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COMPACT SELF-DUAL MANIFOLDS WITH TORUS ACTIONS

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Abstract

We show that a compact self-dual four-manifold with a smooth action of a two-torus and with non-zero Euler characterestic is necessarily diffeomorphic to a connected sum of copies of complex projective planes, and furthermore the self-dual structure is isomorphic to one of those constructed by Joyce in [11]. This settles a conjecture of Joyce [11] affirmatively. Our method of proof is to show, by complex geometric techniques, that the associated twistor space, which is a compact complex threefold with the induced holomorphic action of algebraic two-torus, has a very special structure and is indeed determined by a certain invariant which is eventually identified with the invariant associated with the Joyce's construction of his self-dual manifolds.

1. Introduction

Let m be a non-negative integer and $M = m\boldsymbol{P}^2$ the connected sum of m copies of complex projective plane $\boldsymbol{P}^2$. Suppose that we are given a smooth effective action of the real two torus K on M. In [11] Joyce constructed a series of examples of conformal self-dual metrics on M which are invariant under the given K-action. Further he raised the following conjecture [11, 3.3.4]:

> Let M be a compact connected oriented simply connected four-manifold with a smooth and effective K-action. Suppose that M admits a K-invariant conformal self-dual metric. Then M must be diffeomorphic to a connected sum of

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complex projective planes, and the conformal self-dual metric coincides with one of his self-dual metrics above.

The purpose of this paper is to give an affirmative answer to this conjecture in the following stronger form:

Theorem 1.1. *Let M be a compact connected oriented four-manifold with smooth and effective K-action. Suppose that its Euler-Poincare characterestic $\chi(M)$ is nonzero. Then if M admits a K-invariant self-dual metric, M is diffeomorphic to a connected sum of copies of complex projective planes and the conformal self-dual metric coincides with those constructed in* [11].

Together with the results of Poon [22] this leads to a classification of self-dual conformal structures with two-torus actions in general.

Corollary 1.2. *Let M be a compact connected oriented four-manifold with smooth and effective K-action. If M admits a K-invariant self-dual structure, then either M is conformally flat or M is isomorphic to one of Joyce's examples.*

As is shown in [22, Th. A], when M is locally conformally flat with K-action, it is one of the following standard examples: 1) a manifold covered by a real four-torus, 2) a four-sphere S^4, and 3) a Hopf surface. Note also that compact self-dual manifolds which admit a group of conformal automorphisms of dimension greater than two have been classified by Poon [21].

Now the method of Joyce [11] for constructing the K-invariant self-dual metric is to reduce, by using the K-symmetry, the self-duality equation on M to a certain system of linear partial differential equations on the quotient $N := M/K$, which is identified with the unit disc in the complex plane. Once the underlying smooth K-action on M is fixed, his solutions of the latter equation, i.e., his self-dual metrics, depend on a parameter which is a circular sequence of k points on the boundary of the unit disc considered modulo the action of the conformal automorphism group of the unit disc and the cyclic permutations, where $k = m + 2$.

With this last fact in mind, we start with a general compact orientated self-dual four-manifold M with K-action, and study in detail the complex analytic structure of the associated twistor space Z with the induced action of the complexfication G of K, which is an algebraic two-torus.

Under the assumption of Theorem 1.1 the singular set $\Sigma(Z)$ of the induced K-action on Z, i.e., the points where the stabilizer group is non-trivial, turns out to be a disjoint union of cycles of nonsingular rational curves. We then determine the local G-action at each point of $\Sigma(Z)$. Using these results, we further show that the closures of general G-orbits which intersect a sufficiently small neighborhood of $\Sigma(Z)$ is necessarily analytic and smooth in Z. Together with some global consideration using the Douady space of Z, we then conclude that every G-orbit has an analytic closure and there exists a canonical meromorphic quotient $\overline{f} : Z \to P$ of the G-action on Z, where P is a nonsingular rational curve.

The meromorphic map $\overline{f}$ becomes a holomorphic map $f : \hat{Z} \to P$ after blowing up Z with center a distinguished cycle C of nonsingular rational curves contained in $\Sigma(Z)$. This is in fact the main object of study in this paper. Among other things we show that the map $\overline{f}$ is given by a subsystem of a G-invariant elements of the fundamental system $|-\frac{1}{2}K|$ of Z, which forms a pencil, and that the general members of this system are mutually isomorphic projective smooth toric surfaces whose isomorphism class is explicitly determined by the invariant of the given K-action on M introduced by Orlik and Raymond [18]. At this stage we can already show that M is diffeomorphic to $m\boldsymbol{P}^2$ and that Z is Moishezon, where m is the second betti number of M. $\Sigma(M)$ consists then of a *single* cycle of two-spheres and the quotient $N = M/K$ is diffeomorphic to a closed two disc by [18] as in the case of Joyce.

The real structure σ on Z induces ones on $\hat{Z}$ and P, making f σ-equivariant. The fixed point set R of σ on P is diffeomorphic to S^1 and is considered as the boundary of either of the closed two-discs D which are connected components of $P - R$. It turns out that there exist exactly k singular fibers for f and the corresponding points $a_1, \dots, a_k$ on the base P all lie in the real part R, where $k = m + 2$ as above. Suppose that $a_1, \dots, a_k$ are arragned cyclically on R. Then we consider the sequence of points $(a_1, \dots, a_k)$ (modulo the action of the conformal automorphism group of D and cyclic permutations) as a basic invariant of our twistor space Z (endowed with G-action, the real structure σ and the twistor fibration $t : Z \to M$), and hence, of the original self-dual manifold M with K-action via the inverse twistor correspondence.

We then proceed to show the effectivity of this invariant in the following sense: Suppose that we are given two self-dual conformal structures $[g]$ and $[g']$ on $M = m\boldsymbol{P}^2$ which is invariant under the given K-action on M. Perform the construction above for the corresponding twistor spaces Z and Z' and obtain fiber spaces $f : \hat{Z} \to P$ and $f' : \hat{Z}' \to P'$ respectively. Suppose further that the sets of points on the bases corresponding to the singular fibers are identical with respect to a suitable identification of P and P'. Then there exists a biholomorphic map $j : Z \to Z'$ which is compatible with the real structures and G-actions and which maps twistor lines to twistor lines. We shall show this, first by studying the local structure of the fiber space $f : \hat{Z} \to P$ along each singular fiber, and then by studying its global structure on the basis of this local consideration.

Finally, for the purpose of identifying our invariant $(a_1, \dots, a_k)$ with the Joyce's "invariant" attached to his examples, we first interpret Joyce's invariant as a special case of an invariant which can be defined for any simply connected compact conformal four-manifold with K-action. We then show that in the case of a self-dual manifold with K-action, this conformal invariant and the invariant above arising from the associated twistor space are naturally identified via a certain twistor-like correspondence between the quotient N and one of the discs D in P with boundary R. Combined with the effectivity of the invariant mentioned above this immediately yields Theorem 1.1.

This article is arranged as follows. In §2 we summarize what we need on toric surfaces which will appear in the fibers of the morphism $f : \hat{Z} \to P$ above. In §3 first we recall some of the results of Orlik-Raymond [18] on K-actions on a compact smooth four-manifold M. Furthermore when a K-invariant conformal structure on M is given, we introduce an invariant of this conformal K-action as a circular sequence of points on the boundary of the unit disc on a complex plane modulo certain equivalence.

From §4 on we start our study of a self-dual manifold M with K-action. After summarizing the basic construction and properties of the associated twistor space we determine the singular set $\Sigma = \Sigma(Z)$ of the K-action on Z and the isotropic representation of the stabilizer group at each point of Σ. By using this in §5 we determine the local behaviour of G-orbits near the singular set Σ . From this we deduce that any G-orbit which intersects with a sufficiently small neighborhood of Σ has, up to a finite number of exceptions, a smooth analytic closure in Z. Combined with global consideration using the structure of the space of divisors on Z we obtain in §6 the meromorphic quotient $Z \to P$ and the associated holomorphic fiber space $f : \hat{Z} \to P$ mentioned above. In this section we further study the structure of this fiber space in detail. Especially, we show that there exist exactly k singular fibers $S_{a_i}, a_i \in P$, for f and that a_i are real points with respect to the induced real structure on P.

We then proceed to show that f is determined uniquely by the location of the points a_i. For this purpose we first show in §7 that this is the case at least locally along each singular fiber. Here the deformation theoretic method and the toric method give supplmentary descriptions of the local structure of this fiber space. Then in §8 we prove the global uniqueness of the fiber space f with the given invariant $(a_1, \dots, a_k)$. Finally in §9 we prove the coincidence of this invariant with the conformal invariant defined in §3.

Notation and Convention. Unless otherwise is mentioned, G and K denote respectively the real and algebraic two-torus so that $G = \boldsymbol{C}^* \times \boldsymbol{C}^*$ and $K \cong S^1 \times S^1$. In this paper we often encounter the sentences which contain $\pm$ more than once such as: "$A^\pm$ is contained in $B^\pm$", which should always read as: "A^+ is contained in B^+ and A^- is contained in B^-."

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2. Toric surfaces

(2.1) We recall some basic definitions and results on toric surfaces [17, 13]. Let $G \cong \boldsymbol{C}^* \times \boldsymbol{C}^*$ be an algebraic torus of dimension two. A compact smooth *toric surface* is a compact connected nonsingular projective algebraic surface S on which G acts effectively and algebraically and with a unique open orbit U.

The complement $C := S - U$ then forms a cycle of nonsingular rational curves in the sense that it has the irreducible decomposition $C = C_1 \cup \cdots \cup C_s$, where each C_i is a nonsingular rational curve, $C_i \cap C_j = \emptyset$ if $|i-j| \geq 2$, and C_i and C_{i+1} intersect transversally at exactly one point, with $C_{s+1} := C_1$. Moreover C with reduced structure belongs to the anti-canonical system $|-K_S|$ of S and it is called *the anti-canonical cycle* of the toric surface S. We also refer to the curves C_i as *boundary curves* of S.

Let $a_i = C_i^2$ be the self-intersection number of C_i. Then we may associate with S a weighted circular graph with vertices v_i, one for each C_i, with weights a_i. The isomorphism class of a toric surface is completely determined by the associated circular weighted graph modulo cyclic permutations and reversing the order (cf. [17, Cor. 1.29]).

We also note that by [24, 7.6] the anti-Kodaira dimension $\kappa^{-1}(S)$ of S always satisfies

$$\kappa^{-1}(S) = 2. \tag{1}$$

(2.2) A toric surface is completely described by the associated combinatorial data [17, 1.1,1.2]. Let N be the set of one parameter subgroups of G, which is naturally a free abelian group of rank 2. We may write $G = \boldsymbol{C}^* \otimes_{\boldsymbol{Z}} N$ and any isomorphism $N \cong \boldsymbol{Z}^2$ gives rise to an isomorphism $G \cong \boldsymbol{C}^* \times \boldsymbol{C}^*$. Let $N_{\boldsymbol{R}} = N \otimes_{\boldsymbol{Z}} \boldsymbol{R} \cong \boldsymbol{R}^2$.

A complete *fan* on N, or a complete rational polyhedral decomposition of N, is by definition a collection

$$\triangle = \{\{0\}, \tau_j, \sigma_i\}, \quad 1 \leq i, j \leq s$$

of one-dimensional simplices τ_i and two-dimensional simplices σ_j in $N_{\boldsymbol{R}}$, both defined over the rationals, such that $N_{\boldsymbol{R}} = \cup \sigma_i$ and that the faces of σ_i consist of τ_i and τ_{i+1}, where $\tau_{s+1} = \tau_1$, $\sigma_{s+1} = \sigma_1$ by convention and $\tau_1, \dots, \tau_n$ are numbered so that it is arranged cyclically around the origin in $N_{\boldsymbol{R}}$. For any τ_i there exists a unique primitive element $\rho_i \in \tau \cap N$ such that $\tau_i = \boldsymbol{R}_{\geq 0}\rho_i$. The fan $\triangle$ is then completely recovered from the circular sequence $\{\rho_1, \dots, \rho_s\}$. Conversely, any such sequcence gives rise to a complete fan, if ρ_i are not contained in a fixed halfplane in $N_{\boldsymbol{R}}$, by the above correspondence.

(2.3) Compact smooth toric surfaces are in natural bijecitive correspondence with complete fans in $N_{\boldsymbol{R}}$ such that

$$\text{for every } i,\ \rho_i \text{ and } \rho_{i+1} \text{ form a } \boldsymbol{Z}\text{-basis of } N. \tag{2}$$

If we fix an isomorphism $N \cong \boldsymbol{Z}^2$, and hence, an isomorphism $G \cong \boldsymbol{C}^* \times \boldsymbol{C}^*$, we may write $\rho_i = (m_i, n_i)$ for coprime integers m_i and n_i so that ρ_i is the one parameter subgroup given by $\rho_i(t) = (t^{m_i}, t^{n_i}), t \in \boldsymbol{C}^*$. We assume that ρ_i are arranged counterclockwise in $N_{\boldsymbol{R}} \cong \boldsymbol{R}^2$. Then the condition (2) is equivalent to the condition

$$\begin{vmatrix} m_i & n_i \\ m_{i+1} & n_{i+1} \end{vmatrix} = 1. \tag{3}$$

The correspondence between toric surfaces and fans is determined by the following condition: If a compact smooth toric surface S corresponds to a fan $\triangle$, then the one-parameter subgroup $\rho_i : \boldsymbol{C}^*(t) \to G$ fixes each point of C_i and for any point $x \in U$ we have

$$\lim_{t \to 0} \rho_i(t)(x) \in C_i. \tag{4}$$

We also note the relation [17, p.43 $(*)$]

$$\rho_{i-1} + \rho_{i+1} = -a_i \rho_i. \tag{5}$$

(2.4) Let $\mathrm{Aut}(S, C)$ be the group of automorphisms of S which preserves C, and $\mathrm{Aut}_0(S, C)$ its identity component. Moreover, denote by $\mathrm{Aut}(S, \{C_i\})$ the subgroup of $\mathrm{Aut}(S, C)$ consisting of those elements which preserves each irreducible component of C.

Lemma 2.1. $G = \mathrm{Aut}_0(S, C) = \mathrm{Aut}(S, \{C_i\})$.

Proof. We have the natural inclusions

$$G \subseteq \mathrm{Aut}_0(S, C) \subseteq \mathrm{Aut}(S, \{C_i\}).$$

The Lie algebra of $\mathrm{Aut}_0(S, C)$ is naturally identified with the sheaf $\Theta(-\log C)$ of holomorphic vector fields on S which are tangent to C, while the latter is isomorphic to $O_S \otimes_{\boldsymbol{Z}} N$ (cf. [17, Prop. 3.1]). Thus we have

$$\dim \mathrm{Aut}_0(S, C) = \dim H^0(S, \Theta(-\log C)) = 2$$

and hence the equality $G = \mathrm{Aut}_0(S, C)$.

It remains to see that the latter inclusion is an equality. Let $g \in \mathrm{Aut}(S, \{C_i\})$ be an arbitrary element. It induces an automorphism $a(g) : t \to gtg^{-1}$ of $G, t \in G$. But since g leaves each boundary component fixed, it also fixes the corresponding one parameter subgroup. This implies that the automorphism of N induced by $a(g)$ is trivial, and hence, $a(g)$ itself is trivial. Thus g commutes with every element of G. Let x be an arbitrary element of U and t the unique element of G which sends $g(x)$ to x. Then tg has a fixed point x on U. The commutativity with elements of G implies then that tg fixes any point of U. Thus tg is the identity. Hence $g \in G$ as desired. q.e.d.

(2.5) We are interested in a special class of toric surfaces.

Definition. A compact smooth toric surface S is called *symmetric* if the number s of boundary curves is even, say $s = 2k$, with $k \geq 2$, and the weithts a_i of the weighted circular graph associated to S has the property that $a_i = a_{i+k}$ for all $i, 1 \leq i \leq k$. In this case we call $a_1, \dots, a_k$ the *reduced weights* of S.

For instance if $S_0 := \boldsymbol{P}^1 \times \boldsymbol{P}^1$ with standard toric structure, it is symmetric with $k = 2$ and $a_1 = a_2 = 0$.

Suppose that S is a symmetric toric surface with $s = 2k$. Denote by p_j the intersection point of C_j and $C_{j+1}, 1 \leq j \leq 2k$. We call a pair of points *admissible* if it is of the form (p_i, p_{i+k}) for $1 \leq i \leq k$. We call a blowing up $\widetilde{S} \to S$ *admissible* if its center is an admissible pair of points on S. Then any toric surface obtained from a symmetric toric surface by an admissible blowing-up is again symmetric.

Lemma 2.2. *Every symmetric toric surface is obtained from $S_0 = \boldsymbol{P}^1 \times \boldsymbol{P}^1$ by a finite succession of admissible blowing-ups. In particular if $k \geq 3$, $a_i = C_i^2 < 0$ for all i.*

Proof. We proceed by induction on $k \geq 2$. When $k = 2$, the possible weights for a toric surface S are $a, 0, -a, 0$ for some non-negative integer a [17, Cor. 1.29]. Thus S is symmetric if and only if $a = 0$, i.e., $S = S_0$.

Let S be a toric surface with $s = 2k \geq 6$. Suppose that the lemma is verified for symmetric toric surfaces with smaller s. Since S is obtained from a minimal toric surface by a finite succession of blowing-ups at some intersection points of boundary divisors [17, Th. 1.28], there is a (-1)-curve, say C_i, on S. By symmetry, C_{i+k} also is a (-1)-curve. Then by blowing down these two (-1)-curves we obtain a symmetric toric surface with $s = 2k - 2$. By induction it is obtained from S_0 by a finite succession of admissible blowing-ups. The lemma follows. q.e.d.

For later use we also give a result on the existence of (-1)-curves among the boundary curves considered in the above proof. We call a pair of (-1)-curves of the form (C_j, C_{j+k}) *admissible.*

Lemma 2.3. *Let S be a symmetric toric surface with $s = 2k \geq 6$. Fix a pair of admissible points (p_i, p_{i+k}) on S. Then there eixsts an admissible pair of (-1)-curves on S which do not pass through either of p_i and p_{i+k}.*

Proof. We proceed by induction on $k \geq 3$. If $k = 3$, S is obtained from S_0 by an admissible blowing-up. Then all the weights are -1 and the conclusion of the lemma is true.

In the general case let S be obtained from a symmetric torus surface $\overline{S}$ by an admissible blowing-up. Let (C_j, C_{j+k}) be the admissible pair of (-1)-curves which are the exceptional curves of this blowing up. If they do not pass through either of p_i and p_{i+k}, the proof is done. Suppose not. Then p_i and p_{i+k} are mapped to the center of this admissible blowing-up. By induction there exists on $\overline{S}$ an admissible pair of (-1)-curves which do not pass through these points. Then the proper transforms of these (-1)-curves to S give rise to a desired admissible pair of (-1)-curves.

q.e.d.

(2.6) We give characterizations of symmetric toric surfaces.

Proposition 2.4. *Let S be a toric surface such that the number s of boundary curves is an even number $2k (\geq 4)$. Then the following conditions are equivalent:*

1) *S is symmetric.*

2) *The fan defining S satisfies $\rho_{i+k} = -\rho_i$ for all $i, 1 \leq i \leq k$.*

3) *There exists a holomorphic involution τ of S which interchanges C_i and $C_{i+k}, 1 \leq i \leq k$, and which satisfies*

$$\tau g \tau^{-1} = g^{-1}, \; g \in G. \tag{6}$$

Proof. 3) $\Longrightarrow$ 2): Applying τ to (4) and taking into account the relation (6) we obtain

$$C_{i+k} = \tau(C_i) \ni \lim_{t \to 0} \tau(\rho_i(t)x) = \lim_{t \to 0} \rho_i(t)^{-1}(\tau(x))$$

This implies that $-\rho_i = \rho_{i+k}$. 2) follows.

2) $\Longrightarrow$ 3): If 2) is true, the multiplication by -1 is an isomorphism of the fan of S. Therefore it defines an involution τ of S which maps C_i to C_{i+k} satisfying (6) (cf. [17, Th. 1.13]).

3) $\Longrightarrow$ 1): Clear.

1) $\Longrightarrow$ 3): We proceed by induction on k. When $k = 2$, the involution τ of $S_0 = \boldsymbol{P}^1 \times \boldsymbol{P}^1$ defined by $(z, w) \to (1/z, 1/w)$ has the desired property. In the general case let S be obtained from a symmetric toric surface $\overline{S}$ by an admissible blowing-up. Then by induction there exists an involution $\overline{\tau}$ of $\overline{S}$ satisfying the desired properties. Since $\overline{\tau}$ interchanges the blown-up points p_i and p_{i+k}, it lifts to an involution τ on S inheriting the desired properties from $\overline{\tau}$. q.e.d.

Let τ be as in 3) above. Let $\hat{G}$ be the semidirect product $\hat{G} := G \cdot \langle \tau \rangle$ with respect to the action (6) of τ on G. We have $\hat{G} = G \coprod \tau G$ and any element of τG is an involution and can play the role of τ above. The above lemma shows that for a symmetric toric surface S, $\mathrm{Aut}(S, C)$

always contains a subgroup isomorphic to $\hat{G}$ and any admissible blowing-down to S_0 is $\hat{G}$-equivariant.

(2.7) Given $k = m + 2 \geq 2$, up to isomorphisms there exist at most a finite number $n(k)$ of symmetric toric surfaces; in fact we have $n(2) = n(3) = n(4) = n(5) = 1$ and $n(6) = 3$. More precisely, we obtain the following table:

k	$(-a_1, \dots, -a_k)$	$n(k)$	S
2	(0,0)	1	Del Pezzo
3	(1,1,1)	1	Del Pezzo
4	(1,2,1,2)	1	generalized Del Pezzo
5	(2,1,3,1,2)	1	
	(1,2,3,1,2,3)		
6	(1,3,1,3,1,3)	3	
	(2,1,4,1,2,2)		of maximal type

In general for every k there exists (up to equivalence) a unique weighted circular graph with $2k$ vertices which has $-m$ as a reduced weight; then the whole reduced weights are given (up to cyclic permutations and the inversion of the order) by

$$(-a_1, \dots, -a_k) = (2, \dots, 2, 1, m, 1, 2, \dots, 2)$$

unless $k \leq 4$. In this case we call the corresponding toric surface *of maximal type*. In this sense the surfaces for $2 \leq k \leq 5$ are all of maximal type although it is not indicated in the table. Moreover the number $n(k)$ can be determined explicitly for any k (cf. Proposition 3.4 below).

(2.8) We shall determine equivariant morphisms of a symmetric toric surface onto curves.

Lemma 2.5. *Let S be a symmetric toric surface with boundary components $C_i, 1 \leq i \leq 2k$. Then for each i, $1 \leq i \leq k$, there exists a natural equivariant morphism $\nu_i : S \to P_i$ onto a nonsingular rational curve P_i with the following properties:*

1) *Any proper surjective morphism of S onto a curve coincides with one of ν_i. More generally, any nonsingular rational curve L in S with self-intersection number $L^2 = 0$ is a fiber of ν_i for a unique i.*

2) *C_i and C_{i+k} are mapped isomorphically onto P_i by ν_i.*

3) *With respect to a suitable choice of an affine coordinate z of P_i the singular fibers of ν_i appear precisely over* 0 *and* ∞ *such that:*

   a) *The singular fiber over* 0 *is a positive linear combination of C_j, $i+1 \leq j \leq i+k-1$, while the fiber over ∞ is one of C_j with $i+k+1 \leq j \leq i-1$ (index considered modulo $2k$) where the coefficients of C_j and C_{j+k} coincide for any j.*

   b) *The smooth fibers are orbit closures of the one parameter group ρ_i.*

4) *For any i, $\nu_i \times \nu_{i+1} : S \to \boldsymbol{P}^1 \times \boldsymbol{P}^1$ is a birational morphism which contracts all the components C_j other than C_l and C_{l+k}, $l = i, i+1$.*

Proof. We give the definition of ν_i in two ways.

1) Geometric definition: Fix i. By Lemma 2.3 there exists an admissible blowing-down $u : S \to S_0$ in which all the components C_j other than $C_l, C_{l+1}, l = i, i+1$, are contracted to an admissible pair of points on S_0 and which is otherwise isomorphic. Then the compositions $S \to \boldsymbol{P}^1$ of u with the two projections $S_0 \to \boldsymbol{P}^1$ yields holomorphic surjection ν_i and ν_{i+1}. The properties 2), 3) and 4) are all immediately checked as well as the independency of the definition of the choices of the points p_i or p_{i-1} used in Lemma 2.3.

2) Combinatorial definition: For every $1 \leq i \leq k$ we consider the quotient homomorphism $\chi_i : N \to \overline{N}_i := N/\boldsymbol{Z}\rho_i$. Since ρ_i is primitive, $\overline{N}_i$ is a free abelian group of rank one and χ_i induces a natural map of fans $(N, \triangle) \to (\overline{N}_i, \overline{\triangle})$, where if ξ is a genarator of $\overline{N}_i$, $\overline{\triangle} = \{\boldsymbol{R}_{\geq 0}\xi, 0, -\boldsymbol{R}_{\leq 0}\xi\}$.

Corresponding to χ_i we get a G-equivariant morphism $S \to P_i$ onto the toric projective line P_i, which is easily identified with the above ν_i. The map $\nu_i \times \nu_{i+1} : S \to S_0$ is the equivariant birational morphism corresponding to the “forgetting map” of the fans, where the fan of S_0 is obtained by deleting all the primitive vectors ρ_j other than ρ_l, ρ_{l+1}, $l = i, i+1$.

Proof of the property 1). It suffices to show the latter statement. If $L^2 = 0$, the linear system defined by L gives a morphism $f : S \to \boldsymbol{P}^1$ onto the complex projective line, which is necessarily equivariant with an induced G-action on $\boldsymbol{P}^1$ [23, Satz 1.3]. On the other hand, any equivariant morphism of toric varieties corresponds to a map of fans (cf. [17, Th. 1.13]); however, from the shape of our fan it is immediate that the possible such equivariant morphisms are those given by χ_i.

q.e.d.

(2.9) Let S be a symmetric toric surface. Let $g \to g^*$ be the unique anti-holomorphic involution of G with fixed point set the maximal compact subgroup $K \cong S^1 \times S^1$. Suppose that S admits an anti-holomorphic fixed point free involution satisfying

$$\sigma g \sigma^{-1} = g^*, \quad g \in G. \tag{7}$$

Then the following holds:

Lemma 2.6. *The equivariant morphisms $\nu_i : S \to P_i$ of Lemma* 2.5 *are all σ-equivariant with respect to an induced action of σ on P_i. If there exists a real fiber, then the fixed point set R_i of σ on P_i is diffeomorphic to S^1 and consists of a single K-orbit.*

Proof. Applying σ to (4) and taking into account (7) we obtain

$$\sigma(C_i) \ni \lim_{t \to 0} \sigma(\rho_i(t)(x)) = \lim_{t \to 0} (\rho_i(t)^*)(\sigma(x)) \in C_{i+k}.$$

This implies that

$$\sigma(C_i) = C_{i+k} \tag{8}$$

for any i. Thus the σ-equivariancy of ν_i follows from 3) b) of Lemma 2.5. The second assertion is obvious. q.e.d.

Lemma 2.7. *There exist up to isomorphisms at most four anti-holomorphic involutions on S satisfying the condition* (7)*.*

Proof. On $S_0 = \boldsymbol{P}^1 \times \boldsymbol{P}^1$ we have four natural anti-holomorphic involutions; with respect to affine coordinates (z, w) of S_0 they are $(z, w) \to (\pm 1/\overline{z}, \pm 1/\overline{w})$. Note that these anti-holomorphic involutions are expressed as σs for some element s of order ≤ 2 of K. It is easy to see that these all satisfy the property (7). For any succession of admissible blowing-ups starting from S_0 these involutions lift to the resulting symmetric toric surface. By Lemma 2.2 the existence of four anti-holomorphic involutions which satisfy (7) is thus verified in general.

Next we show that any such anti-holomorphic involution is isomorphic to one of them. If there exist two anti-holomorphic involutions σ and σ' of S satisfying (7), then in view of (8) $h := \sigma^{-1}\sigma'$ is a holomorphic automorphism of S which preserves all the irreducible components of C. Then by Lemma 2.1 we have $h \in G$. Since σ' is an involution, we must have $\sigma h \sigma h = id$; thus $\sigma h \sigma h(x) = h^* h x = x$ for any $x \in S$, or equivalently, with respect to a suitable isomorphism $G \cong \boldsymbol{C}^* \times \boldsymbol{C}^*$ we may write $h = (t_1, t_2)$ with $t_i \in \boldsymbol{R}$. If we set $|h| = (|t_1|, |t_2|)$, we may write $h = s|h|$ with $s = (\pm 1, \pm 1)$. Then

$$a^{-1}\sigma a(x) = \sigma a^2(x) = \sigma|h|(x) \quad \text{for } a = (\sqrt{|t_1|}, \sqrt{|t_2|}).$$

Thus, $\sigma h = \sigma|h|s = a^{-1}(\sigma s)a$. Namely, σh is conjugate to σs, and up to isomorphisms there exist at most four non-isomorphic anti-holomorphic involutions. q.e.d.

(2.10) We may "divide into halves" a symmetric toric surface along each of its k "diagonals" $\overline{p_i p_{i+k}}$. Let S be a symmetric toric surface with reduced weights $(a_1, \ldots, a_k)$. Then for each $i, 1 \le i \le k$, we define a pair of (not necessarily symmetric) projecitve smooth toric surfaces S^+ and S^- as follows. Let $\rho = \{\rho_i, 1 \le i \le 2k\}$ be the set of primitive elements giving the fan of $S = S(\rho)$. Then define for each i

$$S_i^{\pm}(\rho) \tag{9}$$

to be the toric surfaces determined by a sequence of primitive elements

$$\pm(\rho_1, \ldots, \rho_i, -\rho_i + \rho_{i+1}, -\rho_{i+1}, \ldots, -\rho_k) \tag{10}$$

which generates 1-simplices of a complete fan. (Here and in what follows we adopt the following convention. If a sentence contains the notation $\pm$ more than once, it actually consist of two sentences, one for $+$ only and the other for $-$ only. See Convention and Notation in the Introduction.) It is clear that any two adjacent elements in (10) form a $\boldsymbol{Z}$-basis so that the resulting toric surface is smooth and compact. Both surfaces have $k+1$ boundary components, which we shall denote by $(C_1^{\pm}, \ldots, C_i^{\pm}, L_i^{\pm}, C_{i+1}^{\pm}, \ldots, C_k^{\pm})$ respectively. In view of (5), from

$$\rho_i + (-\rho_{i+1}) \qquad = (-1)(-\rho_i + \rho_{i+1}) \tag{11}$$

$$\rho_{i-1} + (-\rho_i + \rho_{i+1}) \quad = -a_i\rho_i - \rho_i = -(a_i + 1)\rho_i \tag{12}$$

the self-intersection numbers are computed as

$$L_i^{\pm 2} = 1 \quad \text{and} \quad C_l^{\pm 2} = a_l + 1, \; l = i, i+1$$

respectively. Therefore the corresponding weighted cirular graphs for $S_i^{\pm}(\rho)$ are the same and is given by

$$(a_1, \ldots a_{i-1}, a_i + 1, 1, a_{i+1} + 1, a_{i+2}, \ldots, a_k) \tag{13}$$

modulo cyclic permutations; in particular $S^+(\rho)$ and $S^-(\rho)$ are mutually isomorphic. We shall note one important property of surfaces $S_i^{\pm}(\rho)$.

Lemma 2.8. *Let S be either of $S_i^{\pm}(\rho)$ of* (9). *Then there exists a toric birational morphism $f : S \to \boldsymbol{P}^2$ which blows down all the boundary curves other than L and the two irreducible components of C which are adjacent to L.*

Proof. We proceed by induction on $k \geq 2$. When $k = 2$, by (13) and Lemma 2.2 $S = \boldsymbol{P}^2$ and there is nothing to prove. Suppose that $k > 2$. Then by Lemma 2.3 there exists an admissible pair of (-1)-curves on $S(\rho)$ which do not pass through neither p_i nor p_{i+k}. Contracting these (-1)-curves we obtain a symmetric toric surface $S(\rho')$ for a suitable ρ' with $(k-1)$ boundary components. One of these (-1)-curves may be naturally identified with a (-1)-curve in S and contracting this curve in S we obtain a surface S' which is identified with either of $S_{i'}^{+}(\rho')$ for some i'. So we may apply the induction hypothesis to finish the proof. (One may also note that the fan generated by the primitive vectors ρ_i, ρ_{i+1} and $-\rho_i + \rho_{i+1}$ is exactly the fan defining a complex projective plane.) q.e.d.

We identify G-equivariantly $S_i^+ = S_i^+(\rho)$ and $S_i^- = S_i^-(\rho)$ transversally along L. We shall call the resulting surface $S_i := S_i^+ \cup_L S_i^+$ with natural G-action a *degenerate* symmetric toric surface. The G-equivariancy implies that C_i^+, L and C_{i+1}^- (resp. C_i^-, L and C_{i+1}^+) intersect (transversally) at one point p_i^+ (resp. p_i^-). We set $b_j^{\pm} = C_j^{\pm} \cap C_{j+1}^{\pm}$, where the indices j are considered cyclically modulo k. We shall call any blowing up of S_i with center a pair of smooth points $\{b_j^+, b_j^-\}, j \neq i$, an admissible blowing up. Then the resulting surface is again a degenerate symmetric toric surface naturally. In particular if $k = 2$, S_i are all isomorphic and is a union of two copies of the toric projective planes. Denote this surface by $\overline{S}$. Then from Lemma 2.8 we deduce immediately the following:

Lemma 2.9. *Any degenerate symmetric toric surface S_i as above is obtained from a finite succession of admissible blowing-ups from $\overline{S}$.*

We also use the following:

Lemma 2.10. *Let $S = S_i^{\pm}(\rho)$, $L = L_i$ and C_j be as above. Then the complete linear system $|-K_S - 2L|$ is non-empty if and only if either of C_i and C_{i+1} has the self-intersection number $3 - k = 1 - m$.*

Proof. When $k = 2$, S is a toric projective plane $\boldsymbol{P}^2$ and the anticanonical cycle $L_0 + L_1 + L_2$ consists of three lines L_j, one of which is our L. Thus $|-K_S - 2L| = |L|$ on $\boldsymbol{P}^2$, which is non-empty and

$L_j^2 = 1 = 3 - 2$. In the general case let $f : S \to \boldsymbol{P}^2$ be as in Lemma 2.8 and suppose that L is the proper transform of, say L_0. If $|-K_S - 2L| \neq \emptyset$, then the successive blowing-ups, of which f is the composition, is done always on the proper transform of one of L_i for a fixed $i(= 1 \text{ or } 2)$. Then the proper transform of this L_i is either C_i or C_{i+1} and it has the desired self-intersection number $3 - k$. It is also easy to see that conversely, if the latter holds, f is obtained precisely as described above. q.e.d.

3. Torus actions and conformal invariants

(3.1) Let M be a connected compact oritented smooth 4-manifold. Suppose that a real two-torus $K \cong S^1 \times S^1$ acts smoothly and effectively on M. This situation has been extensively studied by Orlik-Raymond [18], to which we refer for the details.

We begin with a simple and well-known remark on the fixed point set of our K-action.

Lemma 3.1. *Let M with a K-action be as above. Then the Euler-Poincare characterestic $\chi(M)$ of M is nonnegative, and it is positive if and only if K has a fixed point. In particular the latter is the case when M is simply connected.*

Proof. We first note that the fixed point set F of the action is isolated if it is not empty. Indeed, let $x \in F$ be any point. Then the isotropic representation of K in the normal space N_x at x of the submanifold F in M is faithful. But this is possible only when $\dim N_x = 4$, i.e., x is isolated.

Now if we take a general S^1-subgroup S of K, its fixed point set also coincides with F. Hence we have $\chi(M) = \chi(F) + \chi(M - F)$ with $\chi(M - F) = 0$, S acting on $M - F$ locally freely. Thus, with b_i the i-th betti number of M, we get

$$0 \leq \#F = \chi(F) = \chi(M) = 2 - 2b_1 + b_2, \tag{14}$$

where $\#F$ denotes the cardinality of F. The lemma follows. q.e.d.

(3.2) We call a union

$$B := B_1 \bigcup \cdots \bigcup B_k.$$

of 2-spheres B_i on M a *cycle of two-spheres* if it has the property that $B_i \bigcap B_j = \emptyset$ if $|i - j| \geq 2$, and B_i and B_{i+1}, $1 \leq i \leq k$, intersect transversally at a single point x_i, $B_i \cap B_{i+1} = \{x_i\}$, $1 \le i \le k$, where we consider the indices i and j modulo k so that $B_{k+1} = B_1$.

Let $\Sigma(M)$ be the singular set of the K-action, i.e., the set of points whose stabilizer group is nontrivial.

Lemma 3.2. *Suppose that $\chi(M) \neq 0$. Then $\Sigma(M)$ is a disjoint union of cycles of two-spheres, each of which is of the form $B := B_1 \cup \cdots \cup B_k$ as above such that:*

1) *Each $x_i := B_i \cap B_{i+1}$, $1 \le i \le k$, is a fixed point.*
2) *At each point x of $B_i' := B_i - \{x_{i-1}, x_i\}$ the stabilizer group is a S^1-subgroup K_i of K which is independent of the choice of the point x, where $x_0 = x_k$.*

Moreover, if M is simply connected, B is unique and coincides with $\Sigma(M)$.

Proof. By Lemma 3.1 $\chi(M) > 0$ and there exists a fixed point x. Then the isotropic representation of K at x is a direct sum of two representations, in each of which the representation has a fixed S^1-subgroup as the kernel. The fixed point set of these S^1-subgroups consists of smooth surfaces embedded in M and intersect transversally at x. These surfaces must be diffeomorphic either to S^2 or real projective plane $\boldsymbol{RP}^2$ since it admits an S^1-action with fixed points.

We claim that $\boldsymbol{RP}^2$ does not occur. Indeed, let R be one of the surfaces passing through x and $K_1 \cong S^1$ the stabilizer group at a general point of R. We have the effective action of K/K_1 on R. Suppose now that R is a real projective plane $\boldsymbol{RP}^2$. Then the element $-1 \in K/K_1 \cong S^1$ admits a component H of a fixed point set on R which is homeomorphic to S^1. We may lift -1 to an element s of order two on K. Then $K_1 \times \langle s \rangle (\cong S^1 \times \boldsymbol{Z}_2)$ acts effectively on the two dimensional normal vector space N_x at each point x of H, which is impossible. Thus R is never $\boldsymbol{RP}^2$. Then for any fixed point x we have two embedded two-spheres intersecting transversally at x. Take one of it, say B_1. On B_1 we have exactly one more fixed point, say x_2, other than x_1. Repeat the same consideration at x_2 and obtain another embedded two-sphere B_2 intersecting transversally with B_1 at x_2 and is pointwise fixed by some S^1-subgroup of K. Continuing in this way we finally obtain a sequence of two-spheres $B_1, \ldots, B_k$ each of which is pointwise fixed by some S^1-subgroup of K, where B_k and B_0 intersect transversally at a single point x_k. In this way we obtain a cycle of two-spheres $B := B_1 \cup \cdots \cup B_k$ with the desired property.

2) is well-known (cf. [18]) and the final assertion is proved in [18, §5]. q.e.d.

(3.3) Fix an identification $K = S^1 \times S^1$. We identify S^1 with the multiplicative group of complex numbers of modulus one. For any pair (m, n) of coprime integers we introduce the S^1-subgroup $K(m, n)$ of K by

$$K(m, n) := \{(s, t) \in K = S^1 \times S^1; s^m t^n = 1\}.$$

These groups are mutually different except that we have $K(m, n) = K(-m, -n)$.

Now we consider a general K-action on M with fixed points as above. Let $B = \cup B_i$ be one of the cycles of S^2 contained in the singular set of the action. Then the stabilizer group K_i of a point x belonging to the set B'_i is independent of x and is written as $K_i = K(m_i, n_i)$ for a pair (m_i, n_i) of coprime integers m_i and n_i which are determined up to (simultaneous) inversion of signs. Moreover, these pairs of coprime integers are subject to the condition:

$$\begin{vmatrix} m_i & n_i \\ m_{i+1} & n_{i+1} \end{vmatrix} = \pm 1 \tag{15}$$

coming from the effectivity of the action. Thus, associated to the cycle B one obtains a circular sequence

$$\{\pm(m_1, n_1), \dots, \pm(m_k, n_k)\} \tag{16}$$

of pairs of coprime integers considered up to signs as above. We shall call this sequence (16) the *Orlik-Raymond invariant* of the cycle B (or of M when M is simply connected.)

(3.4) When M is simply connected, also the structure of the quotient is determined (cf. [18, §5]). In this case, as follows from (14), the number $m := k - 2$ equals the second Betti number of M and the set $\{x_1, \dots, x_k\}$ are precisely the set of fixed points of K The quotient $N := M/K$ is naturally homeomorphic to the closed two-disc and in this way N admits a structure of a C^∞ manifold with boundary. Let $\pi : M \to N$ be the quotient map and N_0 the interior of N. Then $M_0 := \pi^{-1}(N_0) = M - B$ is the maximal open subset of M on which K acts freely.

Further, if bN denotes the boundary of N, we have $\pi^{-1}(bN) = B$, and $bN = B/K$. Moreover, the images

$$y_i := \pi(x_i), \quad 1 \le i \le k,$$

of x_i are arranged cyclically on the circle bN with respect to an orientation of bN and then B_i/K is identified with the closed arc on bN connecting y_{i-1} and y_i. In this case the Orlik-Raymond invariant, considered modulo cyclic permutations and reversing the orders, completely determines the isomorphism class of the K-action on M [18].

(3.5) We shall give a supplement to a result of Joyce [11, 3.1]. Namely we show that when M is diffeomorphic to the connected sum $m\boldsymbol{P}^2$ of m-copies of the complex projective plane $\boldsymbol{P}^2$, the number $\iota(m)$ of isomorphism classes of K-actions on M can be explicitly determined.

Proposition 3.3. *Set $j(m) = (2m)!/m!(m+1)!$ for $m \ge 0$, where $0! = 1$. Then $\iota(m)$ for $m > 0$ is given by the formula:*

$$\iota(m) = j(m)/2(m+2) + j((m-1)/3)/3 + h(m),$$

where $h(m) = 3j(m/2)/4$ if m is even and $= j((m-1)/2)/2$ if m is odd, with $j(l) = 0$ if l is not an integer.

Proof. By [11, Prop. 3.1.1] the Orlik-Raymond invariant

$$\{\pm(m_i, n_i); 1 \le i \le k = m+2\}$$

can be normalized as follows: There exists a unique element h of $SL(2, \boldsymbol{Z})$ such that, after changing the identification $K = S^1 \times S^1$ by the automorphism of K induced by h if necessary, one may choose the signs of $\pm(m_i, n_i)$ so that it is arranged couterclockwise in this order in $\boldsymbol{R}^2$ such that

$$(m_1, n_1) = (1, 0) \text{ and } (m_k, n_k) = (0, 1). \tag{17}$$

In this case we automatically have $m_i > 0$ if $i < k$ and

$$m_i n_{i+1} - m_{i+1} n_i = 1, \quad 1 \le i < k. \tag{18}$$

Moreover, by [11, Prop. 3.1.2] given an integer $k \ge 2$ the number of the sequences (16) satisfying the above conditions (17) and (18) is equal to $j(m)$. In his proof this number is in fact identified with the number of subdivisions of a regular k-gon into triangles.

On the other hand, the Orlik-Raymond invariant is determined up to the cyclic permutations and reversing the orders. In the above correspondence counting the number of possible Orlik-Raymond invariants modulo this equivalence relation exactly corresponds to counting the number of the above subdivisions modulo rotations and reflections of the k-gons. This latter numer is indeed computed to be equal to be the number given in the statement of the proposition by Moon and Moser [16] (cf. the reference of Brown's article listed in [11]). Since the Orlik-Raymond invariant determines the action completely, this yields the proposition. q.e.d.

The number $\iota(m)$ also coincides with the number of the isomorphism classes of symmetric toric surfaces.

Proposition 3.4. *For $k \geq 2$ the number $\iota(k-2)$ coincides with the number $n(k)$ of isomorphism classes of symmetric toric surfaces with $2k$ boundary components.*

Proof. If $\{\pm(m_i, n_i); 1 \leq i \leq k\}$ is an Orlik-Raymond invariant, then with respect to a suitable choice of signs (which is unique up to simultaneous inversion of signs) the vectors

$$(n_1, -m_1), \dots, (n_k, -m_k), (-n_1, m_1), \dots, (-n_k, m_k)$$

form a cyclic sequence of primitive vectors arranged counterclockwise in $\boldsymbol{Z}^2$. These give rise to a fan defining a symmetric toric surface by setting $\rho_i = (n_i, -m_i)$ and $\rho_{i+k} = (-n_i, m_i)$ for $1 \leq i \leq k$ (cf. Lemma 4.12 below). This sets a bijective correspondence of the set of Orlik-Raymond invariants and symmetric fans when both are considered modulo the action of $GL(2, \boldsymbol{Z})$. q.e.d.

Remark. One can establish a geometric correspondence between symmetric toric surfaces and $m\boldsymbol{P}^2$ with K-actions.

(3.6) In the rest of this section we shall introduce an invariant for a simply connected conformal four-manifold with K-action. So suppose now that M is simply connected and is endowed with a K-invariant conformal structure $[g]$.

Let $\triangle$ be the open unit disc $\{|z| < 1\}$ in the complex plane $\boldsymbol{C} = \boldsymbol{C}(z)$ with the unit circle $S = \{|z| = 1\}$ as the boundary. We denote by $\mathcal{A}$ the set of (ordered) sequences $(q_1, \dots, q_k)$ of distinct points q_i of S arranged cyclically on S with respect to an orientation of S and considered modulo cyclic permutations. Let H be the group of conformal automorphisms of $\triangle$, any of which extends to a diffeomorphism of the closure $\overline{\Delta}$. H thus acts on the set $\mathcal{A}$ and the quotient $\overline{\mathcal{A}} := \mathcal{A}/H$ is the set in which our invariant for the action belongs. Note that two sequences $(q_1, \dots, q_k)$ and $(q'_1, \dots, q'_k)$ defines the same element of $\overline{\mathcal{A}}$ if and only if there exist a conformal automorphism h of $\triangle$ and an integer l with $1 \le l < k$ such that $h(q_i) = q_{i+l}$ (resp. q_{i-l}) for all i.

We use freely the notation of (3.4). We have a natural conformal structure $[h]$ on N_0. In fact, for any point $y \in N_0$ the corresponding conformal structure on the tangent space $T_y N_0$ is described as follows. Take a point x of $\pi^{-1}(y)$. Let Kx be the orbit of x and $T_x^{\perp}$ the orthogonal complement of the tangent space of Kx in the tangent space of M at x with respect to the given conformal structure. Then via the natural isomorphism $T_x^{\perp} \xrightarrow{\sim} T_y N$ we get an induced conformal structure, which is independent of the choice of the point $x \in \pi^{-1}(y)$.

Since N_0 is of real dimensional two, the conformal structure defines a unique complex structure up to complex conjugation. Hence by the Riemann's mapping theorem there exists a conformal isomorphism of N_0 onto the open unit disc $\triangle$. (From the argument below it follows that N_0 is not isomorphic to $\boldsymbol{C}$.) Let $\alpha : N_0 \to \triangle$ be such a map. We shall prove below the following:

Lemma 3.5. *α extends to a unique homeomorphism $\overline{\alpha}$ of N onto the closed disc $\overline{\triangle}$.*

Then our invariant for a conformal torus action is defined as follows.

Definition. Let $x_1, \dots, x_k,\ k \ge 2$, be the fixed points of the K-action and $y_i = \pi(x_i) \in bN,\ 1 \le i \le k$, be as in (3.4). Then we set

$$q_i = \overline{\alpha}(y_i), \quad 1 \le i \le k.$$

Since any other choice of $\overline{\alpha}$ is obtained by composing an element of H with α, the image $[q_1, \dots, q_k]$ of the cyclic sequence $(q_1, \dots, q_k)$ in $\overline{\mathcal{A}}$ is an invariant of the given K-action.

Example. In the construction of self-dual metrics on $m\boldsymbol{P}^2$ in [11] Joyce first puts a conformal structure on the quotient $N_0 = M_0/K$ by identifying N with the closed unit disc $\overline{\triangle}$ and then specifies k points $\{y_1, \dots, y_k\}$ on the boundary bN, which is to be the parameter of his self-dual metrics. By the very construction of his, this conformal structure on N_0 is precisely the one coming from the conformal structure of M as explained above and the points y_i are exactly the images by $\pi : M \to N$ of the fixed points of K on M. Thus the parameter of Joyce is naturally identified with the conformal invariant of the K-action on M defined above.

(3.7) For the proof of Lemma 3.5 we use the following classical result due to Lindelöf [2, p.86]:

Lemma 3.6. *Let D be a simply connected bounded domain in the complex plane $\boldsymbol{C}$ whose boundary is a Jordan curve. Let $\alpha : D \to \triangle$ be a biholomorphic map onto the unit disc, whose existence is guaranteed by the Riemann's mapping theorem. Then α extends to a homeomorphism up to the boundary of both domains.*

In order to apply this result we prove the following:

Proposition 3.7. *N is realized as a closed subdomain of a simply connected manifold $\hat{N}$ to which the conformal structure $[h]$ on N_0 extends smoothly.*

Proof of Lemma 3.5 *from Proposition* 3.7. By Proposition 3.7 and the Riemann's mapping theorem, there exists a conformal isomorphism of $\hat{N}$ (with the extended conformal structure) onto the unit disc $\triangle$ in $\boldsymbol{C}$, after restricting $\hat{N}$ if necessary. The restriction of this map to N realizes N_0 as a bounded domain in $\boldsymbol{C}$ with smooth boundary so that we can apply Lemma 3.6 to get a desired homeomorphic extension of ϕ. q.e.d.

(3.8) Before going into the proof of Proposition 3.7 we recall some of the general facts in the construction of the quotient $N = M/K$ in the smooth category.

To get $\hat{N}$ we consider N as a C^∞ manifold with corners rather than as a manifold with boundary. Here, in our two dimensional case a *manifold with corners* means a manifold structure which is modelled on open subsets of the first quadrant $\boldsymbol{R}^2_{\geq 0} := \{(x, y); x, y \geq 0\}$ and C^∞ maps i.e., the maps which extend to C^∞ maps of open neighborhoods in $\boldsymbol{R}^2$. Such a structure on N is determined roughly as follows.

For each point $y \in N$ take a point $x \in \pi^{-1}(y)$ and a local slice S at x for the action; S is a locally closed subset of M passing through x, which is mapped homeomorphically by π onto a neighborhood, say P, of y of N, and which is diffeomorphic to an open subset Q of $\boldsymbol{R}^2_{\geq 0}$. Then the homeomorphism $P \to S \to Q$ defines a coordinate neighborhood of N for the structure of a manifold with corners on N. In our case we prove further:

Lemma 3.8. *A slice S as above can be taken to be an open subset of a locally closed submanifold $\hat{S}$ of M such that there exists a diffeomorphism of $\hat{S}$ onto an open subset of $\boldsymbol{R}^2$ which maps S onto an open subset of $\boldsymbol{R}^2_{\geq 0}$.*

Proof. The construction of the slices at $x \in B$ is as follows. We take a K_x-invariant (complex) coordinate neighborhood W of x in M in which the action of K_x is complex linearlized. Let z_1, z_2 be (smooth) complex coordinates around x and write $z_i = u_i + \sqrt{-1}v_i,\ u_i, v_i \in \boldsymbol{R},\ i = 1, 2$. We have two cases:

Case 1. $x \neq x_i$.

In this case, by a suitable choice of z_i the K_x-action takes the form:

$$(z_1, z_2) \to (z_1, sz_2), \quad s \in K_x \cong S^1$$

where the orbits of the induced action of K/K_x is defined by $u_1 = \text{const.}$ and $z_2 = \text{const.}$ We then define a local slice S (resp. its extension $\hat{S}$) to be the intersection of W with the halfplane (resp. plane) $\{v_1 = v_2 = 0, u_2 \geq 0\}$ (resp. $\{v_1 = v_2 = 0\}$).

Case 2. $x = x_i$.

With respect to a suitable isomorphism $K \cong S^1(s) \times S^1(t)$, the K-action takes the form

$$(z_1, z_2) \to (sz_1, tz_2).$$

We define a local slice S (resp. its extension $\hat{S}$) as before to be the intersection of W with the quadrant (resp. the plane) $\{u_i \geq 0,\ v_i = 0,\ i = 1, 2\}$ (resp. $\{v_1 = v_2 = 0\}$).

Clearly in either case the pair $(S, \hat{S})$ meets the requirement of the lemma. q.e.d.

(3.9) The main point in the proof of the existence of a smooth structure on the quotient is that if we take another slice S' at the same point x defined in the same way as above, starting from another K_x-linearlized coordinates, we can construct naturally a diffeomorphism between S and S'.

To understand this, it is convenient to perform the real blowing up $\nu : \widetilde{M} \to M$ along $B = \cup B_i$. Then $\widetilde{M}$ is a smooth 4-manifold with corners to which the K-action naturally lifts. The main advantage in passing to $\widetilde{M}$ is that the action becomes free there. Moreover, since each fiber of ν is in the same K-orbit, we have $N = \widetilde{M}/K$. A slice on M is then nothing but a diffeomorphic image of a slice for the free K-action on $\widetilde{M}$, and if $\widetilde{S}$ and $\widetilde{S}'$ are such slices on $\widetilde{M}$ we have a corresponding deffeomorphism $\widetilde{u} : \widetilde{S} \to \widetilde{S}'$ in the form $\widetilde{u}(\widetilde{s}) = \widetilde{g}(\widetilde{s})\widetilde{s}$ for a unique smooth map $\widetilde{g} : \widetilde{S} \to K$. Since ν is K-equivariant, this induces a diffeomorphism $u : S \to S'$ with $u(s) = g(s)s$, where the smooth map $g : S \to K$ is naturally induced by $\widetilde{g}$.

Now we show

Lemma 3.9. *The above diffeomorphism $u : S \to S'$ extends to a diffeomorphism $\hat{u} : \hat{S} \to \hat{S}'$ between their respective extensions $\hat{S}$ and $\hat{S}'$ as defined in Lemma* 3.8.

Proof. In Case 1 (resp. Case 2), by the construction of $\hat{S}$ in the proof of Lemma 3.8 there exists an element k (resp. k_1, k_2) of K_x of order two such that

$$\hat{S} = S \cup k(S) \quad (resp.\ \hat{S} = S \cup k_1(S) \cup k_2(S) \cup (k_1k_2)(S));$$

in fact in the local expression there we may take

$$k = -1 \quad (\text{resp.} \quad k_1 = (-1, 0),\ k_2 = (0, -1)).$$

Then for $x \in k(S)$ we have

$$g(k^{-1}x)(x) = (kg(k^{-1}x)k^{-1})(x) = kg(k^{-1}x)(k^{-1}x) \in k(S') \subseteq \hat{S}'$$

where $k = k_i,\ i = 1, 2$ in Case 2. Thus the map $g : S \to K$ extends naturally to a continuous map $\hat{g} : \hat{S} \to K$ such that $x \to \hat{g}(x)x$ gives a homeomorphism of $\hat{S}$ onto $\hat{S}'$; in fact it is characterized by the equation

$$\hat{g}(x) = g(kx), \quad x \in \hat{S} - S \quad \text{and} \quad k \quad \text{as} \quad \text{above}. \tag{19}$$

It remains to show that the extended map is smooth on $\hat{S}$. For this purpose we may consider the map g also as a smooth map $g : P \to K$ since $S \to P$ is diffeomorphic. On the other hand, we may consider P also as the quotient $\hat{S}/\langle k \rangle$ (resp. $\hat{S}/\langle k_1, k_2 \rangle$). Then by (19) our map $\hat{g}$ is nothing but the pull-back of the map $g : P \to K$ to $\hat{S}$ by the quotient map, and hence is smooth. q.e.d.

Proof of Proposition 3.7. Using Lemma 3.9 we can patch together the extended slices $\hat{S}$ in Lemma 3.8 and obtain a desired smooth manifold $\hat{N}$ which contains N as a compact subdomain with corners. Restricting $\hat{N}$ if necessary, we may assume that $\hat{N}$ is simply connected.

It remains to show that the conformal structure on N is extendible to $\hat{N}$. For this purpose we have only to show the extendibility locally across any boundary point of N; indeed, then we can use a partition of unity to obtain a global extension to $\hat{N}$.

Let T_π be the relative tangent bundle for the projection $\pi : M_0 \to N_0$, which is a subbundle on M_0 of the tangent bundle TM of M. Let S be any slice and $\hat{S}$ its extention as in Lemma 3.8. We claim that the restriction $T_\pi|S \cap M_0$ extends to a subbundle T' of $TM|\hat{S}$. Indeed, in either of Cases 1 and 2 we have only to define T' to be the subbundle of $TM|\hat{S}$ generated by the two independent sections $\partial/\partial v_1, \partial/\partial v_2$, for T_π is generated on $S \cap M_0$ by the two sections $\partial/\partial v_1$ and $u_2\partial/\partial v_2$ in Case 1 and by $u_1\partial/\partial v_1$ and $u_2\partial/\partial v_2$ in Case 2.

The natural conformal structure on the orthogonal complement ${T'}^\perp$ of T' in $TM|\hat{S}$ defines a desired local extension of the conformal structure on the interior of $P = \pi(S)$ to $\hat{P} := \pi(\hat{S})$. q.e.d.

4. Self-dual manifolds with two-torus action

(4.1) In the first few subsections we summarize basic facts about self-dual manifolds and the associated twistor spaces (cf. [1]). Let (M, g) be a compact connected oriented four-dimensional Riemannian manifold. (M, g) is called *self-dual* if the anti-self-dual part W_- of the Weyl conformal curvature tensor of (M, g) vanishes identitically. Since W_- depends only on the conformal class $[g]$ of g, the notion of self-duality is defined actually for the conformal manifold $(M, [g])$.

For any self-dual conformal manifold $(M, [g])$, one constructs naturally a compact complex manifold Z of dimension three, called the *twistor space* of $(M, [g])$, with the following properties.

1) Z admits a structure of a C^∞ fiber bundle $t : Z \to M$ over M.

2) Each fiber $L_x := t^{-1}(x),\ x \in M$, of t is a complex submanifold of Z and is isomorphic to a complex projective line $\boldsymbol{P}^1$. Moreover, its holomorphic normal bundle N_x in Z is isomorphic to the direct sum $O(1) \oplus O(1)$, where $O(1)$ is the line bundle of degree one on $\boldsymbol{P}^1$.

3) There exists an anti-holomorphic and fixed point free involution σ on Z which leaves each fiber L_x invariant. We call $t : Z \to M$ the *twistor fibration*, each L_x a *twistor line* and σ the *real structure* of Z.

(4.2) We need some information on the construction of Z from $(M, [g])$. Let T be an oriented real four-dimensional vector space with a positive definite inner product g. We denote by $C(T) = C(T, g, \iota)$ the set of isometric complex structures J on T which defines an orientation that is opposite to the given orientation ι of T. Namely, J is an endmorphism of T such that $J^2 = -\text{identity}$, $g(Ju, Jv) = g(u, v)$ for any $u, v \in T$ and for any complex basis e, f of the complex vector space (T, J), the resulting real basis e, Je, f, Jf of T defines the orientation opposite to the original one on T. $C(T)$ is naturally identified with a complex projective line and each twistor line L_x is naturally identified with the set $C(T_x) = C(T_x, g_x, \iota_x)$ for any representative g of $[g]$, where $T_x = T_xM$ is the tangent space of M at x.

To explain these statements we take an orientation preserving isometry $u : T \to \boldsymbol{H}$ of real vector spaces, where $\boldsymbol{H}$ is the space of real quaternions with the oriented orthonormal basis $1, i, j, k$. Denote by $Sp(1)$ the group of unit quaternions. Let

$$Sp(1)_+ \times Sp(1)_- \tag{20}$$

act on $\boldsymbol{H}$ by left and right multiplications;

$$h \to qhq'^{-1}, \quad h \in \boldsymbol{H}, \quad (q, q') \in Sp(1)_+ \times Sp(1)_- \tag{21}$$

where $Sp(1)_\pm$ are copies of $Sp(1)$. If we identity $\boldsymbol{H}$ with $\boldsymbol{R}^4$ naturally, this action gives a natural isomorphism of groups

$$(Sp(1)_+ \times Sp(1)_-)/\langle(-1, -1)\rangle \cong SO(4). \tag{22}$$

Now we consider the set

$$\begin{aligned} C : \ &= \ \{q \in Sp(1)_-; q^2 = -1\} \\ &= \ \{ai + bj + ck; a^2 + b^2 + c^2 = 1, a, b, c \in \boldsymbol{R}\} \\ &\cong \ S^2. \end{aligned} \tag{23}$$

Any element q of this set defines by right multiplications

$$h \to hq^{-1} \tag{24}$$

an isometric complex structure on $\boldsymbol{H}$ which defines an orientation that is opposite to the given one on $\boldsymbol{H}$, and in this way one identifies C with $C(\boldsymbol{H}) = C(T_x)$ naturally.

The identification of this set with a complex projective line is seen as follows. Denote by V_+ (resp. V_-) the complex two dimensional vector space $(\boldsymbol{H}, \times i)$ (resp. $(\boldsymbol{H}, i\times)$), where $\times i$ and $i\times$ denote respectively the complex structures given by the right and the left multiplications by $i \in \boldsymbol{H}$. Note that the natural action of $Sp(1)_\pm$ on $\boldsymbol{H}$ defines a complex representation of these groups on $V_\pm$ respectively.

Let $\pi : V_- - 0 \to C$ be defined by $\pi(h) = h^{-1}ih$. Then π gives a smooth fibration whose fiber over $h^{-1}ih$ is precisely the complex line $\boldsymbol{C}h$ of V_- with the origin deleted. Thus we obtain a natural identification

$$C = (V_- - 0)/\boldsymbol{C}^* =: \boldsymbol{P}(V_-) \tag{25}$$

and the map π is $Sp(1)_-$-equivariant if we let $Sp(1)_-$ act on C by $q \to gqg^{-1}, g \in Sp(1)_-$. Note that the latter action induces an action of $Sp(1)_-/\langle -1 \rangle \cong SO(3)$ on C, and therefore by (22), of $SO(4)$, where the action of $Sp(1)_+$-factor is trivial.

(4.3) Now for simplicity fix any Riemannian metric g in the given conformal class $[g]$ on M although the construction below actually depends only on the conformal class $[g]$. Let $\alpha : P \to M$ be the associated principal bundle of oriented orthonormal frames of M. Then Z is nothing but the fiber bundle over M with typical fiber $\boldsymbol{P}(V_-) = C$ associated to α via the above action of $SO(4)$ on $\boldsymbol{P}(V_-)$

$$Z := P \times_{SO(4)} \boldsymbol{P}(V_-) = P \times_{SO(4)} C.$$

In this way we get a natural interpretation of each fiber L_x as $C(T_x)$ above. Further, the left multiplication by j on $\boldsymbol{H}$ descends to an anti-holomorphic and fixed point free involution on $\boldsymbol{P}(V_-)$ which commutes with the action of $SO(4)$; therefore it globalizes and gives the desired real structure σ on Z.

The complex structure of Z is defined as follows. First of all, the Levi-Civita connection of g defines at any point z of Z a horizontal subspace H_z of the real tangent space T_z of Z, which is mapped isomorphically by the differential t_* onto the tangent space T_x, where $x = t(z)$. We have thus a natural direct sum decomposition

$$T_z = H_z \oplus T_z L_x = T_x \oplus T_z L_x \tag{26}$$

of real vector spaces. Let J_z be the complex structure on T_x corresponding to $z \in L_x = C(T_x)$ and J'_z the natural complex structure of $T_z L_x$ coming from the complex structure of L_x. Then the complex structure on Z is by definition the direct sum of J_z and J'_z with respect to (26) at each point of Z.

With respect to the identification $L_x = \boldsymbol{P}(V_-) = Sp(1)_-/S^1$ the normal bundle N_x of L_x in Z is described as the complex homogeneous vector bundle by

$$N_x = Sp(1)_- \times_{S^1} V_+,$$

where $S^1 \subseteq Sp(1)_-$ is identified with $\{a + bi; a^2 + b^2 = 1, a, b \in \boldsymbol{R}\}$ acting on $V_+ = (\boldsymbol{H}, \times i)$ by right multiplications. The map

$$Sp(1)_- \times_{S^1} V_+ \to V_+, \quad (g, v) \to gv \tag{27}$$

gives rise to a canonical C^∞ trivialization

$$N_x \cong C \times V_+. \tag{28}$$

(4.4) Let $(M, [g])$ be a compact connected self-dual manifold with the associated twistor fibration $t : Z \to M$. Suppose now that a real two-torus $K \cong S^1 \times S^1$ acts on $(M, [g])$ by conformal transformations. Then the K-action on M lifts naturally to a biholomorphic action on the twistor space Z, which in turn extends to a biholomorphic action of the complexfication $G \cong \boldsymbol{C}^* \times \boldsymbol{C}^*$ of K. The lifted K-action clearly sends a real twistor line onto a real twistor line, commutes with the real structure σ and is compatible with the direct sum decomposition (26) at various points of Z. G-action satisfies (7).

We shall use the notation of §2 for the underlying topological action of K on M. In what follows we assume that $\chi(M) \neq 0$ and take and fix once and for all a cycle B of two-spheres which is a connected component of the singular set $\Sigma(M)$ of the action.

(4.5) First we shall determine the singular set $\Sigma = \Sigma(B)$ of the K-action on Z along $t^{-1}(B)$, i.e., the set of points of $t^{-1}(B)$ with non-trivial stabilizer groups.

For this purpose we study for each point x of B the induced action on the twistor line L_x of the stabilizer group K_x at x.

It is convenient to distinguish two cases:

Case A. $x \neq x_i$ for any $i, 1 \leq i \leq k$, so that $x \in B_i'$ for some i. In this case K_x is the S^1-subgroup $K_i := K(m_i, n_i)$ of K.

Case B. $x = x_i$ for some $i, 1 \leq i \leq k$. In this case $K_x = K$.

First we show the following:

Lemma 4.1. *For any point $x \in B$ there exists on L_x exactly two fixed points of the stabilizer group K_x, which are σ-conjugate to each other. If $x \in B'$, the stabilizer group K_z at any other point z of L_x is trivial.*

Proof. There exists a K_x-invariant orthogonal decomposition of T_x into two inequivalent two-dimensional K_x-invariant subspaces

$$T_x = T_1 \oplus T_2. \tag{29}$$

In fact, in Case A we may take T_1 to be the maximal subspace on which K_x acts trivially, i.e., the tangent space of B_i at x, and T_2 to be its orthogonal complement, while in Case B, (29) is nothing but the isotropic representation of $K_x = K = K_i \times K_{i+1}$, where e.g., K_i acts faithfully on T_1 and trivially on T_2, so that T_2 (resp. T_1) is identified with the tangent space of B_i (resp. B_{i+1}) at x.

On the other hand, a point $z \in L_x$ is a fixed point of K_x if and only if the isotropic action of K_x on T_x commutes with the corresponding complex structure J_z on T_x. Thus, if z is a fixed point of K_x, then the complex structure J_z should preserve the direct sum decomposition (29). In fact, among $J_z,\ z \in L_x$, precisely two complex structures leave invariant the decomposition, and if one is denoted by $J = J_{z_0},\ z_0 \in L_x$, the other is given by $-J = J_{\sigma(z_0)}$. Hence, we have proved the existence of precisely two σ-conjugate fixed points on L_x. This proves the first assertion.

For the second assertion, we note that in Case A if K_z is nontrivial at some point of $z \in L_x$, the above argument still works with K_x replaced by K_z, so that J_z still must preserve the decomposition (29). Thus it must be either of $\pm J_{z_0}$ above and z must coincide with one of the fixed points of K_x. q.e.d.

Denote these two K_x-fixed points on L_x by $z^{\pm}$, and when $x = x_i$ we write them $z_i^{\pm}$. Since a point is fixed by K if and only if it is fixed by G, the above lemma immediately implies the following:

Corollary 4.2. *The fixed point set of G on $t^{-1}(B)$ consists of $2k$ isolated points $z_i^{\pm}, 1 \le i \le k$. The two points $z_i^{\pm}$ are σ-conjugate to each other. When M is simply connected, they exhaust all the fixed points of G on Z.*

From Lemma 4.1 we conclude:

Lemma 4.3. *There exist precisely two irreducible nonsingular rational curves $C_i^{\pm}$, $1 \le i \le k$, contained in Σ which are mapped diffeomorphically onto B_i by t. Moreover we have $C_i^+ \cap C_i^- = \emptyset$ and $\sigma(C_i^{\pm}) = C_i^{\mp}$.*

Proof. The fixed point set F_i of K_i on $t^{-1}(B_i)$ is a σ-invariant complex submanifold of Z. From Lemma 4.1 we see readily that it is a nonsingular curve and is an unramified double covering of B_i. Since B_i is simply connected, F_i is a disjoint union of two nonsingular rational curves $C_i^{\pm}$ which are σ-conjugate to each other. q.e.d.

We call twistor lines L_x over points x of B *special* and the twistor lines

$$L_i := t^{-1}(x_i),\ 1 \le i \le k$$

very special. We may assume that $z_i^{\pm} \in C_i^{\pm}$.

Proposition 4.4. *The singular set* $\Sigma(B)$ *of the* K*-action along* $t^{-1}(B)$ *is written as a union of nonsingular rational curves as*

$$\Sigma(B) = (\cup_{i=1}^{k} L_i) \bigcup (\cup_{i=1}^{k} (C_i^{+} \cup C_i^{-})).$$

Moreover, the stabilizer group of each point of $C_i'^{\pm} := C_i^{\pm} - \{z_{i-1}^{\pm}, z_i^{\pm}\}$ *coincides with* K_i. $\Sigma(B)$ *is connected.*

Proof. Since x_i is a fixed point, K acts naturally on L_i, but since the two-torus cannot act on a projective line effectively, L_i is contained in $\Sigma(B)$. The rest follows immediately from Lemmas 4.1 and 4.3. q.e.d.

Remark. The singular set of K-action on Z is thus a disjoint union of $\Sigma(B')$ for various components B' of $\Sigma(M)$. The singular set of G-action on Z is more difficult to identify because of the possible stabilizer group which is isomorphic to the additive group $\boldsymbol{C}$. It finally turns out that the singular set of the G-action also coincides with that of K ; our proof for this, however, will be indirect.

(4.6) We shall give some notational remark. Let C denote the union of $C_i^{\pm}, 1 \le i \le k$;

$$C = \bigcup_{i=1}^{k} (C_i^{+} \cup C_i^{-}).$$

We have two possibilities:

a) C is connected, or

b) C has exactly two connected components which are σ-conjugate to each other.

In Case b) the $\pm$-signs of $C_i^{\pm}$ can be chosen in such a way that

$$C_i^{\pm} \cap C_{i+1}^{\pm} = \{z_i^{\pm}\},\ 1 \le i \le k,\ k+1 = 1$$

and there exist no other intersections among any of the irreducible components of C. Thus there exist two mutually disjoint and σ-conjugate cycles of nonsingular rational curves $C_1^{\pm} \cup \cdots \cup C_k^{\pm}$ in C.

On the other hand, in Case a) the notation $\pm$ is not quite reasonable and it is more convenient to denote the curves $C_i^{\pm}$ by C_i and C_{i+k}, and the points $z_i^{\pm}$ by z_i and z_{i+k} in such a way that

$$C_i \cap C_{i+1} = \{z_i\},\ 1 \le i \le 2k,\ 2k+1 = 1$$

and there exist no other intersections among any two irreducible components of C. C itself forms a cycle of nonsingular rational curves.

However, when we want a universal notation which is valid in both cases we make the following identification of the components

$$C_i = C_i^+,\ C_{k+i} = C_i^-,\ z_i = z_i^+,\ z_{k+i} = z_i^-,\quad 1 \le i \le k.$$

For $k < i \le 2k$ it is sometimes convenient to put $L_i = L_{i-k}$ and $K_i = K_{i-k}$.

In these notations the intersections with the very special twistor lines are given by

$$L_i \cap C_i \cap C_{i+1} = \{z_i\},\quad 1 \le i \le 2k$$

in Case a), where the intersections are transversal. In Case b) for $i = k$ and $2k$ we need obvious modifications.

(4.7) With the above notational conventions $K_i, 1 \le i \le 2k$, is the stabilizer group in K of points of $C_i' := C_i - \{z_{i-1}, z_i\}$. Let $G_i (\cong \boldsymbol{C}^*)$ be the complexfication of $K_i (\cong S^1)$.

Lemma 4.5. *For any point z of C_i', $1 \le i \le 2k$, the stabilizer group G_z in G coincides with G_i.*

Proof. Clearly we have the inclusion $G_i \subseteq G_z$ with dim $G_z = 1$. If $G_z \neq G_i$, G_z/G_i is a non-trivial closed subgroup of $G/G_i \cong \boldsymbol{C}^*$. Since $G_z \cap K = K_i$, it is easy to show that G_z/G_i is an infinite group. Since G_z is discrete in G, G_z/G_i is an infinite cyclic subgroup of G/G_i. Then the $\boldsymbol{C}^*$-orbit $C_i' \cong G/G_z$ must be a complex torus of dimension one, which is absurd. q.e.d.

Let z be any point of C. We next compute the isotropic representation of the stabilizer group G_z on the holomorphic tangent space T_zZ of Z at z.

Proposition 4.6.

1) *Let $z \in C'_i, 1 \leq i \leq 2k$. Then with respect to suitable $\boldsymbol{C}$-linear coordinates u, v, w of T_zZ and an isomorphism $G_z \cong \boldsymbol{C}^*(t)$, G_z acts on T_zZ by*

$$(u, v, w) \to (u, tv, tw),$$

where the tangent space of C'_i at z is given by $v = w = 0$.

2) *Let $z = z_i$ for some $i, 1 \leq i \leq 2k$. Then with respect to suitable $\boldsymbol{C}$-linear coordinates u, v, w of T_zZ and an isomorphism $G_z = G \cong \boldsymbol{C}^*(s) \times \boldsymbol{C}^*(t)$, G_z acts on T_zZ by*

$$(u, v, w) \to (stu, sv, tw),$$

where the tangent space of L_i (resp. C_i, resp. C_{i+1}) at z is given by $v = w = 0$ (resp. $u = v = 0$, resp. $u = w = 0$).

Proof. By what we have explained in (4.4) we see that the complex isotropic representation of K_z on T_zZ is a direct sum of the two complex representations, one on (T_x, J_z) and the other on T_zL_x, where $x = t(z)$ and T_zL_x is the holomorphic tangent space of L_x.

A) First we normalize the representation of K_x on (T_x, J_z). The computation will be done by using a suitable model. Take an orientation preserving isometric $\boldsymbol{C}$-linear isomorphism

$$(T_x, J_z) \cong (\boldsymbol{H}, \times i) \tag{30}$$

which sends the decomposition (29) into $\boldsymbol{H} = \boldsymbol{C} \oplus j\boldsymbol{C}$, and we consider the corresponding action of K_x on the latter. Write an element of $\boldsymbol{H} = \boldsymbol{C} \oplus j\boldsymbol{C} \cong \boldsymbol{C}^2$ as

$$(z, w) = z + jw, \ z, w \in \boldsymbol{C}.$$

It is clear from (30) that in Case A, with respect to a suitable isomorphism $K_x \cong S^1(t)$, the action of K_x on $\boldsymbol{H}$ is given by

$$(z, w) \to (z, tw), \tag{31}$$

and in Case B with respect to the isomorphism $K_i \times K_{i+1} \cong K$ composed with suitable isomorphisms $K_i \cong S^1(s)$ and $K_{i+1} \cong S^1(t)$ the action of K on $\boldsymbol{H}$ is given by

$$(z, w) \to (sz, tw). \tag{32}$$

In both cases the stabilizer group K_x is realized as (a subgroup of) the distinguished maximal torus $T = S^1 \times S^1$ in $U(2) \subseteq SO(4)$.

B) In order to understand the action of K_x on C, it is convenient to consider K_x as the image of a subgroup of the distinguished subtorus $\widetilde{T} := S^1(u) \times S^1(v)$ of $Sp(1)_+ \times Sp(1)_-$ of (20), where S^1 is considered as the subgroup $\{a + bi \in Sp(1); a, b \in \boldsymbol{R}\}$ of $Sp(1)$. The action of $Sp(1)_+ \times Sp(1)_-$ on $\boldsymbol{H}$ is given by (21) and so the induced action of $(u, v) \in \widetilde{T}$ is given by

$$(z, w) \to (uv^{-1}z, (uv)^{-1}w). \tag{33}$$

In view of (31) and (32) this implies that the action of K_x is realized in Case A by the action of the subgroup $\{(u, u)\}$ of $\widetilde{T}$ with

$$t = u^{-2}, \tag{34}$$

and in Case B by the action of the whole group $\widetilde{T}$ via the double covering map $(u, v) \to (s, t) = (uv^{-1}, (uv)^{-1})$, giving the isomorphism $\widetilde{T}/\langle(-1, -1)\rangle \cong T = K$. In particular we get in this case

$$st = v^{-2}. \tag{35}$$

C) Now we detect the action of K_x on T_zL_x. By the isomorphism (30), the twistor line L_x, considered as a set of isometric complex structures on T_x, is mapped bijectively to the set C defined by (23), in which $z \in L_x$ is mapped to $-i \in C$, i.e., J_z corresponds to the right multiplication by i. (Recall that C operates on $\boldsymbol{H}$ by (24).)

We next identify the tangent space $T_{-i}C$ of C at $-i$ and the induced tangential representation of $\widetilde{T}$. Note first that by (25) the holomorphic tangent space of $C = \boldsymbol{P}(V_-)$ at $-i$ is identified with the set of $\boldsymbol{C}$-linear maps $\mathrm{Hom}(\boldsymbol{C}j, \boldsymbol{C})$ since $-i = j^{-1}ij$ and $(V_-/\boldsymbol{C}j) \cong \boldsymbol{C}i = \boldsymbol{C}$. Since the action of $Sp(1)_-$ on C is induced by the right multiplications on V_-, if $f \in \mathrm{Hom}(\boldsymbol{C}j, \boldsymbol{C})$, the action of $v \in S^1(v)$ on f is given by

$$f(wj) \to f(wjv)v^{-1} = v^{-1}f(v^{-1}wj) = v^{-2}f(h),\ wj \in \boldsymbol{C}j,$$

i.e., the action takes the form

$$f \to v^{-2}f.$$

Because of (34) and (35) we get that the action of K_x on $T_{-i}C \cong T_zL_x$ is by multiplication by t in Case A and by st in Case B (cf. (4.2)).

Combining this with (31) and (32) we get the assertion of the proposition after passing to the complexfication G_x of K_x. q.e.d.

(4.8) Example. When $M = S^4$ with standard conformal structure, the associated twistor space is the complex projective 3-space $\boldsymbol{P}^3(z_0 : z_1 : z_2 : z_3)$ and the twistor fibration $t : \boldsymbol{P}^3 \to \boldsymbol{P}^1\boldsymbol{H} = S^4$ is given by

$$(z_0 : z_1 : z_2 : z_3) \to (z_0 + jz_3 : z_1 + jz_2),$$

where S^4 is identified with the right quaternionic projective line $\boldsymbol{P}^1\boldsymbol{H}$. Introduce the complex affine coordinates (u, v) on $S^4 = \boldsymbol{C}^2 \cup \{\infty\}$ so that $(u + jv)(z_0 + jz_3) = z_1 + jz_2$. Define the K-action on $\boldsymbol{P}^3$ and on S^4 respectively by

$$(z_0 : z_1 : z_2 : z_3) \to (z_0 : sz_1 : tz_2 : stz_3) \quad \text{and} \quad (u, v) \to (su, tv)$$

for $(s, t) \in K = S^1 \times S^1$. Then we see readily that π is K-equivariant, in compatible with Proposition 4.6.

Remark. The restriction of t over $\boldsymbol{C}^2 \subseteq S^4$ is nothing but the twistor fibration for $\boldsymbol{C}^2 = \boldsymbol{R}^4$ with flat metric. This has the following general significance. For a general self-dual manifold M and any point $x \in M$ let N_x be the normal bundle of L_x in Z. Then the differential t_* induces a surjection $t_x : N_x \to T_x$ giving a linear isomorphism of each fiber of N_x onto T_x, which is identified with (27). This map t_x may be considered as giving a twistor fibration of T_x with the flat metric g_x, and moreover, as such, it is isomorphic to the above fibration $t^{-1}(\boldsymbol{R}^4) \to \boldsymbol{R}^4$. In this way we may elaborate on the above example so that it gives another proof of Proposition 4.6.

(4.9) By Lemma 4.5 for each i, $1 \leq i \leq 2k$, the stabilizer group G_i of G at points of C_i' is independent of the choice of the points, is a complexfication of K_i and is isomorphic to $\boldsymbol{C}^*$. In particular $G_i = G_{i+k}$.

However, in view of the structure of the isotropic representation in Proposition 4.6 we can specify for each i not only the subgroup G_i of G, but also a one parameter subgroup $\rho_i : \boldsymbol{C}^* \to G$ with image G_i, which amounts to specifying one of the two possible isomorphisms $\boldsymbol{C}^*(t) \to G_i$, by the following rule: For any point z in a sufficiently small neighborhood of a point of C_i we have

$$\lim_{t\to 0} \rho_i(t)z \in C_i. \tag{36}$$

Let N be the free abelian group of rank two of one parameter subgroups of G as in (2.2).

Lemma 4.7. *For any $i, 1 \leq i \leq k$, we have $\rho_i = -\rho_{i+k}$.*

Proof. Take any point x of B'_i and consider the induced action of G_i on the twistor line L_x. For any $z \in L'_x = L_x - \{z^{\pm}\}$ if $\lim_{t\to 0} \rho_i(t)z = z^{\pm}$, then $\lim_{t\to\infty} \rho_i(t)z = z^{\mp}$. From this the lemma follows readily. q.e.d.

5. Analyticity of orbit closures

(5.1) We study first the G-action in a neighborhood of C in Z. Let z be any point on C. Then there exist

1) a K_z-invariant neighborhood U of the origin o in the holomorphic tangent space $T_z = T_z Z$ of Z at z,

2) a K_z-invariant neighborhood V of z in Z, and

3) a K_z-equivariant isomorphism $\phi : U \to V$, namely,

$$\phi(gx) = g\phi(x), \quad g \in K_z, \quad x \in U \tag{37}$$

(cf. [12, Satz 4.4]). We may use linear coordinates u, v, w of T_z as local coordinates of Z at z via ϕ. This local linear description of the K_z-action extends partly to that of the G_z-action.

Lemma 5.1. *Let U_1 be a connected open neighborhood of o which is relatively compact in U and let $V_1 = \phi(U_1)$. Define an open subset A_0 of $G_z \times U_1$ by $A_0 := \{(g, x) \in G_z \times U_1; gx \in U\}$ and let A be the connected component of A_0 containing $K_z \times U_1$. Then for any element (g, x) of A, $g\phi(x)$ belongs to V and the equivariancy* (37) *holds true on A.*

Proof. Let A' be the connected component of the open subset $\{(g, x) \in A; g\phi(x) \in V\}$ of A containing $K_z \times U_1$. The two holomorphic maps $\phi(gx)$ and $g\phi(x)$ from A' to Z are defined and coincide identically on $K_z \times U_1$. Since G_z is a complexfication of K_z, they must also coincide identically also on A'. We show that A' is also closed in A. In fact, let $(g_0, x_0) \in A$ be on the boundary of A'. Take a sequence $(g_n, x_n) \in A'$ converging to (g_0, x_0). We know that $g_n\phi(x_n) = \phi(g_n x_n) \in U$. Then we have

$$g_0\phi(x_0) = \lim_{n\to\infty} g_n\phi(x_n) = \lim_{n\to\infty} \phi(g_n x_n) = \phi(g_0 x_0)$$

with limit taken in Z; since $g_0 x_0 \in U$, we have $g_0\phi(x_0) \in V$, i.e., $(g_0, x_0) \in A'$. Since A is connected, $A = A'$. q.e.d.

(5.2) Let z be any point of C. We describe a local action of G in a neighborhood of z.

We distinguish two cases:

Case A: $z \in C' := C - \{z_i\}$ and Case B: $z = z_i$ for some i.

In each case by using Proposition 4.6 and Lemma 5.1 we see that there exists a neighborhood W of z with the following properties.

Case A. Suppose that $z \in C_i' := C_i - \{z_{i-1}, z_i\}$ for some i. The corresponding one parameter group $\rho_i : \boldsymbol{C}^*(s) \to G$ has the image G_i, and G_i admits a complement $G' = \boldsymbol{C}^*(t)$ (e.g., $G' = G_{i-1}$ or G_{i+1}) and we obtain an isomorphism $G = G_i \times G' = \boldsymbol{C}^*(s) \times \boldsymbol{C}^*(t)$. There exists a coordinate neighborhood W of z in Z with local coordinates u, v, w such that

(38)
$$W = \{(u, v, w); |u - 1| < \varepsilon, |v| < 1, |w| < 1\}, \quad \varepsilon > 0, \quad z = (1, 0, 0),$$

$v = w = 0$ is a defining equation of C in W and the action of $G_i = G_i \times 1$ takes the form

$$(u, v, w) \to (u, sv, sw), \quad |s| \leq 1. \tag{39}$$

Moreover, the action of $t \in G'$ is of the form $(u, v, w) \to (tu, v, w)$ where $|tu - 1| < \varepsilon$.

Case B. $z = z_i$ for some i:

In this case the natural isomorphism

$$\rho_i \times \rho_{i+1} : \boldsymbol{C}^*(s) \times \boldsymbol{C}^*(t) \to G$$

gives natural coordinates of G. There exists a coordinate neighborhood

$$W = \{(u, v, w); |u| < 1, |v| < 1, |w| < 1\}, \tag{40}$$

of z in Z with local coordinates u, v, w around z such that 1) $u = v = 0$ and $u = w = 0$ are defining equations in W of C_i and C_{i+1} respectively, and $v = w = 0$ is one of L_i, and 2) the action of

$$(s, t) \in G \cong \boldsymbol{C}^*(s) \times \boldsymbol{C}^*(t)$$

takes the form

$$(u, v, w) \to (stu, sv, tw), \ |s|, |t| \leq 1. \tag{41}$$

We shall call any of the above neighborhoods W of points of C simply an *admissible* neighborhood.

(5.3) Let W be an admissible neighborhood and set

$$W' := W - (C \cup L),$$

where $L = \cup L_j$. Define subsets W'_{ab} and W''_{01} of W' as follows.

Case A. For any $(a, b) \neq (0, 0)$

$$W'_{ab} := \{(u, v, w) \in W'; av = bw\}.$$

Case B. For any (a, b) with $a \neq 0$

$$W'_{ab} := \{(u, v, w) \in W'; au = bvw\}$$

and

$$W'_{01} := \{(u, v, w) \in W'; v = 0\}, \quad \text{and} \quad W''_{01} := \{(u, v, w) \in W'; w = 0\}.$$

From the description of G-action on W in (5.2) the following lemma is clear.

Lemma 5.2. *The subsets defined above are all connected and closed submanifolds of W' and are contained in a single G-orbit. Moreover, the closure of W'_{ab} contains $C' \cap W$ in Case A.*

Lemma 5.3. *For any point $x \in W'$ the stabilizer G_x at x reduces to the identity.*

Proof. We may assume that x belongs to an admissible neighborhood W of Case A. (If x belongs to some W of Case B, we may find a suitable admissible neighborhood of Case A containing x and contained in W.)

If $g \in G_x$, g fixes any points of the G-orbit of x. In particular by Lemma 5.2 g fixes any point of the subset W'_{ab} containing x, and hence g also fixes the points of $C'_i \cap W$ which is contained in the closure W'_{ab}. However, by Lemma 4.5 the stabilizer group of a point of C'_i is G_i itself; so g must belong to G_i. On the other hand, as follows from (39) and (41), no element of G_i other than the identity e fixes any point of W'. Thus $g = e$. q.e.d.

By Lemma 5.3 the G-action defines a foliation on W' whose leaves we call *local leaves* on W'. Local leaves are precisely the subsets W'_{ab} and W''_{01} of W defined above before Lemma 5.2

Lemma 5.4. *The closure of each local leaf in W is an analytic submanifold of W.*

Proof. The closures are defined in W by the same equations and their locus are again nonsingular. q.e.d.

(5.4) We next consider the behavior of G-orbits on Z along C. Let W and $W' = W - (C \cup L)$ be as above. Let O be a G-orbit on Z which intersects with W'. Then $O \cap W'$ consists of a union of local leaves on W'. We shall show that actually $O \cap W'$ consists of a unique local leaf modulo finite number of exceptions. In this connection we shall call the two local leaves W'_{01} and W''_{01} *special* leaves in Case B.

Lemma 5.5. *Let F and F' be distinct local leaves in W'. Suppose that neither F nor F' is special in Case B. Then they are never contained in the same G-orbit.*

Proof. Suppose that they are contained in one and the same G-orbit. We use the notation of (5.3) and consider the two cases separately.

Case A. Let $z_0 := (1, 0, 0) \in C$ be the center of the admissible neighborhood. There exist points $z \in F$ and $z' \in F'$ with the same u-coordinate 1. By assumption there exists an element $g \in G$ with $gz = z'$. Then we have

$$gsz = sgz = sz' \tag{42}$$

for any $s = (s, 1) \in G_i \subseteq G$, and then letting s tend to zero, we have

$$gz_0 = z_0. \tag{43}$$

This implies that $g \in G_i$ by Lemma 4.5. This, however, is impossible because any element s of G_i with $|s| \leq 1$ preserves each local leaf. (If $g = (s_g, 1)$ satisfies $|s_g| > 1$, consider g^{-1} instead, interchanging the roles of z and z'.)

Case B. The proof is same as in Case A. Since neither F nor F' is special, we can find two points $z \in F$ and $z' \in F'$ with the same w-coordinate, $w_0 \neq 0$ say. Take an element $g \in G$ with $gz = z'$. Then again we have (42) and then have (43) with $z_0 = (0, 0, w_0)$, which implies that $g \in G_i$. This is, however, impossible exactly by the same reasoning as above. q.e.d.

We call a G-orbit O on Z *special* if O contains a special local leaf at some z_i. Clearly special orbits are finite in number ($\leq 4k$). Now we take and fix a small neighborhood U of C in Z which is covered by admissible neighborhoods. We set $U' = U - (C \cup L)$.

Corollary 5.6. *For any non-special G-orbit O in Z with $O \cap U' \neq \emptyset$, the closure of O in U is analytic and smooth.*

Proof. By Lemma 5.5 $O \cap W'$ is a single local leaf in W' and the closure of the latter in W is analytic and smooth by Lemma 5.4. q.e.d.

Remark. Actually, with a little more preparation we can show that the above lemma, and hence the corollary also, is actually true without exceptions by the same method.

(5.5) We turn to considering the closures of G-orbits in the whole Z. We first prove the following:

Lemma 5.7. *Any non-special G-orbit O on Z with $O \cap U' \neq \emptyset$ intersects with any admissible neighborhood of points of C.*

We use the following:

Lemma 5.8. *Let z be any point of $C_i' := C_i - \{z_{i-1}, z_i\}$ (resp. $L_i' := L_i - \{z_i, z_{i+k}\}$) and W an admissible neighborhood of z_{i-1} or z_i (resp. z_i^+ or z_i^-). Then there exist a neighborhood V of z in Z and an element g of G such that $g(V) \subseteq W$.*

Proof. First we consider the case $z \in C_i'$. Take an admissible neighborhood of z in Z with coordinates u, v, w as in (5.2). Then the equation $u = 0$ defines a G_z-invariant 2-dimensional local slice D for the local G-action at z. We consider the induced action of G'

$$G' \times D \to Z$$

which is locally biholomorphic onto an open neighborhood of $C_i' = G'z$ in Z. Since $C_i' \cap W \neq \emptyset$, the lemma follows.

The proof for L_i' is similar. In this case its stabilizer group H_i is the image of the one parameter group $\rho_i \rho_{i+1}^{-1} : \boldsymbol{C}^*(s) \to G$ and it admits a complement $G' = \boldsymbol{C}^*(t)$ (e.g., $= G_i$ or G_{i+k}) so that we have an isomorphism $G \cong H_i \times G' = \boldsymbol{C}^*(s) \times \boldsymbol{C}^*(t)$. There exists a coordinate neighborhood W of z in Z with local coordinates u, v, w around z as in (38) such that $v = w = 0$ is a defining equation of L_i in W, where the action of H_i takes the form

$$(u, v, w) \to (u, sv, s^{-1}w), \quad |s| \leq 1. \tag{44}$$

Moreover, for $t \in G'$ the action is of the form $(u, v, w) \to (tu, v, w)$ where $|tu - 1| < \varepsilon$. So the same argument applies also in this case.

q.e.d.

Proof of Lemma 5.7. Let W_1 be any admissible neighborhood and z_1 an arbitrary point of $C \cap W_1$. Then any non-special local leaf in W_1 intersects with any neighborhood of z_1, as follows from the description of local leaves.

Now let W be an arbitrary admissible neighborhood of a point z of C. Take an admissible neighborhood W_0 of a point z_0 of C which intersects with O. Suppose that $z_0 \in C_i$ and $z \in C_j$. Since $\Sigma = C \cup L$ is connected by Proposition 4.4, we have a finite chain $\{Y_1, \dots, Y_n\}$ of irreducible components of Σ such that $Y_i \cap Y_{i+1} = \{y_i\} \neq \emptyset$ for $1 \leq i \leq n-1$, $Y_1 = C_i$ and $Y_n = C_j$. Then since O is nonspecial, in view of the first remark, by applying succesively the previous lemma we can transform a point of W_0 by elements of G into a point of W. This shows that O also intersects with W. q.e.d.

(5.6) Let O be any non-special G-orbit with $O \cap U' \neq \emptyset$. By Corollary 5.6 the closure $\hat{O}$ of $O \cap U'$ in U is analytic and smooth. We next consider the closure of O in the whole Z. Our basic observation is that $\hat{O}$ looks like a neighborhood of the anti-canonical cycle of a compact smooth toric surface. We shall determine this toric surface first. For each i, $1 \leq i \leq 2k$, we specified in (4.9) a one parameter subgroup $\rho_i : \boldsymbol{C}^*(t) \to G$ determined by the following two conditions

1) $\rho_i(t)$ fixes any element of C_i, and

2) for any sufficiently small neighborhood V_i of C_i in Z we have

$$\lim_{t \to 0} \rho_i(t) z \in C_i, \quad z \in V_i.$$

Also recall by Lemma 4.7 that we have $\rho_{i+k} = -\rho_i$ as elements of N.

Lemma 5.9. *For any element ρ of N which is in the open simplex spanned by ρ_i and ρ_{i+1} the following holds true: Let z be any point of an admissible neighborhood of the point z_i. Then we have $\lim_{t \to 0} \rho(t) z = z_i$. Here the range of indices are $1 \leq i \leq 2k$, $2k+1 = 1$, in* Case a) *of* (4.6) *and $1 \leq i \leq k-1, k+1 \leq i \leq 2k-1$, in* Case b) *of* (4.6) *respectively. Moreover, in* Case b) *the same statment holds true if we replace in the above statements (ρ_i, ρ_{i+1}) and $z_i = C_i \cap C_{i+1}$ by (ρ_k, ρ_1) and $z_k = C_k \cap C_1$ respectively.*

Proof. We only consider the first statement, the supplementary case being treated in the same way. By assumption ρ is written as $\rho = m\rho_i + n\rho_{i+1}$ for some positive integers m and n. By Proposition 4.6, in an admissible neighborhood of z_i, with respect to the isomorphism $\rho_i \times \rho_{i+1} : \boldsymbol{C}^*(t_1) \times \boldsymbol{C}^*(t_2) \to G$ the action of G takes the form $(u, v, w) = (t_1 t_2 u, t_1 v, t_2 w)$, and therefore the induced action of ρ takes the form $\rho(t) = (t^{m+n}x, t^m y, t^n z)$. The assertion is clear from this. q.e.d.

The following lemma is crucial for the construction of our toric surface.

Lemma 5.10. *For* $1 \leq i \leq 2k,\ 2k+1 = 1$, *in* Case a) (*resp. for* $1 \leq i \leq k-1, k+1 \leq i \leq 2k-1$ *in* Case b)) *of* (4.6), *there exists no* $\rho_j, j \neq i, i+1$, *in the closed simplex spanned by* ρ_i *and* ρ_{i+1}. *In* Case b) *the same statement holds true if we replace* (ρ_i, ρ_{i+1}) *and* $j \neq i, i+1$ *in the above statements by* (ρ_k, ρ_1) *and* $j \neq k, 1$ *respectively.*

Proof. We only show the first statement. Suppose that some $\rho_j, j \neq i, i+1$, is between ρ_i and ρ_{i+1}. Take a G-orbit O with $O \cap U' \neq \emptyset$ which is not special. By Lemma 5.7 we can find a point z (resp. y) of $O \cap U'$ which is in an admissible neighborhood of a general point of C'_j (resp. in an admissible neighborhood W of z_i). We can find an element $g \in G$ which maps z to a point y.

Suppose first that $\rho_j \neq \rho_i, \rho_{i+1}$. By Lemma 5.9 our assumption implies that $\lim_{t\to 0} \rho_j(t)y = z_i$, while we have $z_0 := \lim_{t\to 0} \rho_j(t)z \in C'_j$. Since G is commutative, we thus get

$$z_i = \lim_{t\to 0} \rho_j(t)y = \lim_{t\to 0} \rho_j(t)gz = g \lim_{t\to 0} \rho_j(t)z = gz_0.$$

But this is impossible since z_i is a fixed point and $z_0 \in C'_j$.

Suppose next that $\rho_j = \rho_i$, say. In the above proof we have only to replace the equality $\lim_{t\to 0} \rho_j(t)y = z_i$ by the following one

$$v := \lim_{t\to 0} \rho_j(t)y = \lim_{t\to 0} \rho_i(t)y \in C'_i,$$

where the derivation of the contradiction is similar, noting the assumption that $j \neq i, i+1$ and the fact that C'_i and C'_j are different G-orbits. q.e.d.

As an important conclusion of the lemma we obtain

Corollary 5.11. *C is connected.*

Proof. Suppose that C is not connected, i.e., we are in Case b). The lemma says that there exists no other ρ_j's between ρ_i and ρ_{i+1} for any $1 \leq i \leq k, k+1 = 1$. This would lead to the non-existence of $\rho_{k+j}, 1 \leq j \leq k$, which is absurd. q.e.d.

Remark. A. Huckleberry pointed out that once Lemma 5.3 is proved, then the connectedness of C and the analyticity of the closure of the general orbits follows by topological argument using the fact that G has only one end. This could certainly simplify the exposition above and below.

(5.7) By Lemma 5.10 we conclude that

$$\rho_1, \dots, \rho_{2k}, \ \rho_{i+k} = -\rho_i,$$

are arranged on N couterclockwise or clockwise in this order around the origin. To fix an idea let us assume that they are arranged couterclockwise, reversing the numbering if necessary. If we fix an identification $K = S^1 \times S^1$, we have the corresponding identifications $G = \boldsymbol{C}^* \times \boldsymbol{C}^*$ and $N = \boldsymbol{Z}^2$.

Lemma 5.12. *Let $\{\pm(m_i, n_i)\}$ be the Orlik-Raymond invariant of the underlying K-action on M, and $(k_i, l_i) \in \boldsymbol{Z}^2$ the coordinates of ρ_i. Then these satisfy the following equalities for any i; $(k_i, l_i) = \pm(-n_i, m_i)$, $(-l_{i+k}, k_{i+k}) = -(-l_i, k_i)$ and*

$$\begin{vmatrix} k_i & l_i \\ k_{i+1} & l_{i+1} \end{vmatrix} = 1. \tag{45}$$

Proof. The assertions follows readily from the fact that the image G_i of $\rho_i : \boldsymbol{C}^* \to G$ is the complexfication of $K_i = K(m_i, n_i)$, noting the dual description of K_i and G_i via characters and one parameter groups respectively. q.e.d.

Clearly, the one parameter groups $\rho_i, \ 1 \leq i \leq 2k$, define a complete fan $\triangle(\rho)$ on N and we get a (compact smooth) symmetric toric surface

$$S = S(\rho) \tag{46}$$

with the anti-canonical cycle $D = D_1 \cup \cdots \cup D_{2k}, D_i$ corresponding to ρ_i. We identify the open orbit of S with G once and for all (by fixing a point of the open orbit) and consider S as a distinguished compactification of $G = \boldsymbol{C}^* \times \boldsymbol{C}^*$ associated to our G-action on Z.

Lemma 5.13. *Let O be a non-special G-orbit with $O \cap U' \neq \emptyset$. Then its closure $\overline{O}$ in Z is analytic and smooth. $\overline{O}$ is obtained by adding C to O, is G-equivariantly isomorphic to the toric surface $S = S(\rho)$ of (46), and has C as the anti-canonical cycle. In particular such closures are all isomorphic to each other.*

Proof. Let $F = O \cap U$ and $N = \overline{O} - O$. Clearly $O = GF$. Take any point o of N and a sequence $\{x_n\}$ of points of O which converges to o. Fix any point $z \in F$ and write $z_n = g_n z$ for some $g_n \in G$ which is unique by Lemma 5.3. Then after passing to a subsequence if necessary, we can assume that g_n converge to a point $g_0 \in S$ with respect to the compactification $G \subseteq S$ above. If $g_0 \in G$, then $o = \lim x_n = \lim g_n z = g_0 z$ and o is in the orbit O, which contradicts our choice of o.

Thus $g_0 \notin G$ so that $g_0 \in D_i \subseteq S$ for some i. Suppose first that g_0 is not a G-fixed point. With respect to the isomorphism

$$\rho_i \times \rho_{i+1} : \boldsymbol{C}^*(s) \times \boldsymbol{C}^*(t) \cong G,$$

g_0 admits an affine neighborhood in S of the form $\boldsymbol{C}(s) \times \boldsymbol{C}^*(t)$ containing G, where $s = 0$ is a defining equation of D_i. In this neighborhood we may identify g_0 with a point $(0, t_0)$ with $t_0 \neq 0$ and g_n takes the form (s_n, t_n), where $s_n \to 0$ and $t_n \to t_0$ when $n \to \infty$. Thus we may write $g_n = t_n s_n$ with $s_n = (s_n, 1) \in G$ and $t_n = (1, t_n)$ by abuse of notations.

Now coming back to the action on Z, take an admissible neighborhood W of a point of C_i'. Since $z \in F \subseteq O$ and $F \cap W \neq \emptyset$ by Lemma 5.7, there exists an element h of G such that $hz \in W$. Then we have

$$g_n z = t_n s_n h^{-1}(hz) = h^{-1}(t_n s_n hz). \tag{47}$$

Note that $s_n hz$ converge to a point, say b, on C_i. Hence we have

$$o = \lim g_n z = h^{-1} \lim t_n (s_n hz) = h^{-1} t_0 b \in C_i' \tag{48}$$

since C_i' is G-invariant.

Next suppose that g_0 is a fixed point so that $g_0 = D_i \cap D_{i+1}$ for some i. With respect to the isomorphism $\rho_i \times \rho_{i+1} : \boldsymbol{C}^*(s) \times \boldsymbol{C}^*(t) \cong G$, then g_0 admits an affine neighborhood in S of the form $\boldsymbol{C}(s) \times \boldsymbol{C}(t)$ containing G, where $st = 0$ is a defining equation of $D_i \cup D_{i+1}$ and g_0 is identified with the origin $(0, 0)$. Further, g_n takes the form (s_n, t_n) with $s_n, t_n \to 0$ when $n \to \infty$. Write $g_n = t_n s_n$ as before.

Now coming back again to the action on Z take an admissible neighborhood W of z_i. As in the previous case we may find an element h

of G such that $hz \in W$. Then we again have (47), where in this case $t_n s_n hz$ converge to z_i. Hence as in (48) we obtain

$$o = \lim_{n\to\infty} g_n z = h^{-1} t_0 z_i = z_i \tag{49}$$

since z_i is fixed by G.

By (48) and (49) we get that $N \subseteq C$ and by Corollary 5.6 the closure $\overline{O}$ of O in Z is analytic and smooth since $\overline{O} = [O \cap (Z - U)] \cup \hat{O}$, where $\hat{O}$ is the closure of $O \cap U'$ in U. Note also that since F intersecs with any admissible neighborhood we actually have $N = C$.

Finally, the compact nonsingular G-surface $\overline{O}$ is a toric surface with all the data of the boundary divisors same with those of S, D_i corresponding to C_i with the same one parameter group ρ_i. This implies that $\overline{O}$ is isomorphic to $S = S(\rho)$ as a toric surface. q.e.d.

Since the special orbits are finite in number, summarizing what we have obtained we get the following:

Proposition 5.14. *There exist uncountably many G-orbits on Z such that their closures in Z are analytic and smooth. Moreover, they are projecitve nonsingular toric surfaces which are isomorphic to $S(\rho)$ and which contain C as the anti-canonical cycles. In particular, the closures are obtained by adding C to the G-orbits.*

So far we have fixed throughout a connected component B of the singular set $\Sigma(M)$ of the K-action on M and have been dealing exclusively with the G-action along $C \subseteq t^{-1}(B)$. However, we shall see in the next section that B is actually the unique connected component of $\Sigma(M)$.

6. Associated fiber space and invariants

(6.1) In order to show that all the orbit closures are analytic, instead of studying the individual orbit structure more in detail, we apply the method of meromorphic quotient. In fact for the application of the latter, Proposition 5.14 is sufficient. To explain this we begin with the following general setting.

Let Z be a compact connected complex manifold of dimension n. Then there exists a natural structure of a reduced complex space on the set $\mathcal{D}$ of effective divisors on Z [5]. The following property of this space proved in [8] is important for our purpose.

Lemma 6.1. *Any connected component of $\mathcal{D}$ is compact and projective.*

We note that the essential ingredient of the proof is the result of [6] to the effect that, up to finite number of exceptions, every irreducible divisor E on Z is of the form $h^*\overline{E}$, where $h : Z \to Y$ is an algebraic reduction of Z and $\overline{E}$ is a divisor on the projective variety Y. See [8] for the detail.

(6.2) Suppose now that a connected complex Lie group G acts biholomorphically on Z such that general orbits are of codimension one, namely there exists a nonempty Zariski open subset U of Z such that every G-orbit of a point of U is of codimension one. Let $\mathcal{D}^G$ be the fixed point set of the induced G-action on $\mathcal{D}$. For any $d \in \mathcal{D}$ let Y_d be the corresponding effective divisor on Z. Denote by $\mathcal{D}_0^G$ the union of those irreducible components F_α of $\mathcal{D}^G$ such that for a general point d of F_α the corresponding divisor Y_d is reduced and irreducible. Then take an irreducible component F of $\mathcal{D}_0^G$ of positive dimension (if any). F is projective by Lemma 6.1. We consider the universal family of divisors on Z parametrized by F:

$$\begin{tikzcd} Y \arrow[d, "p"'] & \subseteq & F \times Z \arrow[dll, "p_F"] \\ F & & \end{tikzcd} \tag{50}$$

Lemma 6.2. *An irreducible component F of $\mathcal{D}_0^G$ as above is unique* (*if any*) *and is of dimension one. The natural projection $q : Y \to Z$ is bimeromorphic.*

Proof. Y is compact of dimension $n - 1 + \dim F$ and the natural projection $q : Y \to Z$ is surjective. For every $d \in F$, $p^{-1}(d)$ is identified with the corresponding divisor Y_d of Z and is G-invariant. There exists a nonempty Zariski open subset V of F such that for any $d \in V$, Y_d is reduced and irreducible and $Y_d \cap U \neq \emptyset$, and hence, Y_d is a closure of a G-orbit. Since different orbits do not intersect each other, $q^{-1}(U) \cap p^{-1}(V) \to Z$ is injective; it follows that q is bimeromorphic and F is of dimension one. Also by the construction we see that F is the unique irreducible component of $\mathcal{D}_0^G$ of positive dimension. q.e.d.

We call the commutative diagram

$$
\begin{tikzcd}
Y \arrow[r, "q"] \arrow[d, "p"'] & Z \arrow[dl, "pq^{-1}"] \\
F &
\end{tikzcd}
\tag{51}
$$

or simply, the meromorphic map $pq^{-1} : Z \to F$, *the meromorphic quotient* of Z by G, which is canonically associated with the given action of G on Z. In this case Y is nothing but the graph of the meromorphic map pq^{-1} and there exists a natural G-action on the diagram (51), where the action on F is trivial.

(6.3) On the existence of the meromorphic quotient we have the following criterion:

Proposition 6.3. *Let Z be a compact connected complex manifold. Suppose that a connected complex Lie group G acts effectively on Z such that its general orbits are of codimension one and that the closures of uncountably many orbits of codimension one are analytic. Then the meromorphic quotient* (51) *of Z by G exists. Moreover, every orbit of codimension one has an analytic closure in Z and is identified with an irreducible component of a fiber of p.*

Proof. Let $O_t, t \in T$, be a family of G-orbits of codimension one whose closure $\overline{O}_t$ in Z is analytic, where the index set T is uncountable. Let d_t be the point of $\mathcal{D}$ corresponding to the irreducible divisor $\overline{O}_t$ of Z. Since $\overline{O}_t$ is G-invariant, d_t is a point of $\mathcal{D}_0^G$. Moreover, since $\mathcal{D}_0^G$ as well as $\mathcal{D}$ has at most countably many irreducible components, there must exist an irreducible component F of $\mathcal{D}_0^G$ containing infinitely many d_t's. This F must be of positive dimension and the meromorphic quotient exists.

Since Z is nonsingular, there exists an analytic subset A of Z of codimension ≥ 2 such that q is isomorphic over $q^{-1}(Z - A)$. Then by the equivariancy of the diagram (51) every one-codimensional orbit on Z is uniquely lifted to one on Y, which then should be mapped to a point of F. Thus the latter is Zariski open in an irreducible component of a fiber of f; in particular its closure is analytic, and then, as its biholomorphic image in Z, the closure of the original G-orbit in Z also is analytic. q.e.d.

(6.4) We now apply the above consideration to our G-action on the twistor space Z, where G is an algebraic two-torus.

Proposition 6.4. *Let Z be the twistor space with G-action as in* (4.4).

1) *There exists the meromorphic quotient of Z by G :*

$$\begin{array}{ccc} \hat{Z} & \stackrel{\mu}{\rightarrow} & Z \\ f \ \downarrow & & \\ P & & \end{array} \tag{52}$$

where P is a nonsingular rational curve.

2) *Every two-dimensional orbit has an analytic closure in Z and is identified with an irreducible component of a fiber of f.*

3) *The action of σ on Z naturally lifts to one on the diagram* (52). *In particular, P admits a naturally induced real structure.*

Proof. In our case the general orbit has codimension one and the assumption of Proposition 6.3 is fulfilled by Proposition 5.14. Therefore by that proposition we obtain 1) and 2) except the identification of P.

Since a general twistor line is mapped holomorphically and surjectively onto P, P is rational. This implies that for general $a \in P$ the divisors $S_a := \mu(f^{-1}(a))$ are all linearly equivalent. Let Q be the complete linear system containing such S_a's. Then G acts (projective) linearly on Q and it is easily seen that P coincides with a connected component of the fixed point set of this action. Thus it must be a complex projective line in Q since G is an algebraic torus.

The anti-holomorphic involution on $\mathcal{D}$ induced by σ preserves P because of (7). 3) follows. q.e.d.

(6.5) We now study the structure of the diagram (52) more in detail. Note first that the meromorphic map $f\mu^{-1} : Z \to P$ is obtained by the linear pencil V whose members are $S_a := \mu(\hat{Z}_a), a \in P$, where $\hat{Z}_a := f^{-1}(a)$. (For simplicity of notations, however, in what follows we often identify S_a with $\hat{Z}_a$ by μ and use the two notations interchangeably.)

Proposition 6.5.

1) *μ is obtained by the blowing-up of Z with center C. $E := \mu^{-1}(C)$ is isomorphic to $C \times P$ and $f|E$ is identified with the projection to the second factor.*

2) *General members of V are smooth and every smooth member is a projecitve toric surface isomorphic to $S(\rho)$ of* (46) *and has C as its anti-canonical cycle.*

Proof. By Proposition 5.14 there exist uncountably many members of the system V which are closures of G-orbits obtained by adding C to the orbits, and are smooth projecitve toric surfaces isomorphic to $S(\rho)$. Now the meromorphic map $f\mu^{-1}$ is defined by the pencil V and $\hat{Z} \subseteq Z \times P$ is its graph. Thus μ is obtained by the blowing up of the intersection of two general members S_1 and S_2 of this system. In view of the above remark we may take them to be smooth orbit closures containing C and intersecting each other only along C.

Locally their intersection property can be read off in each admissible neighborhood W of a point of C; in Case A the intersections of S_i with W are given by smooth hyperplanes $au = bv$, $(a, b) \neq (0, 0)$ and S_1 and S_2 intersect transversally along $C \cap W$; on the other hand, in Case B the intersections of S_i with W are given by equations of the form $auv = bw, a \neq 0$. Hence we conclude that the base locus of the pencil V is precisely the curve C with multiplicity one, and therefore that μ coincides with the blowing up of Z with center C. The assertion about the structure of E is then clear. In particular every fiber of f contains C naturally; for $a \in P, \hat{Z}_a \cap E = a \times C = C$. This proves 1).

On the other hand, up to finite exceptions every fiber is smooth and it is obtained as a deformations of (projecitve) toric surfaces. Since it is known and easy to show that a deformation of a rational surface is again a rational surface, every smooth fiber of f must be a projecitve toric surface in view of its admitting G-action with open orbit. As it contains C in the complement of the open orbit as a connected component, C must coincide with their anti-canonical cycle. Since the data for the isotropic one parameter subgroups along each component of C, which determines a toric surface, is invariant under deformations, they are indeed isomorphic to each other. q.e.d.

(6.6) On the singularities of $\hat{Z}$ we prove:

Lemma 6.6. *$\hat{Z}$ has $2k$ ordinary double points $r_i, 1 \leq i \leq 2k$, such that $\mu(r_i) = z_i, 1 \leq i \leq 2k$, and $\sigma(r_i) = r_{i+k}$, $1 \leq i \leq k$. $\hat{Z}$ is smooth outside these points. Moreover, $f(r_i) = f(r_{i+k})$ for any i.*

Proof. By Proposition 6.5 μ is the blowing up with center C. Thus $\hat{Z}$ is smooth over the smooth points of C. At the singular points z_i of C, the defining equation of C in an admissible neighborhood W of (40) is of the form $w = uv = 0$. Thus $\hat{Z}$ is described in a neighborhood of $\mu^{-1}(z_i)$ as a hypersurface $w\xi_1 = uv\xi_0$ in $W \times \boldsymbol{P}^1$, with the unique singular point $r_i := \{(0, 0), (1 : 0)\}$, where ξ_i are homogeneous coordinates of

$\boldsymbol{P}^1$. These ordinary double points r_i are thus the only singular points of $\hat{Z}$.

The twistor line L_i is defined in W by $u = v = 0$, and therefore its proper transform $\hat{L}_i$ in $\hat{Z}$ passes through r_i; similarly, it passes through r_{i+k}. Since $\hat{L}_i$ is an orbit closure, it must be mapped to a point by f. Therefore

$$f(r_i) = f(L_i) = f(r_{i+k}), 1 \leq i \leq k.$$

Moreover, the relation $\sigma(z_i) = z_{i+k}$ lifts to $\hat{Z}$ to give $\sigma(r_i) = r_{i+k}$. q.e.d.

Let

$$a_i := f(r_i) = f(r_{i+k}),\ 1 \leq i \leq k.$$

In view of the σ-equivariancy of f and the above lemma, a_i must be a real point of P. In particular σ has a fixed point on P. The fixed point set R is then diffeomorphic to S^1 and $P - R$ consists of two connected components. We may choose a suitable inhomogeneous coordinate z of P so that

$$P = \boldsymbol{C}(z) \cup \{\infty\} \tag{53}$$

and σ takes the form $z \to 1/\overline{z}$ in this coordinate. Then R is given by $|z| = 1$ and $a_i \in R$.

(6.7) Recall that every effective divisor S on Z has a positive intersection number $S \cdot L$ with any twistor line L. We call this number the *degree* of S. S is called an *elementary divisor* if its degree is one, and a *quadratic divisor* when it is two.

Lemma 6.7. *Let S be any member of the pencil V. Then S is a quadratic divisor on Z in the sense defined above.*

Proof. Take any smooth member S of V. We show that $S \cdot L_i = 2$. Note that $L_i - \{z_i, z_{i+k}\}$ is a one-dimensional G-orbit, while one dimensional orbits in S are one of C'_j, $1 \leq j \leq 2k$. Therefore S intersects L_i only at two points z_i and z_{i+k}. In an admissible neighborhood (40) of either of these two points, a defining equation of L_i is $u = v = 0$ and one of S is $w = cuv$ for some nonzero c. Thus the intersection of L_i and S is transversal there. Hence we get $L_i \cdot S = 2$. q.e.d.

Since V contains a real member, it contains also a real smooth member S_a for a general point a of R. Thus by a theorem of Pedersen-Poon [19, Theorem 1] we have the following

Corollary 6.8. *M is diffeomorphic to $m\boldsymbol{P}^2$, where m is the second betti number of M. In particular M is simply connected.*

Proposition 6.9. *Z is Moishezon, i.e., bimeromorphic to a complex projective manifold.*

Proof. Let $r : \widetilde{Z} \to \hat{Z}$ be any resolution of $\hat{Z}$. Consider the resulting holomorphic map $\widetilde{f} : \widetilde{Z} \to P$. It suffices to show that $\widetilde{Z}$ is Moishezon. Any smooth fiber $\widetilde{Z}_a$ is isomorphic to the toric surface $S = S(\rho)$; in particular its anti-Kodaira dimension $\kappa^{-1}(\widetilde{Z}_a) = 2$ by (1). From this follows the Moishezon property of $\hat{Z}$ immediately by applying [27, Prop. 12.2] to the case where D is the anti-canonical divisor of Z in the notation there. q.e.d.

Let $-\frac{1}{2}K$ be the canonical square-root of the anti-canonical line bundle $-K = -K_Z$ of Z defined on any twistor space (cf. [10]).

Proposition 6.10. *The linear system V coincides with the unique positive dimensional linear subsystem $|-\frac{1}{2}K|^G$ of the half-anti-canonical system $|-\frac{1}{2}K|$ consisting only of G-invariant elements.*

Proof. Since both V and $-\frac{1}{2}K$ are real, and $-\frac{1}{2}K \cdot L = 2$, Lemma 6.7 shows that $-\frac{1}{2}K$ and the line bundle $A := [S_a]$ determined by the members of V coincide in the chern class level (cf. [10, §3]). On the other hand, we have $2 \dim H^1(O_Z) = b_1(Z)$ for the Moishezon manifold Z, where b_1 denotes the first betti number. Thus, in view of $b_1(Z) = b_1(M)$ Corollary 6.8 gives $H^1(O_Z) = 0$. It follows that $-\frac{1}{2}K$ and A are isomorphic as line bundles.

The members of V are clearly G-invariant members of $|-\frac{1}{2}K|$. On the other hand, by Proposition 6.4 G-invariant divisors which are not members of V are identified with union of some of the irreducible components of fibers of f and they are discrete. From this the final assertion follows. q.e.d.

(6.8) We shall next detect the singular fibers of f precisely. We use the results of Poon [20] and Pedersen-Poon [19], which we quote as:

Lemma 6.11.

1) ([19, Lemma 2.1]) *Any real and irreducible quadratic divisor on Z is smooth.*

2) ([20, Lemma 1.9]) *Every elementary divisor S on Z is smooth and contains a unique twistor line L. Moreover, if we set $\overline{S} := \sigma(S)$,*

$\overline{S}$ is another elementary divisor in Z such that $S \cap \overline{S} = L$, where the intersection is transversal.

Following our notational convention (cf. (6.5)) we denote the fiber $\hat{Z}_a$ of f by S_a in what follows. Similarly, L_i also denotes its proper transform in $\hat{Z}$.

Proposition 6.12. *The fiber S_a, $a \in P$, of f is smooth if and only if $a \neq a_i$ for any $i, 1 \leq i \leq k$. The points a_i are all distinct and $S_i := S_{a_i}$ is a union of two nonsingular projecitve toric surfaces S_i^+ and S_i^- which intersect transversally along L_i and are mutually σ-conjugate. S_i^+ and S_i^- are isomorphic to the toric surfaces $S_i^+(\rho)$ and $S_i^-(\rho)$ of (9) respectively.*

Proof. By Lemma 6.7 every member S of V is quadratic. Since every effective divisor on Z has a positive intersection number with twistor lines, either S is irreducible or has two irreducible components which are elementary divisors.

Suppose first that S is irreducible. Since every member of V intersects with C, it is a closure of a two dimensional orbit O with $O \cap U' \neq \emptyset$. If O is non-special, by Lemma 5.13 its closure is irreducible and is isomorphic to the smooth toric surface $S(\rho)$.

On the other hand, if O is special, then there exist some $i, 1 \leq i \leq 2k$, and an admissible neighborhood W of z_i as in (40) such that $S \cap U'$ contains either of the plane $u = 0$ or $v = 0$. However, since $u = v = 0$ defines the twistor line L_i and L_i is mapped to $a_i \in P$, we must have $S = S_i$ and both the planes $u = 0$ and $v = 0$ are mapped to the same point a_i by f. Thus, S must contain both the planes $u = 0$ and $v = 0$ since S is assumed to be irreducible. In particular S is not smooth. Now since a_i is a real point, S is σ-invariant. Thus S must be smooth by 1) of Lemma 6.11. This is a contradiction and we conclude that O cannot be special.

Next suppose that S is reducible so that S is a union of two irreducible components S^+ and S^-. This is the case for $S = S_i$ as the previous argument shows. By 2) of Lemma 6.11 $S^{\pm}$ are smooth and if we set $\overline{S}^+ = \sigma(S)$, $\overline{S}^+$ also is an elementary divisor and $L := S^+ \cap \overline{S}^+$ is the unique twistor line contained in S. Now $\overline{S}^+$ is G-invariant as well as S^+, and therefore L also is G-invariant. But the only G-invariant twistor lines in Z are very special twistor lines L_i. Thus L coincides with L_i for some i, and then both S^+ and $\overline{S}^+$ must be mapped to a_i by f; hence $S = S_i$. This implies that S is real and $\overline{S}^+ = S^-$. The transversality of the intersection of S^+ and S^- follows from Lemma 6.11 or directly from the local description (40).

Moreover, S_i never intersects with other twistor lines $L_j, j \neq i$, at its general point since if otherwise S_i would contain L_j since L_j is G-invariant, which is impossible by Lemma 6.11, 2). It follows that $a_i \neq a_j$ if $i \neq j$.

It remains to identify the irreducible components $S_i^{\pm}$ of $S_i, 1 \leq i \leq k$. First of all, since they contain an open orbit of G, they are themselves smooth toric surfaces. (These surfaces are projective since Z is Moishezon.) L_i is the unique common irreducible component of the anti-canonical cycles of both surfaces. L_i is mapped to itself by σ with z_i and z_{i+k} interchanged on it. Further, σ interchanges the two surfaces, their anti-canonical cycles, and interchanges C_j and C_{j+k} for any j. Thus, if C_j is in S^+, then C_{j+k} are in S^- and vice versa. It follows that if C_i is adjacent to L_i in the anti-canonical cycle of S^+, then C_{i+k} is adjacent in S^- to L_i.

From these observations we can readily conclude that the anti-canonical cycle of S_i^+ consists of curves $L_i, C_{i+k+1}, C_{i+k+2}, \ldots, C_{2k}, C_1, \ldots, C_i$ in this order and that of S_i^- consists of $L_i, C_{i+1}, C_{i+2}, \ldots, C_{i+k}$ in this order. So in order to identify these surfaces we have only to identify the one parameter subgroup corresponding to each irreducible component of the anti-canonical cycles. For C_i, this is again ρ_i as is clear from the description (39) of the action along C_i. For L_i, because of the local description (41) of the action at z_i we see readily that it is $\mp(\rho_i - \rho_{i+1})$ on $S_i^{\pm}$. It follows that $S_i^{\pm}$ have the associated weighted circular sequence (13) of $S_i^{\pm}(\rho)$. q.e.d.

(6.9) By Proposition 6.10 and the adjunction formula it follows that the normal bundle of a smooth member S of V in Z is isomorphic to the anti-canonical bundle $-K_S$ of S, while for the irreducible components $S_i^{\pm}$ of the singular members S_i we have

Lemma 6.13. *The normal bundle $[S_i^{\pm}]|S_i^{\pm}$ of $S_i^{\pm}$ in Z is isomorphic to $-K_{S_i^{\pm}} - 2L_i$ on $S_i^{\pm}$ (written additively).*

Proof. We have

$$[S]|S_i^{\pm} = [S_i^+ + S_i^-]|S_i^{\pm} = [C]|S_i^{\pm},$$

where $[C]$ is the line bundle defined by the Cartier divisor C on S_i. On the other hand, we have $[S_i^{\pm}]|S_i^{\mp} = [L_i]$ on $S_i^{\mp}$. From this we have

$$[S_i^{\pm}]|S_i^{\pm} = [C \cap S_i^{\pm}] - [L_i] = (-K_{S_i^{\pm}} - L_i) - L_i = -K_{S_i^{\pm}} - 2L_i$$

since $-K_{S_i^{\pm}} = L_i + C \cap S_i^{\pm}$. q.e.d.

Using the above lemma we shall give a characterization of LeBrun's twistor spaces [14] among our twistor spaces; we shall also determine the dimension of the linear system $|-\frac{1}{2}K|$.

Proposition 6.14. *For a twistor space Z with G-action as above the following three conditions are equivalent.*

1) *The discrepancy $|-\frac{1}{2}K|^G \neq |-\frac{1}{2}K|$ occurs.*
2) *The general member S of V is of maximal type in the sense defined in* (2.7)*.*
3) *Z coincides with one of the twistor spaces of LeBrun* [14] *associated to an S^1-invariant self-dual metric on $m\boldsymbol{P}^2$.*

Moreover, in this case we have

$$\dim|-\frac{1}{2}K| = 9-2m \quad for \quad m \leq 3 \quad and \quad = 3 \quad for \quad m \geq 3.$$

Proof. 1) $\Leftrightarrow$ 2): From the short exact sequence

$$0 \to O_Z \to -\frac{1}{2}K \to -K_S \to 0$$

we obtain the cohomology exact sequence

$$0 \to H^0(Z, O_Z) \to H^0(Z, -\frac{1}{2}K) \to H^0(S, -K_S) \to H^1(Z, O_Z) = 0.$$

Similarly, from the short exact sequence

$$0 \to O_S \to -K_S \to -K_S|C \to 0$$

we get the cohomology exact sequence

$$(54)\quad 0 \to H^0(S, O) \to H^0(S, -K) \to H^0(C, -K_S|C) \to H^1(S, O) = 0.$$

Thus we get $\dim|-\frac{1}{2}K| = \dim|-K_S| + 1 = \dim|-K_S|C| + 2$. Denote in general a symmetric toric surface with $b_2 = 2m+2$ by S_m. S_m is up to isomorphisms unique up to $m = 3$ (cf. (2.7)). Further, S_{m+1} is obtained by blowing up a pair of σ-conjugate fixed points on the anticanonical cycle C of a certain S_m. Moreover, we compute easily that $\dim|-K_S| = 8$ for $m = 0$ and know that $|-K_S|$ is base point free for $m = 0, 1, 2$. Since $|-K_{S_{m+1}}|$ is considered as the set of elements of $|-K_{S_m}|$ which pass through the points of blowing up, if the blown-up points are not contained in the base locus, we get $\dim|-K_{S_{m+1}}| = \dim|-K_{S_m}|-2$, while if the points are in the base locus the dimension does not change. On the other hand, using (54) we can verify that a point p on the anti-canonical cycle C is in the base locus of $-K_S$ if and only if p is on an irreducible component of C with self-intersection number ≤ -3. From all these assertions it is not difficult to check the above statements.

2) $\Leftrightarrow$ 3): By Poon [20, Th. 3.1] 3) holds if and only if there exists an elementary divisor D on Z such that $\dim|D| \geq 1$ on Z. Since G acts on the complete linear system $|D|$ with fixed point, we may assume that D is G-invariant. Then by Propositions 6.4 and 6.10 D is one of $S_i^{\pm}$. By Lemma 6.13 and the fact that $H^1(Z, O_Z) = 0$, $\dim|S_i^{\pm}| \geq 1$ if and only if the complete linear system $|-K_{S_i^{\pm}} - 2L_i|$ on $S_i^{\pm}$ is non-empty. By Lemma 2.10 this is precisely the case where one of the irreducible components of the anti-canonical cycle of $S_i^{\pm}$ which is adjacent to L_i has self-intersection number $1-m$. In view of (13) this is true for some i if and only if the general member S of the system V is of maximal type. q.e.d.

Remark. The equivalent conditions of the proposition depends only on the underlying smooth K-action on M. In fact, the condition 2) depends only on the Orlik-Raymond invariant of the K-action on M.

(6.10) Recall that a twistor line L is called special if $t(L) \in B$, or equivalently $L \cap C \neq \emptyset$. Otherwise we call L *general*. A general twistor line L will be identified with its proper transform in $\hat{Z}$, and will be denoted by the same letter L.

Proposition 6.15. *Let L be a twistor line.*

1) *If L is general, then the restriction $f|L : L \to P$ is a double covering with precisely two mutually σ-conjugate branch points on P. Neither of the branch points is on R.*

2) *If L is special, L is contained in a unique member S_a of V with $a \in R$, and it is never tangent to other members $S_{a'}, a' \neq a$.*

Proof. 1) By Lemma 6.7 the intersection number of L with every fiber of f equals two. Thus f makes L a double covering of P with two branch points. Suppose that one of it, say a, is on R. Since L is tangent to S_a at some point $z \in S_a$ and both L and S_a are real, L is also tangent to S_a at $\sigma(z) \neq z$, which implies that the intersection number $L \cdot S_a \geq 4$. This is a contradiction. Thus the last assertion is proved. Since L is σ-invariant and $f|L$ is σ-equivariant, the two branch points are (non-real and) σ-conjugate to each other.

2) In this case L is in the closure of a G-orbit and hence its proper transform $\hat{L}$ in $\hat{Z}$ is mapped to a point a of P by f, f being G-invariant. Since L is real, a must also be real, i.e., $a \in R$. Finally, if L is not very special, L intersects C transversally at two points other than z_i, at which S_a and $S_{a'}$ intersect transversally for any $a' \neq a$. This proves the last assertion for such special twistor lines. If L is very special, we have $a = a_i$ and the transversality of the intersection is already shown in the proof of Lemma 6.7. q.e.d.

(6.11) Recall that N denotes the quotient M/K and bN its boundary (cf. (3.4)).

Proposition 6.16. *There exists a natural homeomorphism $\phi : bN \to R$ which sends y_i to a_i, and hence, the arcs $y_i y_{i+1}$ to $a_i a_{i+1}, 1 \leq i \leq k$.*

For the continuity part we use the following easy:

Lemma 6.17. *Let X, Y and W be topological spaces, and $h : W \to X$ and $g : W \to Y$ continuous maps with h proper and surjective. Suppose that for every point $x \in X, g(h^{-1}(x))$ is a single point y of Y. Then $u : X \to Y,\ u(x) := y$, is a continuous map.* q.e.d.

Proof of Proposition 6.16. Let $y \in bN$ and take an arbitrary point $x \in B$ over y. Then by Proposition 6.15 the special twistor line $L = L_x$ is contained in S_a for a unique $a = a(x) \in R$. Moreover, since S_a is K-invariant, this point $a(x)$ is invariant if we replace x by tx for any $t \in K$. Hence $a(x)$ depends only on y and we obtain a map $\phi : bN \to R$ with $\phi(y) = a(x)$.

More precisely, if $x = x_i$ for some i, then $L_x = L_i$ and is mapped to a_i and if $x \neq x_i$ for any i, $a(x)$ never coincides with any a_i; in fact since $\sigma(S_i^{\pm}) = S_i^{\mp}$, the only twistor line contained in S_i is L_i. Thus we have

(55)
$$\phi(y_i) = a_i, \quad \text{and if} \quad y \in bN,\ y \neq y_j, \quad \text{then } \phi(y) \neq a_i, 1 \leq i, j \leq k.$$

We proceed to show the bijectivity of ϕ. In view of (55), it suffices to show the bijectivity of $bN - \{y_i\}$ and $R - \{a_i\}$.

Injectivity. Take any point x of $B_i - \{x_{i-1}, x_i\}$ and set $a = a(x)$. By the definition of ϕ it suffices to show that the twistor lines contained in S_a are K-translates of $L = L_x$.

On S_a we consider the equivariant morphism $\nu_i : S_a \to P_i$ of Lemma 2.5. Let G_i be the subgroup of G corresponding to $\pm\rho_i$, which is the complexfication of K_i. Then every G_i-orbit on S_a is contained in a fiber of ν_i. The twistor line $L = L_x$ is invariant by K_i and hence by G_i; thus it is contained in a fiber of ν_i. Since ν_i is G-equivariant and L is not G-invariant, its image is a point of an open orbit on P_i. In particular L is itself a fiber of ν_i. Then any other twistor line contained in S_a is in a fiber of ν_i since it cannot intersect with L.

The linear system which defines ν_i is σ-invariant by Lemma 2.6, and has a real member L. Thus by that lemma the set of real points R_i on P_i consists of a single K-orbit.

Surjectivity. Take any point $a \in R - \{a_i\}$. S_a is then a real smooth quadratic divisor on Z. By the definition of ϕ it suffices to show that S_a contains a special twistor line. Suppose otherwise. Since no twistor line is tangent to S_a unless it is contained in S_a by Proposition 6.15, the twistor fibration induces an unramified double covering $t|S_a : S_a \to M$, which is absurd since M is simply connected by Corollary 6.8. Thus S_a contains a twistor line L. Clearly, L is not general, but not very special. Therefore we may write $L = L_x$ for some $x \in B - \{x_i\}$; this means that $a = a(x)$ is an image of a point of $bN - \{y_i\}$ by ϕ.

Finally, the continuity of ϕ follows from Lemma 6.17 by putting $X = bN, Y = R$ and $W = (t\mu)^{-1}(B) \cap f^{-1}(R)$ there. q.e.d.

Remark. The fact that $S_a, a \in R - \{a_i\}$, contains twistor lines and these are obtained as fibers over real points of a σ-equivariant holomorphic map $S_a \to P$ follows at once from [20, Lemma 1.3]. Here we would like to emphasize the additional information that this map onto P is given exactly by ν_i as long as a is on the arc $a_i a_{i+1}$.

(6.12) Proposition 6.16 shows in particular that the points $a_1, \dots, a_k$ are arranged cyclically in this order on $R \cong S^1$ with respect to an orientation of R.

Definition. With respect to the coordinate (53) the sequence $(a_1, \dots, a_k)$ naturally defines an element $[a_1, \dots, a_k]$ of $\overline{\mathcal{A}}$ which is independent of the choices of the affine coordinate (53), where $\overline{\mathcal{A}}$ is as in (3.6). By construction this invariant depends only on the isomorphism classes of the self-dual manifold $(M,[g])$ with which we have started. We call $[a_1,\dots,a_k]$ the *twistorial invariant* of $(M,[g])$.

Let m be a nonnegative integer and $M = m\boldsymbol{P}^2$, where $0\boldsymbol{P}^2 = S^4$. Fix an effective and smooth K-action on M. Then denote by $\mathcal{C}$ the set of K-equivariant isomorphism classes of K-invariant self-dual conformal structures on M. Then we have the natural map

$$I_t : \mathcal{C} \to \overline{\mathcal{A}} \tag{56}$$

obtained by associating the twistorial invariant to a K-invariant self-dual structure on M.

We shall show in the next two sections that this map is actually injective and show in the last section that this map coincides with the other map

$$I_c : \mathcal{C} \to \overline{\mathcal{A}}$$

given by the conformal invariant defined in (3.6).

(6.13) By a theorem of Hitchin [10] we know that Z is not projecitve when $k-2=m\geq 2$. In our case we have a direct proof of this fact.

Proposition 6.18. *Z is non-projective for $k>3$.*

Proof. Suppose that Z is projective. Then as in [10] we conclude that $-\frac{1}{2}K$ must be ample as it is the unique real line bundle up to multiples. On the other hand, the general orbit closures are members of this system and therefore the intersection of two of them is ample on each member. By Proposition 6.5 such orbit closures are toric surfaces and the intersection is the anti-canonical cycle on each surface. However, such an anti-canonical cycle is ample on a toric surface if and only if $k=2$ or 3 (cf. [17, Prop. 2.21]). q.e.d.

On the other hand, we show in general that Z is not merely a Moishezon manifold, but has a structure of a complete algebraic variety of special type. For each $p_i, 1\leq i\leq 2k$, the local action of the one parameter subgroup $\rho_i+\rho_{i+1}$ with respect to the coordinates (41) at p_i takes the form

$$(x,y,z)\to(sx,sy,s^2z),\ \ s\in\boldsymbol{C}^*, |s|\leq 1.$$

We then set

$$U_i := \{z\in Z; \lim_{s\to 0} sz = p_i\}.$$

We have $p_i, p_{i+k} \in S_i$ and we shall name the two irreducible components $S_i^{\pm}$ of S_i such that $p_i \in S_i^+$ and $p_{i+k} \in S_i^-$. Then by our previous description we check easily the following:

$$U_i = Z - \bigcup_{j \neq i} S_j^-, \quad 1 \leq i \leq k, \text{ and } U_i = Z - \bigcup_{j \neq i-k} S_j^+, \quad k+1 \leq i \leq 2k.$$

Moreover, by the result of [3] we see that for each i there exists an isomorphism of U_i onto the complex affine space $\boldsymbol{C}^3$ which extends to a bimeromorphic map of Z to the natural compactification $\boldsymbol{P}^3$ of $\boldsymbol{C}^3$. (In [3] the result is proved for a compact Kähler manifold, but the proof applies equally well for a Moishezon manifold in general.) Thus we obtain the following:

Proposition 6.19. *Z has a natural structure of a rational complete algebraic variety which is covered by $2k$ copies of $\boldsymbol{C}^3$, the complement of each of which is a union of $k-1$ smooth toric surfaces in Z.*

For $k = 2$, $Z = \boldsymbol{P}^3$ is covered by 4 copies of $\boldsymbol{C}^3$, the complement of each of which is isomorphic to $\boldsymbol{P}^2$. For $k > 2$ the complementary divisor are reducible and are not of normal crossings type. (The author is grateful to M. Pontecorvo for asking him about the rationality of Z.)

7. Local structures along the fibers

(7.1) Let m be a nonnegative integer. Let $M = m\boldsymbol{P}^2$ be the connected sum of m copies complex projective planes for $m > 0$ and set $0\boldsymbol{P}^2 = S^4$. Fix an effective and smooth K-action on M. Suppose that we are given two K-invariant self-dual conformal structures $(M, [g])$ and $(M, [g'])$ on M. Let Z and Z' be the corresponding twistor spaces with the associated fiber spaces (cf. Propositions 6.4 and 6.5):

$$f : \hat{Z} \to P \quad \text{and} \quad f' : \hat{Z}' \to P, \tag{57}$$

where P is the complex projective line with the coordinate (53) and with real structure σ. Then supposing that the points $a_1, \ldots, a_k$ of R corresponding to the singular fibers of f and f' coincide, we shall show that $\hat{Z}$ and $\hat{Z}'$ are isomorphic over P.

By Propositions 6.5 and 6.12 the fibers of f and f' over the same point a of P are isomorphic to each other. In this section, as a first step we prove that f and f' are locally isomorphic along each fiber. Namely we prove:

Theorem 7.1. *Let the notations and assumptions be as above. For every point $a \in P$ there exists a neighborhood U of a in P such that $f^{-1}(U)$ and $f'^{-1}(U)$ are isomorphic as a complex space over U.*

The result is well-known if $a \neq a_i$ for any i, since in this case both f and f' are isomorphic over U as above to the product $U \times S(\rho)$ if U is taken sufficiently small. Therefore we assume that $a = a_i$ in what follows.

There exist two ways for treating the problem, one by a deformation-theoretic method and the other by using the theory of torus embeddings. We first follow the former method and prove the above theorem as a consequence of Propositions 7.10 and 7.11 below. Then by using the latter method we give an explicit description of f (or f') along the fiber over a_i in terms of torus embeddings. Also we shall indicate how the latter method gives another proof of the local uniqueness.

(7.2) We consider the fiber space $f : \hat{Z} \to P$. For any $a_i \in P$ the fiber S_i contains a cycle of nonsingular rational curves $C = a_i \times C$, and consider deformations of the pair (S_i, C) which induces a trivial deformation of C. Our first purpose is to construct explicitly a Kuranishi family of this deformation theory.

For the notational convenience, however, fixing $i, 1 \leq i \leq k$, we write $S = S_i$ and $L = L_i$ in what follows. Accordingly we set $S^{\pm} = S_i^{\pm}$ so that $S = S^+ \cup S^-$ and $L = S^+ \cap S^-$. Let $U^{\pm}$ be the open orbits in $S^{\pm}$ so that $C^{\pm} := S^{\pm} - U^{\pm}$ forms the anti-canonical cycle of $S^{\pm}$. Using the notation of (2.10) we write the irreducible components of $C^{\pm}$ as $(C_1^{\pm}, \ldots, C_i^{\pm}, L_i^{\pm}, C_{i+1}^{\pm}, \ldots, C_k^{\pm})$. Then $B := L \cap C$ consists of two points $p = L \cap C_i^+ = L \cap C_{i+1}^-$ and $q = L \cap C_{i+1}^+ = L \cap C_i^-$. Thus in this notation the one parameter group ρ_i fixes each point of C_i^+ and C_i^-.

For holomorphic maps $h_i : Y_i \to T, i = 1, 2$ over the same base space T by a (*relative*) *isomorphism* of Y_1 to Y_2 over T we shall mean an isomorphism $u : Y_1 \to Y_2$ of complex spaces with $h_2 u = h_1$. We consider deformations of the pair (S, C) which induce trivial deformations of C, i.e., a flat morphism $f : \mathcal{S} \to T$ over an analytic germ $T = (T, o)$ together with a relative divisor $\mathcal{C}$ of $\mathcal{S}$ over T such that $(\mathcal{S}_o, \mathcal{C}_o) = (S, C)$ (fixed isomorphism) and there exists an isomorphism $\phi : C \times T \to \mathcal{C}$ over T, where $(\mathcal{S}_o, \mathcal{C}_o)$ denotes the fiber over o (Thanks are due to K. Ohno for asking the precise meaning of the deformations here, which has eventually led to the correction of the original argument.) We call such a deformation simply a *deformation of the pair* (S, C) and denote it by $f : (\mathcal{S}, \mathcal{C}) \to T$. If $f' : (\mathcal{S}', \mathcal{C}') \to T$ is another deformation over the same space T, the two deformations f and f' are said to be *isomorphic*, if there exists an isomorphism $u : \mathcal{S} \to \mathcal{S}'$ over T which sends $\mathcal{C}$ over $\mathcal{C}'$ (in compatible with the given identifications of the fibers over o with (S, C)). In what follows we shall call the corresponding semiuniversal deformation briefly a *Kuranishi family* of (S, C).

Lemma 7.2. *A Kuranishi family of* (S, C) *exists.*

Proof. Let $u : (\widetilde{\mathcal{S}}, \widetilde{\mathcal{C}}) \to \widetilde{T}$ be a Kuranishi family of (S, C) in the ordinary sense (for which $\widetilde{\mathcal{C}}$ is not necessarily trivial) (cf. [7, Th.8.5] applied to the case of an embedding) and $v : \hat{\mathcal{C}} \to P$ a Kuranishi family of C. We have an induced versal map $b : \widetilde{T} \to P$, i.e., the map induced by the versality of the family v. Then the restriction of u to the inverse image $b^{-1}(o)$ is clearly a Kuranishi family for the deformations of (S, C) in our sense here, where o is the base point of P. q.e.d.

(7.3) We shall consutruct explicitly a deformation of (S, C) which turns out to be a Kuranishi family of (S, C). First we introduce some terminology. Let $f : (\mathcal{S}, \mathcal{C}) \to T$ be a deformation of the pair (S, C). Then $\mathcal{C}$ is a union of divisors $\mathcal{C}_j^{\pm}$ which are (trivial) deformations of $C_j^{\pm}, 1 \leq j \leq k$. Then for any $j \neq i$, $b_j^{\pm} := \mathcal{C}_j^{\pm} \cap \mathcal{C}_{j+1}^{\pm}$ (with j considered cyclically modulo k) defines a holomorphic section of f which does not intersect $L \subseteq S_o = S$. We call any blowing up $\eta : (\widetilde{\mathcal{S}}, \widetilde{\mathcal{C}}) \to (\mathcal{S}, \mathcal{C})$ of $(S, \mathcal{C})$ with center $b_j^+ \coprod b_j^-, j \neq i$, an *admissible blowing up* of $(\mathcal{S}, \mathcal{C})$.

Proposition 7.3. *There exists a natural deformation* $f : (\mathcal{S}, \mathcal{C}) \to \boldsymbol{C}$ *of* (S, C) *parametrized by* $\boldsymbol{C}$ *such that* $(\mathcal{S}, \mathcal{C})$ *admits a natural relative* G*-action over* $\boldsymbol{C}$ *and that for* $t \neq 0$ *the fibers* S_t *are mutually isomorphic nonsingular symmetric toric surface with anti-canonical cycle* C_t *with respect to the induced* G*-action. Moreover,* $\mathcal{S}$ *is nonsingular except at the two points* p *and* q *of* $S = \mathcal{S}_0$*, where* $\mathcal{S}$ *has ordinary double points.*

Proof. First consider the case $k = 2$. In this case $S^{\pm}$ are complex projective planes and S is realized as a reducible quadric $Q := \{\xi_1\xi_2 = 0\}$ in $\boldsymbol{P}^3 = \boldsymbol{P}^3(\xi_1 : \xi_2 : \xi_3 : \xi_4)$ with irreducible components $Q^{\pm} := \{\xi_i = 0\}, i = 1, 2$, where $C =: A$ is the union of four lines $A_{ij} := \{\xi_i = \xi_j = 0\},\ i = 1, 2, j = 3, 4$. Then we consider the one parameter deformation $\xi_1\xi_2 = t\xi_3\xi_4,\ t \in \boldsymbol{C}$ of Q, in which each member $Q_t, t \neq 0$, is a smooth quadric isomorphic to $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ and which contains all the above four lines. Therefore we may consider this deformation, denoted especially by $q : (\mathcal{Q}, \mathcal{A}) \to \boldsymbol{C}$, as a deformation of the pairs

(Q, A) in the sense defined above. With respect to the G-action

$$(\xi_1 : \xi_2 : \xi_3 : \xi_4) \to (\xi_1 : s_1\xi_2 : s_2\xi_3 : s_1s_2\xi_4), \quad (s_1, s_2) \in G = \boldsymbol{C}^{*2}$$

each member $Q_t, t \neq 0$, becomes a symmetric torus surface as in the proposition. It is also immediate to check the final assertion concerning the singularity of Q.

Next we consider the general case. In this case the desired deformation will be obtained by successive blowing-ups of the above family q. By Lemma 2.9 our S is obtained from the above Q for $k = 2$, identified with $\overline{S}$ in that lemma, by a succession of admissible blowing ups. Since our deformation is trivial for C, we can extend the succession of admissible blowing ups sideways to a succession of admissible blowing ups of the families so that starting from the deformation $(\mathcal{Q}, \mathcal{A}) \to \boldsymbol{C}$ for the case $k = 2$ constructed above, we obtain a deformation of (S, C) for any pairs under consideration with the same parameter space $\boldsymbol{C}$ (where the triviality of $\mathcal{C}$ over T is obvious). Moreover, the relative G-action on the family defined above for the case $k = 2$ lifts canonically to any of the deformation obtained in the above way. The final assertion follows from that for the case $k = 2$ since all the blowing-ups occur on the smooth parts of the varieties under consideration. q.e.d.

(7.4) Our purpose is to show the following:

Proposition 7.4. *Let (S, C) be as above. Then the deformation $f : (\mathcal{S}, \mathcal{C}) \to \boldsymbol{C}$ of (S, C) consutructed in Proposition 7.3 is a Kuranishi family of (S, C).*

In view of the construction of f in the proof of Proposition 7.3, we immediately obtain the following:

Corollary 7.5. *Any Kuranishi family $u : (\mathcal{S}, \mathcal{C}) \to \boldsymbol{C}$ of (S, C) is obtained from a Kuranishi family of the pair (Q, A) with $k = 2$ by a finite succession of admissible blowing-ups.*

As for the proof of Proposition 7.4, since the irreducible components of C which intersect with L are not Cartier divisors on S, the cohomological description of the tangent space for our deformation problem seems a bit complicated. So we shall follow some round-about way in showing the proposition. Namely we consider a modified deformation problem for which the cohomological description is easier. For this purpose we prepare some more notations. We define

$$D^{\pm} = C_i^{\pm} \bigcup C_{i+1}^{\pm} \quad \text{and} \quad {C'}^{\pm} = \bigcup_{j \neq i, i+1} C_j^{\pm}.$$

Then we set

$$C' = {C'}^{+} \bigcup {C'}^{-} \quad \text{and} \quad D = D^{+} \bigcup D^{-}.$$

Note that D is a Cartier divisor in S as the inclusion $D \subseteq S$ looks locally at p or q like

$$D = \{xy = z = 0\} \subseteq S = \{xy = 0\} \subseteq \boldsymbol{C}^3.$$

By a D-*deformation* of the pair (S, C) we mean a deformation $f : (\mathcal{S}, \mathcal{C}) \to T$ of (S, C) as above together with an isomorphism $\phi : D \times T \to \mathcal{D}$, where $\mathcal{D} = \cup_{\pm}(\mathcal{C}_i^{\pm} \cup \mathcal{C}_{i+1}^{\pm})$. We shall study the tangent space of this modified deformation theory.

First we recall some general notation and terminology. Let X be a complex space and Y a divisor with normal crossings contained in the smooth locus of X. We shall denote by Ω_X^1 the sheaf of germs of holomorphic 1-forms on X. Then its dual $\Theta_X := \mathcal{H}om_{O_X}(\Omega_X^1, O_X)$ is the holomorphic tangent sheaf of X. $\Omega_X^1(\log Y)$ will denote the sheaf of germs of meromorphic 1-forms with at most logarithmic poles along Y and then its dual $\Theta_X(-\log Y)$ is the sheaf of germs of holomorphic vector fields which are tangent to Y. If D is a Cartier divisor on X, the notations (D) and $(-D)$ are used to indicate $\otimes[D]$ and $\otimes[D]^{-1}$ respectively, where $[D]$ is the line bundle associated to D, e.g.,

$$\Omega_X^1(\log Y)(D) = \Omega_X^1(\log Y) \otimes [D].$$

Note that when $\dim X = 1$, we have

$$\Omega_X^1(\log Y) = \Omega_X^1(Y)$$

and

$$\Theta(-\log Y) = \Theta(-Y).$$

Using these notation the tangent space of the modified deformation theory is identified with the *Ext*-group

$$Ext^1(\Omega_S^1(\log C')(D), O_S)$$

which we shall now compute. (Actually, this space parametrizes the infinitesimal deformations which induce locally analytically trivial deformations of C with globally trivialization of $\mathcal{D}$. But for a cycle of nonsingular rational curves we know that every locally analytically trivial deformation is necessarily globally trivial, and hence, this is a right choice.)

(7.5) With the simplified notations

$$\Xi := \Theta(-\log C')(-D) \quad \text{and} \quad \tau := \mathcal{E}xt^1(\Omega^1_S(\log(C')(D), O_S).$$

the local-to-global spectral sequence of Ext functor gives us the following exact sequence of vector spaces

$$\begin{aligned} 0 \to H^1(S, \Xi) \to Ext^1(\Omega^1_S(\log C')(D), O_S)& \\ \to H^0(S, \tau) \to H^2(S, \Xi).& \end{aligned} \tag{58}$$

Then we prove the following:

Proposition 7.6.

1) *There exist a natural isomorphism*

$$H^1(S, \Xi) \cong H^0(L, \Theta_L(B))(\cong \boldsymbol{C})$$

and we have $H^2(S, \Xi) = 0$.

2) $\tau \cong O_L$.

Corollary 7.7. *The exact sequence* (58) *reduces to*

$$\begin{aligned} 0 \to H^1(S, \Xi) \cong \boldsymbol{C} \to Ext^1(\Omega^1_S(\log C')(D), O_S)& \\ \xrightarrow{\beta} H^0(S, \tau) \cong \boldsymbol{C} \to 0.& \end{aligned} \tag{59}$$

In particular $\dim Ext^1(\Omega^1_S(\log C')(D), O_S) = 2$.

Proof of Proposition 7.6. 1) In order to compute $H^i(S, \Xi)$ we recall the following sheaf exact sequence [4, (5.4)]

$$0 \to \Theta_S \to \bigoplus_{\pm} n^{\pm}_* \Theta_{S^\pm}(-\log L) \to \Theta_L \to 0, \tag{60}$$

where $n^\pm : S^\pm \to S$ are the inclusions. Since C' does not intersect with L and D is a Cartier divisor with $D \cap L = B := \{p, q\}$, from (60) we get

$$0 \to \Xi \to \bigoplus_{\pm} n^{\pm}_* \Xi^{\pm} \to \Theta_L(-B) \to 0, \tag{61}$$

where

$$\Xi^{\pm} = \Theta_{S^{\pm}}(-\log(C^{\pm} \bigcup L))(-D^{\pm}).$$

Note that $\Theta_L(-B) \cong O_L$ so that $H^i(L, \Theta_L(-B)) = 0$ for $i = 1, 2$ and $\cong \boldsymbol{C}$ for $i = 0$. From the cohomology exact sequence associated to (60) we have the following long exact sequence of cohomology and an isomorphism

$$\begin{aligned} & 0 \to H^0(S, \Xi) \to \bigoplus_{\pm} H^0(S^{\pm}, \Xi^{\pm}) \to H^0(L, \Theta_L(-B)) \\ & \qquad \to H^1(S, \Xi) \to \bigoplus_{\pm} H^1(S^{\pm}, \Xi^{\pm}) \to 0 \end{aligned} \tag{62}$$

$$H^2(S, \Xi) \cong \bigoplus_{\pm} H^2(S^{\pm}, \Xi^{\pm}). \tag{63}$$

We next compute $H^i(S^{\pm}, \Xi^{\pm})$. Consider the natural short exact sequence of sheaves

$$0 \to \Xi^{\pm} \to \Theta_{S^{\pm}}(-\log C^{\pm}) \to \bigoplus_{j=i,i+1} \Theta_{C_j^{\pm}}(-B_j^{\pm}) \to 0 \tag{64}$$

where $B_j^{\pm}$ consists of two intersection points of $C_j^{\pm}$ with the other components of $C^{\pm}$. Since $\Theta_{C_j^{\pm}}(-B_j^{\pm})$ are isomorphic to $O_{C_j^{\pm}}$ we have

$$H^i(C_j^{\pm}, \Theta_{C_j^{\pm}}(-B_j^{\pm})) = 0$$

for $i = 1, 2$ and $\cong \boldsymbol{C}$ for $i = 0$. On the other hand, by the sheaf isomorphism ([17, Prop.3.1])

$$\Theta_{S^{\pm}}(-\log C^{\pm}) \cong O_{S^{\pm}} \otimes_{\boldsymbol{Z}} N \cong O_S^{\oplus 2},$$

we have $H^i(S^{\pm}, \Theta_{S^{\pm}}(-\log C^{\pm})) = 0$ for $i = 1, 2$ and $\cong \boldsymbol{C}^2$ for $i = 0$. Then $H^0(S^{\pm}, \Xi^{\pm})$ consists of vector fields associated to the torus action on $S^{\pm}$ which vanishes identically on $C_i^{\pm} \cup C_{i+1}^{\pm}$, which must be identically zero on the whole $S^{\pm}$; namely $H^0(S^{\pm}, \Xi^{\pm}) = 0$. Thus from the cohomology exact sequence associated to (64) we have $H^2(S^{\pm}, \Xi^{\pm}) = 0$ and the following exact sequence of vector spaces

$$\begin{aligned} & 0 \to H^0(S^{\pm}, \Theta_{S^{\pm}}(-\log C^{\pm})) \cong \boldsymbol{C}^2 \\ & \quad \to \bigoplus_{j=i,i+1} H^0(C_j^{\pm}, \Theta_{C_j^{\pm}}(-B_j^{\pm})) \cong \boldsymbol{C}^2 \to H^1(S^{\pm}, \Xi^{\pm}) \to 0. \end{aligned} \tag{65}$$

Therefore we also have $\dim H^1(S^\pm, \Xi^\pm) = 0$. Substituting the results into (62) we get the desired result.

2) Since $\Omega^1_S(\log C')(D)$ is locally free outside L and D is a Cartier divisor on the whole S, τ has support in L and we have

$$\tau \cong \mathcal{E}xt^1(\Omega^1_S(D), O_S) \cong \mathcal{E}xt^1(\Omega^1_S, O_S)(-D).$$

On the other hand, as is well-known the last term is an O_L-module and we have

$$\mathcal{E}xt^1(\Omega^1_S, O_S) \cong N_{L/S^+} \otimes N_{L/S^-} = O_L(2)$$

where $N_{L/S^\pm}$ denotes the normal bundle of L in $S^\pm$. Thus

$$\tau \cong O_L(2)(-B) \cong O_L. \qquad \text{q.e.d.}$$

Remark. From the above argument it also follows that

$$Ext^2(\Omega^1_S(\log C')(D), O_S) = 0.$$

(7.6) We shall construct a Kuranishi family for D-deformation of (S, C). In view the proof of 1) of Proposition 7.6 nonzero elements of $H^1(S, \Xi)$ are described as follows. We set

$$V^+ = S^+ - C^+_{i+1},\ W^+ = S^+ - C^+_i,\ V^- = S^- - C^-_i,\ W^- = S^- - C^-_{i+1}.$$

Then $V := V^+ \cup V^-$ and $W := W^+ \cup W^-$ are open subsets of S and $V \cap W = S - D$. Fix any nonzero element v of $H^0(L, \Theta_L(-B))$. Let $\chi = \chi_i$ be a holomorphic vector field on S coming from the action of the one parameter group ρ_i which restricts to v on L. (The action of ρ_i is nontrivial on L.) Since ρ_i fixes each point of C^+_1 and C^-_k, χ vanishes identically on C^+_1 and C^-_k. Thus $(\chi, 0)$ defines an element of $H^0(V^+, \Xi_1) \bigoplus H^0(V^-, \Xi_2)$ which restricts to v on $L - q$. Similarly $(0, -\chi)$ defines an element of $H^0(W^+, \Xi_1) \bigoplus H^0(W^-, \Xi_2)$ which restricts to v on $L - p$. Then $(\chi, \chi) = (\chi, 0) - (0, -\chi)$ is considered to be an element of $H^0(V^+ \cap V^-, \Xi)$ which defines a nonzero element χ of $H^1(S, \Xi)$ corresponding to to $v \in H^0(L, \Theta_L(-B))$ via the isomorphism of Proposition 7.6.

The corresponding D-deformation parametrized by $\boldsymbol{C}^*$ with the base point $1 \in \boldsymbol{C}^*$ is described as follows. Define a family of complex spaces S_s parametrized by $s \in \boldsymbol{C}^*$ by patching V and W via $\rho_i(s)$;

$S_s = V \cup_{\rho_i(s)} W$. For $s = 1$ we obtain the original space S although clearly all S_s are isomorphic to S. As a D-deformation this family is isomorphic to the trivial deformation $(S, C) \times \boldsymbol{C}^* \to \boldsymbol{C}^*$ of (S, C) together with the non-trivial trivialization $\phi : D \times \boldsymbol{C}^* \to D \times \boldsymbol{C}^*$ given by the condition that $\phi_s | C^-_{i+1} = \rho_i(s) | C^-_{i+1}$ and ϕ_s is the identity on the other irreducible components of D. The infinitesimal deformation at $s = 1$ is then precisely the above element χ up to constants.

We can perform the same construction for the family $f : (\mathcal{S}, \mathcal{C}) \to \boldsymbol{C}$ constructed in Proposition 7.3 using its relative G-action. Fixing a trivialization $\phi : D \times T \to \mathcal{D}$ we may consider f as a D-deformation of (S, C). As in the absolute case we set

$$\mathcal{V}^+ = \mathcal{S} - \mathcal{C}^+_{i+1}, \mathcal{W}^+ = \mathcal{S} - \mathcal{C}^+_i, \mathcal{V}^- = \mathcal{S} - \mathcal{C}^-_i, \text{ and } \mathcal{W}^- = \mathcal{S} - \mathcal{C}^-_{i+1}.$$

Then $\mathcal{V} := \mathcal{V}^+ \cup \mathcal{V}^-$ and $\mathcal{W} := \mathcal{W}^+ \cup \mathcal{W}^-$ are open subsets of $\mathcal{S}$ and $\mathcal{V} \cap \mathcal{W} = \mathcal{S} - \mathcal{D}$. We patch together as before $\mathcal{V}$ and $\mathcal{W}$ via $\rho_i(s)$, $s \in \boldsymbol{C}^*$. In this way we obtain a D-deformation $\hat{f} : (\hat{\mathcal{S}}, \hat{\mathcal{C}}) \to R$ of (S, C) parametrized by $R := \boldsymbol{C} \times \boldsymbol{C}^*$ with the base point $o := (0, 1)$, where the trivialization of $\mathcal{D}$ is obtained by perturbing the original one ϕ by using ρ_i as in the absolute case. By construction the following is clear.

Lemma 7.8. *As a deformation of the pair (S, C) the family $\hat{f}$ is isomorphic to the family induced from $f : (\mathcal{S}, \mathcal{C}) \to T$ via the natural projection $p : R \to T = \boldsymbol{C}$.*

(7.7) Now we are ready to identify a Kuranishi family for D-deformations of (S, C).

Proposition 7.9. *The map $\hat{f} : (\hat{\mathcal{S}}, \hat{\mathcal{C}}) \to R$ of (S, C) gives a Kuranishi family for the D-deformations of (S, C).*

Proof. First we shall show that the tangent space of R at o is naturally identified with the vector space $Ext^1(\Omega^1_S(\log C')(D), O_S)$. Indeed, the tangent space of the subspace $0 \times \boldsymbol{C}^*$ is, by the description of (7.6) in the absolute case, naturally identified with the subspace $H^1(S, \Xi)$ in (39). On the other hand, if $w : T_0\boldsymbol{C} \to Ext^1(\Omega^1_S(\log C')(D), O_S)$ is the natural tangential map (Kodaira-Spencer map) for the original family $f : (S, C) \to \boldsymbol{C}$ with ϕ chosed and fixed, where $T_0\boldsymbol{C}$ is the tangent space of T at 0, the composition of w with the quotient map β in (39) is necessarily surjective. In fact, the image of βw measures the infinitesimal variation of the singularity of $\mathcal{S}$ along L; however by Proposition 7.3 the total space $\mathcal{S}$ of the family f is smooth along L so that the composite map βw cannot be the zero map. Since $H^0(S, \tau) \cong \boldsymbol{C}$, the assertion is verified.

The proposition is deduced from this. However, for the D-deformations we have not found a suitable reference to quote for the existence of a Kuranishi family in general. So we shall proceed in an ad hoc way here. First of all it is not difficult to show that the D-deformation functor has a hull in the sense of [25]; indeed, the proof is basically the same as in the case of the ordinary deformation functor (cf. [28, Th.3.6], [25, (3.7)]) using the criterion of Schlessinger [25, Th.2.11]. Then, since the parameter space of our family is smooth, and the tangent space is identified with the tangent space of the functor, i.e., $Ext^1(\Omega^1_S(\log C')(D), O_S)$, by [28, Th.3.5] this family is formally semiuniversal in the sense that it is semiuniversal for any deformations whose parameter space is zero-dimensional. Then by the same argument as in the proof of [28, Th.3.7] (in the case of ordinary deformation functor) using, e.g., [9, Lemma 2] we conclude that the family is complete (or versal), and hence is a Kuranishi family. q.e.d.

Proof of Proposition 7.4. Let $f' : (\mathcal{S}', \mathcal{C}') \to T'$ be any deformation of the pair (S, C). By taking a trivialization $D \times T' \to \mathcal{D}$ we may consider f' as a D-deformation of the pair (S, C). Then by the previous proposition we have a semiuniversal map $\delta : T' \to R$. Compose this with the projection $p : R \to T$. Then by Lemma 7.8 u is isomorphic as a deformation of (S, C) to the induced family from f via pu. This shows that f is a versal family. Moreover, if $(\widetilde{\mathcal{S}}, \widetilde{\mathcal{C}}) \to \widetilde{T}$ is a Kuranishi family of (S, C), the differential of the versal map $T_o\boldsymbol{C} \to T_o\widetilde{T}$ is not the zero map since otherwise S will have singularity along L. Thus the Kodaira-Spencer map of f is injective and f is a semiuniversal deformation of (S, C) as desired. q.e.d.

We come back to our twistor space Z and consider the associated morphism $f : \hat{Z} \to P$ or more precisely the morphism $f : (\hat{Z}, E) \to P$. Take any $a_i \in R$ and consider the germ f_i of f along the fiber $S_i = S_i^+ \cup S_i^-$ as a deformation of the pair (S_i, C) inducing the trivial deformation of C.

Proposition 7.10. *The map f_i gives rise to a Kuranishi family of (S_i, C) at a_i.*

Proof. Let $f : (\mathcal{S}, \mathcal{C}) \to (T, o)$ be the Kuranishi family of the pair (S_i, C) constructed in Proposition 7.4. We have a versal map $\tau : (P, a_i) \to (T, o)$ which is induced by the versality of the family f. If τ has ramification at a_i, then $\hat{Z}$ should have singularity along the intersection $L_i = S_i^+ \cap S_i^-$. This is a contradiction; so τ must be isomorphic. Thus f_i and f are isomorphic as a deformation of (S_i, C), and hence, f_i itself is a Kuranishi family of (S_i, C). q.e.d.

(7.8) In order to prove Theorem 7.1 we still have to prove the following:

Proposition 7.11. *Let $u : (\mathcal{S}, \mathcal{C}) \to (T, o)$ and $u' : (\mathcal{S}', \mathcal{C}') \to (T, o)$ be two Kuranishi families of the pair (S, C) with the* same *parameter space T. Then there exists an isomorphism $j : (\mathcal{S}, \mathcal{C}) \to (\mathcal{S}', \mathcal{C}')$* over T *which maps each irreducible component $\mathcal{C}_j$ to the corresponding component $\mathcal{C}'_j$. Moreover, if we are given a section $s : T \to \mathcal{S}$ (resp. $s' : T \to \mathcal{S}'$) of u (resp. u') which do not pass through $\mathcal{C}$ (resp. $\mathcal{C}'$), then j can be taken so that $js = s'$ on T.*

Remark. The versality in general implies the existence of such an isomorphism over some automorphism α of T. Our claim is that, in our case, an arbitrary automorphism of the base lifts to an isomorphism of the two families. This would not be automatic in general.

Proof. We first consider the case $k = 2$. In this case $S = S^+ \cup S^-$, $S^\pm \cong \boldsymbol{P}^2$, and the intersection $S^+ \cap S^-$ is a line L. Moreover, the general fiber of u is isomorphic to $\boldsymbol{P}^1 \times \boldsymbol{P}^1$. Our method is to perform some canonical bimeromorphic modifications to both the fiber spaces u and u' and reduce them to the trivial fiber spaces for which the existence of the isomorphism is clear; and then we follow the inverse process to get a desired isomorphism between the original fiber spaces. Now the total space $\mathcal{S}$ has two singular points p and q, which are ordinary double points. We may perform a small resolution $\widetilde{\mathcal{S}} \to \mathcal{S}$ at p and q with the exceptional nonsingular rational curves N_p and N_q respectively. At each point there exist two choices of such small resolutions; our choice is that N_p and N_q are contained in one and the same component of S, say S^+. (In the presence of a section s we take the component which does not intersect with the image of s, noting that the image never intersects with L.) Then the proper transform $\widetilde{S}^-$ of S^- in $\widetilde{\mathcal{S}}$, which is still isomorphic to $\boldsymbol{P}^2$, is exceptional in $\widetilde{\mathcal{S}}$; hence we can blow down $\widetilde{S}^-$ to a smooth point r of a complex manifold $\overline{\mathcal{S}}$:

(66)
$$\begin{tikzcd} \mathcal{S} \arrow[dr, "u"'] & \widetilde{\mathcal{S}} \arrow[l] \arrow[r] \arrow[d] & \overline{\mathcal{S}} \arrow[dl, "\overline{u}"] \\ & T & \end{tikzcd}$$

It turns out that the resulting morphism $\overline{u} : \overline{\mathcal{S}} \to T$ is smooth and is a trivial bundle with fiber $\boldsymbol{P}^1 \times \boldsymbol{P}^1$. We denote by $\overline{A}_i, i = 1, 2$, the proper transforms in $\overline{\mathcal{S}}$ of the two irreducible components of the anti-canonical cycles of S^+ other than L, and by $\overline{N}_p$ and $\overline{N}_q$ those of N_p and N_q respectively. Then one sees that $\overline{N}_p \cap \overline{N}_q = \{r\}$. Furthermore, the cycle of rational curves $\overline{C}_0 := \overline{N}_p + \overline{A}_1 + \overline{A}_2 + \overline{N}_q$ extends to the anti-canonical cycle $\overline{C}_t$ on the general fiber $\overline{S}_t$ so that $\overline{u}$ is considered as a deformation of the pair $(\overline{S}_0, \overline{C}_0)$.

We perform the same construction to the other family $u' : (\mathcal{S}', \mathcal{C}') \to (T, o)$ and obtain a manifold $\widetilde{\mathcal{S}}'$ and a trivial family $\overline{u}' : \overline{\mathcal{S}}' \to T$ considered as a deformation of the pair $(\overline{S}'_0, \overline{C}'_0)$ defined in the same way as $(\overline{S}_0, \overline{C}_0)$. Let $r' \in \overline{\mathcal{S}}'$ be the point corresponding to r.

Now take an isomorphism $\overline{j} : (\overline{\mathcal{S}}, \overline{\mathcal{C}}) \to (\overline{\mathcal{S}}', \overline{\mathcal{C}}')$ over T which sends $\overline{A}_i, \overline{N}_p, \overline{N}_q$ to $\overline{A}'_i, \overline{N}'_p, \overline{N}'_q$ respectively, and hence sends r to r', which is always possible. Then j clearly lifts to an isomorphism $\widetilde{j} : \widetilde{\mathcal{S}} \to \widetilde{\mathcal{S}}'$ of the blown-up manifolds which automatically sends N_p onto N'_p and N_q to N'_q. Hence, after contracting these curves, $\widetilde{j}$ further descends to an isomorphism j between the original fiber spaces $\mathcal{S} \to T$ and $\mathcal{S}' \to T$ which obviously sends each irreducible component of $\mathcal{C}$ to the corresponding components of $\mathcal{C}'$. This is a desired isomorphism.

Finally suppose that we are given sections s and s'. Let $\overline{s}$ and $\overline{s}'$ be the corresponding sections to $\overline{u}$ and $\overline{u}'$. Since $\overline{j}\overline{s}(t)$ and $\overline{s}'(t)$ are in the open orbit of S'_t for any $t \in T$, there exists a unique element $g(t)$ of G such that $g(t)\overline{j}\overline{s}(t) = \overline{s}'(t)$. This $g(t)$ defines a holomorphic map $g : T \to G$. Then replacing $\overline{j}$ by $g\overline{j}$ in the above argument we get a desired isomorphism j.

Next we consider the general case. By Proposition 7.4 and Corollary 7.5 $u : (\mathcal{S}, \mathcal{C}) \to T$ is obtained by a succession $\phi := \phi_1 \cdots \phi_m$ of admissible blowing-ups $\phi_d : \mathcal{S}_d \to \mathcal{S}_{d-1}, d = 1, \ldots, m$, over T such that $\mathcal{S}_m = \mathcal{S}$ and that $(\mathcal{S}_0, \mathcal{C}_0)$ is a Kuranishi family of the pair (Q, A) for the case $k = 2$ where $m = k - 2$ (cf. the proof of Proposition 7.3). The similar statement is true for u', too. Let $\phi'_d : \mathcal{S}'_d \to \mathcal{S}'_{d-1}, d = 1, \ldots, m$, over T be the corresponding admissible blowing ups. Moreover, these blowing-ups are determined by the induced admissible blowing-ups of the central fiber so that we may assume that at each step $(\mathcal{S}_d, \mathcal{C}_d)$ and $(\mathcal{S}'_d, \mathcal{C}'_d)$ are deformations of one and the same pair.

Now by the proof in the case $k = 2$ there exists an isomorphism $j_0 : \mathcal{S}_0 \to \mathcal{S}'_0$ over T which maps the irreducible components of $\mathcal{C}_0$ onto the corresponding components of $\mathcal{C}'_0$, and in the presence of sections maps the image of the section s in $\mathcal{S}_0$ to that of the section s' in $\mathcal{S}'_0$. Then by the definition of admissible blowing-ups it is immediately seen that j_0 lifts to a desired isomorphism j of $\mathcal{S}$ onto $\mathcal{S}'$. q.e.d.

Proof of Theorem 7.1. Immediate from Propositions 7.10 and 7.11.

(7.9) Let $f : X \to Y$ be holomorphic maps of complex spaces. Then a *relative automorphism* of X over Y is an autmorphism $h : X \to X$ of complex spaces such that $f = fh$. We also need the following result in the next section.

Proposition 7.12. *Let $u : (\mathcal{S}, \mathcal{C}) \to T$ be a Kuranishi family of (S, C). Let $\beta : \mathcal{S} \to \mathcal{S}$ be a relative automorphism over T which preserves each irreducible component $\mathcal{C}_i$ of $\mathcal{C}$. Then there exists a unique holomorphic map $h : T \to G$ such that the automorphism $\beta_t : S_t \to S_t$, $t \in T$, is induced by the action of the element $h(t)$.*

Proof. The existence of a unique holomorphic map h defined on $T - \{0\}$ follows from Lemma 2.1 immediately. The problem is to show that it extends across the origin 0.

First consider the case $k = 2$. In this case in view of Proposition 7.4 and the proof of Proposition 7.3 we may assume that u is the family of quadrics in $\boldsymbol{P}^3$ constructed in the proof of Proposition 7.3. Thus $\mathcal{S}$ is naturally embedded in $\boldsymbol{P}^3 \times T$ over T and β is given by a holomorphic map $h : T \to PGL(3)$ which has values in G for $t \neq 0$ with respect to the natural embedding $G \hookrightarrow PGL(3)$ induced by the given action. Since G is closed in $PGL(3), h(0)$ also belongs to G.

In the general case, as in the last part of the proof of the previous proposition we take a bimeromorphic morphism $\phi : \mathcal{S} \to \mathcal{S}_0$ over T. Then β descends to an automorphism $\overline{\beta}$ of $\mathcal{S}_0$ over T preserving the irreducible components of $\mathcal{C}_0$ and with $\overline{\beta}(t), t \neq 0$, induced by the action of $h(t)$. By what we have proved above for the case $k = 2$, h extends across the origin. But then the extended map $h : T \to G$ already meets the requirement of the lemma for the original $\mathcal{S}$, too. q.e.d.

Remark. It is clear that G is the identity component of the group of extendible automorphisms of the fiber S_0. But *a priori* it may contain another components to which β_o belongs as in the case of degeneration of elliptic curves.

(7.10) We shall construct by the method of torus embedding a local model of the degeneration at a_i of the morphism f for any $i, 1 \leq i \leq k$.

Let $N = \boldsymbol{Z}^3$ be the additive group of one parameter subgroups of the algebraic three-torus $\boldsymbol{C}^{*3}$. Decompose N into the direct sum $N = N_2 \oplus N_1 = \boldsymbol{Z}^2 \oplus \boldsymbol{Z}$. Let

$$\rho = \{\rho_1, \dots, \rho_{2k}\},\ \rho_{j+k} = -\rho_j,\ 1 \leq j \leq k,$$

be the set of primitive elements in $N_2 = \boldsymbol{Z}^2$ arranged counterclockwise in this order.

Fixing i take any elements η and η' of N whose last coordinates are equal to 1 and which satisfy the condition

$$\eta - \eta' = \rho_i - \rho_{i+1}. \tag{67}$$

Define a fan $\triangle$ in $N_{\boldsymbol{R}}$ by the following prescription: The one-dimensional simplices are those generated by the primitive vectors η, η' and $\rho_j,\ 1 \leq j \leq 2k$. The two-dimensional simplicies consist of

$$\begin{aligned} &\{\eta, \eta'\},\quad \{\rho_j, \rho_{j+1}\},\ 1 \leq j \leq 2k\ (2k+1 = 1), \\ &\{\eta, \rho_j\},\ j = 1, \dots, i, i+k+1, \dots, 2k, \quad \text{and} \\ &\{\eta', \rho_j\},\ j = i+1, \dots, i+k, \end{aligned}$$

while three-dimensional simplices are given by

$$\{\eta, \rho_1, \rho_2\}, \dots \{\eta, \rho_{i-1}, \rho_i\}, \{\eta, -\rho_{i+1}, -\rho_{i+2}\}, \dots \{\eta, -\rho_k, -\rho_1\}$$

and

$$\{\eta', -\rho_1, -\rho_2\}, \dots \{\eta', -\rho_{i-1}, -\rho_i\}, \{\eta', \rho_{i+1}, \rho_{i+2}\}, \dots \{\eta', \rho_k, \rho_1\}.$$

(Here, for example $\{\eta, \rho_1, \rho_2\}$ denotes the simplex spanned by η, ρ_1, ρ_2.) Besides these, we take the following two three-dimensional quadrilateral cones

$$\{\eta, \eta', \rho_i, \rho_{i+1}\} \quad \text{and} \quad \{\eta, \eta', -\rho_{i+1}, -\rho_i\}$$

as constituents of our fan. It is immediate to see that these data actually define a fan $\triangle$ whose support coincides with the half space

$N_{2\boldsymbol{R}} \oplus N_{1\boldsymbol{R}_{\geq 0}}$. Let X be the corresponding three dimensional toric variety.

Let $\triangle_1 = \{\{0\}, \boldsymbol{R}_{\geq 0}\}$ be a fan on $N_{1\boldsymbol{R}} = \boldsymbol{R}$, which gives the toric affine line $\boldsymbol{C} = \boldsymbol{C}^*(s) \cup \{0\}$. Then the projection $N \to N_1$ gives rise to a map of fans $\psi : (N, \triangle) \to (N_1, \triangle_1)$, which in turn induces a proper morphism of toric varieties $h : X \to \boldsymbol{C}$ (cf. [17, Th. 1.13, Th. 1.15]).

Lemma 7.13. *Given ρ, the fiber space h is, up to $\boldsymbol{C}^{*3}$-equivariant isomorphisms, independent of the choices of η and η' above.*

Proof. It suffices to show that the map of fans ψ is up to isomorphisms independent of the choice of η and η'. Consider the subgroup $H \cong \boldsymbol{Z}^2 \subseteq \mathrm{SL}(3, \boldsymbol{Z})$ consisting of elements of the form

$$g = \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}, \quad \alpha, \beta \in \boldsymbol{Z}.$$

The natural action of the group H on N leaves fixed any element of $N_2 \subseteq N$ and induces the identity on the quotient $N_1 = N/N_2$. Thus if we apply g to our original fan $\triangle$, the change occurs only on η and η' with the difference $\eta - \eta'$ unchanged. On the other hand, the image of $(0,0,1)$ becomes $(\alpha, \beta, 1)$. From this we easily conclude that any two fans constructed according to the above prescription are mutually mapped to each other by some element of H. q.e.d.

(7.11) We show that h gives a local model of f along S_i.

Proposition 7.14. *We consider the morphism $f : \hat{Z} \to P$ as a germ along the fiber $S_i = S_{a_i}$ over a_i and consider the morphism $h : X \to \boldsymbol{C}$ as a germ along the fiber X_0 over $0 \in \boldsymbol{C}$. Then f and h are isomorphic.*

Proof. Let E_j be the irreducible divisor on X corresponding to ρ_j and S (resp. S') the irreducible divisor corresponding to η (resp. η'). Because of (67) X has ordinary double points at the intersection point of the four divisors S, E_i, E_{i+1} and S' (resp. S, E_{i+k}, E_{i+k+1} and S') (cf. [13, pp. 38–39] [17, p. 37, Ex.]). Since the third coordinate of η (resp. η') is equal to one, η, ρ_j and ρ_{j+1} form a $\boldsymbol{Z}$-basis of N so that X is nonsingular at any other point.

Let $\triangle(\rho)$ be the fan in $N_{2\boldsymbol{R}} = \boldsymbol{R}^2$ generated by ρ, which defines the toric surface $S(\rho)$. Then we have the natural inclusion of fans $(N_2, \triangle(\rho)) \hookrightarrow (N, \triangle)$, giving rise to an embedding of the toric surface $S(\rho)$ onto the fiber of h over 1, and hence onto any fiber over

$s \neq 0$. Note that the divisors E_j are of the form $E_{0j} \times \boldsymbol{C}$ on X, where $E_{0j} = E_j \cap h^{-1}(0)$.

The fiber over $s = 0$ consists of two toric surfaces S and S', whose structures are easily read off from the fan $\triangle$. For S, for instance, we set according to [17, Cor.1.7] $N_\eta := N/\boldsymbol{Z}\eta$ and

$$\triangle_\eta := \{\overline{\sigma} = \text{ the image of } \sigma \text{ in } N_{\eta\boldsymbol{R}}; \eta \text{ is a face of } \sigma\}.$$

Then S is isomorphic to the toric surface associated to the fan $(N_\eta, \triangle_\eta)$ and we have

$$\triangle_\eta = \{\overline{\eta}' \text{ and } \overline{\rho}_j,\ j = 1, \ldots, i, i+k+1, \ldots, 2k\},$$

where $\overline{\eta}' = \overline{\eta}' - \overline{\eta} = -\overline{\rho}_i + \overline{\rho}_{i+1}$ because of (67). Therefore S is isomorphic to $S_i^+(\rho)$. Similarly, we get $S' \cong S_i^-(\rho)$ of (9). Finally since they intersect transversally along the curve $L := S \cap S'$ corresponding to the two-simplex $\{\eta, \eta'\}$, it follows that $X_0 \cong S_i = S_{a_i}$.

Thus, if we set $E = \cup_j E_j$ with its fiber $E_0 = \cup_j E_{0j}$ over 0, the map h, regarded as a map $(X, E) \to \boldsymbol{C}$, is considered as giving a local deformation of the pair (X_0, E_0) which is isomorphic to (S_i, C) by the above consideration. Hence we have a versal map $\tau : (\boldsymbol{C}, 0) \to (P, a_i)$ of analytic germs, considering $f : \hat{Z} \to P$ as a Kuranishi family of the pair (S_i, C) by Proposition 7.10. However since X is smooth along the general points of L, τ cannot have ramifcation, i.e., τ is isomorphic. Thus the proposition follows. q.e.d.

In passing we also note the following:

Proposition 7.15. *There exists a relative involution of X over $\boldsymbol{C}$ which interchanges the two irreducible components of the fiber X_0 over the origin $0 \in \boldsymbol{C}$.*

Proof. Lemma 7.13 shows that we may take η to be $(0,0,1)$. Then if $\eta' = (\alpha, \beta, 1)$, the element

$$h = \begin{pmatrix} -1 & 0 & \alpha \\ 0 & -1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SL}(3, \boldsymbol{Z})$$

defines an involution of the fan $(N, \triangle)$ which interchanges the vectors $(0,0,1)$ and $(\alpha, \beta, 1)$. This induces a desired relative automorphism.

q.e.d.

(7.12) Note that the existence of toric description of the Kuranishi family of (S_i, C) as above is actually *a priori* guaranteed by the following result of Mumford (cf. the proof of Theorem in [13, IV,§1]).

Lemma 7.16. *Let $f : A \to B$ be a proper morphism of a complex algebraic variety A of dimension $n+1$ onto a complex smooth algebraic curve B. Suppose that there exists an effective algebraic action of the algebraic torus $\boldsymbol{C}^{*n}$ on A over B. Let $b \in B$ be an arbitrary point. Then there exist a toric variety X of dimension $n+1$ and an equivariant morphism $h : X \to \boldsymbol{C}$ such that the germ of f, considered along the fiber over b, and the germ of h, considered along the fiber over the origin 0 of $\boldsymbol{C}$, are $\boldsymbol{C}^{*n}$-equivariantly and analytically isomorphic, where the action of $\boldsymbol{C}^{*n}$ on X over $\boldsymbol{C}$ is induced by the given torus action (first n factors).*

In fact by applying the lemma to our G-action on $\hat{Z}$ over P at each point $a_i \in R$, we obtain *a priori* the exsitence of a toric description of f along the singular fiber S_i.

Conversely, we can replace the deformation theoretic argument in the first part of this section by the toric argument above in showing the local uniqueness of the fibering of f over a_i. In fact, from the structure of $\hat{Z}$ along S_i we conclude that any of the toric models describing f along S_i must be isomorphic to h in Proposition 7.14. For instance, from the fact that $\hat{Z}$ has ordinary double point at z_i or z_{i+k} we obtain the condition (67), from the fact that $\hat{Z}$ is otherwise smooth along S_i we get that the last coordinates of η and η' should be equal to one, and etc.

Remark. In order to apply Lemma 7.16 we have to check that a suitable neighborhood of S_i is algebraic; in this respect it is not difficult to show that if we set $P_i = \{a \in P; a \neq a_j, j \neq i\}$, $h^{-1}(P_i)$ is quasi-projective over P_i.

8. Effectivity of invariants

(8.1) The purpose of this section to prove the following global isomorphism theorem which implies the effectivity of the twistorial invariants, i.e., the injectivity of the map I_t in (56).

Theorem 8.1. *Let $(M, [g])$, $(M, [g'])$, Z, Z', $f : \hat{Z} \to P$* and *$f' : \hat{Z}' \to P$ be as in* (7.1). *Suppose that the points $a_1, \dots, a_k$ on P corresponding to the singular fibers of f and f' coincide. Then there exists an isomorphism $j : Z \to Z'$ of complex manifolds which is G-*

equivariant, is compatible with real structures and maps twistor lines to twistor lines.

By the (equivariant) inverse twistor correspondence this implies the following

Corollary 8.2. *The original self-dual manifolds* $(M,[g])$ *and* $(M,[g'])$ *are* K*-equivariantly isomorphic.*

We first construct a global isomorphism between the two fiber spaces $f:\hat{Z}\to P$ and $f':\hat{Z}'\to P$ forgetting the additonal structures for a while.

(8.2) As a preliminary result, we first deduce a consequence of Proposition 7.12 on the (global) automorphism group of $\hat{Z}$ over P. Let $E=\mu^{-1}(C)\cong C\times P$ and $E_i,\ 1\leq i\leq 2k$, its irreducible components as in Proposition 6.5.

Lemma 8.3. *Let* $\mathrm{Aut}_P(\hat{Z},\{E_i\})$ *be the group of relative automorphisms of* $\hat{Z}$ *over* P *which preserve any irreducible components* E_i *of* E*. Then the embedding* $G\hookrightarrow\mathrm{Aut}_P(\hat{Z},\{E_i\})$ *induced by the natural* G*-action on* $\hat{Z}$ *is an isomorphism. Moreover, the same conclusion is also true for any fiber space* $f_1:(\hat{Z}_1,\{E'_{1i}\})\to P_1$ *obtained by pulling back* f *by a finite covering* $P_1\to P$ *which is unramified over all* a_i*, where* P_1 *is any compact curve.*

Proof. Let $g\in\mathrm{Aut}_P(\hat{Z},\{E_i\})$ be any element. Let $P_0:=P-\{a_i,1\leq i\leq k\}$. By Lemma 2.1 for any $a\in P_0$ there exists a unique element $h(a)$ of G depending holomorphically on a such that $g|\hat{Z}_a$ is given by the action of $h(a)$. Moreover, by Proposition 7.12 the resulting holomorphic map $h:P_0\to G$ extends to a holomorphic map $\widetilde{h}:P\to G$ such that $g|Y_a$ coincides with the action of the element $\widetilde{h}(a)$ for all $a\in P$. Then $\widetilde{h}$ must be a constant map since G is Stein. This implies that g is an element of G. The proof is completely the same for f_1. q.e.d.

(8.3) In what follows for an auxiliary purpose we fix general twistor lines L and L', one for each $\hat{Z}$ and $\hat{Z}'$, with the same branch points a and $\overline{a}:=\sigma(a)$ on P (cf. (6.10)).

Take and fix a ramified double covering $u:\widetilde{P}\to P$ ramified precisely over a and $\overline{a}$. We set $\widetilde{R}:=u^{-1}(R)$, which is connected and is an unramified double covering of R. We write

$$u^{-1}(a_i)=\{b_i,b_{i+k}\},$$

where $b_1,\dots,b_{2k}$ are arranged cyclically in this order on the circle $\widetilde{R}$.

We consider the pull-back $\widetilde{f} : \widetilde{Z} \to \widetilde{P}$ of f by u, i.e., $\widetilde{Z} = \hat{Z} \times_P \widetilde{P}$ with $\widetilde{f}$ the natural projection. $\widetilde{Z}$ inherits a natural G-action over $\widetilde{P}$ from $\hat{Z}$. Denote the inverse image of E in $\widetilde{Z}$ by $\widetilde{E}(\cong C \times \widetilde{P})$. Since L is a double coverings of P branched exactly over a and $\overline{a}$, the inverse image of L in $\widetilde{Z}$ decomposes into two irreducible components which determine sections of $\widetilde{f}$. We then choose and fix one of its irreducible components $\widetilde{L}$. The branch locus of the double covering map $p : \widetilde{Z} \to \hat{Z}$ is precisely the two nonsingular fibers $\hat{Z}_a := f^{-1}(a)$ and $\hat{Z}_{\overline{a}} := f^{-1}(\overline{a})$. The morphism $\widetilde{f}$ has singular fibers precisely over b_i. $\widetilde{L}$ intersects neither with $\widetilde{E}$ nor with the inverse images of special twistor lines.

We perform the same construction to $f' : \hat{Z}' \to P$ and denote the corresponding objects by putting $'$ on the same letter; for instance its pullback to $\widetilde{P}$ is denoted by $\widetilde{f}' : \widetilde{Z}' \to \widetilde{P}$. By our assumption the general fibers of f and f' are isomorphic to the same toric surface $S(\rho)$. So the irreducible components of $\widetilde{E}$ and $\widetilde{E}'$ may be numbered in such a way that $\widetilde{E}_i$ and $\widetilde{E}'_i$ correspond with respect to a fixed isomorphism of the general fibers. This in particular implies that the one parameter groups corresponding to $\widetilde{E}_i$ and $\widetilde{E}'_i$ are the same and is equal to ρ_i. We then show the following:

Lemma 8.4. *There exists an isomorphism $\widetilde{j} : \widetilde{Z} \to \widetilde{Z}'$ over $\widetilde{P}$ which sends $\widetilde{E}_i$ to $\widetilde{E}'_i, 1 \le i \le 2k$, and $\widetilde{L}$ to $\widetilde{L}'$.*

Proof. Let $I = \mathrm{Isom}_{\widetilde{P}}((\widetilde{Z}, \{\widetilde{E}_i\}, \widetilde{L}), (\widetilde{Z}', \{\widetilde{E}'_i\}, \widetilde{L}'))$ be the analytic space over $\widetilde{P}$ whose fiber $I_q, q \in \widetilde{P}$, parametrizes all the isomorphisms of $(\widetilde{Z}_q, \{\widetilde{E}_{iq}\}, \widetilde{L}_q)$ and $(\widetilde{Z}'_q, \{\widetilde{E}'_{iq}\}, \widetilde{L}'_q))$, namely the isomorphisms of $\widetilde{Z}_q$ to $\widetilde{Z}'_q$ which sends each $\widetilde{E}_{iq}$ to $\widetilde{E}'_{iq}$ and $\widetilde{L}_q$ to $\widetilde{L}'_q$, where $\widetilde{L}_q$ and $\widetilde{L}'_q$ consist of single points. See [26] and [9, 3.1]. We have natural identifications $(\widetilde{Z}_q, \{\widetilde{E}_{iq}\}) = (S_{u(q)}, \{C_{iu(q)}\})$ and $(\widetilde{Z}'_q, \{\widetilde{E}'_{iq}\}) = (S'_{u(q)}, \{C'_{iu(q)}\})$.

Since we know that each smooth fiber of f and f' are isomorphic as a toric surface, the set of such isomorphisms $\mathrm{Isom}((\widetilde{Z}_q, \{\widetilde{E}\}_{iq}), (\widetilde{Z}'_q, \{\widetilde{E}\}'_{iq}))$ is non-empty, and then by using the torus action we conclude that the fiber $I_q = \mathrm{Isom}((\widetilde{Z}_q, \{\widetilde{E}\}_{iq}, \widetilde{L}_q), (\widetilde{Z}'_q, \{\widetilde{E}\}'_{iq}, \widetilde{L}'_q))$ is non-empty either; in fact the latter consists of a single element in view of Lemma 2.1.

Thus the natural projection $i : I \to \widetilde{P}$ is an isomorphism over $\widetilde{P}_0 := \widetilde{P} - \{b_i\}$ and hence the closure I' of $i^{-1}(\widetilde{P}_0)$ in I is an irreducible component of I. On the other hand, at any point of $\widetilde{P}$ we have by Proposition 7.11 a local section to $I \to \widetilde{P}$. This implies that I' is mapped surjectively, and hence isomorphically, onto $\widetilde{P}$. Thus we have obtained a section of $I \to \widetilde{P}$, which in turn gives rise to a desired isomorphism of two fiber spaces. q.e.d.

(8.4) We want the $\widetilde{P}$-isomorphism $\widetilde{j} : \widetilde{Z} \to \widetilde{Z}'$ obtained in Lemma 8.4 to descend to a P-isomorphism between original fiber spaces $f : \hat{Z} \to P$ and $f' : \hat{Z}' \to P$. For this it suffices to show that j is compatible with the covering transformations κ and κ' of the double coverings $\widetilde{Z} \to \hat{Z}$ and $\widetilde{Z}' \to \hat{Z}'$ respectively;

$$\widetilde{j}\kappa = \kappa'\widetilde{j}. \tag{68}$$

For this purpose we first prove the following:

Lemma 8.5. *The covering transformation κ is the unique holomorphic involution of $\widetilde{Z}$ which covers the covering transformation of $\widetilde{P} \to P$ and which induces the identity on the two fibers $\widetilde{Z}_b$ and $\widetilde{Z}_{\overline{b}}$, where $b = u^{-1}(a)$ and $\overline{b} = u^{-1}(\overline{a})$.*

Proof. Suppose that there exist two such involutions, say κ and κ_1. Then $g := \kappa^{-1}\kappa_1$ gives a relative automorphism of $\widetilde{Z}$ over T which induces the identity on $\widetilde{Z}_b$ and $\widetilde{Z}_{\overline{b}}$. In particular g preserves each irreducible component of $\widetilde{E}$. Then by Lemma 8.3 g is induced by the action of an element of G, which will be denoted by the same letter g. Then g must be the identity since $g|\widetilde{Z}_b$ is the identity; therefore $\kappa = \kappa_1$ as desired. q.e.d.

From this we get the desired uniqueness immediately.

Lemma 8.6. *The isomorphism $\widetilde{j} : \widetilde{Z} \to \widetilde{Z}'$ obtained in Lemma 8.4 descends to an isomorphism $\hat{j} : \hat{Z} \to \hat{Z}'$ over P which sends each E_i to E'_i, where E_i and E'_i are the images of $\widetilde{E}_i$ and $\widetilde{E}'_i$ respectively.*

Proof. The composite map $\widetilde{j}^{-1}\kappa'\widetilde{j}$ is a holomorphic involution on $\widetilde{Z}$ with all the properties of Lemma 8.5. Hence, by Lemma 8.5 we have $\widetilde{j}^{-1}\kappa'\widetilde{j} = \kappa$ and (68) follows. The rest of the assertion is clear. q.e.d.

(8.5) So far we have shown that the two fiber spaces f and f' are isomorphic as complex spaces over P. Next we show that such an isomorphism can be taken to commute with real structures. First we show the following:

Lemma 8.7. *Let σ and σ' be anti-holomorphic and fixed point free involutions of $\hat{Z}$ which send the divisors E_j to E_{j+k} for every $j, 1 \leq j \leq k$, and which induce the given real structure on P. Then there exists an automorphism d of $\hat{Z}$ such that $\sigma' = d\sigma d^{-1}$.*

Proof. As usual if we set $v = \sigma'\sigma^{-1}$, v defines a *holomorphic* and *relative* automorphism of $\hat{Z}$ over P. By our assumptions v preserves each irreducible component of E. Hence by Lemma 8.3 we conclude that v is induced by the action of some element g of G. Thus we may write $\sigma' = \sigma g,\ g \in G$.

From the fact that σg is an involution, by the same argument as in Lemma 2.7, after conjugation by a suitable element of G we may assume that g is either an element of order two of K or the unit element. We show, however, that if $g \neq e$ and $k > 2$, σg necessarily has a fixed point on Z and therefore is inappropriate.

Indeed, since g is an involution which is homotopic to the identity, and M is topologically $m\boldsymbol{P}^2$ with $m > 0$ by our assumption, it has always a two dimensional component in its fixed point set on M (cf. the proof of [15, Prop. 1]), which then must be one of B_i. Take a point x of B_i'. Then g has two fixed points on the twistor line L_x; take an isomorphism $L_x \to P$ which sends these two points to 0 and ∞. Take an affine coordinate z of P such that the action of g takes the form $z \to -z$. Then the action on σ on L_x takes the form $z \to -a/\overline{z}$ with $a > 0$. It follows that σg admits a fixed point set $|z| = \sqrt{a}$ on $L_x \subseteq Z$.

When $k = 2$, there exists a unique element g of order two with only isolated fixed points. However, in this case Z is the complex projective space and a direct computation shows easily that σg is conjugate to σ by some element of Aut $\hat{Z}$. The detail is left to the reader. q.e.d.

Lemma 8.8. *The isomorphism $\hat{j} : \hat{Z} \to \hat{Z}'$ over P obtained in Lemma* 8.6 *can be taken so that it is compatible with the real structures $\hat{\sigma}$ on $\hat{Z}$ and $\hat{\sigma}'$ on $\hat{Z}'$.*

Proof. We set $\hat{\sigma}_1 = \hat{j}^{-1}\hat{\sigma}\hat{j}$. Then both $\hat{\sigma}$ and $\hat{\sigma}_1$ are anti-holomorphic involutions satisfying the condition of Lemma 8.7. Hence by that lemma there exists an element $d \in G$ such that $\hat{\sigma} = d\hat{\sigma}_1 d^{-1}$. Then if we replace the original $\hat{j}$ by $\hat{j}d^{-1}$, we get the desired compatibility. q.e.d.

(8.6) Next we verify the G-equivariance of the isomorphism.

Lemma 8.9. *The isomorphism $\hat{j} : \hat{Z} \to \hat{Z}'$ above is G-equivariant.*

Proof. By Lemma 8.3 we have natural identifications

$$G = \mathrm{Aut}(\hat{Z}, \{E_i\})$$

and

$$G = \mathrm{Aut}(\hat{Z}', \{E_i'\}).$$

On the other hand, the map

$$g \to \hat{j} g \hat{j}^{-1}, g \in \mathrm{Aut}(\hat{Z}, \{E_i\}),$$

induces a group isomorphism of $\mathrm{Aut}(\hat{Z}, \{E_i\})$ and $\mathrm{Aut}(\hat{Z}', \{E_i'\})$, which in view of the above identification is considered as an automorphism $\zeta : G \to G$. We have to show that this is the identity; in fact since $\hat{j}$ maps the irreducible components E_i to E_i', the induced automorphism of the group N of one parameter groups reduces to the identity. Thus ζ itself must be the identity, i.e., $\hat{j}$ is G-equivariant. q.e.d.

(8.7) By the above lemmas we have obtained a G-equivariant isomorphism $\hat{j} : \hat{Z} \to \hat{Z}'$ over P which is compatible with the real structures of both spaces. Since it maps E onto E', it descends to an isomorphism $j : Z \to Z'$ of the twistor spaces which is compatible with the real structures of both spaces. Finally the following lemma finishes our proof of Theorem 8.1.

Lemma 8.10. *The isomorphism $j : Z \to Z'$ above sends any twistor line of Z to a twistor line of Z'.*

Proof. Let D be the open subset of the Douady space of Z [5] consisting of points whose corresponding subspaces are nonsingular rational curves with normal bundle isomorphic to $O(1) \oplus O(1)$. Then the real structure σ of Z acts on D as an anti-holomorphic involution and M is considered as a connected component of the fixed point set of this action. Similar statement holds true also for Z' and M' and the corresponding objects will be denoted with the superscript $'$.

Since j is (σ, σ')-equivariant, the induced map $\delta : D \to D'$ sends the fixed point set of σ to the fixed point set of σ'. Now $\hat{j}$ sends the fiber $\hat{Z}_{a_i}$ isomorphically onto $\hat{Z}'_{a_i}$ and hence the corresponding singular locus $\hat{L}_i$ to $\hat{L}_i'$, which are the proper transforms of the twistor lines L_i and L_i' on Z and Z' respectively. This implies that j sends the twistor line L_i onto L_i', and hence $\delta(d_i) = d_i'$, where $d_i \in D$ and $d_i' \in D'$ are the points corresponding to L_i and L_i' respectively. By what we have remarked above, this already shows that σ maps M diffeomorphically onto M'. Namely, the twistor lines are mapped to twistor lines by j. q.e.d.

(8.8) For the reference in the next section we study further the structure of the real part of the fiber space $f : \hat{Z} \to P$, i.e., we consider the structure of the induced fiber space $f_R : \hat{Z}_R \to R$, where $\hat{Z}_R =$

$f^{-1}(R)$. Especially we are interested in the structure of

$$Q := (t\mu)^{-1}(B) \cap \hat{Z}_R \tag{69}$$

and its complement in $\hat{Z}_R$

$$A := \hat{Z}_R - Q.$$

Q is the union of the proper transforms in $\hat{Z}$ of all the special twistor lines on Z. Thus for any $a \in R$ the fiber Q_a is the union of twistor lines contained in $\hat{Z}_a = S_a$. K acts smoothly on $\hat{Z}_R$ over R and the action is free on A.

Lemma 8.11. *Each fiber $A_a, a \in R$, of A has exactly two connected components.*

Proof. 1) When $a \neq a_i$ and a is on the arc $a_i a_{i+1}$, consider the σ-equivariant morphism $\nu_i : S_a \to P_i$ in Lemma 2.6 and write $P_i - R_i = D_i^+ \coprod D_i^-$ as two disjoint open discs $D_i^{\pm}$ where R_i is the real part of P_i. Then

$$Q_a = S_a \cap t^{-1}(B) = \nu_i^{-1}(R_i)$$

and

$$A_a = S_a - \nu_i^{-1}(R_i) = \nu_i^{-1}(D_i^+) \coprod \nu_i^{-1}(D_i^-);$$

thus A_a has two connected components.

2) When $a = a_i$, $S_a = S_i$ and we have $S_i \cap t^{-1}(B) = L_i$. Thus $A_a = S_i - L_i = (S_i^+ - L_i) \cup (S_i^- - L_i)$ and A_a has two connected components. q.e.d.

Nevertheless we have the following:

Lemma 8.12. *A is connected.*

Proof. We consider the quotients $\overline{Z}_R := \hat{Z}_R/K, \overline{Q} := Q/K$ and $\overline{A} := A/K$ over R. We have

$$\overline{A} = \overline{Z}_R - \overline{Q}.$$

For each $a \in R$ we set $\overline{S}_a := \overline{Z}_{Ra} = S_a/K$. For $a \neq a_i$ it is a $2k$-gon, i.e., a polygon with $2k$-edges (together with its interior) considered as a 2-manifold with corners (cf. [17, 1.3]). For $a = a_i$ it is a union of two $(k+1)$-gons with a unique common edge l_i which is the image of the very special twistor line L_i. We may "flatten" this union topologically along the edge l_i and consider it as a $2k$-gon in this case too, regarding l_i as a special diagonal of it.

The boundary $\partial\overline{S}_a$ of $\overline{S}_a$ is naturally identified with the quotient $\overline{C} := C/K$ and the union $\bigcup_a \partial\overline{S}_a$ is a trivial fiber bundle over R with fiber $\overline{C}$. Thus $\overline{C}_i := C_i/K,\ 1 \le i \le 2k$, form boundary edges of each $2k$-gon $\overline{S}_a$ and the images $\overline{z}_i$ of $z_i(= C_i \cap C_{i+1}),\ 1 \le i \le 2k$, form the vertices of the $2k$-gon. For $a = a_i$ the mentioned diagonal l_i connects exactly the vertices $\overline{z}_i$ and $\overline{z}_{i+k}$. Also for $a \neq a_i$, real twistor lines in Q_a are K-equivalent to each other and the quotient Q_a/K gives again a kind of diagonal l_a connecting some points $\overline{z}_{ia}$ and $\overline{z}_{i+k,a}$ on the edges $\overline{C}_i$ and $\overline{C}_{i+k}$ other than the vertices.

Thus, topologically we may consider $\overline{Z}_R \to R$ as the trivial bundle $R \times D \to R$ where D is a closed two-disc, and $\overline{Q}$ is then the union of diagonals $\bigcup_{a\in R} l_a$, and form a bundle over $R = S^1$ with closed interval as a fiber. Since its boundary $\overline{C}$ is connected, $\overline{Q}$ is homeomorphic to a Möbius band; indeed, when a moves from a_i to a_{i+1} on the edge $a_i a_{i+1}$, a diagonal starts from l_i and by way of l_a reaches to l_{i+1}, while both edges of the diagonal which are the points on $\overline{C}_i$ (resp. $\overline{C}_{i+k}$) goes from $\overline{z}_i$ (resp. $\overline{z}_{i+k}$) to $\overline{z}_{i+1}$ (resp. $\overline{z}_{i+k+1}$) via $\overline{z}_{ia}$ (resp. $\overline{z}_{i+k,a}$) for $1 \le i \le k$, with the indices i considered cyclically modulo $2k$. Therefore the complement $\overline{A}$ of $\overline{Q}$ in $\overline{Z}_R$ is connected, and hence its inverse image A also is connected. q.e.d.

(8.9) Let L be a general twistor line, identified with its inverse image in $\hat{Z}$. By Proposition 6.15 the induced map $f|L : L \to P$ is a double covering branched exactly over σ-conjugate points, say a and $\overline{a} \in P - R$. We note that $L \cap \hat{Z}_R \subseteq A$; otherwise it would intersect with some other twistor line in view of the definition of A.

As in (8.3) take a double covering $u : \widetilde{P} \to P$ branced exactly over the two points a and $\overline{a}$ and let $b_1, \dots, b_{2k}$ be points of $\widetilde{R} := u^{-1}(R)$ which are the inverse images of a_i. We set

$$\hat{Z}_0 = \hat{Z} - (E \bigcup (\cup \hat{L}_i))$$

where $\hat{L}_i$ is the proper transform of L_i in $\hat{Z}$. $\hat{Z}_0$ is a G-invariant set on which G acts freely. Then $\widetilde{Z}_0 := \hat{Z}_0 \times_P \widetilde{P}$ is a G-invariant Zariski open subset of the total space of the induced fiber space $\widetilde{Z} := \hat{Z} \times_P \widetilde{P} \to \widetilde{P}$ with the induced G-action. The inverse image of L in $\widetilde{Z}$ decomposes into two irreducible components $L^{\pm}$. The following lemma is used in the proof of the injectivity of a certain map in the next section.

Lemma 8.13. *There exist G-invariant Zariski open subsets $V^{\pm}$ of $\widetilde{Z}$ contained in $\widetilde{Z}_0$ such that $\widetilde{Z}_0 = V^+ \cup V^-$, $V^{\pm} \cong \widetilde{P} \times G$ and $L^{\pm} \subseteq V^{\pm}$*

by a suitable choice of the signs $\pm$.

Proof. By Lemma 8.12 A is connected, while its inverse image $\widetilde{A} := A \times_R \widetilde{R}$ in $\widetilde{Z}$ admits exactly two connected components $\widetilde{A}^{\pm}$. In fact, the pull-back $\widetilde{Q} := \overline{Q} \times_R \widetilde{R} \to \widetilde{R}$ of the Möbius band $\overline{Q} \to R$ becomes a trivial bundle over $\widetilde{R}$. Correspondingly $\widetilde{A}/K = \overline{A} \times_R \widetilde{R}$ gets exactly two components with common boundary $\widetilde{Q}$. This implies that $\widetilde{A}$ also has two connected components. $\widetilde{A}$ is contained in $\widetilde{Z}_0$ and $\widetilde{A}^{\pm}_{b_i} := \widetilde{A}^{\pm} \cap \widetilde{S}_{b_i}$ is Zariski open in $\widetilde{S}_{b_i} := \widetilde{f}^{-1}(b_i)$. We now define

$$V^{\pm} := \widetilde{Z}_0 - \bigcup_{1 \le i \le 2k} \widetilde{A}^{\mp}_{b_i}$$

so that $V^{\pm} \cap \widetilde{A}_{b_i} = \widetilde{A}^{\pm}_{b_i}$. $V^{\pm}$ are clearly G-invariant Zariski open subsets of $\widetilde{Z}$ contained in $\widetilde{Z}_0$ with $\widetilde{Z}_0 = V^+ \cup V^-$.

Since $L^{\pm}_{\widetilde{R}} := L^{\pm} \cap \widetilde{f}^{-1}(\widetilde{R})$ is connected and contained in $\widetilde{A}$, after changing the signs if necessary, we may assume that $L^{\pm}_{\widetilde{R}} \subseteq \widetilde{A}^{\pm}$. Then we have $L^{\pm} \subseteq V^{\pm}$.

Finally the map $L^{\pm} \times G \to V^{\pm}$ induced by the G-action is isomorphic since the G-action on $\widetilde{Z}_0$ is free, G-orbits on $V^{\pm}$ are precisely the fibers of $V^{\pm} \to \widetilde{P}$, and that the intersection of $L^{\pm}$ with each G-orbit is transversal at exactly one point. q.e.d.

9. Coincidence of invariants

(9.1) Let m be a non-negative integer and $M = m\boldsymbol{P}^2$ with an effective and smooth K-action. Suppose that we are given a K-invariant self-dual conformal structure $[g]$ on M. Then the quotient $N = M/K$ is a closed two-disc and we have a cyclic sequence $(y_1, \ldots, y_k)$ of points on the boundary bN which are the images of fixed points of the K-action on M. On the other hand, let Z be the twistor space associated to $(M, [g])$ and $f : \hat{Z} \to P$ the associated fiber space, where P is endowed with the affine coordinate z (53) with σ given by $\sigma(z) = 1/\overline{z}$. Let $a_1, \ldots, a_k$ be the points of the fixed point set R of σ corresponding to the singular fibers of f such that y_i is mapped to a_i by the natural homeomorphism $\phi : bN \to R$ obtained in Proposition 6.16. We set

$$D := \{z \in P; |z| \le 1\}$$

so that R is the boundary of D. In this section we prove that ϕ extends naturally to a diffeomorphism $\psi : N \to D$ which preserves the geometric structures on the interiors D_0 and N_0 respectively: namely the induced complex structure on D_0 and the induced conformal structure of N_0 (cf. (3.6)). Recall that in real dimension two, giving a conformal structure is equivalent to giving a complex structure modulo the complex conjugate. The precise statement of our theorem is as follows.

Theorem 9.1. *There exists a natural homeomorphism $\psi : N \to D$ with the following properties:*

1) *ψ restricts to ϕ on the boundary bN; in particular ψ sends y_i to a_i for each $1 \leq i \leq k$, and*

2) *ψ induces a diffeomorphism $N_0 \to D_0$ of the interiors, with respect to which the conformal structure on N_0 and the holomorphic structure of D_0 modulo its complex conjugate are compatible, i.e., ψ sends the conformal structure of N_0 to one on D_0 which coincides with that defined by the complex structure of D.*

One may think of D_0 as a kind of (mini-mini)-twistor space of N_0. We first deduce Theorem 1.1 from Theorem 9.1.

Proof of Theorem 1.1. By Corollary 6.8 we may assume that $M = m\boldsymbol{P}^2$.

Let $\mathcal{J}$ be the subset of $\mathcal{C}$ of (6.12) consisting of the isomorphism classes of Joyce's self-dual manifolds [11]. We have a diagram

$$\begin{array}{ccc} \mathcal{J} & \xrightarrow{I_c} & \mathcal{A} \\ \cap & & \| \\ \mathcal{C} & \xrightarrow{I_t} & \mathcal{A} \end{array}$$

where I_c and I_t are the maps given by the conformal invariants and the twistorial invariants respectively (cf. (6.12)). This diagram is commutative by Theorem 9.1 and I_t is injective by Theorem 8.1. Thus I_c on the top line is also injective and we have $\mathcal{J} = \mathcal{C}$. This is precisely the assertion of Theorem 1.1. q.e.d.

Proof of Corollary 1.2. Since M admits a K-action, $\chi(M)$ is nonnegative by Lemma 3.1. The case $\chi(M) > 0$ is treated in Theorem 1.1. If $\chi(M) = 0$, the K-action has no fixed points. Thus by [22, Th. 3.6] M must be conformally flat. q.e.d.

The rest of this section is devoted to the proof of Theorem 9.1.

(9.2) First we give the definition of $\psi(y),\ y \in N$, set-theoretically. When $y \in bN$, we just set $\psi(y) = \phi(y)$ with ϕ in (6.11). In case $y \in N_0$, we define $\psi(y) \in D_0$ as follows. Let $x \in M - B$ be an arbitrary point with $\pi(x) = y$ and $L = L_x$ the corresponding general twistor line. By Proposition 6.15 precisely one of the branch points of the double covering $f|L : L \to P$ is in D_0, which we shall denote by $a(x)$. Since S_a are K-invariant, we have $a(x) = a(t(x))$, and hence, we may write $a(x) = a(y)$. Thus we can define

$$\psi(y) = a(x).$$

We show the following:

Lemma 9.2. *$\psi|N_0 : N_0 \to D_0$ is bijective.*

Together with Proposition 6.16 this would imply

Corollary 9.3. *$\psi : N \to D$ is bijective.*

We need the following lemma for the injectivity.

Lemma 9.4. *Let L be a general twistor line. If $g(L)$ is again a twistor line for some $g \in G$, then $g \in K$.*

Proof. If $g(L)$ is a twistor line, we have $(\sigma g \sigma)L = g(L)$. On the other hand, since $\sigma g \sigma^{-1} = g^*$ (cf. (2.9)) we get $g^{-1}g^*(L) = L$. Namely, $|g|^2 := g^{-1}g^* \in G$ stabilizes the point d of the Douady space D_Z of Z corresponding to the submanifold L; $d = t(L) \in M_0 \subseteq D_Z$. Since Z is Moishezon and L is general, the stabilizer group of d is finite, and therefore it is contained in K; $|g|^2 \in K$. Since the K-action on M_0 is free, this implies that $|g|^2 = 1$, i.e., $g \in K$. q.e.d.

Proof of Lemma 9.2. **Surjectivity:** Let a be any point of D_0 and consider the corresponding non-real fiber $S = S_a$ of $f : \hat{Z} \to P$. By Lemma 6.7 t induces a double covering $s : S \to M$, which cannot be unramified since M is simply connected. If $z \in S$ is a ramification point, then the twistor line $L_x,\ x = t(z)$, is tangent to S at z. Thus $a = a(x) = \psi(\pi(x))$.

Injectivity. Suppose that $a = \psi(y) = \psi(y')$ in D_0 for y and y' in N_0. Take any point x and x' in M_0 over y and y' respectively. Then the twistor lines L_x and $L_{x'}$ are both tangent to the fibers S_a and $S_{\overline{a}}$. Take a double covering $\widetilde{P} \to P$ branced exactly at a and $\overline{a}$, and consider the pull-back $\widetilde{f} : \widetilde{Z} \to \widetilde{P}$ of f as in (8.9). Then the inverse images of L_x and

$L_{x'}$ in $\widetilde{Z}$ are decomposed into two irreducible components $L_x^{\pm}$ and $L_{x'}^{\pm}$ respectively and define sections of $\widetilde{f}$. L_x^{+} and $L_{x'}^{+}$ are entirely contained in the Zariski open subset $V^+ \cong G \times \widetilde{P}$ by Lemma 8.13. Since any compact curves in V^+ is of the form $g \times \widetilde{P}$ for some $g \in G$, we see that L_x^+ is a G-translate of $L_{x'}^+$, and then as their G-equivariant images L_x is a G-translate of $L_{x'}$. Thus by Lemma 9.4 L_x further is a K-translate of $L_{x'}$, which implies that $y = y'$. q.e.d.

(9.3) We next check the smoothness of ψ on N_0. Let $M_0 = M - B$ and $Z_0 = t^{-1}(M_0)$. We consider the subset

(70)
$$\Gamma := \{z \in Z_0; L_x \text{ is tangent to } S_a, \quad x = t(z), \quad a = a(x) = f(z)\}.$$

Lemma 9.5. *Γ is a closed submanifold of real codimension two in Z_0 and consists of two connected components Γ^+ and Γ^- which are mapped diffeomorphically onto M_0 by t and which are mutually σ-conjugate.*

Proof. Take any point $z_0 \in Z_0$. Since t is a submersion, we can find real local coordinates $x_1, x_2, x_3, x_4, y_1, y_2$ of Z_0 at z_0 such that t is described by the projection to the first four coordinates and that $z := y_1 + \sqrt{-1}y_2$ gives a holomorphic local coordinate on L_x for any x in a neighborhood of $t(z_0)$. Then with respect to a suitable complex coordinate of P around $f(z_0)$, f is locally considered as a complex-valued smooth function which is holomorphic when restricted to each L_x. Thus a point z in a neighborhood of z_0 belongs to Γ if and only if $(\partial f/\partial z)(z) = 0$. Namely, $g := \partial f/\partial z = 0$ is a defining equation of Γ in a neighborhood of z_0. In particular, Γ is closed in Z_0.

Since the ramification indices at branch points of $L_x \to P$ are always two, at any point z of Γ we have $\partial g/\partial z \neq 0$. Thus the implicit function theorem shows that a neighborhood of z of Γ is mapped diffeomorphically onto a neighborhood of $t(z)$ in M. This shows that Γ is a submanifold of real codimension two in Z_0 and $t|\Gamma : \Gamma \to M$ is locally diffeomorphic. Since each general twistor line has exactly two ramification points, we conclude that $t|\Gamma$ is an unramified double covering.

Let $F := \mu(f^{-1}(R))$. Then F is a closed subset of Z, and $Z - F$ has two connected components

$$Z^+ := \mu(f^{-1}(D_0) - E)$$

and

$$Z^- := \mu(f^{-1}(\sigma(D_0)) - E);$$

namely Z is a disjoint union of F, Z^+ and Z^- with Z^+ and Z^- σ-conjugate to each other. By Proposition 6.15 Γ never intersects with F so that Γ is a disjoint union of $\Gamma^+ := \Gamma \cap Z^+$ and $\Gamma^- = \Gamma \cap Z^-$ with $\sigma(\Gamma^\pm) = \Gamma^\mp$. Since for each general twistor line the corresponding two ramification points are σ-conjugate to each other, each of $\Gamma^\pm$ is mapped surjectively onto M_0. Thus $t|\Gamma^\pm$ is a diffeomorphism onto M_0. q.e.d.

Corollary 9.6. *The restricted map $\psi|N_0 : N_0 \to D_0$ is smooth.*

Proof. The map $\pi t|\Gamma^+ : \Gamma^+ \to N_0$ is a submersion by Lemma 9.5; thus it admits a smooth section $N_0 \to \Gamma^+$. Composing this with the smooth map $f|\Gamma^+ : \Gamma^+ \to D_0$ we obtain ψ, which is smooth. q.e.d.

(9.4) Next we show the following:

Proposition 9.7. *$\psi : N \to D$ is a homeomorphism.*

Since N and D are compact and Hausdorff and ψ is bijective by Corollary 9.3, Proposition 9.7 follows from the next lemma.

Lemma 9.8. *The map $\psi : N \to D$ is continuous.*

Proof. We use Lemma 6.17. We set $X = N$ and $Y = D$. Both are compact. Define

$$W := Q \cup \Gamma^+ \subseteq \hat{Z}$$

where Q is defined by (69) and Γ^+ is a closed submanifold of Z_0 in Lemma 9.5 and is identified with a subset of $\hat{Z}$ via μ^{-1}. We may also consider Q as the proper transform of $(\pi t)^{-1}(bN) = t^{-1}(B)$ in $\hat{Z}$, namely, it is the closure of $\mu^{-1}(t^{-1}(B) - C)$ in $\hat{Z}$. By the definition Q is clearly compact.

Putting on W the relative topology from $\hat{Z}$ we have the continuous maps $h := \pi t \mu|W : W \to X$ and $g := f|W : W \to Y$ with h surjective; moreover by our definition of the map ψ, it is clear that h and g satisfy the condition $\psi(x) = g(h^{-1}(x)), x \in X$. Thus, in view of Lemma 6.17, we have only to show that W is closed in $\hat{Z}$, and hence, is compact.

Let $\hat{\Gamma}^+$ be the closures of Γ^+ in $\hat{Z}$. Clearly, $\hat{\Gamma}^+ \cup Q$ is closed. It suffices to show that $\hat{\Gamma}^+ \cup Q = W$, which in turn would follow if we show that $\hat{\Gamma}^+ - \Gamma^+ \subseteq Q$. Since Γ^+ is closed in $\hat{Z} - \mu^{-1}(t^{-1}(B))$, we must have

$$\hat{\Gamma}^+ - \Gamma^+ \subseteq \mu^{-1}(t^{-1}(B)) = Q \cup \mu^{-1}(C).$$

Now take an arbitrary point $y \in \hat{\Gamma}^+ - \Gamma^+$. Take a sequence of points $y_n \in \Gamma^+$ converging to y. Then $\mu(y_n)$ converge to $\mu(y) =: \widetilde{y}$ in

Z and $x_n := t\mu(y_n)$ converge to $x := t(\widetilde{y})$ in M. Thus $\widetilde{y} \in L_x$ and $y \in \mu^{-1}(L_x)$. On the other hand, since $y \notin \Gamma^+$, we have $x \in B$ and L_x is a special twistor line. Thus $\mu^{-1}(L_x)$ has the form $l_x^+ \cup \hat{L}_x \cup l_x^-$, where $\hat{L}_x$ is the proper transform of L_x in $\hat{Z}$, and $l_x^\pm = \mu^{-1}(z^\pm)$, where $L_x \cap C = \{z^+, z^-\}$.

We have to show that $y \in \hat{L}_x$ since $\hat{L}_x \subseteq Q$. So supposing that $y \notin \hat{L}_x$ and hence that $y \in l_x^+ - \hat{L}_x$ say, we shall derive a contradiction. Since x_n converge to x, we have the convergence $L_{x_n} \to L_x$ as subspaces of Z. Thus by taking the inverse image by μ we see that in a neighborhood of y, L_{x_n} (identified with its inverse image $\mu^{-1}(L_{x_n})$ in $\hat{Z}$) converge to l_x^+, and $y_n \in L_{x_n}$; in particular we get the convergence of the tangent spaces $T_{y_n}L_{x_n} \to T_y l_x^+$. But l_x^+ is transversal, at each of its point z, to the fiber $f^{-1}(f(z))$. Hence, for all sufficiently large n, L_{x_n} should also be transversal at y_n to the fiber of f passing through this point. This contradicts the fact $y_n \in \Gamma^+$. q.e.d.

(9.5) Finally, we study in detail the differential ψ_* of ψ on N_0. Especially we show that it is isomorphic everywhere; this would follow from the submersiveness of $f\mu^{-1}|\Gamma : \Gamma \to P$, in view of the commutative diagram

$$\begin{array}{ccccc} \Gamma & \subseteq & Z & \overset{\mu}{\leftarrow} & \hat{Z} \\ t\,\| & & t\downarrow & & \\ M_0 & \subseteq & M & & \downarrow f \\ \pi\downarrow & & \pi\downarrow & & \\ N_0 & \subseteq & N & \overset{\psi}{\to} & P. \end{array}$$

For this purpose we pass to the complexfication of the situation. This has the advantage that we can work in a purely complex analytic setting. The desired submersiveness of ψ will be proved in Proposition 9.17 below in this complexfied setting.

We start with the complexfication of M. Let $M'^{\boldsymbol{C}}$ be the irreducible component of the Douady space of Z [5] which contains the points corresponding to twistor lines of Z, and $M^{\boldsymbol{C}}$ the Zariski open subset of $M'^{\boldsymbol{C}}$ corresponding to nonsingular rational curves whose normal bundle in Z is of type $O(1) \oplus O(1)$. Then $M^{\boldsymbol{C}}$ is a 4-dimensional complex manifold and M, as the parameter space of twistor lines, is considered as a totally real submanifold of $M^{\boldsymbol{C}}$. Indeed, σ induces an anti-holomorphic involution on $M^{\boldsymbol{C}}$ and M is one of the connected components of its fixed point set. $M^{\boldsymbol{C}}$ also admits a natural G-action. We have the universal family $t_{\boldsymbol{C}} : Z^{\boldsymbol{C}} \to M^{\boldsymbol{C}}$ over $M^{\boldsymbol{C}}$ which fits into the commutative diagram

$$\begin{array}{ccccc} Z & \subseteq & Z^{\boldsymbol{C}} & \stackrel{\alpha}{\to} & Z \\ t\downarrow & & \downarrow t_{\boldsymbol{C}} & & \\ M & \subseteq & M^{\boldsymbol{C}} & & \end{array}$$

where $\alpha : Z^{\boldsymbol{C}} \to Z$ is the natural projection and $\alpha|Z : Z \to Z$ is the identity. The G- and σ-actions lift naturally to $Z^{\boldsymbol{C}}$ making α and $t_{\boldsymbol{C}}$ equivariant. $Z^{\boldsymbol{C}}$ is a 5-dimensional complex manifold and $t_{\boldsymbol{C}}$ is a holomorphic submerion.

For each point $x \in M^{\boldsymbol{C}}$ the fiber $t_{\boldsymbol{C}}^{-1}(x)$ is identified with the corresponding subspace of Z and will be denoted again by L_x. We call such an L_x a *complex twistor line* in general.

We take and fix a G-invariant Zariski open subset $M_0^{\boldsymbol{C}}$ of $M^{\boldsymbol{C}}$ such that $M_0^{\boldsymbol{C}} \cap M = M_0$ and $L_x \subseteq Z-(C \cup L)$ for any $x \in M_0^{\boldsymbol{C}}$ (cf. (5.3)) so that $f\mu^{-1}|L_x : L_x \to P$ is holomorphic and is a double covering; e.g., we may take $M_0^{\boldsymbol{C}} = M^{\boldsymbol{C}} - t(Z'^{\boldsymbol{C}})$, where $Z'^{\boldsymbol{C}} = \{z \in Z^{\boldsymbol{C}}; \dim G_z > 0\}$. Moreover, we have

Lemma 9.9. *The G-action on $M_0^{\boldsymbol{C}}$ is free.*

Proof. Let x be any point of $M_0^{\boldsymbol{C}}$. Then the stabilizer G_x acts on L_x, while the G-action on Z is free at each point of L_x since $L_x \subseteq Z-(C \cup L)$. Thus the action of G_x is free on $L_x \cong \boldsymbol{P}^1$, which implies that G_x reduces to the identity. q.e.d.

(9.6) For any $z \in Z$ the fiber $F(z) := \alpha^{-1}(z) \subseteq Z^{\boldsymbol{C}}$ of α is mapped isomorphically by $t_{\boldsymbol{C}}$ onto its image $M^{\boldsymbol{C}}(z) := t_{\boldsymbol{C}}(F(z))$ in $M^{\boldsymbol{C}}$, which in turn is identified with the subspace of $M^{\boldsymbol{C}}$ parametrizing complex twistor lines L_x with $z \in L_x$. Denote by N_x the normal bundle of L_x in Z.

Lemma 9.10. *$F(z)$ and $M^{\boldsymbol{C}}(z)$ are smooth of dimension two. The map $\alpha : Z^{\boldsymbol{C}} \to Z$ is a submersion.*

Proof. Let $N_x(-z)$ denote the sheaf of sections of N_x vanishing at z. Since $H^1(L_x, N_x(-z)) = 0$ for any $x \in M^{\boldsymbol{C}}(z)$, by standard deformation theory we have that $M^{\boldsymbol{C}}(z)$ is smooth of dimension two. The assertion on $F(z)$, and hence the submersiveness of α also, follows from this. q.e.d.

The inclusions of the tangent spaces at $x \in M^{\boldsymbol{C}}(z)$ are described cohomologically as follows:

$$\begin{array}{ccc} T_xM^{\boldsymbol{C}}(z) & \subseteq & T_xM^{\boldsymbol{C}} \\ \| & & \| \\ H^0(L_x, N_x(-z)) & \subseteq & H^0(L_x, N_x). \end{array} \tag{71}$$

When $x \in M$, $T_xM^{\boldsymbol{C}}$ is identified with the complexfication of the (real) tangent space T_xM. If, further, $x \in M_0$, then since the K-action is free at x, the associated infinitesimal action on M determines a (real) two dimensional subspace $E = E_x$ of T_xM, whose complexfication $E_{\boldsymbol{C}} = E_{\boldsymbol{C}x}$ arises from the infinitesimal G-action on $M^{\boldsymbol{C}}$ at x;

$$\begin{array}{ccc} E & \subseteq & E_{\boldsymbol{C}} \\ \cap & & \cap \\ T_xM & \subseteq & T_xM^{\boldsymbol{C}}. \end{array}$$

Lemma 9.11. *For any $z \in \Gamma$ and $x \in M^{\boldsymbol{C}}(z)$ we have*

$$\dim(T_xM^{\boldsymbol{C}}(z) \cap E_{\boldsymbol{C}}) = 1. \tag{72}$$

Proof. Since $z \notin k(L_x)$ for any $k \in K$ with $k \neq e$, we have $T_xM^{\boldsymbol{C}}(z) \cap E = \{0\}$. It thus suffices to show that $T_xM^{\boldsymbol{C}}(z) \cap E_{\boldsymbol{C}} \neq \{0\}$ since for $T_xM^{\boldsymbol{C}}(z)$ and $E_{\boldsymbol{C}}$ are of dimension two. By the definition of Γ, L_x is tangent to S_a at z, where $a = f(z)$. Since S_a coincides with the G-orbit of z at z, there exists an element v of the Lie algebra Lie G of G whose associated vector field $\underline{v}$ on Z is tangent to L_x at z. Then the section $w \in H^0(L_x, N_x)$ induced from $\underline{v}|L_x$ via the natural projection $TZ|L_x \to N_x$ vanishes at z. However w is identified, with respect to the identification (71), with the element $\delta(v)$ of $T_xM^{\boldsymbol{C}}$ where δ denotes the infinitesimal G-action on $M^{\boldsymbol{C}}$ at x;

$$\begin{array}{ccc} \mathrm{Lie}\, G & \stackrel{\delta}{\to} & T_xM^{\boldsymbol{C}} \\ \downarrow & & a \\ H^0(L_x, TZ|L_x) & \to & H^0(L_x, N_x). \end{array}$$

Thus if we consider $\delta(v) \in T_xM^{\boldsymbol{C}}$ as an element of $H^0(L_x, N_x)$, it vanishes at $z \in L_x$. Then $\delta(v)$ is an element of $T_xM^{\boldsymbol{C}}(z)$ in view of (71), while it is obviously an element of $E_{\boldsymbol{C}}$. Note finally that $\delta(v)$ is nonzero because the G-action on $M^{\boldsymbol{C}}$ is locally free at x by Lemma 9.9.

q.e.d.

(9.7) Set $Z_0^{\boldsymbol{C}} = t_{\boldsymbol{C}}^{-1}(M_0^{\boldsymbol{C}})$. We now define a complex analogue of Γ by

$$\Gamma_{\boldsymbol{C}} := \{z \in Z_0^{\boldsymbol{C}}; L_x \ \text{ is tangent to } \ S_a, \text{ where } \ x = t_{\boldsymbol{C}}(z) \text{ and } a = f\alpha(z)\}$$

which fits into the commutative diagram

$$\begin{array}{ccccc} \Gamma & \subseteq & \Gamma_{\boldsymbol{C}} & \subseteq & Z_0^{\boldsymbol{C}} \\ \downarrow t & & \downarrow t_{\boldsymbol{C}} & & \downarrow t_{\boldsymbol{C}} \\ M_0 & \subseteq & M_0^{\boldsymbol{C}} & = & M_0^{\boldsymbol{C}}. \end{array}$$

Clearly we have $\Gamma = \Gamma_{\boldsymbol{C}} \cap Z$; moreover, Γ and $\Gamma_{\boldsymbol{C}}$ are K- and G-invariant respectively and the induced actions are free by Lemma 9.9; moreover they are σ-invariant. Analogously to Lemma 9.5 we get

Lemma 9.12. *$\Gamma_{\boldsymbol{C}}$ is a σ-invariant closed and smooth divisor in $Z_0^{\boldsymbol{C}}$ and is an unramified double covering of $M_0^{\boldsymbol{C}}$.*

Proof. The ramifcation locus of $v = t_{\boldsymbol{C}} \times (f\mu^{-1}\alpha) : Z_0^{\boldsymbol{C}} \to M_0^{\boldsymbol{C}} \times P$ coincides with $\Gamma_{\boldsymbol{C}}$. Thus it is a closed divisor in $Z_0^{\boldsymbol{C}}$. More precisely, for any point $z_0 \in Z_0^{\boldsymbol{C}}$ we may take holomorphic local coordinates of $M^{\boldsymbol{C}}$ at $x := t_{\boldsymbol{C}}(z_0)$ and (z_1, z_2, z_3, z_4, z) of $Z_0^{\boldsymbol{C}}$ around z_0 such that $t_{\boldsymbol{C}}$ is given by $t_{\boldsymbol{C}}(z_1, z_2, z_3, z_4, z) = (z_1, z_2, z_3, z_4)$. Then $z|L_x$ is a local parameter on L_x and the Jacobian matrix for v is of the form Jac

$$v = \begin{pmatrix} I & 0 \\ * & \partial f/\partial z \end{pmatrix},$$

where I is the identity matrix of size four and we have written $f = f\mu^{-1}\alpha$ for simplicity. Therefore

$$z \in \Gamma_{\boldsymbol{C}} \Leftrightarrow \text{detJac } v(z) = 0 \Leftrightarrow \frac{\partial f}{\partial z}(z) = 0$$

Since the defining function $\partial f/\partial z$ of $\Gamma^{\boldsymbol{C}}$ restricted to L_x is locally of the form qz for some function q with $q(z_0) \neq 0$, $\Gamma_{\boldsymbol{C}}$ is smooth and its intersection with each fiber of $t_{\boldsymbol{C}}$ is transversal.

The final assertion follows from the fact that $L_x \subseteq Z - (C \cup L)$ and $L_x \to P$ is a double covering. q.e.d.

(9.8) Let $\widetilde{E}_{\boldsymbol{C}}$ (resp. $\widetilde{E}$) be the subspace of the tangent space $T_z\Gamma_{\boldsymbol{C}}$ (resp. of $T_z\Gamma$) arising from the infinitesimal action of G on $\Gamma_{\boldsymbol{C}}$ (resp. K on Γ).

Lemma 9.13. *For any* $z \in \Gamma$ *we have* $\widetilde{E}_{\boldsymbol{C}} \cap T_zF(z) = \{0\}$ *and*

$$\dim(T_zF(z) \cap T_z\Gamma_{\boldsymbol{C}}) = 1.$$

Proof. Let v be any element of $\widetilde{E}_{\boldsymbol{C}} \cap T_zF(z)$. Since $v \in T_zF(z)$, we have $\alpha_* v = 0$, while since α is G-equivariant and the G-action is free at $z \in Z_0$, the restriction $\alpha_*|\widetilde{E}_{\boldsymbol{C}}$ must be injective. Thus $v = 0$ and the first assertion is proved.

Since $F(z)$ is of dimension two and $\Gamma_{\boldsymbol{C}}$ is of codimension one, we have $\dim(T_zF(z) \cap T_z\Gamma_{\boldsymbol{C}}) \geq 1$. So assuming that this dimension is equal to two, i.e., that $T_zF(z)$ is contained in $T_z\Gamma_{\boldsymbol{C}}$, we shall derive a contradiction. In fact the natural isomorphism $t_{\boldsymbol{C}*} : T_z\Gamma_{\boldsymbol{C}} \cong T_xM^{\boldsymbol{C}}$, $x = t(z)$, induces isomorphisms $\widetilde{E}_{\boldsymbol{C}} \cong E_{\boldsymbol{C}}$ and $T_zF(z) \cong T_xM^{\boldsymbol{C}}(z)$. Then (72) would imply that $\widetilde{E}_{\boldsymbol{C}} \cap T_zF(z) \neq \{0\}$. This contradicts the first assertion. q.e.d.

Corollary 9.14. *In a neighborhood of* Γ *the induced map* $\alpha|\Gamma_{\boldsymbol{C}} : \Gamma_{\boldsymbol{C}} \to Z$ *is a holomorphic submersion; therefore so is the composite morphism* $f\mu^{-1}\alpha|\Gamma_{\boldsymbol{C}} : \Gamma_{\boldsymbol{C}} \to P$.

Proof. This follows from the above lemma and the fact that the kernel of the differential of $\alpha|\Gamma_{\boldsymbol{C}}$ is precisely $T_zF(z) \cap T_z\Gamma_{\boldsymbol{C}}$ for $z \in \Gamma$. q.e.d.

(9.9) Let x be any point of M_0. $T_xM^{\boldsymbol{C}}$ admits a holomorphic conformal structure which induces the given conformal structure on T_xM. Let $E^\perp$ be the orthogonal complement of E in T_xM with respect to the given conformal structure. Then its complexfication $E^\perp_{\boldsymbol{C}}$ is the orthogonal complement of $E_{\boldsymbol{C}}$ with respect to the holomorphic conformal structure on $T_xM^{\boldsymbol{C}}$. We have the direct sum decomposition

$$T_xM^{\boldsymbol{C}} = E_{\boldsymbol{C}} \oplus E^\perp_{\boldsymbol{C}} \tag{73}$$

which is defined over the reals.

For the computation below it is convenient to work with a concrete model of the space $T_xM^{\boldsymbol{C}} = H^0(L_x, N_x)$ as follows. Fix an oriented (conformal) isometry $T_xM \to \boldsymbol{H}$ inducing the identification $L_x = C$, where $1, i, j, k$ is an oriented orthonormal basis of $\boldsymbol{H}$, such that its complexfication

$$T_xM^{\boldsymbol{C}} \to \boldsymbol{H} \otimes \boldsymbol{C} \tag{74}$$

maps the direct sum decomposition (73) into the natural decomposition

$$\boldsymbol{H} \otimes \boldsymbol{C} = (\boldsymbol{C} \oplus \boldsymbol{C}i) \oplus (\boldsymbol{C}j \oplus \boldsymbol{C}k), \tag{75}$$

where one should not confuse $i \in \boldsymbol{H}$ with $\sqrt{-1} \in \boldsymbol{C}$; in particular 1 and i are linearly independent over $\boldsymbol{C}$.

Correspondingly, we take a (real) model of the normal bundle $N = N_x$ as $C \times \boldsymbol{H}$ as in (28). In this identification the fiber N_q over $q \in C$ of $N \to C$ is identified with the complex vector space $\boldsymbol{H}$ with complex structure given by the right multiplication by q^{-1}. Thus, when one considers an element w of $\boldsymbol{H} \otimes \boldsymbol{C}$ as a section of N_x with respect to the identifications (71) and (74), at any point $q \in C = L_x$ the value $w(q) \in N_q \cong \boldsymbol{H}$ is obtained by replacing any $\sqrt{-1}$ in the expression of w via (75) by $q^{-1} = -q$. Now we have the following:

Lemma 9.15. *Let $x \in M_0$ be as above. Then we have $\Gamma \cap L_x = \{\pm i\}$ with respect to our identifications $L_x = C$ above.*

Proof. Consider an element of $E_{\boldsymbol{C}}$ as a holomorphic section of N_x via the isomorphisms (73) and (71). Let $z \in \Gamma \cap L_x$. By Lemma 9.11 and (71) there exists a nonzero element $\beta \in E_{\boldsymbol{C}}$ which vanishes at z. Let z correspond to $q \in C$ with respect to our identification $L_x = C$. We may write $\beta = u + \sqrt{-1}v$ for some $u, v \in \boldsymbol{R} + \boldsymbol{R}i$. Now for a point $q \in C$, β vanishes at q as a section of N_x if and only if $u - vq = 0$, or equivalently, $u - vsi - jvt = 0$ if $q = si + jt, s \in \boldsymbol{R}, t \in \boldsymbol{C}$, namely, $vt = 0$ and $u = vsi$. This implies that $t = 0$; thus $z = q = \pm i$. From this follows the assertion since $\Gamma \cap L_x$ consists of two points. q.e.d.

The method of the above proof also yields the following:

Lemma 9.16. *Let $x \in M_0$ and $z \in \Gamma \cap L_x$. Then we have*

$$\dim(T_x M^{\boldsymbol{C}}(z) \cap E_{\boldsymbol{C}}^{\perp}) = 1.$$

Proof. The element $\pm\sqrt{-1}j + k \in \boldsymbol{C}j + \boldsymbol{C}k = E_{\boldsymbol{C}}^{\perp}$ vanishes at $\pm i \in C$ when considered as a section of N_x. Thus by Lemma 9.15 either of $\pm\sqrt{-1}j + k$ belongs to $T_x M^{\boldsymbol{C}}(z)$, and hence, $T_x M^{\boldsymbol{C}}(z) \cap E_{\boldsymbol{C}}^{\perp} \neq \emptyset$. On the other hand, $\dim T_x M^{\boldsymbol{C}} \cap E_{\boldsymbol{C}}^{\perp} \leq 1$ by Lemma 9.11. The lemma follows. q.e.d.

Remark. The method of the proof of Lemma 9.15 also shows the following: *Let J be the complex structure of $T_x M$ corresponding to $z \in L_x$. Let $T_z^{\pm} M^{\boldsymbol{C}}$ be the $\pm\sqrt{-1}$-eigenspaces of J on $T_x M^{\boldsymbol{C}}$. Then* *we have the natural identifications* $T^+M^{\boldsymbol{C}} = T_xM^{\boldsymbol{C}}(\overline{z})$ *and* $T^-M^{\boldsymbol{C}} = T_xM(z)$. *Thus we have the identical direct sum decompositions*

$$T_xM^{\boldsymbol{C}} = T_xM^{\boldsymbol{C}}(z) \oplus T_xM^{\boldsymbol{C}}(\overline{z}) = T_z^-M^{\boldsymbol{C}} \oplus T_z^+M^{\boldsymbol{C}}.$$

This in turn yields the following identification of the normal bundle N_x: $(N_x)_z = T_xM(\overline{z})$ for each $z \in L_x$.

(9.10) Using Lemma 9.15 we now prove the anticipated result:

Proposition 9.17. *For any point* z *of* Γ *the composite map* $f\mu^{-1}|\Gamma : \Gamma \to P$ *is submersive at* z. *In fact, the differential* $(f\mu^{-1})_*$ *induces a real isomorphism of* $\widetilde{E}^\perp$ *onto* $T_a^{\boldsymbol{R}}P$, *where* $\widetilde{E}^\perp = (t|\Gamma)_*^{-1}(E^\perp)$, $a = f\mu^{-1}(z)$ *and* $T_a^{\boldsymbol{R}}P$ *denotes the real tangent space of* P *at* a.

Proof. Note first the following facts: $\alpha_* : T_z\Gamma \to T_zZ$ is injective, the kernel of $(f\mu^{-1})_*$ at z is T_zS_a and α_* maps $\widetilde{E}_{\boldsymbol{C}}$ isomorphically onto T_zS_a; indeed, first two assertions are clear and the last follows from the fact that α is G-equivariant and S_a is G-homogeneous locally at z. In particular $\widetilde{E}_{\boldsymbol{C}}$ is mapped into the kernel of f_*. Thus, in view of $f\mu^{-1}|\Gamma = f\mu^{-1}\alpha|\Gamma$, our task is to show that

$$\alpha_*T_z\Gamma \cap T_zS_a = \alpha_*\widetilde{E}.$$

Indeed, by dimensional consideration we then deduce that $(f\mu^{-1}\alpha)_*$ induces an isomorphism $\widetilde{E}^\perp \to T_a^{\boldsymbol{R}}P$.

Suppose now that $\alpha_*T_z\Gamma \cap T_zS_a$ is strictly larger than $\alpha_*\widetilde{E}$, namely that there exists a (real) subspace $\widetilde{E}'$ of $T_z\Gamma$ of dimension ≥ 3 which contains $\widetilde{E}$ and is mapped into T_zS_a by α_*. Since $\alpha_*(\widetilde{E}_{\boldsymbol{C}}) = T_zS_a$, any element d of $\widetilde{E}'$ is then written in $T_z\Gamma_{\boldsymbol{C}}$ in the form

$$d = b + c, \quad b \in \widetilde{E}_{\boldsymbol{C}}, \quad c \in \operatorname{Ker}\alpha_* = T_zF(z). \tag{76}$$

Passing to $T_xM^{\boldsymbol{C}}$, with respect to the natural isomorphisms $(T_z\Gamma_{\boldsymbol{C}}, T_z\Gamma) \cong (T_xM^{\boldsymbol{C}}, T_xM)$ and $T_zF(z) \cong T_xM^{\boldsymbol{C}}(z)$ we may consider (76) as a relation in $T_xM^{\boldsymbol{C}}$ with $d \in E' := t_{\boldsymbol{C}*}(\widetilde{E}')$, $b \in E_{\boldsymbol{C}}$ and $c \in T_xM^{\boldsymbol{C}}(z)$. Since $\dim E' \geq 3$, $E' \cap E^\perp \neq \{0\}$. We consider (76) in the case $0 \neq d \in E^\perp$. We shall derive a contradiction from this.

For this purpose we use (74) and (75) to identify d as an element of $\boldsymbol{R}j + \boldsymbol{R}k$ and b as an element of $\boldsymbol{C} + \boldsymbol{C}i$. We interpret these elements as sections of N_x on L_x by the isomorphism (71). By Lemma 9.15 we have $z = \pm i$ under the identification $L_x = C$ there. Then, considering $d = b + c$ as a relation among sections, and evaluating at $z = \pm i$, we get $d(z) = b(z)$ as an element of $(N_x)_z = \boldsymbol{H}$. This is a contradiction because if $b = u + \sqrt{-1}v$, where $u, v \in \boldsymbol{R} + \boldsymbol{R}i$, then $b(z) = u - zv = u \mp iv \in \boldsymbol{C} + \boldsymbol{C}i$, while we have $0 \neq d(z) = d \in \boldsymbol{R}j + \boldsymbol{R}k$ by our choice of d. q.e.d.

Corollary 9.18. *The differential ψ_* of ψ is isomorphic at every point y of N_0. It is identified with the composite of natural isomorphisms $T_yN_0 \cong E^\perp \cong \widetilde{E}^\perp \cong T_a^{\boldsymbol{R}}P$.*

Proof. For any point $y \in N_0$, we may canonically identify the tangent space of N_0 at y with the orthogonal complement $E^\perp$ of E in T_xM, where x is an arbitrary point over y. Then ψ_* is identified with the real isomorphism $E^\perp \to T_a^{\boldsymbol{R}}P$ in Proposition 9.17. q.e.d.

(9.11) For the proof of Theorem 9.1 it remains to show that the conformal structure of N_0 and the complex structure of D_0 are compatible with respect to the diffeomorphism ψ. For this purpose we need the following two lemmas.

Lemma 9.19. *For any $z \in \Gamma$ the restriction to $E_{\boldsymbol{C}}^\perp$ of the composite map*

$$(f\mu^{-1}\alpha)_*(t_{\boldsymbol{C}}|\Gamma_{\boldsymbol{C}})_*^{-1} : T_xM^{\boldsymbol{C}} \to T_z\Gamma_{\boldsymbol{C}} \to T_aP,\ x = t(z),\ a = f\mu^{-1}(z)$$

is surjective and its kernel is $E_{\boldsymbol{C}}^\perp \cap T_xM^{\boldsymbol{C}}(z)$.

Proof. First we consider $(f\mu^{-1}\alpha)_* : T_z\Gamma_{\boldsymbol{C}} \to T_aP$. Let V be its kernel. By Corollary 9.14 it is three dimensional and contains both $\widetilde{E}_{\boldsymbol{C}}$ and $T_zF(z) \cap T_z\Gamma_{\boldsymbol{C}}$. By Lemma 9.13 the latter is one dimensional and they have zero intersection. Hence we have

$$V = \widetilde{E}_{\boldsymbol{C}} \oplus (T_zF(z) \cap T_z\Gamma_{\boldsymbol{C}}).$$

On the other hand, by the isomorphism $t_{\boldsymbol{C}*} : T_z\Gamma_{\boldsymbol{C}} \cong T_xM^{\boldsymbol{C}}$, $\widetilde{E}_{\boldsymbol{C}}$ is mapped onto $E_{\boldsymbol{C}}$. The surjectivity of $E_{\boldsymbol{C}}^\perp \to T_aP$ already follows.

If we let I be the image of $T_zF(z) \cap T_z\Gamma_{\boldsymbol{C}}$ in $T_xM^{\boldsymbol{C}}$, the kernel of our composite map is a direct sum $E_{\boldsymbol{C}} \oplus I = E_{\boldsymbol{C}} \oplus I'$, where I' denotes the orthogonal projection of I onto $E_{\boldsymbol{C}}^\perp$ in (73). We know that $I \subseteq T_xM^{\boldsymbol{C}}(z)$, and as follows from Lemmas 9.11 and 9.16 $T_xM^{\boldsymbol{C}}(z)$ is compatible with the direct sum decomposition (73); therefore I' must coincide with $E_{\boldsymbol{C}}^\perp \cap T_xM^{\boldsymbol{C}}(z)$. The lemma follows. q.e.d.

Lemma 9.20. *Let W be a 2-dimensional real vector space with (positive definite) inner product. Let $W^{\pm}$ be the (non-trivial) isotropic subspaces of the complexfication $W^{\boldsymbol{C}}$ with the complexified inner product. Then $W^{\pm}$ are complex conjugate to each other and we have $W^{\boldsymbol{C}} = W^{+} \oplus W^{-}$. Moreover, the projections $W^{\boldsymbol{C}} \to W^{\pm}$ induces a real isomorphism $W \to W^{\pm}$, with respect to which the conformal structure on W and the complex structures on $W^{\pm}$ are compatible, i.e., the conformal structure on W is hermitian with respect to the complex structures of $W^{\pm}$.*

Proof. We take an orthonormal basis of W, with respect to which we identify W with $\boldsymbol{R}^2$ and $W^{\boldsymbol{C}}$ with $\boldsymbol{C}^2$. The complexified quadratic form takes the form $\boldsymbol{C}^2 \ni (z, w) \to z^2 + w^2$ and we may identify $W^{\pm} = \{z \pm iw = 0\}$. This shows the first assertion.

Moreover, the projection $W^{\boldsymbol{C}} \to W^{\pm}$ restricted to W is identified with the map $\boldsymbol{R}^2 \ni (x, y) \to (x \pm iy, \pm i(x \pm iy)) \in W^{\pm} \subseteq \boldsymbol{C}^2$. Then the pull-back by this map of the hermitian structure $z\overline{z} + w\overline{w}$ on $\boldsymbol{C}^2$ to $\boldsymbol{R}^2$ is just $2(x^2 + y^2)$. Thus the latter is hermitian on W with respect to the complex structure induced from $W^{\pm}$. q.e.d.

(9.12) *Proof of Theorem* 9.1. Fix a Riemannian metric in the given conformal class $[g]$. Let y be any point of N_0 and x a point of M_0 over N_0. Let Q be the null-cone of the complex conformal structure of $T_x M^{\boldsymbol{C}} = H^0(L_x, N_x)$ consisting of sections vanishing at some point. By Lemma 9.20 applied to $W = E^{\perp}$, $Q \cap E^{\perp}_{\boldsymbol{C}}$ is a union of two σ-conjugate isotropic subspaces. One of this is $E^{\perp}_{\boldsymbol{C}} \cap T_x M^{\boldsymbol{C}}(z)$ by Lemma 9.16 and the σ-conjugate of this space is $E^{\perp}_{\boldsymbol{C}} \cap \overline{T}_x M^{\boldsymbol{C}}(z) = E^{\perp}_{\boldsymbol{C}} \cap T_x M^{\boldsymbol{C}}(\overline{z})$. We consider the following commutative diagram describing the differential ψ_*:

$$\begin{array}{ccccc} T_y N_0 & \cong & E^{\perp} & \cong & T_a^{\boldsymbol{R}} P \\ & & u \downarrow & & \downarrow \\ & & E^{\perp}_{\boldsymbol{C}}/(T_x M^{\boldsymbol{C}}(z) \cap E^{\perp}_{\boldsymbol{C}}) & \cong & T_a P, \end{array}$$

where the top and bottom horizontal isomorphisms are due to Proposition 9.17 and Lemma 9.19 respectively, while the vertical real isomorphisms are the natural ones. We have to compare the conformal structure on $T_y N_0$ and the complex structure on $T_a P$ and this is reduced to proving the compatibility of the corresponding structures on both hand sides of the map

$$u : E^{\perp} \to E^{\perp}_{\boldsymbol{C}}/(T_x M^{\boldsymbol{C}}(z) \cap E^{\perp}_{\boldsymbol{C}}).$$

This however follows from Lemma 9.20, where $W = E^{\perp}$, $W = E^{\perp}_{\boldsymbol{C}}$,

$$W^{+} = T_x M^{\boldsymbol{C}}(z) \cap E^{\perp}_{\boldsymbol{C}}$$

and

$$W^{-} = T_x M^{\boldsymbol{C}}(\overline{z}) \cap E^{\perp}_{\boldsymbol{C}}.$$

q.e.d.

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