \circ \phi_i : X \to \mathbb{R}$ converges to $\frac{1}{\beta_{\Sigma_p}(\epsilon)} \sum_j d(x^j, \cdot)$ uniformly on any compact set. Therefore, since $(\frac{1}{r} X, p) \to (K_p, o_p)$ as $r \to 0$, we have

$$\begin{aligned} \lim_{r \to 0} \lim_{i \to \infty} f_i \circ \phi_i \circ \psi_r &= \lim_{r \to 0} \frac{1}{\beta_{\Sigma_p}(\epsilon)} \sum_j d(x^j, \psi_r(\cdot)) \\ &= \frac{1}{\beta_{\Sigma_p}(\epsilon)} \sum_j d(\iota(\xi^j), \cdot). \end{aligned}$$

This completes the proof. q.e.d.

Lemma 3.2. *If $k = 1$ or 2, and if* $\operatorname{diam}(\Sigma_p) < \pi$, *the function* $\bar{f}$ *takes a strictly local maximum at the vertex* o_p *of* K_p.

Proof. If $k = 1$, the lemma is trivial. Assume $k = 2$. The directionally derivative of the function $\bar{f}$ with the direction $v \in \Sigma_{o_p}(K_p)$ is

$$v(\bar{f}) = -\frac{\sin \operatorname{diam}(\Sigma_p)}{\operatorname{diam}(\Sigma_p)} < 0,$$

which proves the lemma. q.e.d.

From now on we assume the following:

Assumption 3.3. The function $\bar{f}$ takes a strictly local maximum at the vertex o_p of K_p.

Assume that $\epsilon > 0$ is small enough compared to the point $p \in X$ and that $0 < r \ll \epsilon$. The precise conditions for ϵ will be exposed in the proof of Lemma 3.5 below. The assumption together with Lemma 3.1 directly implies the following:

Lemma 3.4. *For every large i there is a point $\hat{p}_i \in M_i$ where f_i takes a local maximum such that $d(p_i, \hat{p}_i) \to 0$ as $i \to \infty$.*

We define the metric annulus

$$A(x; r_1, r_2) = B(x, r_2) - \operatorname{int} B(x, r_1)$$

for $r_1 < r_2$ and a point x in a metric space. Letting r be small and i large, we may assume that $\hat{p}_i$ as in Lemma 3.4 exists and satisfies $d(p_i, \hat{p}_i) \ll r$, and that the annulus $A(p_i; r/1000, 2r)$ contains no critical points of $d(p_i, \cdot)$ (resp. $d(\hat{p}_i, \cdot)$). Denote by q_i one of the critical points of $d(\hat{p}_i, \cdot)$ in $B(p_i, r)$ which are furthest from $\hat{p}_i$ if it exists, and set

$$\delta_i = d(\hat{p}_i, q_i).$$

Notice that $B(p_i, r) \simeq D^n$ if such q_i does not exist. Clearly, $\lim_{i\to\infty} \delta_i = 0$. Therefore, if i is large enough compared to a given $\lambda > 1$, the balls $B(p_i, r)$, $B(\hat{p}_i, r)$, and $B(\hat{p}_i, \lambda\delta_i)$ are all homeomorphic each other. By replacing with a subsequence of (M_i, p_i), it may be assumed that the rescaled pointed manifold $(\frac{1}{\delta_i} M_i, \hat{p}_i)$ converges to a noncompact pointed Alexandrov space (Y, y_0) of nonnegative curvature. The following lemma is important.

Lemma 3.5. *We have* $\dim Y \geq k + 1$.

Proof. Taking a subsequence if necessarily, we assume that $q_i \in \frac{1}{\delta_i}M_i$ tends to some point $z \in Y$ under the convergence $(\frac{1}{\delta_i}M_i, \hat{p}_i) \to (Y, y_0)$. Since q_i is a critical point of $d(\hat{p}_i, \cdot)$, the point z is a critical point of $d(y_0, \cdot)$. For any fixed number $a > 1$, set $R_i^j = d(\hat{p}_i, x_i^j) - a\delta_i$ and $B_i^j = B(x_i^j, R_i^j)$. Taking a subsequence, we assume that for each j, B_i^j converges to some closed subset B^j of Y as $i \to \infty$. Since the function $(d(x_i^j, \cdot) - R_i^j)/\delta_i = d(B_i^j, \cdot)/\delta_i$ on $M_i - B_i^j$ tends to the function $d(B^j, \cdot)$ on $Y - B^j$ and since $\hat{p}_i$ takes a local maximum of f_i, the point y_0 takes a local maximum of the function

$$f = \frac{1}{\beta_{\Sigma_p}(\epsilon)} \sum_j d(B^j, \cdot).$$

For each j, let $y_0 b^j$, $b^j \in \partial B^j$, be a minimal segment from y_0 to B^j which is a limit of $\hat{p}_i x_i^j - \mathrm{int}\, B_i^j$. The direction v^j of $y_0 b^j$ at Σ_{y_0} satisfies

$$\angle(v^j, v^{j'}) \geq \tilde{\angle} b^j y_0 b^{j'} \geq \lim_{i \to \infty} \tilde{\angle} x_i^j \hat{p}_i x_i^{j'} \geq \epsilon/2 \qquad \text{for all } j \neq j'.$$

Since z is a critical point of $d(y_0, \cdot)$, we have $\tilde{\angle} y_0 z b^j \leq \pi/2$ and hence $\angle z y_0 b^j \geq \tilde{\angle} z y_0 b^j \geq \pi/2 - \arcsin(1/a)$. Therefore, fixing a direction u to z at Σ_{y_0} we have

$$\angle(u, v^j) \geq \pi/2 - \arcsin(1/a) \qquad \text{for all } j.$$

Since f takes a local maximum at y_0, it follows that

$$0 \geq u(f) = -\frac{1}{\beta_{\Sigma_p}(\epsilon)} \sum_j \cos \angle(u, v^j),$$

which implies that the number of j's with $v^j \in A_a$ is not less than $\beta_{\Sigma_p}(\epsilon)/2$, where we set

$$A_a = A(u; \pi/2 - \arcsin(1/a), \pi/2 + \arcsin(1/a)).$$

Therefore, $\beta_{A_a}(\epsilon/2) \geq \beta_{\Sigma_p}(\epsilon)/2$. Taking $a \to \infty$ yields

$$\beta_{\partial B(u, \pi/2)}(\epsilon/2) \geq \beta_{\Sigma_p}(\epsilon)/2.$$

Consider the map which assigns to each $x \in \partial B(u, \pi/2)$ the direction at u of a minimal geodesic joining u and x. It follows from the Alexandrov convexity that this map is expanding, i.e., distance nondecreasing. Since the curvature of $\Sigma_u(\Sigma_{y_0})$ is ≥ 1, there is an expanding map from $\Sigma_u(\Sigma_{y_0})$

to the $(\ell-2)$-dimensional unit sphere $S^{\ell-2}(1)$, $\ell := \dim Y$. Combining these two expanding maps, we have $\beta_{\Sigma_p}(\epsilon)/2 \le \beta_{\partial B(u,\pi/2)}(\epsilon/2) \le \beta_{S^{\ell-2}(1)}(\epsilon/2)$. Since the order of $\beta_{\Sigma_p}(\epsilon)$ (resp. $\beta_{S^{k-2}(1)}(\epsilon)$) as $\epsilon \to 0$ is exactly ϵ^{1-k} (resp. ϵ^{2-k}), there is an $\epsilon_p > 0$ depending only on p such that $\beta_{\Sigma_p}(\epsilon_p)/2 > \beta_{S^{k-2}(1)}(\epsilon_p/2)$. We may take $\epsilon = \epsilon_p$. Thus we obtain $\beta_{S^{k-2}(1)}(\epsilon/2) < \beta_{S^{\ell-2}(1)}(\epsilon/2)$, which implies $k < \ell$. q.e.d.

We put the above results together into the following:

Lemma 3.6 (Key Lemma). *Assume that the dimensions satisfy $n = \dim M_i = 3$, $k = \dim X = 1$ or 2, and that $\operatorname{diam}(\Sigma_p) < \pi$ for a point $p \in X$. Then, there exists a small number $r_p > 0$ such that if $B(p_i, r)$ for a number $0 < r \le r_p$ is not homeomorphic to D^3, there are sequences $\hat{p}_i \in B(p_i, r)$ and $\delta_i \to 0$ satisfying the following (1)–(3).*

(1) *$d(p_i, \hat{p}_i) \to 0$ as $i \to \infty$; in particular, $\hat{p}_i$ converges to p in the convergence $(M_i, p_i) \to (X, p)$.*

(2) *For the limit (Y, y_0) of any convergent subsequence of $(\frac{1}{\delta_i} M_i, \hat{p}_i)$ we have*

$$k + 1 \le \dim Y \le 3.$$

(3) *If $S_i \subset M_i$, $S \subset Y$ are compact subsets such that S_i converges to S under the convergence $(\frac{1}{\delta_i} M_i, \hat{p}_i) \to (Y, y_0)$, then for every sufficiently large $R > 0$ we have $B(p_i, r) \simeq B(S_i, R\delta_i)$ for all i large enough compared to R.*

Proof. (1) and (2) are the direct consequences of the discussion above.

We will prove (3). Since Y is noncompact and of nonnegative curvature, $(\frac{1}{R} Y, y_0)$ converges to the limit cone of Y as $R \to \infty$. Therefore, for a $\mu > 1$ and a sufficiently large $R > 0$, the rescaled annulus $\frac{1}{R\delta_i} A(\hat{p}_i; \mu^{-1} R\delta_i, \mu R\delta_i)$ in M_i is d_{GH}-close to the annulus $A(o_\infty; \mu^{-1}, \mu)$ in the limit cone for i large, where o_∞ is the vertex of the limit cone. This together with a standard argument of critical point theory proves $B(\hat{p}_i, R\delta_i) \simeq B(S_i, R\delta_i)$. This completes the proof. q.e.d.

Next we shortly discuss the ideal boundary of Y, where Y is as in Key Lemma 3.6. The ideal boundary $Y(\infty)$ of Y and the Tits metric $\angle_\infty$ on $Y(\infty)$ were defined in [2] (cf. [22], [32]). Let K be the asymptotic cone of Y defined as the pointed Gromov-Hausdorff limit of $(\epsilon Y, y_0)$ as $\epsilon \to 0$ for a point $y_0 \in Y$. Then K is the Euclidean cone over $(Y(\infty), \angle_\infty)$.

Lemma 3.7. *There is an expanding map $\Sigma_p \to Y(\infty)$.*

Proof. For a fixed $\epsilon > 0$ and for each $x \in \partial B(p, \epsilon)$, take a minimal geodesic γ_i from $\hat{p}_i$ to x_i, where x_i is a point in $\partial B(p_i, \epsilon)$ converging to x. Passing to a subsequence, we may assume that $\frac{1}{\delta_i}\gamma_i$ converges to a geodesic ray γ_x from y_0 under the convergence $(\frac{1}{\delta_i}B(\hat{p}_i, r), \hat{p}_i) \to (Y, y_0)$. Thus we have a map $\varphi_\epsilon : \partial B(p, \epsilon) \to Y(\infty)$ defined by $\varphi_\epsilon(x) := \gamma_x(\infty)$. Since the lower bound of the sectional curvature of $\frac{1}{\delta_i}M_i$ goes to zero, we get

$$\begin{aligned} \angle_\infty(\gamma_x(\infty), \gamma_y(\infty)) &= \lim_{t\to\infty} 2\sin^{-1}\left(\frac{d(\gamma_x(t), \gamma_y(t))}{2t}\right) \\ &\geq 2\sin^{-1}\left(\frac{d(x, y)}{2\epsilon}\right) > 0. \end{aligned}$$

Letting $\epsilon \to 0$, we obtain an expanding map $\varphi : \Sigma_p \to Y(\infty)$. q.e.d.

4. Seifert bundle structure

Let a sequence of pointed complete orientable three-manifolds (M_i, p_i) with $K \geq -1$ converge to a pointed complete Alexandrov space (X, p) of dimension two with respect to the pointed Gromov-Hausdorff convergence. In this section, we study the topology of a neighborhood of p_i in the case when p is an interior singular point of X and define a compatible Seifert bundle structure on the neighborhood.

Let $p \in \operatorname{int} X$ be an interior singular point of X, and $r = r_p$ a fixed small positive number given in Key Lemma 3.6. We may assume that $B(p, 10r) - \{p\} \subset X_\delta$, where $\delta = \delta_2$ is a constant given in Fibration Theorem 2.2. Applying Fibration Theorem 2.2 to the convergence $B(p_i, r) \to B(p, r)$, we see that

$$\partial B(p_i, r) \simeq S^1 \times S^1. \tag{4.1}$$

Lemma 4.1. *$B(p_i, r)$ is homeomorphic to $S^1 \times D^2$ for large i.*

Proof. We prove the lemma by contradiction. Suppose that it does not hold. Passing to a subsequence, we may assume that all $B(p_i, r)$ are not homeomorphic to $S^1 \times D^2$. Note that the assumption in Key Lemma 3.6 is satisfied because of (4.1). Let $\hat{p}_i \in M_i$ and $\delta_i \to 0$ be as in Key Lemma 3.6. Then the limit (Y, y_0) of a subsequence of $(\frac{1}{\delta_i}B(\hat{p}_i, r), \hat{p}_i)$ has dimension three. The space Y contains an essential information on the topology of $B(p_i, r)$. We note that by Theorem 9.6 the topology of Y is determined by its soul S.

Assertion 4.2. S is a circle.

Suppose this assertion for a moment. Then we can obtain the topological type of $B(p_i, r)$ as follows. By Theorem 9.6, Y is isometric to the form $Y = (\mathbb{R} \times N^2)/\mathbb{Z}$, where N^2 is a nonnegatively curved Alexandrov surface homeomorphic to $\mathbb{R}^2$. Let a compact set $S_i \subset M_i$ converges to $S \subset Y$ under the convergence $(\frac{1}{\delta_i} B(\hat{p}_i, r), \hat{p}_i) \to (Y, y_0)$. For a large R, we then have

$$B(p_i, r) \simeq B(S_i, R; \tfrac{1}{\delta_i} M_i) \simeq B(S, R; Y) \simeq S^1 \times D^2, \tag{4.2}$$

where the second $\simeq$ follows from Stability Theorem 2.4 and the third $\simeq$ follows from Corollary 9.7. q.e.d.

Proof of Assertion 4.2. This is done by an argument similar to the above. If $\dim S = 0$, then $B(p_i, r) \simeq B(y_0, R\delta_i) \simeq D^3$. However this is impossible because of (4.1). Next suppose that $\dim S = 2$. Then by Theorem 9.6, Y would be isometric to the normal bundle $N(S)$ of S. It turns out that the ideal boundary $Y(\infty)$ of Y consists of at most two points. However this is impossible because of Lemma 3.7. Therefore we have $\dim S = 1$ and hence S is a circle. q.e.d.

Our next purpose is to study the limit of the universal covering spaces $\pi_i : \widetilde{B}(p_i, r) \to B(p_i, r)$ to define the Seifert bundle structure on $B(p_i, r)$. This will immediately provide the proof of Theorem 0.2. To do this, we need a rescaling argument.

Let $\tilde{p}_i \in \widetilde{B}(p_i, r)$ be a point over p_i, and $\Gamma_i \simeq \mathbb{Z}$ the deck transformation group. Since our argument is by contradiction, we can take a subsequence if necessary. For $\epsilon_i = d_{GH}(B(p_i, r), B(p, r))$, take a sequence $r_i \to 0$ such that $\epsilon_i / r_i \to 0$. Passing to a subsequence, we may assume that $(\frac{1}{r_i} \widetilde{B}(p_i, r), \tilde{p}_i, \Gamma_i)$ converges to a triple (Z, z, G) with respect to the pointed equivariant Gromov-Hausdorff convergence ([13]), where Z is a simply connected, complete Alexandrov space with nonnegative curvature.

Proposition 4.3. *Under the situation above, we have:*

(1) *There exists a locally trivial fibration*

$$f_i : A(p_i; r_i/2, r) \to A(p; r_i/2, r)$$

satisfying the conclusion of Fibration Theorem 2.2.

(2) $(\frac{1}{r_i} B(p_i, r), p_i)$ *converges to* (K_p, o_p).

(3) *Z is isometric to a product $Z_0 \times \mathbb{R}$ and G is isomorphic to $\mathbb{Z}_\mu \times \mathbb{R}$ for some integer $\mu \le [2\pi/L(\Sigma_p)]$, where $Z/G \equiv Z_0/\mathbb{Z}_\mu \equiv K_p$ (isometric).*

(4) *The space Z_0 is isometric to a flat cone, say $Z_0 = K(S^1_\ell)$ with cone angle $\ell \le 2\pi$. Thus the generator γ of $\mathbb{Z}_\mu$ is given by*

$$\gamma(re^{\ell\theta i}) = re^{\ell(\theta+\nu/\mu)i},$$

where $(\mu,\nu) = 1$ and we make an obvious identification

$$K(S^1_\ell) = \{re^{\ell\theta i} \,|\, 0 \le \theta \le 1, r \ge 0\}.$$

(5) *$(\frac{1}{r_i}\widetilde{B}(p_i,r_i), \tilde{p}_i)$ converges to $(B(z_0,1)\times\mathbb{R}, z)$ under the convergence $(\frac{1}{r_i}\widetilde{B}(p_i,r),\tilde{p}_i) \to (Z,z)$, where $\widetilde{B}(p_i,r_i) = \pi_i^{-1}(B(p_i,r_i))$ and z_0 is the vertex of the cone Z_0.*

Proof. Since

$$d_{GH}\left(\tfrac{1}{r_i}A(p_i;r_i/2,r), \tfrac{1}{r_i}A(p;r_i/2,r)\right) \le \epsilon_i/r_i,$$

and the δ_2-strain radius of $\frac{1}{r_i}A(p;r_i/2,r)$ is greater than a constant independent of i, (1) follows from Fibration Theorem 2.2. (2) is clear from the choice of r_i.

Using the limit G-action, one can construct a line in Z. It follows from the splitting theorem that Z is isometric to a product $Z_0 \times \mathbb{R}$. Since G is a Lie group ([14]), by using Lemma 3.10 of [13] it is possible to take a subgroup Γ_i' of Γ_i such that:

(1) $(\frac{1}{r_i}\widetilde{B}(p_i,r),\tilde{p}_i,\Gamma_i')$ converges to (Z,z,G_0), where G_0 is the identity component of G.

(2) $\Gamma_i/\Gamma_i' \simeq G/G_0$ for large i.

Since $Z/G \equiv K_p$ is of dimension two, $\dim Z - \dim G = 2$. If G_0 were trivial, $G \simeq \mathbb{Z}$ and $\dim Z = 2$. It turns out that Z is isometric to one of $\mathbb{R}^2$, $[0,\ell]\times\mathbb{R}$ and $[0,\infty)\times\mathbb{R}$. It is now an easy exercise to show that none of those cases implies that Z/G is a flat cone K_p, a contradiction. Thus $\dim G_0 = 1$ and $\dim Z = 3$. It follows from Stability Theorem 2.4 that Z_0 is a complete open Alexandrov surface homeomorphic to $\mathbb{R}^2$. If $G_0 \simeq S^1$, then $G \simeq \mathbb{Z}\times S^1$ and Z/G cannot be a flat cone, a contradiction. Hence $G_0 \simeq \mathbb{R}$. It is now easy to show that $G \simeq \mathbb{Z}_\mu \times \mathbb{R}$ and $Z/G \equiv Z_0/\mathbb{Z}_\mu \equiv K_p$. Thus Z_0 is a flat cone with cone angle $\mu \times L(\Sigma_p)$, and all the conclusions of the proposition follow. q.e.d.

Lemma 4.4. *There exists a topological Seifert bundle structure on $B(p_i, r)$ of orbit type (μ, ν) over $B(p, r)$ which is compatible to the circle bundle structure on $A(p_i; r_i/2, r)$ defined by f_i.*

Proof. For any $x \in \partial B(p_i, r/2)$ consider the fibre $f_i^{-1}(x)$.

Sublemma 4.5. *$f_i^{-1}(x)$ represents a generator of Γ_i', where Γ_i' is as in the proof of Proposition* 4.3.

Proof. We put $U = B(x, \ell r/10)$. Any non-trivial geodesic loop at $x_i \in f_i^{-1}(x)$ of length $\leq \ell r/100$ is contained in $f_i^{-1}(U)$. Since $f_i^{-1}(U) \simeq U \times S^1$, it follows that Γ_i' is contained in the image H_i of the inclusion homomorphism $\pi_1(f_i^{-1}(U)) \to \Gamma_i$. Conversely Proposition 4.3(3) implies that $H_i \subset \Gamma_i'$. Therefore $H_i = \Gamma_i'$. q.e.d.

Consider $B_i' = \widetilde{B}(p_i, r_i)/\Gamma_i' \simeq D^2 \times S^1$. Let γ_i be a generator of Γ_i, and γ_i' the generator of $\Gamma_i/\Gamma_i' \simeq \mathbb{Z}_\mu$ represented by γ_i. Let C_i be a path on $\partial B_i'$ joining a point $x \in \partial B_i'$ to $\gamma_i' x$ in a suitable direction. Then Proposition 4.3(4) implies that the union of $(\gamma_i')^k(C_i)$, $k = 0, \dots, \mu - 1$, is a loop rotating ν-times in the meridian direction. Hence the $\mathbb{Z}_\mu$-action on B_i' defines a Seifert bundle structure on $B(p_i, r_i)$ which is isomorphic to the Seifert bundle structure defined by the standard $\mathbb{Z}_\mu$-action on $D^2 \times S^1$:

$$\tau_{\mu\nu}(re^{i\theta}, e^{i\phi}) = (re^{i\,(\theta + \frac{2\nu}{\mu}\pi)}, e^{i\,(\phi + \frac{2\pi}{\mu})}).$$

Thus we can put the topological Seifert bundle structure on $B(p_i, r_i)$ of orbit type (μ, ν) which is compatible to the circle bundle structure f_i on $A(p_i; r_i/2, r)$. This completes the proof of Lemma 4.4. q.e.d.

Proof of Theorem 0.2. Take finitely many points $\{p_1, \dots, p_m\}$ such that $X - \{p_1, \dots, p_m\} \subset X_\delta$. Let r be so small that the balls $B(p_j, 10r)$, $1 \leq j \leq m$, are disjoint. Applying Lemma 4.4 to each $B(p_j, r)$, we complete the proof of Theorem 0.2. q.e.d.

Remark 4.6. Let X be as in Theorems 0.2 or 0.3. Applying the Gauss-Bonnet Theorem (Proposition 14.1) and the volume comparison to X yields that the Euler characteristic of X satisfies

$$2 \geq \chi(X) \geq -v_{-1}^2(D)/2\pi,$$

where $v_{-1}^2(D)$ denotes the volume of a D-ball in the hyperbolic plane. Moreover, as a consequence of Theorem 0.2 and 0.3, the number of singular fibres of M_i (resp. of $\mathrm{Seif}_i(X)$ in Theorem 0.3) is at most $4 + v_{-1}^2(D)/\pi$ (see Corollary 14.3).

5. Topology near boundary

In this section, we consider the case when the limit space X is a two-dimensional Alexandrov space with boundary. The argument in the previous section shows that the part M_i' of M_i converging to a part X_0 of X away from the boundary ∂X is a Seifert fibred space over X_0. Hence the essential point of the proof is to describe the topology of $M_i - M_i'$. Actually we prove that it is homeomorphic to $\partial X \times D^2$.

For reader's convenience, we give the definition of twisted bundles over surfaces. For the details, see [21]. Let $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$ and $I = [0,1]$. A twisted S^1-bundle $\text{Mö}\,\tilde{\times} S^1$ over a Möbius band Mö is defined as the quotient space $(S^1 \times I \times S^1)/\tau$, where τ is the involution of $S^1 \times I \times S^1$ defined by $\tau(e^{i\theta}, t, e^{i\eta}) = (e^{i(\theta+\pi)}, 1-t, e^{-i\eta})$. Let N be a non-orientable surface and $\hat{N}$ the orientable double cover of N with the nontrivial deck transformation σ on $\hat{N}$. Then a twisted I-bundle $N \tilde{\times} I$ over N is defined as the quotient space $(\hat{N} \times I)/\tau$, where τ is the involution of $\hat{N} \times I$ defined by $\tau(x,t) = (\sigma(x), 1-t)$. Note that $\partial\,\text{Mö}\,\tilde{\times} S^1 \simeq T^2$, $\partial N \tilde{\times} I \simeq \hat{N}$ and $\text{Mö}\,\tilde{\times} S^1 \simeq K^2 \tilde{\times} I$, where K^2 denotes a Klein bottle.

Proof of Theorem 0.3. For a small $\epsilon > 0$ we take $\{x_1, \dots, x_N\}$ contained in a fixed connected component C of ∂X such that:

(1) x_j is adjacent to x_{j-1}.

(2) $L(\Sigma_x) > \pi - \epsilon$ for any $x \in C - \{x_1, \dots, x_N\}$.

(3) There exist positive numbers r and $\delta \ll r$ such that:

   (a) $\tilde{\angle} x_j y z > \pi - \epsilon$ for every $y \in B(x_j, r)$ and for some $z \in X$.

   (b) $\tilde{\angle} x_j x x_{j+1} > \pi - \epsilon$ for every point $x \in B(\widehat{x_j x_{j+1}}, \delta) - B(x_j, r) - B(x_{j+1}, r)$, where $\widehat{x_j x_{j+1}}$ is the arc joining x_j and x_{j+1} in C.

   (c) If $\operatorname{diam}(\Sigma_{x_j}) = \pi$, then $\tilde{\angle} x_{j-1} x x_{j+1} > \pi - \epsilon$ for every point $x \in B(\widehat{x_{j-1} x_{j+1}}, \delta) - B(x_{j-1}, r) - B(x_{j+1}, r)$.

Now suppose $d_{GH}(M_i, X) < \epsilon_i$ and $\epsilon_i \ll \delta$. For a fixed j we take $p_i, p_i' \in M_i$ such that p_i and p_i' converge to x_j and x_{j+1} respectively under the convergence $M_i \to X$. Let B_i and B_i' be C^∞-approximations of $B(p_i, r)$ and $B(p_i', r)$ respectively. Let C_i' be a compact domain which converges to $B(\widehat{x_j x_{j+1}}, \delta)$, C_i the closure of $C_i' - B_i - B_i'$, and N_i the closure of $\partial C_i - B_i - B_i'$. Applying Fibration Theorem 2.2 to a neighborhood of $\partial B(\widehat{x_j x_{j+1}}, \delta)$, we can take such C_i' that for every $x \in (B_i \cup B_i') \cap N_i$

$$|\,\angle(\xi_1(x), \xi_2(x)) - \pi/2\,| < \tau(r) + \tau(r|\delta) + \tau(r, \delta|\epsilon),$$

where ξ_1 and ξ_2 denote the unit normal vector fields to $\partial(B_i \cup B_i')$ and N_i respectively, and $\tau(r_1, \dots, r_k|\epsilon)$ a function depending on r_i, ϵ satisfying $\lim_{\epsilon\to 0} \tau(r_, \dots, r_k|\epsilon) = 0$ for fixed r_i. Thus both ∂B_i and $\partial B_i'$ meet N_i transversally, and $B_i \cap N_i \simeq S^1$, $B_i' \cap N_i \simeq S^1$. It follows from Fibration Theorem 2.2 that

$$N_i \simeq S^1 \times I.$$

We next show that $C_i \simeq D^3$. Let γ_i be a geodesic in $B_i \cup C_i \cup B_i'$ converging to a geodesic joining x_j and x_{j+1}. Now we consider the functions

$$f_i = d(\gamma_i, \cdot), \qquad g_i = d(p_i, \cdot) - d(p_i', \cdot).$$

Note that f_i is regular on $f_i^{-1}([\,\delta/100, \delta\,])$ and the gradient of f_i is almost perpendicular to N_i. Note also that g_i is regular on C_i. Set $F_i = f_i^{-1}([\,0, \delta/2\,]) \cap g^{-1}(0)$ and denote by H_i the set consisting of all flow curves of the gradient of g_i contained in C_i through F_i. Clearly,

$$H_i \simeq F_i \times I.$$

Note that the gradient of f_i is almost perpendicular to that of g_i on $f_i^{-1}([\,\delta/100, \infty)) \cap C_i$. It follows that $\partial F_i \simeq S^1$. By the generalized Margulis lemma ([13]), $\pi_1(F_i) \simeq \pi_1(H_i)$ is almost nilpotent, and therefore by the orientability, $F_i \simeq D^2$.

It is easy to construct a smooth vector field V_i on a neighborhood of $C_i - H_i$ such that:

(1) $V_i = \operatorname{grad} f_i$ outside a small neighborhood of $\partial B_i \cup \partial B_i'$.

(2) V_i is tangent to $\partial B_i \cup \partial B_i'$.

(3) f_i is strictly decreasing along the flow curves of V_i.

Thus we have

$$C_i \simeq H_i \simeq D^3.$$

Next we show that $B_i \simeq D^3$. Suppose that $\operatorname{diam}(\Sigma_{x_j}) = \pi$. Let $\hat{C}_i'$ be a compact domain which converges to $B(\widehat{x_{j-1} x_{j+1}}, \delta)$, and $\hat{C}_i$ the closure of $\hat{C}_i' - B_i - B_i'$. Applying the previous argument, we obtain that $\hat{C}_i \simeq D^3$. It is now easy to see that $B_i \simeq \hat{C}_i \cap B_i \simeq \hat{C}_i \simeq D^3$. If $\operatorname{diam}(\Sigma_{x_j}) < \pi$, take a point $p_i'' \in M_i$ converging to x_{j-1}. For the points

p_i and p_i'' we construct a compact domain C_i' in the same way as the construction of C_i. From Fibration Theorem 2.2, $\partial B_i - C_i - C_i' \simeq S^1 \times I$, which implies that

$$\partial B_i \simeq S^2. \tag{5.1}$$

Then we show

Assertion 5.1. $B_i \simeq D^3$.

Proof. This is done by an argument similar to the proof of Lemma 4.1 as follows. Suppose that it does not hold. Passing to a subsequence, we may assume that all B_i are not homeomorphic to D^3. We may assume that the assumption in Key Lemma 3.6 is satisfied. Let $\hat{p}_i \in B_i$ and $\delta_i \to 0$ be as in Key Lemma 3.6. Then the limit (Y, y_0) of a subsequence of $(\frac{1}{\delta_i} B_i, \hat{p}_i)$ has dimension three. We show that the soul S of Y is a point. If $\dim S = 1$, then S is a circle and $B_i \simeq B(S,R) \simeq S^1 \times D^2$. However this is impossible because of (5.1). Next suppose that $\dim S = 2$. Then Y would be isometric to the normal bundle $N(S)$ of S. It turns out that the ideal boundary $Y(\infty)$ of Y consists of at most two points. However this is impossible because we have an expanding map $\Sigma_{x_j} \to Y(\infty)$ as before. Thus S is a point and we see that $B_i \simeq B(S,R) \simeq D^3$. q.e.d.

Now we change the notation. Let $p_j^i \in M_i$ be a point converging to x_j. Let B_j^i and C_j^i denote B_i and C_i respectively. Then the previous argument shows that

$$A_i = \bigcup_{j=1}^{N} (B_j^i \cup C_j^i) \simeq \partial X \times D^2.$$

By Theorem 0.2, $M_i - A_i$ is homeomorphic to a Seifert fibred space, say $\mathrm{Seif}_i(X)$ over $X - A \simeq X$, where $A = \cup_{i=1}^N B(x_j, r) \cup B(\widehat{x_{j-1}x_j}, \delta)$. Thus

$$M_i \simeq \mathrm{Seif}_i(X) \cup \partial X \times D^2,$$

where the identification is made by $\{$the fibre over $x \in \partial X\} = \{x\} \times \partial D^2$. This completes the proof of Theorem 0.3. q.e.d.

For the proof of Corollary 0.4, it suffices to prove the following

Proposition 5.2. *Let M be a closed orientable three-manifold and X a compact surface with boundary such that*

$$M \simeq \mathrm{Seif}(X) \cup \partial X \times D^2,$$

for a Seifert fibred space Seif(X) *over* X*, where the fibre over* $x \in \partial X$ *is identified with* $\{x\} \times \partial D^2$*. Let* g *and* k *denote the genus and the number of components of* ∂X*. Then we have*

$$M \simeq S^3 \# \underbrace{S^2 \times S^1 \# \cdots \# S^2 \times S^1}_{f(g,k)} \# L(\mu_1, \nu_1) \# \cdots \# L(\mu_\ell, \nu_\ell),$$

where $f(g,k)$ *and* (μ_j, ν_j) *are as in Corollary* 0.4.

We need a lemma.

Lemma 5.3. *Let* M *be an orientable three-manifold containing a surface* S *homeomorphic to* S^2 *such that* $M - S$ *is connected. Then:*

(1) M *has a decomposition,* $M \simeq S^2 \times S^1 \# N$*.*

(2) *Let* P *be the result of cutting of* M *along* S*. Then* $N = \mathrm{Cap}(P)$*, where* $\mathrm{Cap}(P)$ *denotes the closed three-manifold obtained from* P *by attaching* D^3 *along their boundary spheres.*

Proof. The first part follows from for instance Lemma 3.8 in [21]. The second part is an easy exercise. q.e.d.

Proof of Proposition 5.2. The proof is done by induction on $m = g + k + \ell$. First we suppose that X is orientable. If $m = 1$, then $k = 1$ and $g = \ell = 0$, and we have that $M \simeq D^2 \times S^1 \cup S^1 \times D^2$, where $(x, y) \in \partial D^2 \times S^1$ is identified with $(x, y) \in S^1 \times \partial D^2$. Hence $M \simeq S^3$.

Next we consider the case $m = 2$.

Case I. $g = k = 1$, $\ell = 0$.

Let $T^2 = S^1 \times S^1$, $B = \{(e^{i\theta}, e^{i\varphi}) \mid -\epsilon < \theta, \varphi < \epsilon\}$. We identify $X \simeq T^2 - B$. Consider the two curves, $\gamma_1(\theta) = (e^{i\theta}, 1)$, $\gamma_2(\varphi) = (1, e^{i\varphi})$, $\epsilon \le \theta, \varphi \le 2\pi - \epsilon$. Let S be the part of M "over γ_1";

$$S \simeq \gamma_1 \times S^1 \cup \partial\gamma_1 \times D^2 \simeq S^2.$$

Note that $M - S$ is connected. Then we have the decomposition $M = S^2 \times S^1 \# M'$, where $M' = \mathrm{Cap}(P_i)$, and P_i is the result of cutting M along S. Let S' be the part of M over γ_2, which is homeomorphic to S^2 as before. We may assume that $S' \subset M'$. Note that $M' - S'$ is connected. Then we have a decomposition $M' = S^2 \times S^1 \# M''$, where $M'' = \mathrm{Cap}(Q_i)$, and Q_i is the result of cutting M' along S'. Now one can verify that $M'' = \mathrm{Cap}(Q_i) \simeq S^3$. Thus $M \simeq S^2 \times S^1 \# S^2 \times S^1$.

Case II. $g = 0$, $k = \ell = 1$.

Let (μ_1, ν_1) be the orbit type of the unique singular orbit in $\mathrm{Seif}(X)$. Then from the definition of lens spaces, we have $M \simeq L(\mu_1, \nu_1)$.

Now consider the general case. Let γ be a path joining two points of ∂X which divides X into two compact domains X_1, X_2 in such a way that each X_j contains at least one of handles, boundary components and singular loci of X. The part of M over γ is homeomorphic to S^2. If we denote by M' and M'' the part of M over X_1 and X_2 respectively, then we have $M = M' \# M''$. By applying the induction to M' and M'' we obtain the required form for M.

We next consider the case when X is non-orientable. Suppose first $g = k = 1$, $\ell = 0$. Then $M = \text{Mö}\,\tilde{\times} S^1 \cup S^1 \times D^2$, where $(x, y) \in \partial\,\text{Mö}\,\tilde{\times} S^1$ is identified with $(x, y) \in S^1 \times \partial D^2$. Let γ be a path in Mö cutting Mö open to a disk. Let S be the part of M over γ, which is homeomorphic to S^2. Note that $M - S$ is connected. Then by a similar argument, we can conclude that $M \simeq S^2 \times S^1$. The rest of the inductive argument follows in the same way. q.e.d.

Corollary 5.4. *Let M_i and X be as in Theorem* 0.3*. Then the set of homeomorphism classes of M_i is finite.*

Proof. This follows from Theorem 0.3 and Proposition 5.2. q.e.d.

6. Collapsing to a closed interval

For a Riemannian manifold M with boundary, we denote by $\mathrm{dbl}(M)$ the double of M, i.e., the gluing of two copies of M along their boundaries by the original identification.

Proof of Theorem 0.5. Let M_i be a sequence of closed orientable three-manifolds with $K \geq -1$ collapsing to a closed interval $[0, \ell]$. By Fibration Theorem 2.2, we have

$$M_i = B_i \cup A_i \cup C_i,$$

where $A_i \simeq F_i \times [0, 1]$, B_i and C_i are metric balls Gromov-Hausdorff close to the endpoints of $[0, \ell]$ and F_i is homeomorphic to S^2 or T^2. If B_i is not homeomorphic to D^3, then by Key Lemma 3.6, we have a sequence $\hat{p}_i$ converging to the end point 0 of $[0, \ell]$, and $\delta_i \to 0$ such that for a subsequence, $(\frac{1}{\delta_i} B_i, \hat{p}_i)$ converges to a pointed noncompact Alexandrov space (Z, z_0) with nonnegative curvature, where $\dim Z \geq 2$. In what follows, assuming that B_i is not homeomorphic to D^3, we analyze the topology of B_i from the information on the fibre data F_i, the dimension and the boundary data of Z.

Case I. $F_i \simeq S^2$.

We have to show that B_i is homeomorphic to D^3 or $P^3 - \operatorname{int} D^3$. Assume that B_i is not homeomorphic to D^3. If $\dim Z = 3$, Z has no boundary. Let S be a soul of Z. If $\dim S = 1$, then $B_i \simeq S^1 \times D^2$, a contradiction to $\partial B_i \simeq S^2$. If $\dim S = 2$, we see that $S \simeq P^2$ and Z is isometric to a flat line bundle $P^2 \tilde{\times} \mathbb{R}$. Therefore $B_i \simeq P^2 \tilde{\times} I \simeq P^3 - \operatorname{int} D^3$.

Next we consider the case $\dim Z = 2$.

We claim

Assertion 6.1. *Z is isometric to the double $\operatorname{dbl}([0,\infty) \times [0,\infty)) \cap \{(x,y) \mid y \leq h\}$.*

Proof. First we show that Z has non-empty boundary. If Z has empty boundary, then take a large metric ball D around z_0. It turns out from Fibration Theorem 2.2 that ∂B_i is homeomorphic to $S^1 \times S^1$, a contradiction. Since ∂B_i is connected, Z has one end. It follows from Corollary 14.4 that Z is homeomorphic to $[0,\infty) \times \mathbb{R}$. Note that B_i is a Seifert fibred space over D. If B_i has no singular fibre, then Theorem 0.3 implies that $B_i \simeq D^3$, a contradiction. Thus B_i has a singular fibre, say one over $z \in Z$. Theorem 0.3 shows that z is an essential singular point. Hence Corollary 14.4 yields the conclusion. q.e.d.

We put

$$D = \operatorname{dbl}([0,\infty) \times [0,\infty)) \cap \{(x,y) \mid x \leq 1, y \leq h\} \subset Z,$$
$$D_1 = \operatorname{dbl}([0,\infty) \times [0,\infty)) \cap \{(x,y) \mid x \leq 1, y \leq h_1\}$$

for some $h_1 < h$. Then we obtain a Seifert fibration $f_i : B_i' \to D$ for a closed domain $B_i' \simeq B_i$. Let $N_i = \operatorname{Cap}(B_i') = B_i' \cup_{S^2} D^3$. Note that the closure of $N_i - f_i^{-1}(D_1)$ is homeomorphic to $S^1 \times D^2$. It is then easy to see that

$$N_i \simeq S^1 \times D^2 \cup S^1 \times D^2 \simeq L(2,1) \simeq P^3.$$

Thus $B_i \simeq P^3 - \operatorname{int} D^3$ as required.

Case II. $F_i \simeq T^2$.

We have to show that B_i is homeomorphic to $S^1 \times D^2$ or Mö $\tilde{\times} S^1$. Let us first assume $\dim Z = 3$. If the soul S of Z has dimension zero, the Soul Theorem would imply that $B_i \simeq D^3$, a contradiction. If $\dim S = 1$, we have $B_i \simeq S^1 \times D^2$. If $\dim S = 2$, then Z must be isometric to a flat twisted line-bundle $K^2 \tilde{\times} \mathbb{R}$ since $\partial B_i \simeq T^2$. Therefore $B_i \simeq K^2 \tilde{\times} I \simeq$ Mö $\tilde{\times} S^1$.

Next suppose that $\dim Z = 2$. We first consider the case when Z has empty boundary. If the soul of Z is not a point, in view of the connectedness of ∂B_i, Z must be isometric to a Möbius strip, and therefore $B_i \simeq$ Mö $\tilde{\times} S^1$. If the soul of Z is a point, then Z is homeomorphic to a plane. Note that B_i is a Seifert fibred space over a large metric ball $D \subset Z$. The Cohn-Vossen formula then implies that the number r of singular orbits in B_i is at most two. If $r = 1$, then $B_i \simeq S^1 \times D^2$. If $r = 2$, then a straightforward argument shows that B_i is homeomorphic to $K^2 \tilde{\times} I \simeq$ Mö $\tilde{\times} S^1$.

Suppose next that Z has non-empty boundary. If ∂Z is compact, then Corollary 14.4 implies that Z is isometric to $S^1 \times [0, \infty)$. It follows from Theorem 0.3 that $B_i \simeq S^1 \times D^2$. If ∂Z is noncompact, it follows from a way similar to the previous argument that $Z \simeq [0, \infty) \times \mathbb{R}$ and that the number r of singular orbits in B_i is at most one. If $r = 1$, then $B_i \simeq P^3 - D^3$. If $r = 0$, then $B_i \simeq D^3$. In any case, we have a contradiction to $\partial B_i \simeq T^2$. This completes the proof of Theorem 0.5.

q.e.d.

Proof of Corollary 0.7. Rescale the metric of M_i so that the diameter of the new metric is equal to one. Passing to a subsequence, we may assume that M_i with the new metric converges to a nonnegatively curved Alexandrov space Y with positive dimension. If $\dim Y \leq 2$ or $\pi_1(M_i)$ is infinite, then Theorems 0.2, 0.3 and 0.5 together with the result of [13] mentioned in Introduction imply that a finite cover of M_i is homeomorphic to either S^3, $S^1 \times S^2$, T^3 or a nilmanifold. If $\dim Y = 3$, then Stability Theorem 2.4 yields that M_i is homeomorphic to Y. q.e.d.

Proof of Corollary 0.9. This is done by contradiction. Suppose that the corollary does not hold. Then we have a sequence $M_i \in \mathcal{M}(3, D)$ of closed, prime three-dimensional Riemannian manifolds with infinite fundamental groups such that the volume of M_i goes to zero as $i \to \infty$ and that M_i does not admit a geometric structure. We may assume that each M_i is orientable. Passing to a subsequence, we may assume that M_i collapses to a compact Alexandrov space X. If X is a point, it follows from [13] together with the infiniteness assumption on the fundamental groups that M_i admits a geometric structure modelled on either $S^1 \times \mathbb{R}$, $\mathbb{R}^3$ or Nil, a contradiction. If $\dim X = 1$ or 2, then Theorems 0.2, 0.3 and 0.5 together imply that M_i is homeomorphic to a Seifert fibred space or a infrasolvmanifold, and hence admits a geometric structure (see [29]). q.e.d.

In view of the above proof, the infiniteness assumption on the fundamental groups in Corollary 0.9 can be replaced by a lower diameter bound. Namely we have the following corollary by the same argument.

Corollary 6.2. *For any positive numbers $\delta \le D$, there exists a constant $\epsilon = \epsilon(\delta, D) > 0$ such that if a closed, prime three-manifold admits a Riemannian metric contained in $\mathcal{M}(3, D)$ with diameter $\ge \delta$ and volume $< \epsilon$, then it admits a geometric structure modelled on one of the seven geometries except H^3.*

Proof of Corollary 0.10. This is done by contradiction. Suppose that the corollary does not hold. Then we have a sequence $M_i \in \mathcal{M}(3, D)$ of pairwise non-homeomorphic closed three-dimensional Riemannian manifolds admitting no geometric structures. We may assume that each M_i is orientable. Passing to a subsequence, we may assume that M_i converges to a compact Alexandrov space X. By Stability Theorem 2.4 we only have to consider the collapsing case $\dim X \le 2$. If $\dim X = 0$, we can rescale the metric of M_i so that the new metric has diameter $= 1$. Thus we may consider that $\dim X = 1$ or 2. Theorems 0.2, 0.3 and 0.5 then imply that M_i is homeomorphic to a Seifert fibred space or a infrasolvmanifold, and hence admits a geometric structure. q.e.d.

7. Construction of collapsing metrics

In this section, we prove Theorem 0.6 by constructing collapsing metrics together with some examples. First we show that an infinite sequence of pairwise non-homeomorphic prism manifolds collapses to a closed interval under $K = 1$.

Example 7.1. Let M be a prism manifold S^3/Γ, where $\Gamma \subset SO(4)$ is one of the following two types (see [36]):

Type 1) Γ is generated by

$$\gamma_1 = \begin{pmatrix} R(1/m) & 0 \\ 0 & R(r/m) \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 0 & I \\ R(2\ell/n) & 0 \end{pmatrix},$$

where n is even, $(n(r-1), m) = 1$, $r \not\equiv r^2 \equiv 1 \bmod m$, $(\ell, n/2) = 1$ and

$$R(\theta) = \begin{pmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}.$$

Type 2) Γ is generated by

$$\sigma_1 = \begin{pmatrix} R(\frac{u+v}{uv}) & 0 \\ 0 & R(\frac{v-u}{uv}) \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where v is even, $(u, v) = 1$.

For Type 1), take distinct prime numbers p and q and consider the group Γ_{pq} of Type 1) defined by $m = p$, $r = p-1$, $n = 2q$ and $\ell = 1$. The group generated by γ_1 and the group generated by γ_2^2 converge to distinct circle groups. This implies that the subgroup Λ_{pq} generated by γ_1 and γ_2^2 has index two in Γ_{pq} and converges to T^2. Since the limit of the $\mathbb{Z}_2$-action on $S^3(1)/\Lambda_{pq}$ induced by γ_2 changes the orientation of $[\,0, \pi/2\,]$, we conclude that $S^3(1)/\Gamma_{pq}$ converge to $[\,0, \pi/4\,]$ under $K \equiv 1$.

For Type 2), consider a sequence Γ_p of groups of Type 2) defined by $u = p$, an odd prime number and $v = p+1$. Then similarly one can verify that as $p \to \infty$, $S^3(1)/\Gamma_p$ converges to $[\,0, \pi/4\,]$ under $K \equiv 1$.

The case $\dim X = 2$ is covered by the following two examples. We denote by $D^n(\epsilon)$ and $S^{n-1}(\epsilon) = \partial D^n(\epsilon)$ the n-disk and the $(n-1)$-sphere of radius ϵ.

Proof of Theorem 0.6. (I) Let M^n be a Seifert fibred space over an $(n-1)$-dimensional smooth compact orbifold X without boundary. Then it admits a local S^1-action, which defines a pure-polarized F-structure on M^n. Therefore M^n admits a sequence of metrics which collapses to X with bounded curvature $|K| \le \Lambda$ for some constant $\Lambda > 0$. See [10] for details.

(II) Let N^3 be a Seifert fibred space over two-dimensional smooth compact orbifold X with boundary. We suppose that X has a product metric $\partial X \times [\,0, \delta\,)$ near the boundary ∂X. As in (I), one can construct a Riemannian metric h_ϵ on N such that:

(a) (N, h_ϵ) collapses to X with $|K| \le \Lambda$.

(b) Near the boundary (N, h_ϵ) is the product of a collar neighborhood of ∂X and $S^1(\epsilon)$.

Let D^2_ϵ denote a disk with a metric such that:

(c) The diameter of D^2_ϵ is less than 10ϵ.

(d) The curvature of D^2_ϵ is nonnegative and its metric is a product metric $S^1(\epsilon) \times [\,0, \delta\,)$ near the boundary.

If (M, g_ϵ) denotes the union $(N, h_\epsilon) \cup (\partial X \times D^2_\epsilon)$ glued along their boundaries, then it converges to X under a lower curvature bound $K \geq -\Lambda$. Note that $\lim_{\epsilon\to 0} \sup K_{g_\epsilon} = +\infty$.

(III) Let M, $X = [\,0, \ell\,]$, $F = S^2$ or $= T^2$, A, B, C be as in Theorem 0.6. Namely, $F = \partial B = \partial C$, $A = F \times [\,0, \ell\,]$ and $M \simeq B \cup_\varphi C$, where $\varphi : \partial B \to \partial C$ is the gluing map.

Let T^2_ϵ be the flat torus of square of length ϵ. Let F_ϵ denote $S^2(\epsilon)$ or T^2_ϵ, and $A_\epsilon = F_\epsilon \times [\,0, \ell\,]$.

Case (1) $F = S^2$.

Then M is homeomorphic to S^3, P^3 or $P^3 \# P^3$. Let $D^3(\epsilon)$ be a three-disk with a metric satisfying conditions similar to (c), (d) in (II). Let B_ϵ and C_ϵ be either $D^3(\epsilon)$ or a projective space $P^3 - \operatorname{int} D^3$ with a disk removed equipped with a metric satisfying conditions similar to (c), (d) in (II). In either case, $(M, g_\epsilon) = B_\epsilon \cup A_\epsilon \cup C_\epsilon$ is a smooth Riemannian manifold and converges to $[\,0, \ell\,]$ under $K \geq 0$.

Case (2) $F = T^2$.

First consider the following

Case (2-i) $(B, C) = (S^1 \times D^2, S^1 \times D^2)$.

Then M is homeomorphic to either S^3, $S^2 \times S^1$ or a lens space. If $M \simeq S^2 \times S^1$, then the union $(M, g_\epsilon) = S^1(\epsilon) \times D^2(\epsilon) \cup A_\epsilon \cup S^1(\epsilon) \times D^2(\epsilon)$ is a smooth Riemannian manifold with nonnegative curvature which converges to $[\,0, \ell\,]$ as $\epsilon \to 0$.

Lemma 7.2. *Let T^m act effectively on a compact smooth manifold M^n, and g a T^m-invariant metric on M^n. Then there exists a sequence of T^m-invariant metrics g_i on M^n such that (M^n, g_i) collapses to $(M^n/T^m, \bar{g})$ under a lower curvature bound $K \geq -\Lambda$, where $\bar{g}$ is the quotient metric.*

Proof. Let g_i be the metrics constructed in Example 1.2(c) in [37] so that (M^n, g_i) collapses to $(M^n/T^m, \bar{g})$ under a lower curvature bound $K \geq -\Lambda$. Then the commutativity of T^m implies the T^m-invariance of g_i. q.e.d.

Let S^3/Γ be any lens space and $T^2 \subset SO(4)$ the maximal torus. By Lemma 7.2, we have a sequence g_i of T^2-invariant metrics on S^3 collapsing to $S^3(1)/T^2 = [\,0, \pi/2\,]$ under $K \geq -\Lambda$. Note that g_i descends to a metric $\bar{g}_i$ on S^3/Γ. Thus, $(S^3/\Gamma, \bar{g}_i)$ collapses to $[\,0, \pi/2\,]$ under $K \geq -\Lambda$.

Case (2-ii) $(B, C) = (S^1 \times D^2, \text{Mö}\,\tilde{\times} S^1)$.

In this case we show that M is homeomorphic to either $S^1 \times S^2$, $P^3 \# P^3$ or a prism manifold.

Case (2-ii-a) $\varphi(\partial D^2\text{-factor}) = S^1\text{-factor}$.

In this case we come to the situation of Propositon 5.2. Namely it is the case when the base surface X is a Möbius band with no singular points ($g = k = 1, \ell = 0$). It follows that $M \simeq S^1 \times S^2$. Let $B_\epsilon = S^1(\epsilon) \times D^2(\epsilon)$ and $C_\epsilon = (\text{Mö}\,\tilde{\times} S^1, h_\epsilon)$ such that ∂C_ϵ is isometric to T^2_ϵ, $\mathrm{diam}(C_\epsilon) < 10\epsilon$ and $K_{h_\epsilon} = 0$. Consider now the union $(M, g_\epsilon) = B_\epsilon \cup A_\epsilon \cup C_\epsilon$. Since $\varphi : T^2_\epsilon \to T^2_\epsilon$ is an isometry in this case, (M, g_ϵ) is a smooth Riemannian manifold and converges to the closed interval $[\,0, \ell\,]$ under $K \geq 0$.

Case (2-ii-b) The case other than Case (2-ii-a).

In this case M is a Seifert fibred space over P^2, where the number r of singular fibres satisfies $r \leq 1$. If $r = 0$, namely, $\varphi(S^1\text{-factor}) = S^1\text{-factor}$, $\varphi(\partial D^2\text{-factor}) = \partial\text{Mö-factor}$, then $M \simeq P^2 \tilde{\times} S^1 = P^3 \# P^3$. Consider the union $M_\epsilon = B_\epsilon \cup A_\epsilon \cup C_\epsilon$, where B_ϵ and C_ϵ are as in Case (2-ii-a). Note that $\varphi : T^2_\epsilon \to T^2_\epsilon$ is an isometry in this case. Hence M_ϵ is a smooth Riemannian manifold and converges to the closed interval $[\,0, \ell\,]$ under $K \geq 0$.

If $r = 1$, it follows (see [24]) that M is a prism manifold.

Case (3) $(B, C) = (\text{Mö}\,\tilde{\times} S^1, \text{Mö}\,\tilde{\times} S^1)$.

In this case, M is doubly covered by a T^2-bundle over S^1. In particular M admits a geometric structure modelled on $\mathbb{R}^3$, Nil or Sol (see [29]). We show that every such $M \simeq B \cup_\varphi C$ actually admits a sequence of metrics collapsing to a closed interval with bounded curvature $|K| \leq \Lambda$.

First consider the monodoromy matrix $J \in SL(2, \mathbb{Z})$ induced by the homomorphism φ_* on the first homology group of the torus. If J can be represented by

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},$$

then M is finitely covered by T^3 ($n = 0$) or a nilmanifold ($n \neq 0$). Suppose now that $n \neq 0$. Then M is a circle bundle over the Klein bottle

$$K^2 = \text{Mö} \cup_{S^1} \text{Mö}\,.$$

It follows from a straightforward argument that a suitable double cover $\hat{M}$ admits a sequence of metrics g_i such that:

(a) $\lim_{i\to\infty} |K_{g_i}| = 0$.

(b) $(\hat{M}, g_i)$ collapses to S^1.

(c) g_i is invariant under the non-trivial deck transformation of $\hat{M} \to M$.

Thus g_i descends to a metric on M collapsing to a closed interval.

For a specific example, see Example 7.3.

Finally consider the other case that J is of hyperbolic type:

$$J \sim \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

where $t \neq 0$. In this case, M is doubly covered by a solvmanifold $\hat{M}$, which is a T^2-bundle over a circle. It is standard to construct a sequence of metrics on $\hat{M}$ such that:

(a) $|K_{g_i}| \leq \Lambda$.

(b) $(\hat{M}, g_i)$ converges to S^1.

(c) g_i is invariant under the non-trivial deck transformation of $\hat{M} \to M$.

Thus g_i descends to a metric on M converging to a closed interval. This completes the proof of Theorem 0.6. q.e.d.

Example 7.3. Let us consider the Heisenberg group N consisting of all 3×3 real upper triangular matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

with the left invariant metric

$$g_\epsilon = dx^2 + \epsilon^2 dy^2 + \epsilon^4 (dz - x dy)^2.$$

Let Λ be the integer lattice of N and γ the isometry of (N, g_ϵ) defined as

$$\gamma(x, y, z) = (-x, y + 1, -z).$$

Then the group Γ generated by Λ and γ is discrete and contains Λ as a normal subgroup of index two. It is easily verified that as $\epsilon \to 0$, $(N/\Gamma, g_\epsilon)$ collapses to a closed interval under $\lim_{\epsilon \to 0} |K_{g_\epsilon}| = 0$.

Corollary 7.4. *Let M_i be a convergent sequence in $\mathcal{M}(3, D)$ of closed orientable three-dimensional Riemannian manifolds with finite fundamental groups. Suppose that the limit X of M_i has boundary as an Alexandrov space. Then M_i is homeomorphic to either S^3, a lens space or a prism manifold.*

Proof. This follows from the discussion above and Corollary 0.4.

q.e.d.

8. Comparison with bounded curvature collapsing

In the bounded curvature case, the collapsing phenomena are well understood (see [12], [11], [8], etc.). Since we know no reference for the following result however, we give a proof.

Proposition 8.1. *Let M_i, $i = 1, 2, \dots$, be a sequence of closed n-dimensional Riemannian manifolds with $|K| \le 1$, $\mathrm{diam}(M_i) \le D$ converging to a space X of dimension $(n-1)$. Then:*

(1) *M_i is a Seifert fibred space over X for large i.*

(2) *If each M_i is orientable, then the Alexandrov space X has no boundary.*

Proposition 8.1 (2) explains a difference between the bounded curvature collapsing and the lower curvature collapsing.

Proof. We follow an argument in [12]. Let $B(r)$ denote the metric ball $B(0, r; \mathbb{R}^n)$. In particular, we use the notation $B = B(1)$ and $B' = B(2)$. For $p \in X$ and $p_i \in M_i$ with $p_i \to p$, let g_i denote the pullback metric on B' of the metric of M_i via the exponential map $f_i = \exp_{p_i} : B' \to M_i$. Let Γ_i denote the pseudogroup of isometric imbeddings $\gamma : (B, g_i) \to (B', g_i)$ such that $f_i \circ \gamma = f_i$. Then $(B, g_i)/\Gamma_i$ is isometric to $B(p_i, 1; M_i)$. Passing to a subsequence, we may assume that $((B, g_i), \Gamma_i)$ converges to $((B, g_\infty), G)$, where g_∞ is a $C^{1,\alpha}$-metric, G is a pseudogroup of isometric imbeddings $\gamma : (B, g_\infty) \to (B', g_\infty)$ and $(B, g_\infty)/G$ is isometric to $B(p, 1; X)$. Note that G is locally isomorphic to a Lie group ([12]). We consider the isotropy group $I_p = \{g \in G \mid g\bar{p} = \bar{p}\}$, where $\bar{p}$ denote the origin of B. Since $\dim G = 1$, I_p should be finite. Otherwise I_p would contain G_0, the identity component of G, and the orbit $G\bar{p}$ would be a finite set, a contradiction. Then it is straightforward ([12]) to show that there exists a $\delta > 0$ such that $B(p, \delta; X)$ is isometric to $(B(\delta), g_\infty)/I_p G_0$. Let V be the $(n-1)$-plane in $T_{\bar{p}}B$ perpendicular to $G_0\bar{p}$. Then I_p acts on V as linear isotoropy representation. If U is a small ball in V around the origin, then U/I_p is almost isometric to $(B(\delta), g_\infty)/I_p G_0 \equiv B(p, \delta; X)$. If each M_i is orientable, Γ_i preserves the orientation, and so does G. This implies that I_p preserves the orientation of U and hence X has empty boundary.

By [12], we have a map $f : M_i \to X$ such that $f^{-1}(p) \simeq S^1/I_p \simeq S^1$. Thus M_i is foliated by circles and I_p is a cyclic group, say $\mathbb{Z}_m$. Note that I_p is lower semicontinuous with respect to p, namely $I_q \subset I_p$ for any q sufficiently close to p. This implies that there exists a small neighborhood D of p in X such that an m-fold cyclic covering of $f^{-1}(D)$ is isomorphic to the product $D \times S^1$ as S^1-foliated manifolds. q.e.d.

Remark 8.2. By using the argument above, we can also prove Theorem 0.5 in the case when $|K| \le 1$ and the limit is a closed interval. We leave it as reader's exercise.

By the construction of metrics in Section 7 and the following examples, we obtain that *for any given fibre and collapsing data a collapsing satisfying the data actually occurs in the bounded curvature.* To see this, it suffices to consider the situation that a sequence of closed orientable three-manifolds M_i collapses to a closed interval with $|K| \le 1$. In this case, we have the fibre and collapsing data: $F = T^2$, (B, C) is one of $(S^1 \times D^2, S^1 \times D^2)$, $(S^1 \times D^2, \text{Mö}\,\tilde{\times} S^1)$ and $(\text{Mö}\,\tilde{\times} S^1, \text{Mö}\,\tilde{\times} S^1)$.

Lemma 8.3. *Let (T^2, g_0) be the flat torus of rectangle with side lengths a and b, and let (T^2, g_1) be the flat torus of parallelogram with the same sidelengths a, b and angle θ. Then there exists a metric g on*

$T^2 \times [\,0,1\,]$ *such that* $|K_g| \le \tau(\cos\theta)$ *and*

$$g|_{T^2\times t} = \begin{cases} g_0 & \text{near } t=0, \\ g_1 & \text{near } t=1. \end{cases}$$

Proof. Putting $\epsilon = \cos\theta$, choose a monotone non-decreasing function $\psi : [\,0,1\,] \to \mathbb{R}$ with

$$\psi(t) = \begin{cases} 0 & \text{near } t=0, \\ \epsilon & \text{near } t=1, \end{cases}$$

and $\sup\{\,|\psi'(t)|, |\psi''(t)\,\} < \tau(\epsilon)$. Let $T^2 = \mathbb{R}^2/\Gamma$, where $\Gamma = a\mathbb{Z} \times b\mathbb{Z}$. Using the canonical coordinates $(x,y,t) \in T^2 \times [\,0,1\,]$, we define the metric g by

$$g = dt^2 + dx^2 + \psi(t)dxdy + dy^2.$$

Then a standard calculation shows that $|K_g| \le \tau(\epsilon)$. q.e.d.

Example 8.4. First consider the collapsing of $S^2 \times S^1$ with possible data on (B,C). Let Γ_i, $i \ge 1$, be the subgroup of $\mathrm{Isom}(S^2(1) \times \mathbb{R})$ generated by

$$\gamma(x,t) = (R(1/i)x, t + 1/i^2),$$

where

$$R(\theta) = \begin{pmatrix} \cos 2\pi\theta & -\sin 2\pi\theta & 0 \\ \sin 2\pi\theta & \cos 2\pi\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $(S^2(1)\times\mathbb{R})/\Gamma_i$ is homeomorphic to $S^2 \times S^1$ and converges to $[\,0,\pi\,]$ under $0 \le K \le 1$. In this case, $(B,C) = (S^1 \times D^2, S^1 \times D^2)$.

In the above construction, we slightly change the metric of $S^2(1)$ so that the new metric, say g, has the product metric $S^1(1) \times (-\epsilon,\epsilon)$ near the equator $S^1(1) \subset S^2(1)$. Note that $((S^2,g)\times\mathbb{R})/\Gamma_i$ still converges to a closed interval under $0 \le K \le 1$. Next we consider $P_i = ((S^2_+, g) \times \mathbb{R})/\Gamma_i$, where S^2_+ denotes the closed upper hemisphere. The boundary of P_i is isometric to the flat torus of parallelogram with side lengths $a_i = \sqrt{4\pi^2/i^2 + 1/i^4}$, $b_i = 1/i$ and angle θ_i, where $\lim_{i\to\infty}\theta_i = \pi/2$. Note that a_i (resp. b_i) corresponds to the ∂D^2-factor (resp. the S^1-factor) of $\partial P_i \simeq \partial D^2 \times S^1$. Choose a flat metric h_i on $\text{Mö}\,\tilde{\times} S^1$ such that:

(1) The boundary of $(\text{Mö}\,\tilde{\times} S^1, h_i)$ is isometric to the flat torus of rectangle with side lengths a_i and b_i.

(2) The length of S^1-factor is a_i.

(3) The length of ∂ Mö-factor is b_i.

Let $(T^2 \times [\,0,1\,], g_i)$ be the metric constructed in Lemma 8.3 for (a_i, b_i, θ_i). Then the union $(\text{Mö}\,\tilde{\times} S^1, h_i) \cup (T^2 \times [\,0,1\,], g_i) \cup P_i$ defined in an obvious way is a smooth Riemannian manifolds homeomorphic to $S^2 \times S^1$ (see Proposition 5.2) and collapses to a closed interval under $-\tau(\cos\theta_i) \le K \le 1$. In this case, $(B, C) = (S^1 \times D^2, \text{Mö}\,\tilde{\times} S^1)$.

Next take a flat metric k_i on $\text{Mö}\,\tilde{\times} S^1$ such that:

(1) The boundary of $(\text{Mö}\,\tilde{\times} S^1, k_i)$ is isometric to the flat torus of rectangle with side lengths a_i and b_i.

(2) The length of ∂ Mö-factor is a_i.

(3) The length of S^1-factor is b_i.

Then the union $(\text{Mö}\,\tilde{\times} S^1, k_i) \cup (T^2 \times [\,0,1\,], g_i) \cup P_i$ defined in an obvious way is a smooth Riemannian manifolds homeomorphic to $P^3 \# P^3$ and collapses to a closed interval under $-\tau(\cos\theta_i) \le K \le 1$. In this case, $(B, C) = (S^1 \times D^2, \text{Mö}\,\tilde{\times} S^1)$.

By our argument, we can summarize the results on both lower curvature collapsingand bounded curvature collapsing of oriented three-manifolds in Table 1.

Part 2. Classification of complete open Alexandrov three-spaces of nonnegative curvature

The argument in Section 3 shows that the geometry of complete open Alexandrov spaces of dimension three is important to obtain an essential topological information on a small neighborhood of the manifold near the singular point of the limit space. In Part 2, we give a classification of three-dimensional complete open Alexandrov spaces with nonnegative curvature extending the Cheeger-Gromoll classification ([9]) of three-dimensional complete open Riemannian manifolds with nonnegative curvature.

X	$\|K\| \leq 1$	$K \geq -1$	fibre data
$\partial X^2 = \emptyset$	Seifert bundle	Seifert bundle	$F = S^1$
$\partial X^2 \neq \emptyset$		$S^3 \# S^2 \times S^1 \# \cdots$ $\# L(\mu_1, \nu_1) \# \cdots$	$F = S^1$
S^1		$S^2 \times S^1$	$F = S^2$
	Flat, Nil, Sol	Flat, Nil, Sol	$F = T^2$
$[\,0, \ell\,]$		S^3, P^3, $P^3 \# P^3$	$F = S^2$
	$S^3/\mathbb{Z}_p$, $S^2 \times S^1$	$S^3/\mathbb{Z}_p$, $S^2 \times S^1$	$B, C = S^1 \times D^2$, $F = T^2$
	$S^2 \times S^1$, $P^3 \# P^3$, prism manifolds	$S^2 \times S^1$, $P^3 \# P^3$, prism manifolds	$B = S^1 \times D^2$, $C = \text{Mö}\,\tilde{\times} S^1$, $F = T^2$
	Flat, Nil, Sol	Flat, Nil, Sol	$B, C = \text{Mö}\,\tilde{\times} S^1$, $F = T^2$
point	Flat, Nil	$S^2 \times S^1$, $P^3 \# P^3$, Flat, Nil, $\sim S^3/\Gamma$	

Table 1: Collapsing in dimension three

9. Examples, results and rigidity

First we recall the basic construction in [9] for complete open Riemannian manifold of nonnegative curvature. This can be done with the same procedure for Alexandrov spaces. Let X be an n-dimensional complete noncompact Alexandrov space with nonnegative curvature. For a geodesic ray $\gamma : [\,0, \infty\,) \to X$, consider the Busemann function $b_\gamma : X \to \mathbb{R}$ defined by

$$b_\gamma(x) = \lim_{t \to \infty} t - d(x, \gamma(t)).$$

It is straightforward to see

Lemma 9.1. *b_γ is convex.*

Now we consider the Busemann function associated with a point $p \in X$ defined by $b(x) = \sup_\gamma b_\gamma(x)$, where γ runs over all the geodesic rays emanating from p. The function b is convex and the sublevel sets $b^{-1}(\,-\infty, a\,]$ are compact.

Let $C \subset X$ be a closed totally convex subset, and ∂C denote the boundary of C as an Alexandrov space. We consider the distance function $\rho_C = \mathrm{dist}(\partial C, \cdot)$ on C.

Lemma 9.2 ([26]). *ρ_C is concave on C.*

Let μ be the minimum of b and put $C^0 = b^{-1}(\mu)$. Note that the totally convex set $b^{-1}(\mu)$ has empty interior. If C^0 has boundary, consider the distance function ρ_{C^0} on C^0, and put C^1 to be the maximum set. By iteration, we have a sequence of finitely many non-empty compact totally convex sets:

$$C^0 \supset C^1 \supset C^2 \cdots \supset C^k,$$

where $n > \dim C^0$, $\dim C^i > \dim C^{i+1}$ and C^k has no boundary. Then a soul S of X is defined as $S = C^k$. It was proved in [26] that X is homotopy equivalent to S.

By the following example, the Soul Theorem ([9]) does not hold for three-dimensional Alexandrov spaces. See [26] for such an example in 5-dimension.

Example 9.3. Let $\varphi_1(x, y) = (-x, -y)$ and $\varphi_2(x, y) = (x, -y)$ be the isometric involutions of $\mathbb{R}^2$ and consider the product $X = [0, 1] \times \mathbb{R}^2$. The pairs (φ_i, φ_j) acts on ∂X in such a way that φ_i acts on $\{0\} \times \mathbb{R}^2$ and φ_j acts on $\{1\} \times \mathbb{R}^2$. Now we consider the quotient spaces

$$X_1 = X/(\varphi_1, \varphi_1), \qquad X_2 = X/(\varphi_1, \varphi_2), \qquad X_3 = X/(\varphi_2, \varphi_2).$$

It is easy to verify that each X_i, $1 \le i \le 3$, is a complete open Alexandrov space with nonnegative curvature and without boundary. Note that each X_i has one point soul $S = \{(1/2, 0, 0)\}$, and that the normal bundle $N(S)$ is homeomorphic to $\mathbb{R}^3$. Clearly we have

$$X_1 \simeq K_1(P^2) \cup_{D^2} K_1(P^2), \qquad X_2 \simeq K(P^2), \qquad X_3 \simeq \mathbb{R}^3,$$

where $K_1(P^2)$ is the open unit cone over the projective plane P^2 and D^2 is a disk in $\partial K_1(P^2)$. Note also that X_1 is isometric to $\mathbb{R}^3/\Gamma$, where Γ is the discrete subgroup of isometries of $\mathbb{R}^3$ generated by $\gamma(x, y, z) = -(x, y, z)$ and $\sigma(x, y, z) = (x + 2, y, z)$.

We recall that in the case when X is a Riemannian manifold, X is isometric to the normal bundle $N(S)$ if the soul S is of codimension one (see [9]). As the discussion below shows, this does not hold for Alexandrov spaces even in the three-dimensional case.

Let S be a compact nonnegatively curved Alexandrov surface without boundary, and $p_1, \dots, p_k \in S$ essential singular points of S. Let us consider a three-dimensional Alexandrov space of nonnegative curvature (if it exits), denoted by $L(S; p_1, \dots, p_k)$ or simply $L(S; k)$, satisfying the following :

(1) S is isometrically imbedded as a totally convex set of $L(S; p_1, \dots, p_k)$.

(2) $\{p_1, \dots, p_k\}$ are the set of topological singular points of $L(S; p_1, \dots, p_k)$, and hence the space of directions at p_i is homeomorphic to P^2.

(3) The space of directions at each $x \in S - \{p_1, \dots, p_k\}$ is isometric to the spherical suspension over $\Sigma_x(S)$.

(4) The normal bundle $\mathcal{N}$ of $S - \{p_1, \dots, p_k\}$ in $L(S; p_1, \dots, p_k)$ has a locally product metric.

(5) The normal exponential map $\exp : \mathcal{N} \to L(S; p_1, \dots, p_k)$ carries each ray in the fibre from the zero section to a geodesic ray in $L(S; p_1, \dots, p_k)$.

(6) There is a unique geodesic ray γ_i from p_i perpendicular to every direction in $\Sigma_{p_i}(S)$, and $L(S; p_1, \dots, p_k) - \exp(\mathcal{N})$ consists of γ_i, $1 \le i \le k$.

We note that S is a soul of $L(S; p_1, \dots, p_k)$.

Example 9.4. We take $S_1 \simeq P^2$, $S_2 \simeq S^2$, $S_4 \simeq S^2$ with nonnegative curvature. We assume that each S_i has i essential singular points. Note that S_4 is isometric to $\mathrm{dbl}([\,0, a\,] \times [\,0, b\,])$ (see Proposition 14.4). Then one can define the spaces $L(S_i; i)$ in an obvious way.

Proposition 9.5. *If* $k \ge 1$, $L(S; p_1, \dots, p_k)$ *is isometric to one of* $L(S_1; 1)$, $L(S_2; 2)$ *or* $L(S_4; 4)$ *in Example* 9.4.

Proof. Since p_i is an essential singular point of S, it follows from Corollary 14.3 that $k \le 4$. Note that the union of fibers in $\mathcal{N}$ over a small circle around p_i is isometric to a flat Möbius strip with respect to the induced metric. It follows from a cutting and gluing argument that if $S \simeq S^2$ (resp. $S \simeq P^2$), then $k = 2$ or 4 (resp. $k = 1$). q.e.d.

The main purpose of Part 2 is to prove the following

Theorem 9.6 (Generalized Soul Theorem). *Let X be a three-dimensional complete noncompact Alexandrov space with nonnegative curvature and without boundary, and S a soul of X. Then:*

(1) *If $\dim S = 0$, then X is homeomorphic to $\mathbb{R}^3$, or the cone $K(P^2)$ over the projective plane P^2, or $X_1 = \mathbb{R}^3/\Gamma$ in Example 9.3.*

(2) *If $\dim S = 1$, then X is isometric to a quotient $(\mathbb{R} \times N)/\Lambda$, where N is an Alexandrov space with nonnegative curvature homeomorphic to $\mathbb{R}^2$ and Λ is an infinite cyclic group. The Λ-action is diagonal; Λ acts on $\mathbb{R}$ by translation and on N by isometries fixing a point of N. In particular, X is homeomorphic to an N-bundle over a circle.*

(3) *If $\dim S = 2$, then X is isometric to either the normal bundle $N(S)$ of S in X or one of types $L(S_i, i)$, $i = 1, 2, 4$, in Proposition 9.5.*

The metric of $N(S)$ in Theorem 9.6 is defined in an obvious way.

The proof of Theorem 9.6 (1) is given in Sections 10-13.

The following corollary is the direct consequence of Theorem 9.6 together with the Morse theory given in Section 10.

Corollary 9.7. *Under the assumption of Theorem 9.6, suppose further that X is a topological manifold. Then:*

(1) *X is homeomorphic to the normal bundle $N(S)$.*

(2) *The topology of the closed ball $B = B(S, r)$ around S is determined as follows:*

   (a) *If $\dim S = 0$, then B is homeomorphic to D^3.*

   (b) *If $\dim S = 1$, then B is homeomorphic to $S^1 \times D^2$ or a solid Klein bottle.*

   (c) *If $\dim S = 2$, then B is homeomorphic to the normal I-bundle $N_1(S)$ of S in X.*

Theorem 9.6 (2) and a part of (3) are the special cases of the following:

Theorem 9.8. *Let X be an n-dimensional complete noncompact Alexandrov space with nonnegative curvature and empty boundary, and S a soul of X.*

(1) *If* $\dim S = 1$*, then* X *is isometric to a quotient* $(\mathbb{R} \times N)/\Lambda$*, where* N *is an Alexandrov space with nonnegative curvature whose soul is a point and* $\Lambda \simeq \mathbb{Z}$*. The* Λ*-action is diagonal;* Λ *acts on* $\mathbb{R}$ *by translation and on* N *by isometries fixing a point of* N*. Thus* X *is homeomorphic to an* N*-bundle over a circle.*

(2) *If* $\dim S = n - 1$ *and* X *is a topological manifold, then* Σ_p *is the spherical suspension over* $\Sigma_p(S)$ *for every* $p \in S$ *and* X *is isometric to the normal bundle* $N(S)$ *of* S *in* X*.*

Proof of Theorem 9.8 (1). Suppose that $\dim S = 1$ and let Λ be the deck transformation group of the universal covering $\pi : \widetilde{X} \to X$. Then S is isometric to a circle and $\Lambda \simeq \mathbb{Z}$. It is easy to see that $\pi^{-1}(S)$ is totally convex set isometric to $\mathbb{R}$. The splitting theorem then implies that $\widetilde{X}$ is isometric to a product $\mathbb{R} \times P$. Let $p_1 : \Lambda \to \mathrm{Isom}(\mathbb{R})$, $p_2 : \Lambda \to \mathrm{Isom}(P)$ be the projections. Then $p_1(\Lambda)$ acts on $\mathbb{R}$ by translation. If $\pi^{-1}(S)$ corresponds to $\mathbb{R} \times \{p\}$, then the point p is a fixed point of $p_2(\Gamma)$. This completes the proof of Theorem 9.8(1). q.e.d.

In the Riemannian case, the Berger Comparison Theorem (cf. [7]) was used for the proof of Theorem 9.8(2). It is unknown if the Berger Comparison Theorem holds for Alexandrov spaces. Hence we need another argument for the proof.

For the proof of Theorem 9.8(2), we consider the situation that C^0 is of dimension $n-1$. In what follows, we put $C = C^0$ for simplicity.

We say that a direction $\xi \in \Sigma_p = \Sigma_p(X)$ at $p \in C$ is *normal* to C if $\angle(\xi, v) = \pi/2$ for all $v \in \Sigma_p(C)$.

Lemma 9.9. *Suppose that* $C = C^0$ *has dimension* $n - 1$*.*

(1) *Every point of* C *has at most two normal directions to* C*.*

(2) *For a point* $p \in \mathrm{int}\, C$ *and a normal direction* $\xi \in \Sigma_p(X)$*, there exists a locally isometric covering map*

$$\Sigma_\xi(\Sigma_p(X)) \to \Sigma_p(C),$$

of order $r \le 2$*, where:*

(a) $r = 1$ *if and only if* p *has two normal directions to* C *and* $\Sigma_p(X)$ *is isometirc to the spherical suspension over* $\Sigma_p(C)$*.*

(b) $r = 2$ *if and only if* p *has exactly one normal direction to* C*. In this case,* p *is an essential singular point of* X *and* $\Sigma_p(X) - \Sigma_p(C)$ *is connected.*

Proof. For $p \in C$, let $\xi \in \Sigma_p(X)$ be normal to C. The Alexandrov convexity implies that for every $v_1, v_2 \in \Sigma_p(C)$, ξ, v_1 and v_2 are the vertices of a geodesic triangle isometric to a comparison triangle in the unit sphere. Let $v \in \Sigma_p(C)$ be a regular point of $\Sigma_p(C)$. It follows that $\xi'_v \in \Sigma_v(\Sigma_p(X))$ is perpendicular to $\Sigma_v(\Sigma_p(C))$. Thus if there were three normal directions to C at p, we would have three directions in $\Sigma_v(\Sigma_p(X))$ perpendicular to $\Sigma_v(\Sigma_p(C))$, a contradiction.

By the argument above, we have a locally isometric covering map $\pi : \Sigma_\xi(\Sigma_p(X)) \to \Sigma_p(C)$ of order $r \leq 2$. The rest of the proof is now clear. q.e.d.

Setting $X^t = b^{-1}(-\infty, \mu + t]$, we have the filtration $\{X^t\}_{t \geq 0}$ by compact totally convex sets such that:

(1) $X^s = \{x \in X^t \mid d(x, \partial X^t) \geq t - s\}$ for $s \leq t$.

(2) $X^0 = C^0$.

A point $p \in C$ is called a *one-normal point* (resp. a *two-normal point*) if Σ_p contains exactly one (resp. two) normal direction to C. Recall that if $p \in C$ is a one-normal point, then it is an essential singular point of X.

Proposition 9.10. *Under the assumption of Theorem* 9.8*, suppose further that* $\dim C = n - 1$*. Let* $p, q, r, s \in X$ *be such that:*

(1) $p, r \in \partial X^t$*, and* $q, s \in C$*.*

(2) $d(p, q) = d(r, s) = t$*.*

(3) *A minimal geodesic* pr *does not meet* C*.*

Then p, q, r, s *are the vertices of a totally geodesic flat rectangle.*

Proof. First we show that for every $p \in \partial X^t$ and $q \in C$ with $d(p, q) = t$, $\Sigma_p(X)$ is the spherical suspension over $\partial \Sigma_p(X^t)$. From the basic construction, there is a geodesic ray γ starting from q, through p and perpendicular to C. Let p_1 be the intersection point of γ with $\partial X^{t'}$ for a $t' > t$. Since p is the foot of both q and p_1 to ∂X^t, we see that $\angle(q'_p, \partial \Sigma_p(X^t)) \geq \pi/2$, $\angle((p_1)'_p, \partial \Sigma_p(X^t)) \geq \pi/2$. Since $\angle(q'_p, (p_1)'_p) =$

π, it follows from the splitting theorem applied to K_p that Σ_p is the spherical suspension over $\partial\Sigma_p(X^t)$.

For given p, q, r, s, let $\gamma : [0, a] \to X$ be a minimal geodesic joining p to s. We consider the concave function $f(u) = d(\gamma(u), \partial X^t)$. Put $\alpha = \angle(\dot\gamma(0), \partial\Sigma_p(X^t))$, where $\dot\gamma(0)$ denotes the direction at $\gamma(0)$ represented by γ.

Assertion 9.11. $f'(0) = \sin\alpha$.

Proof. For arbitrary $\epsilon > 0$, take a geodesic $\sigma(u)$ emanating from p such that

$$\angle(\dot\sigma(0), \partial\Sigma_p) < \epsilon, \qquad |\angle(\dot\gamma(0), \dot\sigma(0)) - \alpha| < \epsilon.$$

Consider now the function $g(u) = d(\gamma(u), \sigma(u\cos\alpha))$. We note that

$$g'(0) = \sin\angle(\dot\gamma(0), \dot\sigma(0)).$$

It suffices to show that $|f'(0) - g'(0)| < 2\epsilon$. Clearly, $f(u) \le g(u) + \epsilon u$ and hence $f'(0) \le g'(0) + \epsilon$. Suppose that $f(u_n) \le u_n(g'(0) - 2\epsilon)$ for some sequence $u_n \to 0$. Let p_n be a point of ∂X^t which is closest from $\gamma(u_n)$. Then we would have $\tilde\angle\gamma(u_n)pp_n \le \angle(\dot\gamma(0), \dot\sigma(0)) - 2\epsilon$, and that $\angle\gamma(u_n)pp_n \le \angle(\dot\gamma(0), \dot\sigma(0)) - \epsilon$ for large n. Taking a subsequence if necessary, we may assume that the direction $v_n = (p_n)'_p$ converges to a direction $v \in \Sigma_p$. It turns out from the lower semi-continuity of angle that $\angle(\dot\gamma(0), v) \le \angle(\dot\gamma(0), \dot\sigma(0)) - \epsilon$, and hence $\angle(\dot\gamma(0), \dot\sigma(0)) \ge \alpha + \epsilon$, a contradiction. q.e.d.

We put $b = d(q, s)$ and consider the triangle $\triangle p'q's'$ on $\mathbb{R}^2$ such that $d(p', q') = t$, $d(q', s') = b$ and $\angle p'q's' = \pi/2$. Set $a' = d(p', s')$, $\alpha' = \angle p's'q'$, $\theta' = \angle q'p's'$, and $\theta = \angle qps = \pi/2 - \alpha$. Since $\angle pqs = \pi/2$, we have $a' \ge a = d(p, s)$. It follows from the concavity of f that

$$f'(0) \ge \frac{t}{a} \ge \frac{t}{a'}.$$

Thus from the previous assertion, we obtain that

$$\alpha \ge \alpha' \qquad \text{and} \qquad \theta \le \theta'. \tag{9.1}$$

Consider now a comparison triangle $\tilde\triangle pqs$ in $\mathbb{R}^2$ and put $\tilde\theta = \tilde\angle qps$, $\tilde\alpha = \tilde\angle psq$. Since we may assume for our purpose that $t > b$, it follows from an obvious consideration with $a' \ge a > t$ that $\alpha' \le \tilde\alpha \le \pi/2$, $\theta' \le \tilde\theta$ and hence

$$\theta' = \theta = \tilde\theta, \quad \alpha' = \tilde\alpha = \alpha, \quad a = a' \quad \text{and} \quad \tilde\angle pqs = \pi/2. \tag{9.2}$$

It follows from the rigidity argument(cf.[30]) that $\triangle pqs$ spans a totally geodesic flat triangle isometric to $\tilde{\triangle} pqs$. Furthermore, $f'(0) = t/a$. It follows from the concavity of f that $f(u) = tu/a$ for all u. Let x_u and y_u be the points on ∂X^t and qs respectively such that $f(u) = d(\gamma(u), x_u)$ and $d(q, y_u) = ub/a$. Then it follows together with the comparison argument that $d(x_u, y_u) \le d(x_u, \gamma(u)) + d(\gamma(u), y_u) \le t$. Thus γ lies on the minimal connections from the points of qs to ∂X^t.

By repeating the argument above for x_u, y_u, r, s in place of p, q, r, s, we conclude that the set of minimal connections $x_u y_u$, $0 \le u \le a$, provides a totally geodesic flat rectangle. q.e.d.

Proof of Theorem 9.8 (2). Since $\Sigma_p \simeq S^{n-1}$ for every $p \in S$, it follows from Lemma 9.9 that p is a two-normal point, and hence the conclusion follows from Lemma 9.9(2) and Proposition 9.10. q.e.d.

Proof of Theorem 9.6 (3). If S contains no one-normal point, then X is isometric to $N(S)$ by Proposition 9.10. Now let $p_1, \ldots, p_k$ be the one-normal points of S. Proposition 9.10 implies that X can be written as $X = L(S; p_1, \ldots, p_k)$. Then the conclusion follows from Proposition 9.5. q.e.d.

10. Preliminaries on Morse theory

In the rest of Part 2, we shall prove Theorem 9.6(1).

We need the Morse theory for distance functions on Alexandrov spaces.

A map $\pi : E \to B$ is a *topological submersion* if for each $p \in E$ there are a neighborhood U of p in the fibre $\pi^{-1}(\pi(p))$, a neighborhood N of $\pi(p)$ in B and a topological imbedding $\varphi : U \times N \to E$ onto a neighborhood of p such that $\pi \circ \varphi$ is the projection $U \times N \to N$. The map φ is a *product chart* about U for π, and the image $\varphi(U \times N)$ is a *product neighborhood* around p.

A finite dimensional topological space Y is said to be a *WCS-space* if it satisfies the following (1) and (2):

(1) Y is a stratified space, i.e., it has a stratification

$$Y \supset \cdots \supset Y^{(n)} \supset \cdots \supset Y^{-1} = \phi,$$

such that $Y^{(n)} - Y^{(n-1)}$ is a topological n-manifold without boundary.

(2) For each $x \in Y^{(n)} - Y^{(n-1)}$ there is a cone C with vertex v and a homeomorphism $\rho : \mathbb{R}^n \times C \to Y$ onto an open neighborhood of x in Y such that $\rho^{-1}(Y^{(n)}) = \mathbb{R}^n \times \{v\}$.

Theorem 10.1 ([33]). *Let $\pi : E \to B$ be a topological submersion, and $F = \pi^{-1}(x_0)$ the fibre over a point x_0. We assume that F is a WCS-space.*

(1) *For given compact sets A_1, A_2 of F and for open neighborhoods U_i of A_i in F, let $\varphi_i : U_i \times N_i \to E$ be product charts about U_i for π. Then there exists a product chart $\varphi : U \times N \to E$ about an open set $U \supset A_1 \cup A_2$ in F such that*

$$\varphi = \begin{cases} \varphi_1 & \text{near } A_1 \times \{x_0\}, \\ \varphi_2 & \text{near } (A_2 - U_1) \times \{x_0\}. \end{cases}$$

(2) *If π is proper in addition, then $F \hookrightarrow E \xrightarrow{\pi} B$ is a locally trivial fibre bundle.*

Let $f = (f_1, \dots, f_m) : U \to \mathbb{R}^m$ be a map on an open set U of X defined by $f_i(x) = d(A_i, x)$ for compact subsets $A_i \subset X$. The map f is said to be (c, ϵ)-*regular* at $p \in U$ if there is a point $w \in X$ such that:

(1) $\angle((A_i)'_p, (A_j)'_p) > \pi/2 - \epsilon$.

(2) $\angle(w'_p, (A_i)'_p) > \pi/2 + c$.

Theorem 10.2 ([26]). *Let X be an Alexandrov space with curvature bounded below, $U \subset X$ an open subset, and $f : U \to \mathbb{R}^m$ (c, ϵ)-regular at each point of U. If ϵ is small compared with c, then we have:*

(1) *f is a topological submersion.*

(2) *If f is proper in addition, then the fibres of f are WCS-spaces. Hence f is a locally trivial fibre bundle over its image.*

We simply say that f is regular on U if it satisfies the assumption in Theorem 10.2.

In what follows, we use the notation in Section 9. In particular, X denotes a complete open Alexandrov space with nonnegative curvature without boundary.

We first observe the following simple:

Lemma 10.3. *For any compact set K in C^0, let $f = d(K, \cdot)$. Then f is regular on $X - C^0$.*

Proof. For any $t > b(x)$ there is a point $y \in \partial X^t$ such that $d(x, y) = t - b(x)$. Then it follows from the total convexity that $\angle yxK > \pi/2$. q.e.d.

Lemma 10.4. *If $\dim C^0 = 0$, then X is homeomorphic to the tangent cone K_p, where $\{p\} = C^0$.*

Proof. This follows from Lemma 10.3 and Theorems 10.2 and 2.4. q.e.d.

The purpose of the rest of this section is to obtain the topological type of X^t for any small $t > 0$ in the case when $\dim X = 3$ and the soul S is a point. From now on we assume that X is of dimension three and the dimension of C^0 is one or two. We put $C = C^0$ for simplicity.

In the Riemannian case, the distance function from the soul S has no critical points outside S, which implies that any complete open Riemannian manifold with nonnegative curvature is diffeomorphic to the normal bundle over its soul ([9]). For Alexandrov spaces however, we have the following counterexamples:

Example 10.5. Let $X = \mathrm{dbl}([0, a] \times [0, \infty))$. Then the distance function from the soul point has two critical points $(0, 0)$ and $(a, 0)$.

Example 10.6. We construct a complete open Alexandrov space X with nonnegative curvature consisting of some building blocks each of which is a convex polyhedron in the (x, y, z)-space. Let C be a convex polygon on the (x, y)-plane. We denote by $\{e_i\}$, $1 \le i \le k$, the edges of ∂C. The largest building block is $C \times [0, \infty)$. Let ℓ_i be the ray starting from $(p_i, 1)$ and parallel to the positive direction of z-axis, where p_i is a point on $\mathbb{R}^2 - \mathrm{int}\, C$ sufficiently close to the midpoint of e_i. Let B_i be the convex hull of the union $\ell_i \cup e_i \times [0, \infty)$, and B_i' the identification space $B_i \cup_{E_i} B_i$, where $E_i = \partial B_i - \mathrm{int}\, e_i \times (0, \infty)$. Note that B_i' has nonnegative curvature and its boundary consists of two copies, say F_i, F_i', of $e_i \times [0, \infty)$. Let K denote the quotient space of the disjoint union of $C \times [0, \infty)$ and B_i', $1 \le i \le k$, where $e_i \times [0, \infty)$ is identified with F_i. Finally we take the double $X = \mathrm{dbl}(K)$. Since K has nonnegative curvature, so does X (see [26]). Note that $C^0 = C$ and the soul of X is an interior point of C.

To treat such cases as in the previous example, it is convenient to consider the following notion of pseudo-gradient flows.

For an open set $U \subset X$, a continuous local $\mathbb{R}$-action ψ on U is called *a local flow* on U, and denoted by $\psi(x,s)$ for $(x,s) \in U \times \mathbb{R}$ as long as it can be defined. For a continuous function f on X, a local flow ψ on U is a *gradient* for f on an open subset $V \subset U$ if $f(\psi(x,s))$ is strictly decreasing in s as long as $\psi(x,s) \in V$. A local gradient flow ψ on U for f is a *pseudo-gradient* if it is a gradient outside a compact subset of U.

Later we shall consider local pseudo-gradient flows for the following functions on X:

$$f(x) = d(C,x), \qquad f_\epsilon(x) = d(C_\epsilon, x),$$

where

$$C_\epsilon = \{x \in C \,|\, d(\partial C, x) \geq \epsilon\}.$$

By Theorem 10.2, we have the following lemma in a similar way to Lemma 10.3.

Lemma 10.7. *For every positive numbers ϵ and δ, there exists a local gradient flow $\psi(x,s)$ for f_ϵ on a neighborhood of $f_\epsilon^{-1}([\,\epsilon+\delta,\infty\,))$ which provides a homeomorphism $f_\epsilon^{-1}([\,\epsilon+\delta,\infty\,)) \simeq f_\epsilon^{-1}(\epsilon+\delta) \times [\,\epsilon+\delta,\infty\,)$.*

The idea of the proof of Theorem 9.6 (1) is to push a small neighborhood of ∂C into $\mathcal{N} \subset X$ by using a pseudo-gradient flow of f_ϵ for small ϵ, where $\mathcal{N}$ is the normal bundle over $\operatorname{int} C$ with essential singular points removed and assumed to be imbedded in X as in Section 9. Examples 10.5 and 10.6 suggest that the difficulty in the construction of such a pseudo-gradient flow occurs near ∂C. In the next section, we shall study the local topological structure near ∂C.

11. Local structure at ∂C

Let X be as in Theorem 9.6 and S the soul of X which is a point. Since f_ϵ may have critical points on ∂C, we need to understand the topology of a small neighborhood of a point of ∂C. In this section, we assume that C has dimension two and non-empty boundary. First note that C is homeomorphic to D^2.

The following lemma is an easy consequence of Corollary 14.4.

Lemma 11.1. *The number of one-normal points in* $\operatorname{int} C$ *is less than or equal to two.*

Example 11.2. Let C_k be a nonnegatively curved Alexandrov surface homeomorphic to D^2 and having distinct essential singular points

$p_1, \ldots, p_k$ in $\operatorname{int} C_k$ $(k \leq 2)$. Then we construct a three-dimensional complete noncompact Alexandrov space $L'(C_k; p_1, \ldots, p_k)$ in a similar way to the construction of $L(S_k; k)$ in Section 9. Note that the boundary of $L'(C_1; p_1)$ (resp. of $L'(C_2; p_1, p_2)$) is a non-trivial (resp. trivial) line bundle over ∂C_1 (resp. over ∂C_2). In the fibre over every point $p \in \partial C_k$, we naturally identify the two rays emanating from p, and obtain a complete open nonnegatively curved Alexandrov space $L(C_k; p_1 \ldots, p_k)$ without boundary such that $C = C_k$ and that the topological singular points are $p_1, \ldots, p_k$.

Proposition 11.3. *Every point p of X except the one-normal points in* $\operatorname{int} C$ *is a manifold-point of X, in other words, Σ_p is homeomorphic to a sphere.*

From Proposition 9.10 and the proof of Lemma 10.3, we know that $\operatorname{diam}(\Sigma_p) > \pi/2$ for every point $p \in X - \partial C$ and hence Σ_p is homeomorphic to a sphere ([26]). Note that from the basic construction, $\operatorname{diam}(\Sigma_p) \geq \pi/2$ for every $p \in C$ (if $\dim C \geq 1$). Thus for the proof of Proposition 11.3, we only have to care a point $p \in \partial C$ with $\operatorname{diam}(\Sigma_p) = \pi/2$.

Let Σ be a two-dimensional compact Alexandrov surface with curvature ≥ 1 and without boundary. Suppose that $\operatorname{diam}(\Sigma) = \pi/2$. For a subset $B \subset \Sigma$ such that $\hat{B} = \{x \in \Sigma \,|\, d(B, x) = \pi/2\}$ is non-empty, we consider $A_1 = \hat{B}$ and $A_2 = \hat{A}_1$. Then we have $\hat{A}_2 = A_1$ (see [15] for details).

Proposition 11.4. *Let Σ, A_1 and A_2 be as above. Then we have:*

(1) *If both A_1 and A_2 are contractible, then Σ is homeomorphic to a sphere.*

(2) *If one of A_1 and A_2 is not contractibe, then Σ is isometric to the projective plane,*

$$(\textit{the spherical suspension over } S^1_\ell)/\mathbb{Z}_2,$$

where the length ℓ of the circle S^1_ℓ is less than or equal to 2π and Σ has constant curvature $K = 1$ outside the possible singular vertex.

Proof. First we note that from the Alexandrov convexity:

(a) A_i are convex sets.

(b) Any distinct three points $x, y \in A_i$ and $z \in A_j$, $i \neq j$ span a geodesic triangle isometric to $\tilde{\triangle} xyz$ in $S^2(1)$.

(c) For any $x \in \Sigma - (A_1 \cup A_2)$, minimal geodesic segments from x to A_1 and to A_2 make an angle greater than $\pi/2 + c$ for some $c = c(x) > 0$.

We show that

Assertion 11.5. $\dim A_1 + \dim A_2 \leq 1$.

Proof. Suppose that the assertion does not hold. Then $\dim A_1 = \dim A_2 = 1$. Let x_i be any interior point of A_i. Using (b) above, we see that a neighborhood of a minimal geodesic segment $x_1 x_2$ has constant curvature $= 1$. Then applying a standard parallel translation technique along $x_1 x_2$ together with curvature $K = 1$, we would have a contradiction to $d(A_1, A_2) = \pi/2$. q.e.d.

Making use of Theorem 10.2 together with (c) above, we obtain that $\Sigma - B(A_1, \epsilon) - B(A_2, \epsilon)$ is homeomorphic to $\partial B(A_1, \epsilon) \times [0, 1]$ for any small $\epsilon > 0$. This implies the conclusion (1). If A_1 is a point and A_2 is a circle, in view of the above (b), it is easy to see that $B(A_2, \epsilon)$ is homeomorphic to a Möbius band and that Σ is isometric to the required one. q.e.d.

Lemma 11.6. *If $p \in \partial C$ is a two-normal point, then* $\operatorname{diam}(\Sigma_p) > \pi/2$.

Proof. Let ξ_0, ξ_1 be the directions at p normal to C, v any point of $\Sigma_p(C)$, and ξ_t a minimal geodesic segment in Σ_p joining ξ_0 and ξ_1. Suppose that $\operatorname{diam}(\Sigma_p) = \pi/2$. Applying the Alexandrov convexity to the triangle $\triangle v \xi_0 \xi_1$, we have $\angle(v, \xi_t) \geq \pi/2$ and hence $\angle(v, \xi_t) = \pi/2$ by the assumption. It turns out that ξ_t are normal at p, a contradiction to Lemma 9.9. q.e.d.

Proof of Proposition 11.3. We only have to consider a point $p \in \partial C$ with a unique normal ξ to C. Let us consider subsets of $\Sigma = \Sigma_p$, $B = \Sigma_p(C)$, $A_1 = \{\xi\} = \hat{B}$ and $A_2 = \hat{A}_1$. By Proposition 11.4, it suffices to show that A_2 is a segment. Suppose that A_2 is a circle. Then Proposition 11.4 implies that the length ℓ of A_2 is less than or equal to π. On the other hand, from construction, $K(\Sigma_p(C))$ is totally convex in K_p and hence in $K(A_2)$. This is however impossible since $\ell \leq \pi$.

q.e.d.

12. Deformation of local flows

Let X and C be as in the previous section, and $p \in \partial C$. From the filtration $\{X^t\}_{t\geq 0}$, we obtain the filtration $\{K_p^t\}_{t\geq 0}$ of K_p by totally convex sets, where K_p^t is the limit of $X^{t/n}$ under the convergence $(nX, p) \to (K_p, o_p)$. We put $C_\infty = K_p^0$, and

$$C_{\infty\epsilon} = \{x \in C_\infty \,|\, d(\partial C_\infty, x) \geq \epsilon\},$$

which is the limit of $nC_{\epsilon/n}$ under the convergence $(nX, p) \to (K_p, o_p)$. Note that C_∞ is isometric to the flat cone over the segment $\Sigma_p(C)$ and that Proposition 9.10 holds for K_p^t and C_∞ in place of X^t and C.

We shall consider the function

$$f_{\infty\epsilon}(x) = d(C_{\infty\epsilon}, x), \qquad x \in K_p.$$

Lemma 12.1. *Given $p \in \partial C$, there exist positive numbers ϵ_p and δ_p such that for any ϵ and $\delta \leq \delta_p$ with $\epsilon/\delta \leq \epsilon_p$:*

(1) *f_ϵ is regular on $U(p,\epsilon,\delta) - V(p,\epsilon,\delta)$.*

(2) *(f_ϵ, d_p) is $(c, \tau(\epsilon/\delta))$-regular on $(U(p,\epsilon,\delta) - V(p,\epsilon,\delta)) \cap B(p,\delta/10)^c$.*

where $c > 0$ is a uniform constant and

$$U(p,\epsilon,\delta) = B(\partial C, \epsilon/10) \cap B(p,\delta), \quad V(p,\epsilon,\delta) = B(\partial C, \epsilon/100) \cap B(p,\delta).$$

Proof. (1) is clear. We set

$$U_{\infty\epsilon} = B(\partial C_\infty, \epsilon/10) \cap B(o_p, 1), \qquad V_{\infty\epsilon} = B(\partial C_\infty, \epsilon/100) \cap B(o_p, 1).$$

By a simple convergence argument, for the proof of (2) it suffices to prove that $(f_{\infty\epsilon}, d_{o_p})$ is $(c, \tau(\epsilon))$-regular on $(U_{\infty\epsilon} - V_{\infty\epsilon}) \cap B(o_p, 1/10)^c$ for a uniform constant $c > 0$ and a small $\epsilon > 0$. For every

$$x \in (U_{\infty\epsilon} - V_{\infty\epsilon}) \cap B(o_p, 1/10)^c,$$

let $y \in \partial C_\infty$, $z \in \partial C_{\infty\epsilon}$ and $u \in C_\infty$ be the nearest points of ∂C_∞, of $\partial C_{\infty\epsilon}$ and of C_∞ respectively from x. First note that $\tilde\angle o_p x z > \pi/2 - \tau(\epsilon)$, $\tilde\angle z x a > \pi/2 - \tau(\epsilon)$, where a is a point on the ray from o_p through x with $d(o_p, a) > d(o_p, x)$. We consider the following three cases.

Case 1. $d(y, u) \geq \epsilon/1000$.

Let $b \in \partial C_\infty$ and $v \in C_\infty$ be such that $d(o_p, b) = d(o_p, y) + d(y, b)$ and $ybvu$ forms a square in C_∞. Now observe that the normal bundle $\mathcal{N}$ over $\operatorname{int} C_\infty$ is naturally imbedded in K_p. Let y_1, b_1 and v_1 be points in $\mathcal{N}$ such that $uybvxy_1b_1v_1$ forms a parallelepiped in $\mathcal{N}$. Then we have

$$\tilde{\angle} zxb_1 > \pi/2 + c_1, \qquad \tilde{\angle} o_p xb_1 > \pi/2 + c_1,$$

for some uniform constant $c_1 > 0$. This implies that (f_ϵ, d_{o_p}) is $(c_1, \tau(\epsilon))$-regular at x.

Case 2. $d(y, u) \le \epsilon/1000$ and $\tilde{\angle} xzy \ge 1/100$.

Let x_1 be the point on $xz \cap \bar{\mathcal{N}}$ such that $\tilde{\angle} zux_1 = \pi/2$ ($x_1 = x$ if $y \ne u$). Let x_2 be the point on the ray from o_p through x_1 such that $(x_2)'_u$ is a direction at u normal to C_∞. We show that $\angle x_1ux_2 < \tau(\epsilon)$. Let q be the point on the ray from o_p through u with $d(o_p, q) = 2d(o_p, u)$. Obviously,

$$\tilde{\angle} o_p xq > \pi - \tau(\epsilon), \quad \tilde{\angle} o_p zq > \pi - \tau(\epsilon),$$
$$|\tilde{\angle} o_p xz - \pi/2| < \tau(\epsilon), \quad |\tilde{\angle} o_p zx - \pi/2| < \tau(\epsilon).$$

It follows from Corollary 5.7 in [4] that

$$|\tilde{\angle} o_p x_1 z - \pi/2| < \tau(\epsilon), \quad |\tilde{\angle} q x_1 z - \pi/2| < \tau(\epsilon),$$

which implies that $\angle x_1ux_2 < \tau(\epsilon)$.

Take $v \in \bar{\mathcal{N}}$ such that $d(u, v) = d(u, x_1) + d(x_1, v)$ and $d(x, x_1)/d(v, x)$ is sufficiently small. Then $\tilde{\angle} zxv > \pi/2 + c_2$. Let w be a point such that w'_x is a midpoint between v'_x and a'_x. Then we have

$$\tilde{\angle} zxw > \pi/2 + c'_2, \qquad \tilde{\angle} o_p xw > \pi/2 + c'_2,$$

for some uniform constant $c'_2 > 0$. This implies that (f_ϵ, d_{o_p}) is $(c, \tau(\epsilon))$-regular at x.

Case 3. $d(y, u) \le \epsilon/1000$ and $\tilde{\angle} xzy \le 1/100$.

Note that $y = u$ in this case. Let δ_1 be a small positive number. Take a small $\epsilon > 0$ so that for every $x \in U_{\infty\epsilon} \cap A(o_p; 1/10, 1)$and $y \in \partial C_\infty \cap A(o_p; 1/10, 1)$ with $d(x, y) = d(x, \partial C_\infty)$, there exists a point v such that $\tilde{\angle} yxv > \pi - \delta_1$. Then in the above situation we have $\tilde{\angle} zxv > \pi/2 + c_3$. Taking w as in Case 2, we obtain the required regularity of (f_ϵ, d_{o_p}). q.e.d.

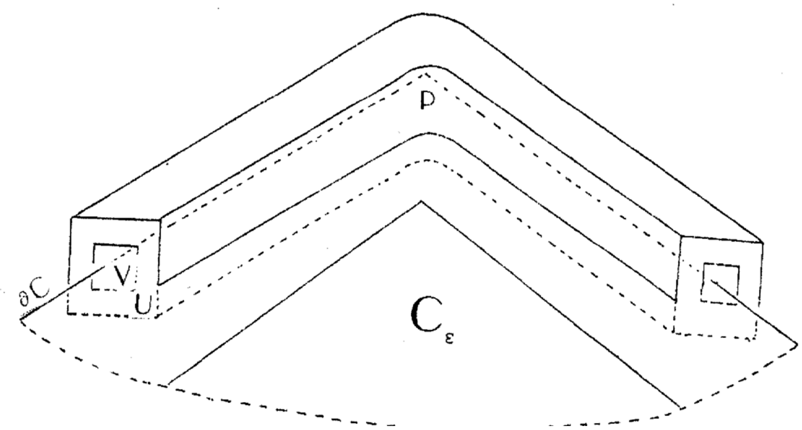

Figure 1: Canonical neighborhoods pair

Definition 12.2. We say that (U, V) is a *canonical neighborhoods pair* of a point $p \in \partial C$ for $\delta \gg \epsilon > 0$ (see Figure 1) if:

(1) $U \supset V$ are neighborhoods of p with $U \cap \partial C = V \cap \partial C = B(p, \delta) \cap \partial C$.

(2) (f_ϵ, d_p) is $(c, \tau(\epsilon/\delta))$-regular on $(U - V) \cap B(p, \delta/10)^c$.

(3) There is a pseudo-gradient flow ψ on U for f_ϵ.

(4) There is a homeomorphism $h : U \cap \partial C \times (I \times J - I' \times J') \to U - V$, where $I = [\,s_0, s_1\,]$, $I' = [\,s_0', s_1'\,]$, $s_0 < s_0'$, $s_1 > s_1'$ and $J \supset\supset J'$ are closed intervals, such that for each $x \in U \cap \partial C$:

   (a) $h(\{x\} \times I \times \{t\})$ provides the flow curve of ψ for each $t \in J - J'$, that is,
   $$\psi(h(x, s_1, t), s) = h(x, s_1 - s, t).$$

   (b) $h(\{x\} \times \{s_1\} \times J) \subset f_\epsilon^{-1}(21\epsilon/20)$, $h(\{x\} \times \{s_0\} \times J) \subset f_\epsilon^{-1}(19\epsilon/20)$.

   (c) $h(\{x\} \times \{s_1'\} \times J') \subset f_\epsilon^{-1}(51\epsilon/50)$, $h(\{x\} \times \{s_0'\} \times J') \subset f_\epsilon^{-1}(49\epsilon/50)$.

   (d) Each ψ-flow curve in $(U - V) \cap B(p, \delta/10)^c$ is contained in a d_p-level set.

(e)

$$\partial U - h((\operatorname{int} B(p,\delta)\cap \partial C)\times \partial(I\times J)) \subset \partial B(p,\delta),$$
$$\partial V - h((\operatorname{int} B(p,\delta)\cap \partial C)\times \partial(I'\times J')) \subset \partial B(p,\delta).$$

Lemma 12.3. *Each point $p\in\partial C$ has a canonical neighborhoods pair for arbitrarily small $\delta \gg \epsilon > 0$.*

For the proof, we need the following

Lemma 12.4. *Let D^2 be the standard unit disk in $\mathbb{R}^2$ with $\partial D^2 = S^1$ the union $I_0\cup I_1$ of closed hemi-circles. Let ψ_0 and ψ_1 be local flows on D^2 such that:*

(1) *The flow curves start from I_0 and reach I_1.*

(2) *ψ_i leaves $I_0\cap I_1$ fixed.*

Then there exists a continuous family of flows ψ_t on D^2 satisfying the conditions above and joining ψ_0 and ψ_1.

Proof. Let r be the reflection about the line through $I_0\cap I_1$. Let ϕ_0 be the canonical straight flow on D^2 starting from $x\in I_0$ and reaching $r(x)\in I_1$. Now consider a flow ψ on D^2 satisfying (1), (2). Making use of the flow curves of ϕ_0 and ψ, we can think of ψ as such a homeomorphism $f = f_\psi$ of D^2 that the restriction to I_0 is the identity. Now let ψ' be the straightening of ψ. Namely, it can be defined as the natural flow formed by straight line segments from $x\in I_0$ to $\psi(x,s)\in I_1$. Put $h = f_{\psi'}^{-1}\circ f_\psi$. Since h is the identity on ∂D^2, it is isotopic to the identity of D^2 while keeping ∂D^2 fixed. Let h_t, $0\le t\le 1$, be the isotopy. The isotopy $f_{\psi'}\circ h_t$ induces a deformation ψ_t of flows from ψ to ψ'. For the given flows ψ_0 and ψ_1, it is now obvious that ψ_0' is deformable to ψ_1'. This completes the proof. q.e.d.

Proof of Lemma 12.3. Let δ_p and ϵ_p be as in Lemma 12.1. Then for $\epsilon<\delta$ with $\delta\le\delta_p$ and $\epsilon/\delta\le\epsilon_p$, consecutive use of Theorem 10.1 together with Lemma 12.1 provides neighborhoods $U\supset V$ of p and a gradient flow ψ on $U-V$ for f_ϵ satisfying the conditions (1), (2) and (4) in Definition 12.2. For the proof, it suffices to extend ψ to a pseudo-gradient flow on U for f_ϵ. By Proposition 11.3, we may assume that $U\supset V$ satisfy (*e*). Let E_0 and E_1 be the two component of $\partial V - h((\operatorname{int} B(p,\delta)\cap\partial C)\times\partial(I'\times J'))\subset\partial B(p,\delta)$. Since $\partial V\simeq S^2$, it follows from Proposition 11.3 and the generalized Schoenflies Theorem

([3]) that $V \simeq I' \times J' \times [0,1]$, where E_i corresponds to $I' \times J' \times \{i\}$. Let ϕ_i be a flow on E_i starting from $\{s'_1\} \times J' \times \{i\}$, reaching $\{s'_0\} \times J' \times \{i\}$ and extending ψ, $i = 0, 1$. Then by Lemma 12.4, we have a flow ϕ on V extending ϕ_i and ψ restricted to $\partial V \cap \operatorname{int} B(p, \delta)$. The flow defined by the union of ψ and ϕ gives a required flow on U. q.e.d.

Proposition 12.5. *There exist an $\epsilon_0 > 0$, a neighborhood U of ∂C and a pseudo-gradient flow ψ on U for f_{ϵ_0} together with a homeomorphism $h : \partial C \times I \times J \to U$ such that for each $x \in \partial C$:*

(1) *$h(\{x\} \times I \times \partial J)$ gives flow curves of ψ, that is,*

$$\psi(h(x, s_1, t_j), s) = h(x, s_1 - s, t_j).$$

(2) $h(\{x\} \times \{s_1\} \times J) \subset f_{\epsilon_0}^{-1}(21\epsilon_0/20)$.

(3) $h(\{x\} \times \{s_0\} \times J) \subset f_{\epsilon_0}^{-1}(19\epsilon_0/20)$.

Here, $I = [s_0, s_1]$, $J = [t_0, t_1]$.

Proof. By a straightforward compactness argument using Lemma 12.1, we have finitely many points $p_1, \dots, p_N$ of ∂C such that for some $\delta_i \le \delta_{p_i}$:

(1) $\{B(p_i, \delta_i)\}$ covers ∂C.

(2) $\{B(p_i, \delta_i/10)\}$ is disjoint.

For each p_i, take $\epsilon_i \ll \delta_i$ with $\epsilon_i/\delta_i \le \epsilon_{p_i}$ as in Lemma 12.1, and choose a small ϵ_0 with $\epsilon_0/\min\{\delta_i\} \le \min\{\epsilon_{p_i}\}$. Let (U_i, V_i) be a canonical neighborhoods pair of p_i for $\epsilon_0 \ll \delta_i$, and ψ_i the pseudo-gradient flow on $U_i - V_i$ for f_{ϵ_0} as in Definition 12.2. We denote by U' the union of U_i. To prove the proposition, we have to deform those local flows ψ_i to obtain a local flow on a neighborhood of ∂C. Suppose that p_i is adjacent to p_j. We use the deformation theory to obtain a pseudo-gradient flow ψ_{ij} on a neighborhood of $\partial C \cap (U_i \cup U_j)$ which differs from ψ_i and ψ_j only on a neighborhood of a compact set of $\partial C \cap U_i \cap U_j$.

We put $g_i = d(p_i, .)$ which is regular on $U_i - B(p_i, \delta_i/10)$. Let T be the component of $\partial C \cap U_i \cap U_j - B(p_i, \delta_i/10) - B(p_j, \delta_j/10)$ on which (f_{ϵ_0}, g_i) is regular. Let $h_i : U_i \cap \partial C \times (I \times J - I' \times J') \to U_i - V_i$ be as in Definition 12.2 (4). We consider a pair (U, V) with $U \subset U_i$, $V \subset V_i$ defined by

$$U - V = h_i(T \times (I \times J - I' \times J')).$$

By Definition 12.2 (4)-(d), we have a homeomorphism $\hat{h} : U - V \to K \times (I \times J - I' \times J')$ such that $pr \circ \hat{h} = (g_i, f_{\epsilon_0})$, where $I = [\,19\epsilon_0/20, 21\epsilon_0/20\,]$, $I' = [\,49\epsilon_0/50, 51\epsilon_0/50\,]$, $K = [\,r, R\,]$, and $pr : K \times I \times J \to K \times I$ is the projection. Here we make an identification for simplicity

$$U - V \equiv T \times (I \times J - I' \times J') \equiv K \times (I \times J - I' \times J').$$

Recall that the flow ψ_i restricted to $U - V$ is gradient for f_{ϵ_0} and that the flow curves lie on the g_i-level sets. Take $r < r_1 < R_1 < R$. For simplicity, we set

$$U(r_1, R_1) = U \cap g_i^{-1}([\,r_1, R_1\,]).$$

Assertion 12.6. We can take a pseudo-gradient flow ψ_{ij} on U which is gradient for f_{ϵ_0} outside V satisfying

$$\psi_{ij} = \begin{cases} \psi_i & \text{on} \quad U(r, r_1) \\ \psi_j & \text{on} \quad U(R_1, R). \end{cases} \tag{12.1}$$

Proof. We have to join the flow ψ_i on $U(r, r_1)$ and the flow ψ_j on $U(R_1, R)$. Let $r_1 < r_2 < R_1$. By Theorem 10.1, we have a flow ψ_j' on $U(r_1, R)$ such that

$$\psi_j' = \begin{cases} \psi_i & \text{on} \quad U(r_1, r_2) - V_0 \\ \psi_j & \text{on} \quad U(R_1, R) \cup (U(r_1, R) \cap V), \end{cases} \tag{12.2}$$

where $V \subset V_0 \subset U$. Let K_1 be the union of ψ_j'-flow curves starting from $\{R_1\} \times \{21\epsilon_0/20\} \times J$ to $\{f_{\epsilon_0} = 19\epsilon_0/20\}$. Put

$$K_2 = \{r_2\} \times I \times J, \qquad K_3 = [\,r_2, R_1\,] \times \{21\epsilon_0/20\} \times J.$$

Let K_4 be the domain on $\{f_{\epsilon_0} = 19\epsilon_0/20\}$ bounded by the curves $\{r_2\} \times \{19\epsilon_0/20\} \times J$, $[\,r_2, R_1\,] \times \{19\epsilon_0/20\} \times \partial J$ and F, where $F = K_1 \cap \{f_{\epsilon_0} = 19\epsilon_0/20\}$. Retaking ϵ_0 smaller and r_1, r_2, r_3 suitably if necessary, we may assume that $K_4 \simeq D^2$ and $g_i(K_1) \subset (r_2, R)$. Let D be the domain bounded by $K_1, \dots, K_4$, and $[\,r_2, R_1\,] \times I \times \partial J$. Note that $D \simeq D^3$ and observe that we have the ψ_i-flow on K_2 and the ψ_j'-flow on K_1. By Lemma 12.4, we can construct a new flow ψ_{ij} on D such that

$$\psi_{ij} = \begin{cases} \psi_i & \text{on} \quad K_2 \\ \psi_j & \text{on} \quad K_1 \cup V. \end{cases} \tag{12.3}$$

Now the flows ψ_i, ψ_{ij}, and ψ'_j provide a required flow on U. q.e.d.

Repeating the above procedure finitely many times, we obtain a pseudo-gradient flow for f_{ϵ_0} on a neighborhood of ∂C as required. q.e.d.

Proof of Theorem 9.6 (1). Let $k \le 2$ be the number of one-normal points in $\operatorname{int} C$. It follows from the previous proposition that

$$\begin{aligned} X &\simeq f_{\epsilon_0}^{-1}([\,0, 21\epsilon_0/20\,)) \\ &\simeq f_{\epsilon_0}^{-1}([\,0, 19\epsilon_0/20\,)) \\ &\simeq \begin{cases} \mathbb{R}^3 & \text{if } k = 0, \\ K(P^2) & \text{if } k = 1, \\ X_1 = \mathbb{R}^3/\Gamma & \text{if } k = 2. \end{cases} \end{aligned}$$

This completes the proof of Theorem 9.6 (1) in the case when $\dim C = 2$. q.e.d.

13. The case of $\dim C = 1$

Let X be as in Theorem 9.6 with $\dim C = 1$. As Example 9.3 shows, X is not necessarily a topological manifold near ∂C. Hence we cannot apply the deformation theory developed in the previous section to a neighborhood of ∂C.

Proof of Theorem 9.6 *in the case* $\dim C = 1$. Let q,q' be the boundary points of the geodesic segment C. Under the convergence $(nX, q) \to (K_q, o_q)$, let C converge to C_∞, and consider the functions

$$\begin{aligned} f &= d(C_\infty, \cdot), \qquad & g &= d(o_q, \cdot), \\ f_n &= d_n(C, \cdot), \qquad & g_n &= d_n(q, \cdot). \end{aligned}$$

where d_n is the n-times rescaling of the original metric of X. Notice that f and g are regular on $K_q - C_\infty$ and $K_q - \{o_q\}$ respectively.

Lemma 13.1. *There exist n_0, $\epsilon_0 > 0$ such that for every $n \ge n_0$ and $\epsilon \le \epsilon_0$, we have a homeomorphism*

$$H : (f, g)^{-1}([\,0, \epsilon\,] \times [\,0, 1\,]) \to (f_n, g_n)^{-1}([\,0, \epsilon\,] \times [\,0, 1\,])$$

with $f_n \circ H = f$.

Proof. Observe that there exist n_0, $\epsilon_0 > 0$ and $c_0 \gg \epsilon_0$ such that for every $\epsilon \le \epsilon_0$, $n \ge n_0$ and for every $x \in (f_n, g_n)^{-1}((0,\epsilon] \times [1/2, 1])$ there is a point w with $d_n(w, C) \ge c_0$ satisfying

$$\tilde{\angle} qxw > \pi/2 + c, \quad \tilde{\angle} Cxw > \pi/2 + c, \quad \tilde{\angle} qxC > \pi/2 - \tau(\epsilon),$$

for some uniform constant $c > 0$. For instance, the point w can be taken as follows: Take $y \in C$ and y_1 with $d_n(x, y) = d_n(x, C)$, $\tilde{\angle} yxy_1 > \pi - \tau(\epsilon)$ and $d_n(y, y_1) \gg \epsilon$. Take a point z such that $\tilde{\angle} qxz > \pi - \tau_n$ and $d_n(x, z) = d_n(x, y_1)$, where $\lim \tau_n = 0$. Then the mid point w of $y_1 z$ satisfies the above conditions. In particular, (f_n, g_n) and (f, g) are $(c, \tau(\epsilon))$-regular on $(f_n, g_n)^{-1}((0,\epsilon] \times [1/2, 1])$ and $(f, g)^{-1}((0,\epsilon] \times [1/2, 1])$ respectively.

Applying Theorem B and its complement in [26], we then have a homeomorphism

$$h_{11} : (f, g)^{-1}([\epsilon/2, \epsilon] \times [1/2, 1]) \to (f_n, g_n)^{-1}([\epsilon/2, \epsilon] \times [1/2, 1])$$

such that $(f_n, g_n) \circ h_{11} = (f, g)$. The $(c, \tau(\epsilon))$-regularities of (f_n, g_n) and (f, g) enables us to extend h_{11} to a homeomorphism

$$\hat{h}_1 : (f, g)^{-1}((0, \epsilon] \times [1/2, 1]) \to (f_n, g_n)^{-1}((0, \epsilon] \times [1/2, 1])$$

with $(f_n, g_n) \circ \hat{h}_1 = (f, g)$. Now it is easy to extend h_1 to a homeomorphism

$$h_1 : (f, g)^{-1}([0, \epsilon] \times [1/2, 1]) \to (f_n, g_n)^{-1}([0, \epsilon] \times [1/2, 1]),$$

with $(f_n, g_n) \circ h_1 = (f, g)$. We put

$$K_i = (f, g)^{-1}([0, \epsilon/2^{i-1}] \times [1/2^i, 1/2^{i-1}]),$$
$$K_i^n = (f_n, g_n)^{-1}([0, \epsilon/2^{i-1}] \times [1/2^i, 1/2^{i-1}]).$$

Noting $2K_i$ and $2K_i^n$ are isometric to K_{i-1} and K_{i-1}^{2n} respectively, we can inductively construct homeomorphisms $h_i : K_i \to K_i^n$ such that $(f_n, g_n) \circ h_i = (f, g)$ and $h_i = h_{i-1}$ on $K_i \cap K_{i-1}$. Thus we obtain a homeomorphism $h : \cup_{i=1}^{\infty} K_i \to \cup_{i=1}^{\infty} K_i^n$ with $(f_n, g_n) \circ h = (f, g)$. Note that f and f_n are regular on $(f, g)^{-1}((0, \epsilon] \times [0, 1])$ and $(f_n, g_n)^{-1}((0, \epsilon] \times [0, 1])$ respectively. We put

$$L_i = (f, g)^{-1}([\epsilon/2^i, \epsilon/2^{i-1}] \times [0, 1/2^i]),$$
$$L_i^n = (f_n, g_n)^{-1}([\epsilon/2^i, \epsilon/2^{i-1}] \times [0, 1/2^i]).$$

By Theorem 10.1 and the complement of Theorem B in [26], there exists n_1 such that for each $n \geq n_1$, we have a homeomorphism $k_1 : L_1 \to L_1^n$ such that:

(1) $f_n \circ k_1 = f$.

(2) k_1 is compatible with h.

Now with the use of Theorem 10.1, we can inductively construct homeomorphisms $k_i : L_i \to L_i^n$ such that:

(3) $f_n \circ k_i = f$.

(4) k_i is compatible with $h, k_1, \ldots, k_{i-1}$.

Thus we have a homeomorphism $k : \cup_{i=1}^{\infty} L_i \to \cup_{i=1}^{\infty} L_i^n$ with $f_n \circ k = f$, and h and k define a homeomorphism

$$H : (f, g)^{-1}([\,0, \epsilon\,] \times [\,0, 1\,]) \to (f_n, g_n)^{-1}([\,0, \epsilon\,] \times [\,0, 1\,])$$

with $f_n \circ H = f$. q.e.d.

Since obviously $B(o_q, 1) \simeq (f, g)^{-1}([\,0, \epsilon\,] \times [\,0, 1\,])$, it follows that $(f_n, g_n)^{-1}([\,0, \epsilon\,] \times [\,0, 1\,]) \simeq K_q$. Thus,

$$(f_1, g_1)^{-1}([\,0, \epsilon/n\,] \times [\,0, 1/n\,]) \simeq K_q,$$

for all sufficiently large n. Similarly we have

$$(f_1, g_1')^{-1}([\,0, \epsilon/n\,] \times [\,0, 1/n\,]) \simeq K_{q'},$$

where $g_1' = d(q', \cdot)$.

For simplicity, we put $\epsilon_1 = \epsilon/n$, $\delta_1 = 1/n$ and $R = d(q, q')/2$. Taking a larger n if necessary, we may also assume that (f_1, g_1) (resp. (f_1, g_1')) is $(c, \tau(\epsilon))$-regular on $(f_1, g_1)^{-1}(\{\epsilon_1\} \times [\,\delta_1, R\,])$ (resp. on $(f_1, g_1')^{-1}(\{\epsilon_1\} \times [\,\delta_1, R\,]))$. It follows that for any $R_1 \leq R$

$$\begin{aligned}
&(f_1, g_1)^{-1}([\,0, \epsilon_1\,] \times [\,\delta_1, R_1\,]) \\
&\qquad \simeq (f_1, g_1)^{-1}([\,0, \epsilon_1\,] \times \delta_1) \times [\,\delta_1, R_1\,] \\
&\qquad \simeq D^2 \times [\,\delta_1, R_1\,] \\
&\qquad \simeq D^3.
\end{aligned}$$

Similarly, we have

$$(f_1, g_1')^{-1}([\,0, \epsilon_1\,] \times [\,\delta_1, R_1\,]) \simeq D^3.$$

We put

$$E(\epsilon_1, \delta_1) = f_1^{-1}[\,0, \epsilon_1\,] - \operatorname{int} B(q, \delta_1) - \operatorname{int} B(q', \delta_1).$$

Since $B(q, \delta_1) \cap E(\epsilon_1, \delta_1) \simeq D^2$ and $B(q', \delta_1) \cap E(\epsilon_1, \delta_1) \simeq D^2$, it suffices to show that $E(\epsilon_1, \delta_1) \simeq D^3$. Let z be the midpoint of C. We may assume that

$$B(z, 10\delta_1) \simeq D^3.$$

Note that:

(1) $F_q(\epsilon_1, \delta_1) := (f_1, g_1)^{-1}([\,0, \epsilon_1\,] \times [\,\delta_1, R - \delta_1\,]) \simeq D^3$.

(2) $F_{q'}(\epsilon_1, \delta_1) := (f_1, g_1')^{-1}([\,0, \epsilon_1\,] \times [\,\delta_1, R - \delta_1\,]) \simeq D^3$.

(3) $G(\epsilon_1, \delta_1) := E(\epsilon_1, \delta_1) - \operatorname{int} F_q(\epsilon_1, \delta_1) - \operatorname{int} F_{q'}(\epsilon_1, \delta_1) \subset B(z, 10\delta_1)$ has boundary homeomorphic to $D^2 \cup S^1 \times I \cup D^2 \simeq S^2$.

It follows from the generalized Schoenflies Theorem ([3]) that $G(\epsilon_1, \delta_1) \simeq D^3$. Therefore

$$E(\epsilon_1, \delta_1) \simeq F_q(\epsilon_1, \delta_1) \cup_{D^2} G(\epsilon_1, \delta_1) \cup_{D^2} F_{q'}(\epsilon_1, \delta_1) \simeq D^3.$$

This completes the proof. q.e.d.

Remark 13.2. In the case when X is a Riemannian manifold, one can use the flow curves of a gradient-like vector field of $d_q - d_{q'}$ to conclude that $E(\epsilon_1, \delta_1) \simeq D^3$. In our case however, it is not certain if one can apply the above argument to a function of type $d_q - d_{q'}$ on a general Alexandrov space.

14. Appendix: Total curvature on Alexandrov surfaces

In this section, we explain the total curvature and the Gauss-Bonnet Theorem on Alexandrov surfaces, originally studied by Alexandrov [1]. We also formulate the Cohn-Vossen Theorem and investigate Alexandrov surfaces admitting essential singular points, which are needed for the proof of Theorem 0.5.

Throughout this section, let X be an Alexandrov surface of curvature $\geq a \in \mathbb{R}$ and $\mathcal{H}^2$ the Hausdorff measure over X. Recall that such an X is a two-dimensional topological manifold possibly with boundary. A *polygonal region of* X is by definition a subset of X whose (topological) boundary is a union of finitely many broken geodesics. The *rotation* $\kappa(\partial D)$ *of the* (*topological*) *boundary* ∂D *of a polygonal region* $D \subset X$ is defined by

$$\kappa(\partial D) := \sum_{x \in \partial D} (\pi - \angle_x D),$$

where $\angle_x D$ denotes the inner angle of D at $x \in \partial D$, which is equal to π if x is not a vertex of D, so that the sum here is indeed a finite sum. Fixing one of the two sides of a broken geodesic $\sigma = x_0 x_1 \dots x_k$ in X, we define the *rotation* $\kappa(\sigma)$ *of* σ *with respect to the chosen side* in the same manner.

Let $\blacktriangle$ denote the open disk domain bounded by a triangle $\triangle$ in X. The *total curvature* (or *excess*) $\omega(\blacktriangle)$ *of* $\blacktriangle$ *in* X is defined by

$$\omega(\blacktriangle) := \alpha + \beta + \gamma - \pi,$$

where α, β, γ are the inner angles of $\blacktriangle$ at its three vertices. Let $D \subset \operatorname{int} X$ be a relatively compact polygonal open region and find a triangulation of D with triangles $\{\triangle\}$. Then, the *total curvature* (or *total excess*) $\omega(D)$ *of* D is defined by

$$\omega(D) := \sum_{\triangle} \omega(\blacktriangle) + \sum_{x \in V \cap \operatorname{int} D} (2\pi - L(\Sigma_x)),$$

where V is the set of vertices of the triangulation of D. According to [1] (see also [23]), the total curvature ω is independent of the triangulation and extends to a signed Radon measure over X with the following properties:

(1) For any $D \subset \operatorname{int} X$ as above, we have the Gauss-Bonnet formula:

$$\omega(D) + \kappa(\partial D) = 2\pi\chi(D),$$

where $\chi(D)$ denotes the Euler characteristic of D.

(2) Any $\mathcal{H}^2$-measurable subset of X is ω-measurable and we have

$$\omega \geq a\mathcal{H}^2,$$

so that $\omega - a\mathcal{H}^2$ is a (nonnegative) Radon measure.

(3) The restriction of ω onto ∂X is

$$\omega|_{\partial X} = 0.$$

(4) For any minimal segment xy in X,

$$\omega|_{xy-\{x,y\}} = 0.$$

(5) For any $x \in \operatorname{int} X$,

$$\omega(\{x\}) = 2\pi - L(\Sigma_x).$$

We now introduce the *rotation measure κ over ∂X*. Let c be a subarc of ∂X from $p \in \partial X$ to $q \in \partial X$. Take a division $\{p = x_0, x_1, \dots, x_m = q\}$ of the arc c, where $x_0, \dots, x_m$ are points lying on c in this order. We obtain a broken geodesic $\sigma := x_0 x_1 \dots x_m$, which approximates c. Choose the side of σ for which we can measure the inner angle. It then follows that

$$\kappa(\sigma) = \sum_{i=1}^{m-1} (\pi - \angle x_{i-1} x_i x_{i+1}).$$

Clearly, $\kappa(\sigma)$ is nonnegative. If a point $y \in c$ is taken to be between x_{k-1} and x_k for a k and if $\blacktriangle$ is the open disk domain surrounded by $\triangle x_{k-1} y x_k$, then

$$\begin{aligned} \kappa(x_0 \dots x_{k-1} y x_k \dots x_m) &= \kappa(\sigma) - \angle_{x_{k-1}} \blacktriangle - \angle_{x_k} \blacktriangle - \angle_y \blacktriangle + \pi \\ &= \kappa(\sigma) - \omega(\blacktriangle). \end{aligned}$$

Therefore, for any subdivision $\{p = y_0, y_1, \dots, y_n = q\}$ of $\{x_0, \dots, x_m\}$ and for $\tau := y_0 y_1 \dots y_n$, we have

$$\kappa(\tau) = \kappa(\sigma) - \omega(E_{\sigma,\tau}),$$

where $E_{\sigma,\tau}$ denotes the union of open disk domains of X between σ and τ. If the subdivision $\{y_0, \dots, y_n\}$ is getting finer and finer, then τ tends to c and hence $E_{\sigma,\tau}$ to the open domain, say E_σ, bounded by σ and c, so that

$$\lim_{\tau \to c} \kappa(\tau) = \kappa(\sigma) - \omega(E_\sigma).$$

We define $\kappa(c)$ to be the above and call this the *rotation of* c. It follows that $\kappa(c)$ is nonnegative and independent of σ. A standard measure construction argument yields that the rotation κ extends to the (nonnegative) Radon measure over ∂X.

For a polygonal region $D \subset X$, we set $\hat{\partial} D := \partial D \cup (\partial X \cap D)$ (where ∂D is the topological boundary of D). The rotation κ is naturally defined over $\hat{\partial} D$ as a signed Radon measure. We now extend the Gauss-Bonnet formula to the case where D may touch ∂X.

Proposition 14.1 (The Gauss-Bonnet Theorem). *For any relatively compact polygonal open region $D \subset X$ we have*

$$\omega(D) + \kappa(\hat{\partial} D) = 2\pi\chi(D).$$

In particular, if X is compact,

$$\omega(X) + \kappa(\partial X) = 2\pi\chi(X).$$

Proof. An easy calculation using $\kappa(c) = \kappa(\sigma) - \omega(E_\sigma)$. q.e.d.

Remark that for the Gauss-Bonnet formula $\omega(D)+\kappa(\hat{\partial} D) = 2\pi\chi(D)$ to hold, the region D has to be an *open* subset, because $\omega(\partial D) > 0$ may happen.

The total curvature ω over X is a signed Radon measure which splits into two nonnegative Radon measures ω_+ and ω_- over X such that $\omega = \omega_+ - \omega_-$. When a Borel subset $D \subset X$ is not relatively compact, it may happen that $\omega_+(D) = \omega_-(D) = \infty$, in which case $\omega(D)$ is not defined. The rotation measure κ over the boundary $\hat{\partial} D$ of a polygonal region $D \subset X$ also splits into two nonnegative Radon measures $\kappa_\pm$ such that $\kappa = \kappa_+ - \kappa_-$. When $\hat{\partial} D - \partial X$ is unbounded, $\kappa(\hat{\partial} D)$ is not necessarily defined as well. Notice that κ over ∂X is nonnegative and $\kappa(\partial X) \in [\,0, \infty\,]$ is always defined.

We say that a topological manifold is *finitely connected* if it can be contracted to a compact submanifold (with boundary) by strong deformation retraction.

Proposition 14.2 (The Cohn-Vossen Theorem). *If $\omega(D)$ and $\kappa(\hat{\partial} D)$ are both defined for a finitely connected polygonal open region $D \subset X$, then we have*

$$2\pi\chi(D) - \pi\chi(\hat{\partial} D) - \omega(D) - \kappa(\hat{\partial} D) \geq 0.$$

In particular, if X is finitely connected and if $\omega(X)$ is defined, then

$$2\pi\chi(X) - \pi\chi(\partial X) - \omega(D) - \kappa(\partial X) \geq 0.$$

Notice here that $\chi(\hat{\partial} D)$ is equal to the number of components of $\hat{\partial} D$ homeomorphic to $\mathbb{R}$.

Proof. The proof is by the same discussion as for Riemannian manifolds (see [31]). q.e.d.

Corollary 14.3. *If X is compact, the number of essential singular points in $X - \partial X$ is at most $4 - a\,\mathcal{H}^2(X)/\pi$.*

Proof. Denoting by E the set of essential singular points in $X - \partial X$, we have

$$\begin{aligned}\pi\,\#E &\le \omega(E) = \omega(X) - \omega(X - E)\\ &\le 2\pi\chi(X) - a\,\mathcal{H}^2(X) \le 4\pi - a\,\mathcal{H}^2(X).\end{aligned}$$

q.e.d.

Corollary 14.4. *Let X be a two-dimensional Alexandrov space of nonnegative curvature. Then, the following hold.*

(1) *X is homeomorphic to either $\mathbb{R}^2$, $[\,0, +\infty\,) \times \mathbb{R}$, S^2, P^2, D^2, or isometric to $[\,0, \ell\,] \times \mathbb{R}$, $[\,0, \ell\,] \times S^1(r)$, $[\,0, +\infty\,) \times S^1(r)$, $\mathbb{R} \times S^1(r)$, $\mathbb{R} \times S^1(r)/\mathbb{Z}_2$, a flat torus, or a flat Klein bottle for some $\ell, r > 0$.*

(2) $\operatorname{int} X$ *contains at most four essential singular points, and denoting by n the number of essential singular points in* $\operatorname{int} X$, *we have the following for some $\ell, h > 0$.*

  (a) *If $n \ge 1$, X is either homeomorphic to $\mathbb{R}^2$, S^2, P^2, D^2, or isometric to* $\operatorname{dbl}([\,0, \infty\,) \times [\,0, \infty\,)) \cap \{\,(x, y) \mid y \le h\,\}$.

  (b) *If $n \ge 2$, X is either homeomorphic to S^2, or isometric to* $\operatorname{dbl}([\,0, \infty\,) \times [\,0, h\,])$, $\operatorname{dbl}([\,0, \infty\,) \times [\,0, h\,]) \cap \{\,(x, y) \mid x \le \ell\,\}$, *or* $\operatorname{dbl}([\,0, \ell] \times [\,0, h\,])/\mathbb{Z}_2$.

  (c) *If $n \ge 3$, X is homeomorphic to S^2.*

  (d) *If $n = 4$, X is isometric to* $\operatorname{dbl}([\,0, \ell\,] \times [\,0, h\,])$.

Proof. Since ∂X can be approximated by broken geodesics as is seen before, there is a triangulation of X with countably many geodesic triangles $\{\triangle\}$ such that $\operatorname{int} X \subset \bigcup_{\triangle} \blacktriangle$, where $\blacktriangle$ is the open disk domain bounded by $\triangle$. Recall that

$$\omega(X) = \sum_{\triangle} \omega(\blacktriangle) + \sum_{x \in V \cap \operatorname{int} X} (2\pi - L(\Sigma_x)),$$

where V is the set of vertices of the triangulation. The nonnegativity of the curvature of X implies that $L(\Sigma_x) \leq 2\pi$, $\omega(\blacktriangle) \geq 0$, and that $\omega(\blacktriangle) = 0$ if and only if $\blacktriangle$ is isometric to a triangular disk domain of $\mathbb{R}^2$. Therefore, we have $\omega(X) \geq 0$ and the equality holds if and only if X is flat everywhere. If X is flat and $\kappa(\partial X) = 0$, then it is easy to observe that ∂X is totally geodesic. Thus, applying the Gauss-Bonnet or Cohn-Vossen Theorem to X proves (1).

To prove (2), we suppose that $\operatorname{int} X$ contains n different essential singular points $x_1, \dots, x_n$. Then, it follows that $\omega(X) \geq n\pi$ and that the equality holds if and only if $X - \{x_1, \dots, x_n\}$ is flat and if $L(\Sigma_{x_i}) = \pi$ for all i. This together with the Gauss-Bonnet or Cohn-Vossen Theorem completes the proof. q.e.d.

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