Duality for Witt–divisorial sheaves

Niklas Lemcke

Abstract. We adapt ideas from Ekedahl [Eke84] to prove a Serre-type duality for Witt-divisorial sheaves of $\mathbb{Q}$–Cartier divisors on a smooth projective variety over a perfect field of finite characteristic. We also explain its relationship to Tanaka's vanishing theorems [Tan20].

Contents

Introduction

Kodaira Vanishing and its generalizations have been crucial in the development of the Minimal Model Program (MMP) in characteristic zero. However, as is well-known, they do not hold in positive characteristic. Tanaka [Tan20] proposed a Kodaira-like vanishing theorem which holds for ample divisors in positive characteristic.

Theorem 0.1. ([Tan20], cf. Theorem 2.5) *Let k be a perfect field of characteristic $p > 0$, and $X \xrightarrow{\phi} Spec\ k$ be an N–dimensional smooth projective variety. If A is an ample $\mathbb{Q}$–Cartier divisor on X, then*

(i) $H^j(X, W\mathcal{O}_X(-A)) = p^t$*–torsion, for some t, for any $j < N$.*

(ii) $\bullet\ R^i\phi_*\,\mathcal{H}om_{W\mathscr{O}_X}(W\mathscr{O}_X(-A),W\Omega_X^N)_{\mathbb{Q}}=0$
     $\bullet\ H^i(X,W\Omega_X^N\underset{W\mathscr{O}_X}{\otimes}W\mathscr{O}_X(A))=0$ *for any* $i>0$, *if* A *is Cartier.*

Interestingly, the proof of (i) is easier than the proof of (ii). Ideally we would want the theorem to hold for nef and big invertible sheaves, but this is not yet known. The purpose of this paper is to establish a duality property for the Witt–divisorial sheaf $W\mathscr{O}_X(D)$ associated to a $\mathbb{Q}$–Cartier divisor D on X.

In [Eke84], Ekedahl introduces a duality functor D, and eventually constructs an isomorphism ([Eke84, Theorem III: 2.9])

$$D(R\Gamma(W\Omega_X^{\bullet}))(-N)[-N]\cong R\Gamma(W\Omega_X^{\bullet}),$$

where $(-N)$ and $[-N]$ denote shifts in module and complex degree, respectively. He then shows that

$$D(R\Gamma(W\Omega_X^{\bullet}))\cong R\operatorname{Hom}_R(R\Gamma(W\Omega_X^{\bullet}),\widetilde{R}),$$

in $D(R)$, where R is the Raynaud Ring (a non-commutative W-algebra), and $\widetilde{R}$ is a certain R–module. Where Ekedahl uses the Raynaud ring R, we use the similar Cartier-Dieudonné-ring $W[F,V]=:\omega$.

Theorem 0.2. (Cf. Theorem 3.10) *Let X be a smooth projective variety over a perfect field k of characteristic $p>0$, and D be a $\mathbb{Q}$–Cartier divisor on X. Then*

$$\prod_{t\in\mathbb{Z}}R\phi_*R\lim_n R\,\mathcal{H}om_{W_n\mathscr{O}_X}(W_n\mathscr{O}_X(p^tD),W_n\Omega_X^N)$$

$$\cong R\operatorname{Hom}_{\omega}\left(\bigoplus_{t\in\mathbb{Z}}R\phi_*W\mathscr{O}_X(p^tD),\widetilde{\omega}[-N]\right),$$

for a certain left–ω–module $\widetilde{\omega}$.

This allows us to recover Tanaka's vanishing theorem, as well as to make the (possibly) non–vanishing torsion somewhat more explicit.

1. Notation

Fix the following notations and conventions:

• A variety over k is a separated integral scheme of finite type over k.

• Throughout this paper we define $X \xrightarrow{\phi} S = \operatorname{Spec} k$, where k is a perfect field of characteristic $p > 0$, and X is assumed to be a smooth projective variety.

• If C is a complex, $C[i]$ denotes C shifted by i in complex degree.

• If M_n is an inverse system, then $\lim_n M_n$ denotes the inverse limit.

• For a module M, we write $M_{\mathbb{Q}} := M \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$.

2. Preliminaries

This section serves to recall some definitions and known results.

2.1. Tanaka's vanishing

The original Kodaira Vanishing is closely related to Hodge decomposition. Hodge decomposition in turn resembles the slope decomposition of crystalline cohomology in terms of the de Rham-Witt complex. This motivates the attempt at finding a useful vanishing theorem in the context of de Rham-Witt.

Definition 2.1. (Teichmüller lifts of line bundles, cf. [Tan20]) For a ring A, any element $a \in A$ can be naturally identified with an element in $W(A)$ by

$$A \longrightarrow W(A)$$

$$a \longmapsto \underline{a} := (a, 0, 0, ...).$$

This $\underline{a}$ is called the *Teichmüller representative* of a. For an invertible sheaf $\mathscr{F}$ on X defined by local transition functions (f_{ij}), Tanaka defines the *Teichmüller lift* $\underline{\mathscr{F}}$ of an invertible $\mathscr{O}_X$–module to be the invertible $W\mathscr{O}_X$–module given by the Teichmüller representatives of the transition functions $(\underline{f_{ij}})$. The *truncated* Teichmüller lift is defined by

$$\underline{\mathscr{F}}_{\leq n} := W_n \mathscr{O}_X \underset{W\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}.$$

Definition 2.2. (Witt-divisorial sheaves, cf. [Tan20]) Alternatively, the *Witt-divisorial sheaf associated to* an $\mathbb{R}$–divisor D is defined by

$$\Gamma_V(W\mathscr{O}_X(D)) := \{(\phi_0, \phi_1, ...) \in W(K(X)); \operatorname{div}(\phi_n) + p^n D|_V \geq 0\}.$$

As Tanaka shows, for a Cartier divisor on a reasonably nice scheme, these two notions are equivalent, since $W\mathscr{O}(D)|_U = \underline{f}W\mathscr{O}_X|_U$ for U affine open in X and f a local equation for D on U (cf. [Tan20, Proposition 3.12]). $W_n\mathscr{O}_X(D)$ is a coherent $W_n\mathscr{O}_X$–module (cf. [Tan20, Proposition 3.8]).

The following two propositions due to Tanaka [Tan20] will be used frequently throughout this paper, often without explicit reference.

Proposition 2.3. (*Cf.* [*Tan20, Proposition* 3.15]) *Let D be an $\mathbb{R}$–divisor on X. Then, for any $0 \le e, 0 < m \le n$, there is an isomorphism*

$$R\,\mathcal{H}om_{W_n\mathscr{O}_X}((F^e)_*W_m\mathscr{O}_X(D), W_n\Omega_X^N) \cong (F^e)_*\,\mathcal{H}om_{W_m\mathscr{O}_X}(W_m\mathscr{O}_X(D), W_m\Omega_X^N)$$

in $D(W\mathscr{O}_X - \mathfrak{mod})$.

Proposition 2.4. (*Cf.* [*Tan20, Proposition* 4.9 *and Lemma* 2.10]) *Let D be an $\mathbb{R}$–divisor on X. Let M be a coherent $W_n\mathscr{O}_X$–module such that the induced map $M(U) \to M_\xi$ is injective for any non–empty open subset $U \subset X$, where M_ξ denotes the stalk of M at the generic point ξ of X. Then the induced $W\mathscr{O}_X$–module homomorphism*

$$\mathcal{H}om_{W_n\mathscr{O}_X}(W_n\mathscr{O}_X(D), M) \xrightarrow{\theta} \mathcal{H}om_{W\mathscr{O}_X}(W\mathscr{O}_X(D), M)$$

is an isomorphism.

$W_n\Omega_X^N$ and $gr^n W\Omega_X^N$ are two such $W_n\mathscr{O}_X$–modules.

Theorem 2.5. (Tanaka, cf. [Tan20, Theorem 1.1]) *Let k be a perfect field of characteristic $p > 0$, and X be an N–dimensional smooth projective variety over k. If A is an ample $\mathbb{Q}$-Cartier divisor on X, then there exists s_0 such that for all $s_0 < s$,*

(i)   • $H^j(X, W_n\mathscr{O}_X(-sA)) = 0$ *for any $j < N, n \in \mathbb{N}$,*
   • $H^j(X, W\mathscr{O}_X(-sA)) = 0$ *for any $j < N$,*
   • $H^j(X, W\mathscr{O}_X(-A)) = p^t$–*torsion, for some t, for any $j < N$.*

(ii)   • $R^i\phi_*\,\mathcal{H}om_{W\mathscr{O}_X}(W\mathscr{O}_X(-A), W\Omega_X^N)_\mathbb{Q} = 0$ *for any $0 < i$,*
   • $R^i\phi_*(W_n\mathscr{O}_X(sA) \otimes W_n\Omega_X^N) = 0$ *for any $0 < i, n \in \mathbb{N}, A$ Cartier,*
   • $R^i\phi_*(W\mathscr{O}_X(sA) \otimes W\Omega_X^N) = 0$ *for any $0 < i, A$ Cartier,*
   • $R^i\phi_*(W\mathscr{O}_X(A) \otimes W\Omega_X^N) = 0$ *for any $0 < i, A$ Cartier.*

Remark 2.6. The theorem appears to suggest a Serre-type duality. This duality would be asymmetric in the sense that torsion from (i) does not appear in (ii). Note that the proof of (i) is simple relative to that of (ii). So ideally duality would recover (ii) from (i), potentially facilitating the proof of the theorem for nef and big D.

3. Duality

3.1. Duality Theorem

Proposition 3.1. *Let $\mathscr{F}$ be an invertible $\mathscr{O}_X$–module. For any $n>0$,*

$$(3.1) \qquad W_n\Omega_X^N \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n} \cong R\mathcal{H}om_{W_n\mathscr{O}_X}(\underline{\mathscr{F}}_{\leq n}^\vee, W_n\Omega_X^N)$$

such that, in particular,

$$(3.2)$$
$$H^i(X, W_n\Omega_X^N \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n}) \cong \mathrm{Hom}_{W_n}(H^{N-i}(X, \underline{\mathscr{F}}_{\leq n}^\vee), W_n) \text{ for any } i \geq 0, n > 0.$$

Proof. We have

$$W_n\Omega_X^N \otimes \underline{\mathscr{F}}_{\leq n} \cong \mathcal{H}om_{W_n\mathscr{O}_X}(W_n\mathscr{O}_X, W_n\Omega_X^N \otimes \underline{\mathscr{F}}_{\leq n})$$
$$\cong \mathcal{H}om_{W_n\mathscr{O}_X}(\underline{\mathscr{F}}_{\leq n}^\vee, W_n\Omega_X^N),$$

where the second isomorphism holds because $\underline{\mathscr{F}}_{\leq n}$ is locally free, and so $-\otimes \underline{\mathscr{F}}_{\leq n}^\vee$ is fully faithful. Since $R^i\mathcal{H}om(\underline{\mathscr{F}}_{\leq n}^\vee, W_n\Omega_X^N)=0$ for any $0<i$ (by local freeness), Equation 3.1 holds. To show Equation 3.2, take global sections of the derived push-forward.

$$\Gamma_S(R\phi_*(W_n\Omega_X^N \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n})) \cong \Gamma_S(R\phi_* R\mathcal{H}om_{W_n\mathscr{O}_X}(\underline{\mathscr{F}}_{\leq n}^\vee, W_n\Omega_X^N))$$
$$\cong \Gamma_S(R\mathcal{H}om_{W_n\mathscr{O}_S \cong W_n}(R\phi_*(\underline{\mathscr{F}}_{\leq n}^\vee), W_n[-N]))$$
$$\cong \mathrm{Hom}_{W_n}((R\phi_*\underline{\mathscr{F}}_{\leq n}^\vee)[N], W_n),$$

where W_n is the constant sheaf, the second isomorphism is due to Coherent Duality and [Eke84, I, Theorem 4.1], and the third isomorphism is due to W_n being an injective W_n–module. In particular for all i there are isomorphisms

$$H^i(X, W_n\Omega_X^N \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n}) \cong \mathrm{Hom}_{W_n}(H^{N-i}(X, \underline{\mathscr{F}}_{\leq n}^\vee), W_n). \quad \square$$

We now attempt passing to the limit. First, recall the following result.

Lemma 3.2. (*Chatzistamatiou, Rülling, cf.* [CR11, *Lemma* 1.5.1]) *Let $(X, \mathscr{O}_X)$ be a ringed space and $E=(E_n)$ a projective system of $\mathscr{O}_X$–modules (indexed by integers $1\leq n$). Let $\mathscr{B}$ be a basis of the topology of X. We consider the following two conditions*:
(1) *For all $U \in \mathscr{B}$, $H^i(U, E_n)=0$ for any $i, 1\leq n$.*
(2) *For all $U \in \mathscr{B}$, the projective system $(H^0(U, E_n))_{n\geq 1}$ satisfies the Mittag–Leffler condition.*

Then

• *If E satisfies condition* (1), *then* $R^i \lim_n E_n = 0$ *for any* $2 \leq i$.

• *If E satisfies conditions* (1) *and* (2), *then* $R^i \lim_n E_n = 0$ *for any* $1 \leq i$, *i.e. E is* lim−*acyclic*.

Lemma 3.3. *For $\mathscr{F}$ an invertible sheaf of $\mathscr{O}_X$−modules,*

$$(3.3) \qquad W\Omega_X^N \underset{W\mathscr{O}_X}{\otimes} \underline{\mathscr{F}} \cong \lim_n (W_n\Omega_X^N \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n}) \cong R\lim_n (W_n\Omega_X^N \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n}).$$

Proof.

$$\begin{aligned}
W\Omega_X^N \underset{W\mathscr{O}_X}{\otimes} \underline{\mathscr{F}} &\cong \lim_n W_n\Omega_X^N \underset{W\mathscr{O}_X}{\otimes} \underline{\mathscr{F}} \\
&\cong \lim_n (W_n\Omega_X^N \underset{W\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}) \\
&\cong \lim_n (W_n\Omega_X^N \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n}).
\end{aligned}$$

Take the exact sequence (cf. [Ill79]) of $W_{n+1}\mathscr{O}_X$−modules

$$0 \longrightarrow \mathrm{gr}^n W\Omega_X^N \longrightarrow W_{n+1}\Omega_X^N \longrightarrow W_n\Omega_X^N \longrightarrow 0,$$

where $\mathrm{gr}^n W\Omega_X^N$ is coherent. Tensoring with $\underline{\mathscr{F}}$ over $W\mathscr{O}_X$ yields an exact sequence

$$0 \longrightarrow \mathrm{gr}^n W\Omega_X^N \underset{W_{n+1}\mathscr{O}_X/p}{\otimes} \underline{\mathscr{F}}_{\leq n+1}/p$$
$$\longrightarrow W_{n+1}\Omega_X^N \underset{W_{n+1}\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n+1} \longrightarrow W_n\Omega_X^N \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n} \longrightarrow 0.$$

For any $x \in X$, let U_x be an affine open neighborhood of x. Then

$$H^1(U_x, \mathrm{gr}^n W\Omega_X^N \underset{W_{n+1}\mathscr{O}_X/p}{\otimes} \underline{\mathscr{F}}_{\leq n+1}/p) = 0$$

by coherence, and therefore

(i) $H^i(U_x, W_n\Omega_X^N \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n}) = 0$ for any $i > 0$, again by coherence,

(ii) $H^0(U_x, W_{n+1}\Omega_X^N \underset{W_{n+1}\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n+1}) \to H^0(U_x, W_n\Omega_X^N \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n})$

is surjective for all $n > 0$.

In this fashion a basis $\mathscr{U}$ for the topology of X can be found, such that the above two properties hold for all $U \in \mathscr{U}$, and so by Lemma 3.2, Equation 3.3 holds. $\square$

The following 'twisting' lemma is the reason for the asymmetry between the vanishing theorems.

Lemma 3.4. (Twisting) *Let D be a $\mathbb{Q}$–Cartier divisor on X such that $p^t D$ is a $\mathbb{Z}$–divisor for some positive integer t. Then*

$$\mathcal{H}om_{W\mathscr{O}_{X,\mathbb{Q}}}(W\mathscr{O}_X(D)_{\mathbb{Q}}, W\Omega^N_{X\mathbb{Q}}) \cong F_* \mathcal{H}om_{W\mathscr{O}_{X,\mathbb{Q}}}(W\mathscr{O}_X(pD)_{\mathbb{Q}}, W\Omega^N_{X\mathbb{Q}}).$$

Furthermore, if D is $\mathbb{Z}$–Cartier, then

$$W\Omega^N_X \underset{W\mathscr{O}_X}{\otimes} W\mathscr{O}_X(D) \cong (F)_*(W\Omega^N_X \underset{W\mathscr{O}_X}{\otimes} W\mathscr{O}_X(pD)).$$

Proof. The proof is due to Tanaka, see for example [Tan20, Theorem 4.2 (4) and Theorem 4.13]. It is repeated here for the reader's convenience.

We have an induced isomorphism

$$(X, W\mathscr{O}_{X,\mathbb{Q}}) \xrightarrow{\;F\;\sim\;} (X, W\mathscr{O}_{X,\mathbb{Q}}).$$

In particular, $F^* \circ F_* = F_* \circ F^* = 1$. We obtain the following chain of isomorphisms:

$$
\begin{aligned}
\mathcal{H}om_{W\mathscr{O}_{X,\mathbb{Q}}}(W\mathscr{O}_X(D)_{\mathbb{Q}}, W\Omega^N_{X,\mathbb{Q}}) &\cong F_* F^* \mathcal{H}om_{W\mathscr{O}_{X,\mathbb{Q}}}(W\mathscr{O}_X(D)_{\mathbb{Q}}, W\Omega^N_{X,\mathbb{Q}}) \\
&\cong F_* \mathcal{H}om_{F^* W\mathscr{O}_{X,\mathbb{Q}}}(F^* W\mathscr{O}_X(D)_{\mathbb{Q}}, F^* W\Omega^N_{X,\mathbb{Q}}) \\
&\cong F_* \mathcal{H}om_{W\mathscr{O}_{X,\mathbb{Q}}}(W\mathscr{O}_X(pD)_{\mathbb{Q}}, F^* F_* W\Omega^N_{X,\mathbb{Q}}) \\
&\cong F_* \mathcal{H}om_{W\mathscr{O}_{X,\mathbb{Q}}}(W\mathscr{O}_X(pD)_{\mathbb{Q}}, W\Omega^N_{X,\mathbb{Q}}).
\end{aligned}
$$

This proves the first statement.

For the second statement, recall that the Frobenius homomorphism

$$W\Omega^N_X \xrightarrow{\;F\;} F_* W\Omega^N_X$$

is an isomorphism of $W\mathscr{O}_X$–modules (cf. [Tan20, Theorem 2.9]). Therefore

$$
\begin{aligned}
W\Omega^N_X \underset{W\mathscr{O}_X}{\otimes} W\mathscr{O}_X(D) &\cong F_*(W\Omega^N_X) \underset{W\mathscr{O}_X}{\otimes} W\mathscr{O}_X(D) \\
&\cong F_*(W\Omega^N_X \underset{W\mathscr{O}_X}{\otimes} F^* W\mathscr{O}_X(D)) \\
&\cong F_*(W\Omega^N_X \underset{W\mathscr{O}_X}{\otimes} W\mathscr{O}_X(pD)).
\end{aligned}
$$

where the second isomorphism is the projection formula. $\square$

We can now observe a first, tenuous duality between Theorem 2.5 (i) and (ii).

Proposition 3.5. *Let D be a $\mathbb{Q}$–Cartier divisor on X such that $p^t D$ is a $\mathbb{Z}$–divisor for some positive integer t. Then*

$$R^i \lim_n R^j \phi_* \, \mathcal{H}om_{W_n \mathcal{O}_X}(W_n \mathcal{O}_X(D), W_n \Omega_X^N) = 0 \text{ for any } 0 < i, j \in \mathbb{N}.$$

Suppose there exists t such that

$$H^j(X, W_n \mathcal{O}_X(p^t D)) = 0 \text{ for any } 0 < n, j < N.$$

Then

$$R^i \phi_* \, \mathcal{H}om_{W \mathcal{O}_X}(W \mathcal{O}_X(D), W \Omega_X^N)_{\mathbb{Q}} = 0 \text{ for any } 0 < i.$$

If further D is Cartier, then

$$H^i(X, W \Omega_X^N \underset{W \mathcal{O}_X}{\otimes} W \mathcal{O}_X(-D)) = 0 \text{ for any } 0 < i.$$

Proof. Set $E_n := \mathcal{H}om_{W_n \mathcal{O}_S}(R^j \phi_* W_n \mathcal{O}_X(D), W_n)$. Since $S = \operatorname{Spec} k$, we have $H^i(S, E_n) = 0$ for any $0 < i$. By Lemma 3.2 then $R^i \lim_n E_n = 0$ for any $1 < i$. $W_n X$ is a proper scheme, so by coherence $H^j(X, W_n \mathcal{O}_X(D))$ and E_n are finite, hence Artinian W_n–modules. But a projective system of Artinian W_n–modules satisfies the Mittag–Leffler condition, and so the first statement holds by Lemma 3.2, Coherent Duality and [Eke84, I, Theorem 4.1]. By the Twisting Lemma we have

$$
\begin{aligned}
&R\phi_* \, \mathcal{H}om_{W \mathcal{O}_{X,\mathbb{Q}}}(W \mathcal{O}_X(D)_{\mathbb{Q}}, W \Omega_{X\mathbb{Q}}^N) \\
&\cong (F_S^t)_* R\phi_* \, \mathcal{H}om_{W \mathcal{O}_X}(W \mathcal{O}_X(p^t D), W \Omega_X^N)_{\mathbb{Q}} \\
&\cong (F_S^t)_* \left(R\lim_n \operatorname{Hom}_{W_n}(H^N(X, W_n \mathcal{O}_X(p^t D)), W_n) \right)_{\mathbb{Q}} \\
&\cong (F_S^t)_* \left(\lim_n \operatorname{Hom}_{W_n}(H^N(X, W_n \mathcal{O}_X(p^t D)), W_n) \right)_{\mathbb{Q}}.
\end{aligned}
$$

This proves the second statement.

For the third statement write $\mathcal{F} := \mathcal{O}_X(-D)$ and consider the derived push–forward of $W \Omega_X^N \otimes \underline{\mathcal{F}}$:

$$
\begin{aligned}
R\phi_*(W \Omega_X^N \underset{W \mathcal{O}_X}{\otimes} \underline{\mathcal{F}}) &\cong (F_S^t)_*(R\phi_*(W \Omega_X^N \underset{W \mathcal{O}_X}{\otimes} \underline{\mathcal{F}}^{p^t})) && \text{(by Lem. 3.4)} \\
&\cong (F_S^t)_* R\phi_*(\lim_n (W_n \Omega_X^N) \underset{W \mathcal{O}_X}{\otimes} \underline{\mathcal{F}}^{p^t}) \\
&\cong (F_S^t)_* R\lim_n R\phi_*(W_n \Omega_X^N \underset{W_n \mathcal{O}_X}{\otimes} \underline{\mathcal{F}}_{\leq n}^{p^t}) && \text{(by Lem. 3.3)} \\
&\cong (F_S^t)_* R\lim_n R\phi_*(R\,\mathcal{H}om_{W_n \mathcal{O}_X}(\underline{\mathcal{F}}_{\leq n}^{-p^t}, W_n \Omega_X^N)) && \text{(by Prop. 3.1)} \\
&\cong (F_S^t)_* R\lim_n \mathcal{H}om_{W_n \mathcal{O}_S}(R^N \phi_* \underline{\mathcal{F}}_{\leq n}^{-p^t}, W_n),
\end{aligned}
$$

for large enough t, where the last isomorphism is again due to Ekedal [Eke84, Theorem 4.1]. The third statement then follows analogously to the proof of the second statement. $\square$

Remark 3.6. While not the same, the proof of Proposition 3.5 is quite similar in spirit to those of [Tan20]. One might therefore view it as a mere reformulation of his theorems from a duality–oriented viewpoint.

We now attempt to establish a more general duality in the spirit of Ekedahl [Eke84]. A crucial ingredient to Ekedahl's result was the isomorphism in $D(W[d])$:

$$R_n \overset{L}{\underset{R}{\otimes}} R\Gamma_S(W\Omega_X^\bullet) \cong R\Gamma_S(W_n\Omega_X^\bullet).$$

We will employ a similar property to our case.

Define ω to be the Cartier-Dieudonné ring $W_\sigma[F, V]$, that is the (non-commutative) W-algebra generated by V and F, subject to the relations

$$aV = V\sigma(a), Fa = \sigma(a)F \text{ for any } a \in W; VF = FV = p,$$

where σ is the Frobenius map on W, induced from that on k. While as a set ω is equal to $(\bigoplus_i WV^i) \oplus (\bigoplus_j WF^j)$, it is a non-commutative ring with an evident left–W–module structure. It follows from the definition (and the fact that $k^p = k$) that every element of ω can be uniquely described by a sum

$$\sum_{0 < i} a_{-i}V^i + \sum_{0 \leq j} b_j F^j, a_i, b_j \in W.$$

Let

$$\omega_n := \omega / V^n \omega,$$

which is a (W, ω)–bimodule, since $V^n\omega$ is a sub–left–W–module of ω and a right-ω–ideal generated by V^n. We then have, as sets,

$$\omega_n = \bigoplus_{0 < i < n} a_{-i}V^i \oplus \bigoplus_{0 \leq j} b_j F^j, a_{-i} \in W_{n-i}, b_j \in W_n.$$

This yields two sets of left–ω–module homomorphisms: an obvious restriction map $\omega_n \xrightarrow{\pi} \omega_{n-1}$, as well as an injective map $\omega_{n-1} \xrightarrow{\varrho} \omega_n$, both induced by the respective maps R and $\varrho = \{\text{multiplication by } p\}$ on $W_\bullet$.

Lemma 3.7. *Let A be a k-algebra. Then $W(A)$ has a natural structure of left–ω–modules and there is an isomorphism of left–W–modules*

$$\omega_n \overset{L}{\underset{\omega}{\otimes}} W(A) \cong W_n(A).$$

For a sheaf of left–ω–modules M on X,

$$\omega_n \overset{L}{\underset{\omega}{\otimes}} R\Gamma(M) \cong R\Gamma(M_n),$$

where $M_n := M/V^n M \cong \omega_n \underset{\omega}{\otimes} M$.

Proof. The left–ω–module structure on $W(A)$ is given by

$$\omega \times W(A) \xrightarrow{\qquad \cdot \qquad} W(A)$$

$$(\Sigma_i a_i V^i + \Sigma_j b_j F^j, w) \longmapsto \Sigma_i a_i V^i(w) + \Sigma_j b_j F^j(w).$$

To compute the derived tensor product

$$D(\omega - \mathfrak{lmod}) \xrightarrow{\omega_n \overset{L}{\underset{\omega}{\otimes}} \cdot} D(\mathfrak{ab}),$$

take a projective resolution $P^\bullet$ of ω_n:

$$0 \longrightarrow \omega \xrightarrow{V^n \cdot} \omega \longrightarrow \omega_n \longrightarrow 0.$$

This complex of right–ω–modules, when tensored with $W(A)$, yields a complex $P^\bullet \otimes_\omega W(A)$:

$$0 \longrightarrow W(A) \xrightarrow{V^n} W(A) \longrightarrow 0.$$

To see that this represents $W_n(A)$ simply observe that the map induced by $\omega \xrightarrow{V^n \cdot} \omega$ via the tensor product is precisely the n-fold Verschiebungs-map on $W(A)$:

$$W(A) \xrightarrow{\ \sim\ } \omega \underset{\omega}{\otimes} W(A) \xrightarrow{\ V\ } \omega \underset{\omega}{\otimes} W(A) \xrightarrow{\ \sim\ } W(A)$$

$$a \longmapsto 1 \otimes a \longmapsto V \otimes a \longmapsto V \cdot a = V(a).$$

Analogously, the action $F\cdot$ on $W(A)$ induced via the tensor product is the familiar Frobenius map F.

Moreover, since ω_n is a left–W–module, so is $\omega_n \otimes_\omega^L W(A)$. Lastly, to see that the $D(\mathfrak{ab})$–isomorphism is in fact in $D(W - \mathfrak{lmod})$, simply observe that the left–W–module structures on both sides coincide via the isomorphism.

For the second statement, let $M \in (X, \omega)$, that is M is a sheaf of left–ω–modules on X. Let $P^\bullet$ be the projective resolution of ω_n

$$0 \longrightarrow \omega \xrightarrow{V^n \cdot} \omega \longrightarrow \omega_n \longrightarrow 0.$$

Then, since P^i is flat for all i,

$$\omega_n \otimes_\omega^L R\Gamma(M) \cong P^\bullet \underset{\omega}{\otimes} R\Gamma(M) \cong R\Gamma(P^\bullet \underset{\omega}{\otimes} M) \cong R\Gamma(\omega_n \otimes_\omega^L M) \cong R\Gamma(M_n). \quad \square$$

Lemma 3.8. *The ring ω has a natural $\mathbb{Z}$–grading given by F and V:*

$$\omega = \left(\bigoplus_{0<i} WV^i \right) \oplus \left(\bigoplus_{0\leq j} WF^j \right).$$

Let D be a $\mathbb{Q}$–Cartier divisor on X, and write $\omega(D) := \bigoplus_{t\in\mathbb{Z}} W\mathscr{O}_X(p^t D)$. Then $\omega(D)$ is a sheaf of graded left–ω–modules, and

$$\omega_n \overset{L}{\underset{\omega}{\otimes}} \omega(D) \cong \omega_n(D) := \bigoplus_{t\in\mathbb{Z}} W_n\mathscr{O}_X(p^t D).$$

By Lemma 3.7 then

$$\omega_n \overset{L}{\underset{\omega}{\otimes}} R\Gamma_X(\omega(D)) \cong R\Gamma_X(\omega_n(D)).$$

Proof. We have the following maps F and V:

$$W\mathscr{O}_X(D) \overset{F}{\longrightarrow} F_* W\mathscr{O}_X(pD)$$
$$F_* W\mathscr{O}_X(pD) \overset{V}{\longrightarrow} W\mathscr{O}_X(D).$$

That is, by definition, since

$$V^n(W\mathscr{O}_X(p^t D)) \subset W\mathscr{O}_X(p^{t-n}) \text{ and } F^n(W\mathscr{O}_X(p^t D)) \subset W\mathscr{O}_X(p^{t+n}D),$$

$\omega(D)$ is in fact a sheaf of $\mathbb{Z}$–graded left–ω–modules. The last statement follows from Lemma 3.7 and the fact that $W_n\mathscr{O}_X(D) \cong W\mathscr{O}_X(D)/V^n((F^n)_* W\mathscr{O}_X(p^n D))$. $\square$

Proposition 3.9. *As left–W–modules, $\omega \cong (\bigoplus_i W) \oplus (\bigoplus_j W)$. Similarly, as left–$W_n$–modules, $\omega_n \cong (\bigoplus_{i<n} W_{n-i}) \oplus (\bigoplus_j W_n)$. It follows that*

$$\mathrm{Hom}_{W_n}(\omega_n, W_n) \cong \left(\bigoplus_{0<i<n} W_{n-i} \right) \oplus \left(\prod_{0\leq j} W_n \right)$$

as W_n–modules. With the left–ω–module structure induced by the right–structure on ω_n, there is an isomorphism
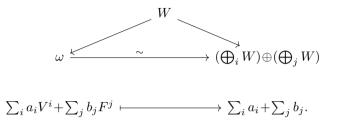
$$\mathrm{Hom}_{W_n}(\omega_n, W_n) \cong \bigoplus_{0<i<n} F^i W_{n-i} \oplus \prod_{0\leq j} F^{-j} W_n$$

of left–ω–modules.

Proof. Since $k = k^p$, elements $a \in \omega$ can be uniquely written as

$$a = \sum_i a_i V^i + \sum_j b_j F^j, \, a_i, b_j \in W.$$

The natural identification is clearly additive and bijective:

$$
\begin{array}{ccc}
& W & \\
\swarrow & & \searrow \\
\omega \xleftarrow{\hspace{2cm} \sim \hspace{2cm}} & & (\bigoplus_i W) \oplus (\bigoplus_j W)
\end{array}
$$

$$\sum_i a_i V^i + \sum_j b_j F^j \longmapsto \sum_i a_i + \sum_j b_j.$$

It is W–linear (on the left), since the left–W–module structure of ω is simply multiplication on the left. Analogously, $\omega_n \cong (\bigoplus_{i<n} W_{n-i}) \oplus (\bigoplus_j W_n)$ as left–W–modules. Therefore in $(W - \mathfrak{mod})$,

$$
\begin{aligned}
\mathrm{Hom}_{W_n}(\omega_n, W_n) &\cong \left(\bigoplus_{0<i<n} \mathrm{Hom}_{W_n}(W_{n-i}, W_n) \right) \\
&\oplus \left(\prod_{0 \leq j} \mathrm{Hom}_{W_n}(W_n, W_n) \right) \\
&\cong \left(\bigoplus_{0<i<n} W_{n-i} \right) \oplus \left(\prod_{0 \leq j} W_n \right) =: \widetilde{\omega}_n.
\end{aligned}
$$

(3.4)

The right–ω–module structure on ω_n induces a structure of (graded) left–ω–modules on $\mathrm{Hom}_{W_n}(\omega_n, W_n)$. For any $\alpha \in \widetilde{\omega}_n$,

$$V \cdot (\omega_n \xrightarrow{\alpha} W_n) = \omega_n \xrightarrow{\alpha \circ (\cdot V)} W_n,$$

$$F \cdot (\omega_n \xrightarrow{\alpha} W_n) = \omega_n \xrightarrow{\alpha \circ (\cdot F)} W_n.$$

Let

$$\alpha = \bigoplus_i \alpha_i \in \bigoplus_{0 \leq i < n} \mathrm{Hom}_{W_n}(W_{n-i} V^i, W_n) \cong \bigoplus_{0 \leq i < n} W_{n-i},$$

$$\beta = \prod_j \beta_j \in \prod_{0 \leq j} \mathrm{Hom}_{W_n}(W_n F^j, W_n) \cong \prod_{0 \leq j} W_n.$$

Under the above isomorphisms, $\alpha_i \in W_{n-i}$ corresponds to the map in $\mathrm{Hom}(W_{n-i}, W_n)$ which takes 1 to that element in W_n which corresponds to α_i under the isomorphism $W_{n-i} \xrightarrow{p^i} \mathrm{im}(p^i) \subset W_n$. That is, it takes 1 to $p^i \alpha_i \in W_n$. Similarly, $\beta_j \in W_n$ corresponds to the map in $\mathrm{Hom}(W_n, W_n)$ which takes 1 to β_j.

Let $\alpha_k = 0$ for any $k \neq i$, $\beta_l = 0$ for any $l \neq j$. That is α is zero outside of $W_{n-i}V^i \subset \omega_n$, and β is zero outside of $W_n F^j \subset \omega_n$. Then

$$
\begin{aligned}
V \cdot \alpha &= p^{-i+1}((V \cdot \alpha_i)(1)) = p^{-i+1}((\alpha_i \circ (\cdot V))(1)) \\
&= p^{-i+1} p^i \alpha_i = p \alpha_i \in W_{n-i+1} \text{ for any } 0 \leq i, \\
V \cdot \beta &= (V \cdot \beta_j)(1) = (\beta_j \circ (\cdot V))(1) \\
&= \beta_j(p) = p \beta_j \in W_n \text{ for any } 0 < j; \\
F \cdot \alpha &= p^{-i-1}((F \cdot \alpha_i)(1)) = p^{-i-1}((\alpha_i \circ (\cdot F))(1)) \\
&= p^{-i-1}(\alpha_i(p)) = p^{-i-1}(pp^i \alpha_i) = R(\alpha_i) \in W_{n-i-1} \text{ for any } 0 \leq i \\
F \cdot \beta &= (F \cdot \beta_j)(1) = (\beta_j \circ (\cdot F))(1) \\
&= \beta_j(1) = \beta_j \in W_n \text{ for any } 0 < j,
\end{aligned}
$$

where R is the natural restriction map $W_k \xrightarrow{R} W_{k-1}$. There is thus an isomorphism of left–ω–modules

$$
\begin{aligned}
\widetilde{\omega}_n &\cong \bigoplus_{0 < i < n} F^i W_{n-i} \oplus \prod_{0 \leq j} F^{-j} W_n \\
&\cong \bigoplus_{0 < i < n} F^i W_{n-i} \oplus \prod_{0 \leq j} V^j p^{-j} W_n,
\end{aligned}
$$

where the graded left–ω–module structure on the latter is the natural one, that is multiplication on the left. $\square$

Note that the injective left–W–linear maps $\omega_{n-1} \xrightarrow{\varrho} \omega_n$ form a direct system. $\mathrm{Hom}_{W_n}(\omega_n, W_n[-N])$ then form an inverse system (cf. [Eke84, III.2.3.*]) with boundary maps π defined by the commutativity of the diagram

(3.5)
$$
\begin{array}{ccc}
\mathcal{H}om_{W_n}(\omega_n, W_n[-N]) & \xrightarrow{\varrho^*} & \mathcal{H}om_{W_n}(j_{n,*}\omega_{n-1}, W_n[-N]) \\
\downarrow{\scriptstyle \pi} & \nearrow{\scriptstyle \varrho_*} & \\
j_{n,*}\,\mathcal{H}om_{W_{n-1}}(\omega_{n-1}, W_{n-1}[-N]). &
\end{array}
$$

Here $W_{n-1}S \xrightarrow{j_n} W_n S$ is the natural immersion. There exist unique such maps π because, by coherent duality and the fact that $W_{n-1} \cong j_n^! W_n$, ϱ_* is an isomorphism.

We can now prove the main duality theorem.

Theorem 3.10. *Let X be a smooth projective variety over a perfect field k of characteristic $p>0$, and D be a $\mathbb{Q}$–Cartier divisor on X. Write*

$$\breve{\omega} := \prod_{j\in\mathbb{Z}} F^j W$$

Then

$$\prod_{t\in\mathbb{Z}} R\phi_* R\lim_n R\,\mathcal{H}om_{W_n\mathscr{O}_X}(W_n\mathscr{O}_X(p^tD), W_n\Omega_X^N)$$
$$\cong R\operatorname{Hom}_\omega\left(R\phi_*\omega(D), \breve{\omega}[-N]\right).$$

Proof.

$$\prod_{t\in\mathbb{Z}} R\phi_* R\lim_n R\,\mathcal{H}om_{W_n\mathscr{O}_X}(W_n\mathscr{O}_X(p^tD), W_n\Omega_X^N)$$
$$\cong R\lim_n \prod_{t\in\mathbb{Z}} R\phi_* R\,\mathcal{H}om_{W_n\mathscr{O}_X}(W_n\mathscr{O}_X(p^tD), W_n\Omega_X^N) \qquad\text{(e.g. [Stacks, 0BKP])}$$
$$\cong R\lim_n R\,\mathcal{H}om_{W_n\mathscr{O}_S}\left(\bigoplus_{t\in\mathbb{Z}} R\phi_* W_n\mathscr{O}_X(p^tD), W_n[-N]\right) \qquad\text{(by [Eke84, Thm. 4.1])}$$
$$\cong \lim_n R\operatorname{Hom}_{W_n}\left(\omega_n \overset{L}{\underset{\omega}{\otimes}} R\Gamma_X(\omega(D)), W_n[-N]\right). \qquad\text{(by Lem. 3.8 and Prop. 3.5)}$$

By derived tensor–hom adjunction (see for instance [Yek19, Proposition 14.3.18]) we have an isomorphism

$$(3.6) \qquad R\lim_n R\operatorname{Hom}_{W_n}\left(\omega_n \overset{L}{\underset{\omega}{\otimes}} R\Gamma_X(\omega(D)), W_n[-N]\right)$$
$$\cong R\lim_n R\operatorname{Hom}_\omega\left(R\Gamma_X(\omega(D)), R\operatorname{Hom}_{W_n}(\omega_n, W_n[-N])\right)$$

in $D(W-\mathfrak{bimod})$.

Let $0\neq w\in\operatorname{Hom}(W_{n-i}, W_n)\cong W_{n-i}$. Since π is induced by the commutative Diagram 3.5, $\pi(w)$ is the unique map such that the following diagram commutes:

$$\begin{array}{ccc} W_{n-i} & \xrightarrow{\;w\;} & W_n \\ {\scriptstyle\varrho}\uparrow & & {\scriptstyle\varrho}\uparrow \\ W_{n-i-1} & \xrightarrow{\;\pi(w)\;} & W_{n-1} \end{array}.$$

This unique map is $R(w)$ (since for $\tau\in W_{n-i-1}$, $\varrho(R(w)(\tau))=w\varrho(\tau)\in W_n$). The induced maps $W_n\xrightarrow{\pi} W_{n-1}$ are therefore precisely the term-wise restriction maps R.

Taking the limit yields an isomorphism of right–W_n–modules

$$\varprojlim_n \mathrm{Hom}_{W_n}(\omega_n, W_n) \cong \varprojlim_n \left\{ \left(\bigoplus_{0 < i < n} W_{n-i} \right) \oplus \left(\prod_{0 \leq j} W_n \right) \right\}$$
$$\cong \prod_{j \in \mathbb{Z}} W,$$

whose ω–module structure is given by that on the $\widetilde{\omega}_n$ (cf. Proposition 3.9). That is as sets

$$\varprojlim_n \mathrm{Hom}_{W_n}(\omega_n, W_n) \cong \prod_{j \in \mathbb{Z}} F^{-j} W =: \widetilde{\omega}.$$

with the obvious structure of graded left–ω–modules. The result then follows from Equation 3.6. $\square$

3.2. Computation and application to vanishing

In this section we consider divisors D such that

$$R^j \Gamma_X(W \mathscr{O}_X(p^t D)) = 0 \text{ for any } j < N, \text{ large enough } t,$$
$$R^N \Gamma_X(W \mathscr{O}_X(p^t D)) = \text{torsion–free for large enough } t.$$

In particular this is the case for D such that $-D$ is ample (cf. [Tan20, Theorem 4.14 and 5.3, Step 5]). For such D the following finiteness lemma holds.

Lemma 3.11.

$$R^j \Gamma_X(W \mathscr{O}_X(p^t D)) \cong R^j \Gamma_X(W_n \mathscr{O}_X(p^t D))$$

for any $j < N, n_0 \leq n$, where n_0 depends on t, for any t.

Proof. By assumption, for large enough n the short exact sequence of $W \mathscr{O}_X$–modules

$$0 \longrightarrow F_*^n W \mathscr{O}_X(p^{t+n} D) \xrightarrow{V^n} W \mathscr{O}_X(p^t D) \xrightarrow{R} W_n \mathscr{O}_X(p^t D) \longrightarrow 0$$

induces exact sequences of W–modules

$$0 \longrightarrow 0 \xrightarrow{V^n} R^j \Gamma_X(W \mathscr{O}_X(p^t D)) \xrightarrow{R} R^j \Gamma_X(W_n \mathscr{O}_X(p^t D)) \longrightarrow 0$$

for all $j < N$. $\square$

The following corollary shows that the dual's cohomology's vanishing depends on the V–torsion of the cohomology.

Corollary 3.12. *Under the above assumptions,*

$$R^i \phi_* \, \mathcal{H}om_{W\mathscr{O}_X}(W\mathscr{O}_X(p^t D), W\Omega_X^N))$$
$$\cong R^{i-2}\Gamma_X\left(R^1 \varprojlim_n \mathcal{H}om_{W\mathscr{O}_X}(W\mathscr{O}_X(p^t D), W_n\Omega_X^N)\right)$$

for any $0 < i, t \in \mathbb{Z}$.

　　If $p^t D$ *is* $\mathbb{Z}$*–Cartier or* $i=1$, *this is equal to zero. Otherwise it is torsion.*

Proof. First we will show that

$$(3.7) \quad \prod_{t\in\mathbb{Z}} R^i \phi_* R\varprojlim_n R\,\mathcal{H}om_{W_n\mathscr{O}_X}(W_n\mathscr{O}_X(p^t D), W_n\Omega_X^N)$$
$$\cong h^i\left(\varprojlim_n R\operatorname{Hom}_{W_n}\left(\omega_n \overset{L}{\underset{\omega}{\otimes}} R\Gamma_X(\omega(D)), W_n[-N]\right)\right) = 0 \text{ for any } 0 < i.$$

There is a spectral sequence

$$E_{n,2}^{p,q} := \omega_n \otimes_\omega^{L^p} R^q \Gamma_X(\omega(D)) \Longrightarrow \omega_n \otimes_\omega^{L^{p+q}} R\Gamma_X(\omega(D)) =: E_n^{p+q}.$$

The sequence degenerates at the second sheet, and so we have an exact sequence

$$0 \longrightarrow E_{n,2}^{0,j} \longrightarrow E_n^j \longrightarrow E_{n,2}^{-1,j+1} \longrightarrow 0.$$

We now consider the two outer terms $E_{n,2}^{0,j} \cong \omega_n \otimes R^t\Gamma(\omega(D))$ and $E_{n,2}^{-1,j+1} \cong R^{j+1}\Gamma(\omega(D))[V^n]$ (where $M[V^n]$ denotes the V^n–torsion of a left–ω–module M) separately.

　　By tensor–hom adjunction we have an isomorphism

$$\varprojlim_n \operatorname{Hom}_{W_n}\left(\omega_n \underset{\omega}{\otimes} R^{-i}\Gamma_X(\omega(D)), W_n[-N]\right)$$
$$\cong \varprojlim_n \operatorname{Hom}_\omega\left(R^{-i}\Gamma_X(\omega(D)), \operatorname{Hom}_{W_n}(\omega_n, W_n[-N])\right)$$
$$\cong \varprojlim_n \operatorname{Hom}_\omega\left(R^{N-i}\Gamma_X(\omega(D)), \widetilde{\omega}_n\right)$$
$$\cong \operatorname{Hom}_\omega\left(R^{N-i}\Gamma_X(\omega(D)), \widetilde{\omega}\right) = 0 \text{ for any } 0 < i$$

in $D(W-\mathfrak{bimod})$, where the final equality follows from the assumption on torsion.

　　Consider now the right outer term.

$$\varprojlim_n \operatorname{Hom}_{W_n}\left(\ker\left(R^{N-i+1}\Gamma_X(\omega(D)) \xrightarrow{V^n} R^{N-i+1}\Gamma_X(\omega(D))\right), W_n\right)$$
$$\cong \prod_t \varprojlim_n \operatorname{Hom}_{W_n}\left(\ker\left(R^{N-i+1}\Gamma_X(W\mathscr{O}_X(p^{t+n}D)) \xrightarrow{V^n} R^{N-i+1}\Gamma_X(W\mathscr{O}_X(p^t D))\right), W_n\right).$$

$R^j \Gamma_X(W\mathscr{O}_X(p^t D))$ is torsion–free for large enough t and any j. So only finitely many objects of each inverse system are non–zero, wherefore the inverse limit is also zero.

Since $\lim \mathrm{Hom}(E_{n,2}^{0,j}, W_n) = 0 = \lim \mathrm{Hom}(E_{n,2}^{-1,j+1}, W_n)$ for any $j < N$,

$$\lim_n \mathrm{Hom}_{W_n}(E_n^j, W_n) = 0 \text{ for any } j < N.$$

We have now shown that Equation 3.7 holds. Unfortunately, we do not know whether

$$R^1 \lim_n \mathcal{H}om_{W_n \mathscr{O}_X}(W_n \mathscr{O}_X(D), W_n \Omega_X^N) = 0,$$

and thus whether

$$\mathcal{H}om_{W\mathscr{O}_X}(W\mathscr{O}_X(D), W\Omega_X^N) \cong R\lim_n R\,\mathcal{H}om_{W_n \mathscr{O}_X}(W_n \mathscr{O}_X(D), W_n \Omega_X^N)$$

holds for D not $\mathbb{Z}$–Cartier. In order to describe the left hand side using Theorem 3.10, we therefore need to consider another spectral sequence. Write $\mathcal{H}_{n,t} := \mathcal{H}om_{W\mathscr{O}_X}(W\mathscr{O}_X(p^t D), W_n \Omega_X^N)$. There is a spectral sequence

$$(3.8) \qquad E_{t,2}^{p,q} := R^p \phi_* R^q \lim_n \mathcal{H}_{n,t} \Longrightarrow R^{p+q} \phi_* R\lim_n \mathcal{H}_{n,t} =: E_t^{p+q}.$$

Since $R^i \lim \mathcal{H}_{n,t} = 0$ for any $1 < i$ (condition (1) of Lemma 3.2 is satisfied), page two of the spectral sequence contains only two nonzero rows, $q = 1$ and $q = 0$. Consequently it degenerates at page three, and

$$\mathrm{Fil}^n E_t^n \cong E_{t,3}^{n,0} \cong E_{t,2}^{n,0} / d(E_{t,2}^{n-2,1}).$$

We obtain a long exact sequence

$$(3.9) \qquad ... \longrightarrow E_{t,2}^{n-2,1} \longrightarrow E_{t,2}^{n,0} \longrightarrow E_t^n \longrightarrow E_{t,2}^{n-1,1} \longrightarrow ....$$

We have $E_t^n = 0$ for $0 < n$. Therefore $E_{t,2}^{1,0} \cong E_{t,2}^{-1,1} = 0$, and $E_{t,2}^{i,0} \cong E_{t,2}^{i-2,1}$ for $1 < i$. Due to the Twisting Lemma 3.4,

$$(E_{t,2}^{i,0})_{\mathbb{Q}} = (E_{t+s,2}^{i,0})_{\mathbb{Q}} = 0,$$

for large enough $s \in \mathbb{N}$. $\square$

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Niklas Lemcke
Department of Mathematics
School of Science and Engineering
Waseda University
Ohkubo 3-4-1
Shinjuku
JP-169-8555
Tokyo
Japan
numberjedi@akane.waseda.jp