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Enveloping algebras with just infinite Gelfand-Kirillov dimension

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Abstract. Let $\mathfrak{g}$ be the Witt algebra or the positive Witt algebra. It is well known that the enveloping algebra $U(\mathfrak{g})$ has intermediate growth and thus infinite Gelfand-Kirillov (GK-) dimension. We prove that the GK-dimension of $U(\mathfrak{g})$ is *just infinite* in the sense that any proper quotient of $U(\mathfrak{g})$ has polynomial growth. This proves a conjecture of Petukhov and the second named author for the positive Witt algebra. We also establish the corresponding results for quotients of the symmetric algebra $S(\mathfrak{g})$ by proper Poisson ideals.

In fact, we prove more generally that any central quotient of the universal enveloping algebra of the Virasoro algebra has just infinite GK-dimension. We give several applications. In particular, we easily compute the annihilators of Verma modules over the Virasoro algebra.

1. Introduction

Let $\mathbb{K}$ be a field of characteristic zero, and let W be the *Witt algebra*, which has $\mathbb{K}$-basis

$$\{e_n : n \in \mathbb{Z}\},$$

with the Lie bracket

$$[e_i, e_j] = (j-i)e_{i+j}.$$

We let W_+, the *positive Witt algebra*, be the Lie subalgebra of W spanned by $\{e_n : n \geqslant 1\}$.

The Witt algebra is a central quotient of the *Virasoro algebra*, *Vir*, which has $\mathbb{K}$-basis

$$\{e_n : n \in \mathbb{Z}\} \cup \{c\},$$

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and Lie bracket

$$[e_i, e_j] = (j-i)e_{i+j} + \frac{i^3-i}{12}\delta_{i+j,0}c, \quad c \text{ central}.$$

The Virasoro algebra and its representations are ubiquitous in conformal field theory.

These algebras were testing examples for the fundamental and important question of whether there is an infinite-dimensional Lie algebra with a (left and right) noetherian enveloping algebra. This question has been asked by many people, including Ralph Amayo and Ian Stewart [AS74, Question 27, p. 396], Ken Brown [Bro07, Question B], Jacques Dixmier, and Victor Latyshev, and conjecturally has a negative answer; see [SW14, Conjecture 0.1]. Recently it was shown by the second named author and Chelsea Walton [SW14] that the conjecture holds for the Lie algebras above: that is, $U(W_+)$, $U(W)$, and $U(Vir)$ are not left or right Noetherian. The question is still unsolved in full generality.

However, the two-sided ideal structure of these enveloping algebras is extremely sparse, and it seems possible that they satisfy the ascending chain condition for two-sided ideals, a property sometimes known as being *weakly Noetherian*. Further, two-sided ideals of W and W_+ are known to be large, and Petukhov and the second named author have conjectured:

Conjecture 1.1. ([PS17, Conjecture 1.2]) The universal enveloping algebra $U(W_+)$ has just infinite GK-dimension; that is, if I is a non-zero ideal of $U(W_+)$, then $U(W_+)/I$ has polynomial growth.

(In this paper, we use *polynomial growth* as a synonym for finite GK-dimension.)

This conjecture was proved in [PS17] in the particular case that the ideal I is generated by quadratic expressions in the e_i. In this paper we establish the conjecture in full, and generalise our arguments to prove that $U(W)$ and indeed any central factor of $U(Vir)$ has just infinite GK-dimension in the sense above. Our main result is:

Theorem 1.2. (See Theorems 3.2, 4.1 and 5.3) *The algebras $U(W_+)$, $U(W)$, and $U(Vir)/(c-\varkappa)$, for any $\varkappa \in \mathbb{K}$, have just infinite GK-dimension. In particular, Conjecture 1.1 holds.*

We note that these algebras are all finitely generated and as many authors have noticed [KKM83], [Smi76] and [Ufn78] have intermediate growth (and thus infinite GK-dimension). The natural set of normal words is monomials of the form $e_{i_1}, \ldots, e_{i_k}, i_1 \leqslant i_2 \ldots \leqslant i_k$. Here the number of monomials with $i_1 + i_2 + \ldots + i_k = n$ is in fact the number of partitions of n, which was shown by Hardy to be bounded by $e^{c\sqrt{n}}$, an estimate later dramatically improved by Ramanujan.

We also consider the induced Poisson structures on the symmetric algebras $S(W_+)$, $S(W)$, and $S(Vir)$, and prove:

Theorem 1.3. (See Theorems 2.1 and 5.5) *Let I be a proper Poisson ideal of $S(W_+)$. Then $S(W)/I$ has polynomial growth. Similar statements hold for $S(W)$ and for $S(Vir)/(c-\varkappa)$, for any $\varkappa\in\mathbb{K}$.*

The Lie algebra *Vir* has a triangular decomposition, and so the classical notion of a *Verma module* makes sense. As an application of Theorem 1.2, we easily compute annihilators of Verma modules for *Vir*:

Theorem 1.4. (See Corollary 6.2) *Let M be a Verma module over Vir with central charge $\varkappa$. Then the annihilator of M in $U(Vir)$ is the ideal $(c-\varkappa)$. As a result, for any $\varkappa\in\mathbb{K}$ the algebra $U(Vir)/(c-\varkappa)$ is primitive.*

Theorem 1.4 is an unpublished result of Wallach [WS13].

We also obtain:

Proposition 1.5. (See Proposition 6.4) *The algebras $U(W)$, $U(W_+)$, and $U(Vir)$ all satisfy the ascending chain condition for completely prime ideals.*

Proposition 1.6. (See Proposition 6.5) *Let $U(\mathcal{W})$ be either $U(W_+)$, $U(Vir)$, or $U(Vir)/(c-\varkappa)$ for some $\varkappa\in\mathbb{K}$. Then any epimorphism of $U(\mathcal{W})$ is an isomorphism.*

This answers a question of Rowen and Small [RS17, Section 4].

The Witt algebra W is a simple, graded, Lie algebra of polynomial growth. Such algebras were famously classified by Mathieu [Mat92]. It is interesting to ask which of these Lie algebras have enveloping algebras with just infinite GK-dimension. This is the subject of ongoing research.

Methods. As notation, we will write the symmetric algebra $S(W_+)=\mathbb{K}[x_1, x_2, \ldots]$, where x_i corresponds to $e_i\in W_+$. Likewise,

$$S(W)=\mathbb{K}[\ldots,x_{-1},x_0,x_1,\ldots] \quad \text{and} \quad S(Vir)=\mathbb{K}[\ldots,x_{-1},x_0,x_1,\ldots,c].$$

The main idea in the proofs of both Theorems 1.2 and 1.3 is to show that if g is a nonzero element of $S(W_+)$ or $S(W)$, then for 'almost all' monomials m in the x_i (more precisely, for long enough monomials on big enough letters), the Poisson ideal generated by g contains an element with leading term m. (See Lemma 2.2.) In the same way, we show that if g is a nonzero element of $U(W_+)$ or $U(W)$, then for almost all monomials m in the variables e_i, the two-sided ideal generated by g contains an element with leading term m. (See Lemmata 3.1 and 4.4.)

We summarise the argument for $S(W_+)$. Let I be the Poisson ideal generated by $0\neq g\in S(W_+)$. We introduce the following ordering on monomials in the x_i.

Denote by $\deg(m)=i_1+...+i_k$ the *degree* of a monomial $m=x_{i_1}...x_{i_k}$, and let the *length* of m be $|m|=|x_{i_1}...x_{i_k}|=k$.

Then the ordering on the set of commutative monomials in the x_i is defined as follows. For two monomials m_1 and m_2, we write $m_1<m_2$ if

- $|m_1|<|m_2|$ or
- $|m_1|=|m_2|$ and $\deg(m_1)<\deg(m_2)$ or
- $|m_1|=|m_2|$, $\deg(m_1)=\deg(m_2)$ and m_1 is less than m_2 with respect to the left-to-right lexicographical order when both m_1 and m_2 are written in increasing order: $m_1=x_{i_1}...x_{i_k}$, $m_2=x_{j_1}...x_{j_k}$ with $i_1\leqslant i_2\leqslant...\leqslant i_k$, $j_1\leqslant j_2\leqslant...\leqslant j_k$.

We show in Lemma 2.2 that all sufficiently long monomials on sufficiently 'big' letters can be written, modulo I, as a sum of smaller monomials. Here a letter is called *big* if it is bigger than $n=\{\max(2i+1)\,|\,x_i \text{ occurs in } g\}$. By this means we are able to introduce a 'normal form' for monomials from $S(W_+)$. Namely, any element of $S(W_+)$ can be written, modulo I, as a linear combination of monomials of the form $m=uv$, where u is a monomial on the finite set of letters $x_1,...x_{n-1}$, and v is a monomial of restricted length on the set of big letters. A similar normal form works for $U(W_+)$.

In the case of W (more generally, central quotients of *Vir*) the normal form is slightly different. Let $0\neq J\lhd U(W)$. Any element of $U(W)$ can be written, modulo J, as a linear combination of monomials $m=u_1vu_2$, where v is a monomial on the finite set of letters $e_{1-n},...e_{n-1}$, u_1 is a monomial of bounded length on letters smaller than e_{-n}, and u_2 is a monomial of bounded length on letters larger than e_n.

Counting the growth of these normal monomials gives us a polynomial estimate which bounds the growth of the quotient algebra. In the case of the full Witt algebra the growth counting is somewhat more involved, since the usual degree function $\deg(u)=i_1+...+i_k$ will not supply us a finite filtration on $U(W)/J$. See Section 4.2 for the details of how this issue is resolved.

Notation. Throughout we fix the following notation. We denote the set of non-negative integers by $\mathbb{N}$. If R is a ring and $g\in R$, the two-sided ideal generated by g is denoted (g). If R is a Poisson algebra, the Poisson ideal generated by g is denoted $\{(g)\}$.

Let $S(\mathcal{W})$ denote either $S(W_+)$, $S(Vir)$, or $S(Vir)/(c-\varkappa)$ for some $\varkappa\in\mathbb{K}$. Likewise, let $U(\mathcal{W})$ be either $U(W_+)$, $U(Vir)$, or $U(Vir)/(c-\varkappa)$. Our convention is that $e_1,e_2,...$ denote elements of $U(\mathcal{W})$, and that $x_1,x_2,...$ are the corresponding elements of $S(\mathcal{W})$. Monomials $x_{i_1}x_{i_2}...x_{i_k}\in S(\mathcal{W})$ are usually written in increasing order (if not specified otherwise), that is $i_1\leqslant i_2\leqslant...\leqslant i_k$. The same shape $e_{i_1}e_{i_2}...e_{i_k}$, where $i_1\leqslant i_2\leqslant...\leqslant i_k$, of a monomial is considered normal in $U(\mathcal{W})$, and such a monomial is called *standard*.

2. The symmetric algebra of the positive Witt algebra

Our main goal in this section is to prove the following theorem, which gives the first statement of Theorem 1.3:

Theorem 2.1. *Let I be a nonzero Poisson ideal of $S(W_+)$. Then $A=S(W_+)/I$ has polynomial growth.*

Let $X=\{x_1, x_2, ...\}$, so $S(W_+)=\mathbb{K}[X]$, with Poisson bracket induced by defining $\{x_i, x_j\}=(j-i)x_{i+j}$. We equip $S(W_+)$ with the degree grading given by setting $\deg(x_j)=j$. Thus the *degree* of a monomial $m=x_{i_1}...x_{i_k}$ is $\deg(m)=i_1+...+i_k$. Of course, we have the natural grading by the *length* of monomials as well, which we write as $|x_{i_1}...x_{i_k}|=k$.

The key technique in the proof is the following *reduction formula* for elements of $A=S(W_+)/I$ in the case that $I=\{(g)\}$ is the Poisson ideal generated by a single, nonzero, degree-homogeneous polynomial g.

Lemma 2.2. *Let $I=\{(g)\}$ be the Poisson ideal in $\mathbb{K}[X]=S(W_+)$ generated by a nonzero degree-homogeneous polynomial $g\in\mathbb{K}[X]$. There exist positive integers k and n such that every monomial $m=x_{j_1}...x_{j_k}$ such that $j_\ell\geqslant n$ for $1\leqslant\ell\leqslant k$ satisfies*

$$m=h+\sum c_s m_s, \tag{2.1}$$

where $h\in I$ is degree-homogeneous with $\deg(h)=\deg(m)$, the sum is finite, $c_s\in\mathbb{K}$, and the m_s are monomials of degree $\deg(m)$ such that for each s, either $|m_s|<k$ or $|m_s|=k$ and $i<n$ for at least one of the letters x_i featuring in m_s.

To prove Lemma 2.2, we introduce the following two orderings on the set $[X]$ of commutative monomials in X. For two monomials m_1 and m_2, we write $m_1<m_2$ if

- $|m_1|<|m_2|$ or
- $|m_1|=|m_2|$ and $\deg(m_1)<\deg(m_2)$ or
- $|m_1|=|m_2|$, $\deg(m_1)=\deg(m_2)$ and m_1 is less than m_2 with respect to the left-to-right lexicographic order when both m_1 and m_2 are written in *increasing* order: $m_1=x_{i_1}...x_{i_k}$, $m_2=x_{j_1}...x_{j_k}$ with $i_1\leqslant i_2\leqslant...\leqslant i_k$, $j_1\leqslant j_2\leqslant...\leqslant j_k$.

Similarly, we write $m_1\prec m_2$ if

- $|m_1|<|m_2|$ or
- $|m_1|=|m_2|$ and $\deg(m_1)<\deg(m_2)$ or
- $|m_1|=|m_2|$, $\deg(m_1)=\deg(m_2)$ and m_1 is less than m_2 with respect to the left-to-right lexicographical order when both m_1 and m_2 are written in *decreasing* order: $m_1=x_{i_1}...x_{i_k}$, $m_2=x_{j_1}...x_{j_k}$ with $i_k\leqslant i_{k-1}\leqslant...\leqslant i_1$, $j_k\leqslant j_{k-1}\leqslant...\leqslant j_1$. (Equivalently, we may write m_1 and m_2 in increasing order and compare them with the right-to-left lexicographic order.)

Note, that in the ordering $<$ we compare the smallest letters first, and in the ordering $\prec$ we compare the biggest letters first. It is easy to see that both orderings are well-orderings on $[X]$ compatible with multiplication.

Proof of Lemma 2.2. Our ordering $<$ satisfies the descending chain condition, and thus Lemma 2.2 can easily be obtained by repeated application of the following sublemma:

Sublemmma. *There exist positive integers k and n such that every monomial $m\in[X]$ of length k with $m\geqslant x_n^k$ satisfies*

$$m=h+\sum c_s m_s,$$

where $h\in I$ is degree-homogeneous with $\deg(h)=\deg(m)$, *the sum is finite*, $c_s\in\mathbb{K}$, *and the m_s are monomials of degree* $\deg(m)$ *such that $m_s<m$ for each s.*

To prove the sublemma, let $\overline{g}$ be the leading monomial of g with respect to $\prec$, and let $k=|\overline{g}|=|g|$. Without loss of generality we can assume that $\overline{g}$ features in g with coefficient 1. We write $\overline{g}$ in an increasing way: $\overline{g}=x_{i_1}...x_{i_k}$ with $i_1\leqslant i_2\leqslant...\leqslant i_k$. Pick any positive integer n such that $n\geqslant 2i_k+1$. We shall show that these n and k satisfy the required conditions.

Let $m\in[X]$ of length k be such that $m\geqslant x_n^k$. Then $m=x_{j_1}...x_{j_k}$ with $n\leqslant j_1\leqslant ...\leqslant j_k$. If $a\in\mathbb{Z}_{\geqslant 1}$, let $d_a:\mathbb{K}[X]\to\mathbb{K}[X]$ be the derivation defined by $d_a(u)=\{u,x_a\}$, extending via the Leibniz rule. Note that d_a is a graded derivation: if applied to a degree-homogeneous polynomial f, then $d_a(f)$ is degree-homogeneous of degree $\deg(d_a(f))=\deg(f)+a$.

Consider

$$h=d_{j_1-i_k}d_{j_2-i_{k-1}}...d_{j_k-i_1}(g).$$

Since $g\in I$ and I is a Poisson ideal, $h\in I$. Further, as the d_a are graded and g is degree-homogenous, so is h. The proof will be complete if we verify that

$$h=cm+\sum c_s m_s,$$

where $c\neq 0$, $c_s\in\mathbb{K}$, and the m_s are monomials such that $m_s<m$ for each s.

Let us apply the sequence of derivations $d_{j_1-i_k}d_{j_2-i_{k-1}}...d_{j_k-i_1}$ to a monomial u occurring in g. By the Leibniz rule we get a sum of monomials with coefficients obtained by prescribing which of the derivations acts on each letter of u.

Note (assuming that $d_a(u)\neq 0$) that $d_a(u)$ has the same length as u. Thus monomials in h obtained from monomials in g of length $<k$ are themselves of length $<k$ and therefore are smaller than m with respect to $<$.

Suppose now that u has length k. Then there are two options: either different differentials act on letters in different places in the monomial u or this is not the case.

We call the first of these ways *permutational* and the second *non-permutational*. Monomials in h obtained from u in a non-permutational way will have at least one letter unchanged and therefore will have at least one letter x_i with $i \leqslant i_k < n$. Hence such monomials of h are again smaller than m with respect to $<$.

It remains to consider monomials of h obtained from monomials in g of length k in a permutational way. For each monomial $u = x_{p_1} \dots x_{p_k}$ with $p_1 \leqslant \dots \leqslant p_k$ occurring in g and each permutation $\sigma \in \mathfrak{S}_k$, we obtain a monomial w of h given by

$$w = x_{j_1 - i_k + p_{\sigma(k)}} \dots x_{j_k - i_1 + p_{\sigma(1)}},$$

occurring with coefficient

$$\prod_{\ell=1}^{k} (j_\ell - i_{k-\ell+1} - p_{\sigma(k-\ell+1)}).$$

Since $u \preceq \overline{g}$, thus $p_{\sigma(k)} \leqslant p_k \leqslant i_k$. Hence $j_1 - i_k + p_{\sigma(k)} \leqslant j_1$ and the equality $j_1 - i_k + p_{\sigma(k)} = j_1$ holds if and only if $p_{\sigma(k)} = p_k = i_k$. If $j_1 - i_k + p_{\sigma(k)} < j_1$, then $w < m$ and we are done. If $j_1 - i_k + p_{\sigma(k)} = j_1$, we have $p_{\sigma(k)} = p_k = i_k$. Since $u \preceq \overline{g}$ and $p_{\sigma(k)} = p_k = i_k$, we have $p_{\sigma(k-1)} \leqslant p_{k-1} \leqslant i_{k-1}$. Hence $j_2 - i_{k-1} + p_{\sigma(k-1)} \leqslant j_2$ and the equality $j_2 - i_{k-1} + p_{\sigma(k-1)} = j_2$ holds if and only if $p_{\sigma(k-1)} = p_{k-1} = i_{k-1}$. Repeating the procedure, we see that $w \leqslant m$ and that $w = m$ only if $u = \overline{g}$ and the permutation σ satisfies $i_{\sigma(s)} = i_s$ for $1 \leqslant s \leqslant k$.

Now, for each such σ, since $n \geqslant 2i_k + 1$, we claim that the coefficient of the monomial m is a positive integer. Indeed, the coefficient is a product of factors of the form $j_\ell - i_{k-\ell+1} - p_{\sigma(k-\ell+1)}$, which are positive since $i_{k-\ell+1} + p_{\sigma(k-\ell+1)} \leqslant 2i_k < n \leqslant j_\ell$. Thus the coefficient with which m occurs in h is nonzero. The other monomials in h are $<m$. This completes the proof of the sublemma and thus of the lemma. □

Note that we used in this proof only that I is a module over $\mathbb{K}$ defined by the bracket multiplication (in other words a submodule of $S(W_+)$ under the adjoint action of W_+).

We now prove Theorem 2.1. The key point of the proof is that, thanks to Lemma 2.2, $A = S(W_+)/I$ is spanned by the set of monomials m in $[X]$ which admit a factorization $m = m_1 m_2$, where m_1 is a monomial in $x_1, x_2, \dots, x_{n-1}$ and $|m_2| < k$. Thus to estimate the growth of A it suffices to count such monomials.

Proof of Theorem 2.1. As $S(W_+)$ is finitely graded by degree, it is standard (see [KL00, Proposition 6.6]) that it suffices to show that $S(W_+)/I$ has polynomial growth if I is a nonzero degree-graded Poisson ideal, and it clearly suffices to consider the case that I is the Poisson ideal generated by a single nonzero degree-homogenous element g. Let k and n be the numbers produced by applying Lemma 2.2 to g, and let S be the set of all monomials m in $[X]$ which admit a factorisation $m{=}m_1m_2$, where m_1 is a monomial in $x_1, x_2, \ldots, x_{n-1}$ and $|m_2|{<}k$.

By Lemma 2.2, each monomial $m{\in}[X]\setminus S$ can be written, modulo I, as a linear combination of monomials of degree $\deg(m)$, each of which either has length strictly less than $|m|$ or has length $|m|$ and features strictly fewer x_i with $i{\geqslant}n$. Applying this observation repeatedly, we see that every monomial $m{\in}[X]\setminus S$ can be written, modulo I, as a linear combination of monomials from S of the same degree as m. We will call such presentation a *normal form* of m. Hence the image of S in $A{=}\mathbb{K}[X]/I$ spans A, and for fixed $N{\in}\mathbb{N}$, the number of monomials in S of degree not exceeding N provides an upper bound for $\dim\{u{\in}A \mid \deg u{\leqslant}N\}$. As A is finitely $\mathbb{N}$-graded by degree, it is standard that the growth of this dimension bounds $\operatorname{GKdim} A$.

It remains to estimate the number $p(N)$ of elements of S of degree at most N. Clearly $p(N){\leqslant}q(N)r(N)$, where $q(N)$ is the number of monomials in $x_1, \ldots, x_{n-1}$ of degree at most N and $r(N)$ is the number of monomials in $x_1, x_2, \ldots$ of degree at most N and length at most $k{-}1$. First, $q(N)$ does not exceed the number of monomials in $x_1, \ldots, x_{n-1}$ of length at most N, which is $\binom{N+n}{N}$. Thus there is a positive constant c so that $q(N){\leqslant}cN^n$ for all N. On the other hand, in a degree at most N monomial of length at most $k{-}1$ in $x_n, x_{n+1}, \ldots$ there are no more than N options for each letter and therefore $r(N){\leqslant}N^{k-1}$ for all N. Hence $p(N){\leqslant}cN^{k+n-1}$.

Hence $\operatorname{GKdim}(A){\leqslant}k{+}n{-}1$, and A has polynomial growth, as required. ☐

We note that Theorem 2.1 is also proved in [PS17] (see Corollary 2.13), with a much less constructive proof.

To end the section, we fix a positive integer k and consider the symmetric power $S^k(W_+)$. Let $g{\in}S^k(W_+)$ be a nonzero degree-homogeneous element, and let I be the W_+-submodule of $S^k(W_+)$ generated by g. As noted after the proof of Lemma 2.2, the reduction formula in Lemma 2.2 still applies to $S^k(W_+)/I$, and the argument in the proof of Theorem 2.1 now gives that $\operatorname{GKdim} S^k(W_+)/I{\leqslant}k{-}1$. As $S^k(W_+)$ clearly has GK-dimension k, we obtain:

Proposition 2.3. *As a W_+-module, $S^k(W_+)$ is GK k-critical.*

For $k{=}2$, this was shown in [PS17, Corollary 4.15].

3. The universal enveloping algebra of the positive Witt algebra

That $U(W_+)$ has just infinite GK-dimension follows from Theorem 2.1 using the Poisson GK-dimension defined in [PS17] and [PS17, Theorem 3.19]. However, a direct proof, which we give here, is also straightforward; the techniques of Section 2 apply also to $U(W_+)$.

We begin by giving a noncommutative version of the reduction formula of Lemma 2.2. Our result is more general than needed here, for later use when considering quotients of $U(W)$.

By the Poincaré-Birkhoff-Witt theorem, $U(W)$ has a basis of monomials $e_{i_1}e_{i_2}...e_{i_k}$ with $i_1\leqslant i_2\leqslant ...\leqslant i_k$. We call such monomials *standard*.

Lemma 3.1. *Let $0\neq g\in U(W)$, and let $I=U(W_+)gU(W_+)$ be the $U(W_+)$-sub-bimodule of $U(W)$ generated by g. Then there exist positive integers k and n and an integer ℓ so that every standard monomial $m=e_{j_1}...e_{j_k}$ with $n\leqslant j_1\leqslant ...\leqslant j_k$ satisfies*

$$m=h+\sum c_t m_t, \tag{3.1}$$

where $h\in I$, the sum is finite, $c_t\in\mathbb{K}$, and the m_t are standard monomials so that for each t, we have $i\geqslant\ell$ for all letters e_i featuring in m_t, and either $|m_t|<k$ or $|m_t|=k$ and $i<n$ for at least one of the letters featuring in m_t. Further, if h is degree-homogeneous, then $\deg(m)=\deg(h)=\deg(m_t)$ for all t.

Proof. Let $k=|g|$. Writing g as a sum of standard monomials, let e_ℓ be the smallest letter occurring in g and define n' so that $e_{n'}$ is the largest letter in g. Let $n=2|n'|+1$.

There are well-defined monomial orderings $<$ and $\prec$ on standard monomials in $U(W)$, defined just as the corresponding orderings on commutative monomials $x_{i_1}...x_{i_k}$ in the previous section. Note that $<$ does not satisfy the descending chain condition because W has no least element, but the induced order on standard monomials in letters $\geqslant\ell$ does satisfy d.c.c. Thus, as in the proof of Lemma 2.2, it suffices to show that we can rewrite m, modulo I, as a linear combination of standard monomials in letters $\geqslant\ell$, each of which are $<m$.

For any $a\in\mathbb{Z}$, let $\partial_a=[\_\_,e_a]$ as a linear operator from $U(W)\to U(W)$. Recall that length defines a filtration on $U(W)$ whose associated graded ring is $S(W)$; for $f\in U(W)$ let $\mathrm{gr}(f)$ be the element of $S(W)$ associated to f, so $x_i=\mathrm{gr}(e_i)$. For any $p\in U(W)$ and any $a\in\mathbb{Z}$, we have

$$\mathrm{gr}\,\partial_a(p)=d_a(\mathrm{gr}(p)) \quad \text{if } d_a(\mathrm{gr}(p))\neq 0, \tag{3.2}$$

where $d_a=\{\_\_,x_a\}$ as in the proof of Lemma 2.2.

Let $\overline{g}$ be the $\prec$-leading standard monomial in g and write $\overline{g}=e_{i_1}...e_{i_k}$ with $i_1\leqslant ...\leqslant i_k$. Let

$$h=\partial_{j_1-i_k}\partial_{j_2-i_{k-1}}...\partial_{j_k-i_1}(g).$$

Since for any letter x_p featuring in $\mathrm{gr}(g)$, and for any d_a which is applied to x_p, by our choice of n we have $a>p$, as in the proof of Lemma 2.2. Thus, just as in that proof, and using (3.2),

$$\mathrm{gr}(h)=d_{j_1-i_k}d_{j_2-i_{k-1}}...d_{j_k-i_1}(\mathrm{gr}(g))=c\,\mathrm{gr}(m)+\sum c_t\,\mathrm{gr}(m'_t),$$

where $c\neq 0$, the sum is finite, $c_t\in\mathbb{K}$, and the m'_t are standard monomials so that $m'_t<m$ for all t. Thus the length of $h-m-\sum c_t m'_t$ is strictly smaller than k, so $h-m$ is a linear combination of standard monomials which are all strictly $<m$.

Since $a>0$ for all ∂_a we have applied, only letters $\geqslant\ell$ occur in h. Finally, as the ∂_a are graded linear maps, if g is degree-homogenous so is h. □

Theorem 3.2. *Let I be a nonzero two-sided ideal in $U(W_+)$. Then $A=U(W_+)/I$ has polynomial growth.*

Proof. All essential points of the proof occur the proof of Theorem 2.1.

As before, since $U(W_+)$ is finitely graded by degree, we may assume that $I=(g)$ is the ideal generated by a single nonzero degree-homogenous element g. Let $k=|g|$.

Let S be the set of standard monomials m which admit a factorisation $m=m_1m_2$, where m_1 is a standard monomial in $e_1, ..., e_{n-1}$ and $|m_2|<k$. It follows from the reduction formula in Lemma 3.1 that $U(W_+)/I$ is spanned by the image of S. The same counting argument as in the proof of Theorem 2.1 shows that $U(W_+)/I$ has polynomial growth. □

Theorem 3.2 gives the first part of Theorem 1.2, dealing with $U(W_+)$.

4. The enveloping algebra of the full Witt algebra

In this section, we consider the enveloping algebra of the full Witt algebra, and show that it has just infinite GK-dimension. It clearly suffices to show:

Theorem 4.1. *Let $I=(g)$ be a two-sided ideal in $U(W)$ generated by one nonzero element $g\in U(W)$. Then $A=U(W)/I$ has polynomial growth.*

Throughout the section, we fix the meanings of g, I, and A as in the statement of Theorem 4.1. Let $\pi: U(W)\to A$ be the natural map.

Now, $U(W)$ is finitely generated, say by $\{e_{-2}, e_{-1}, e_1, e_2\}$, and thus so is A. Since the growth of A is controlled by the growth of any finite filtration, we are free to choose one that is convenient, but it will be a little bit more complicated this time to choose the right one. The problem is that unlike the situation for W_+, the usual degree function $\deg(e_i)=i$ does not give us a finite grading on $U(W)$; note that in the proofs of Theorems 2.1 and 3.2, the finiteness of the degree grading played a crucial role. Moreover, although of course there are many finite filtrations on $U(W)$, it is not necessarily clear how to choose one which induces a filtration on the quotient with polynomial growth.

Thus we will need to find an appropriate degree function which will give us a well-controlled finite filtration on A. We will see that a degree function of the form δ_C, defined by $\delta_C(e_{i_1}...e_{i_k})=|i_1|+...+|i_k|+C$, where C is a constant, does the job.

4.1. A spanning set for A

Our first step is to construct a set of standard monomials in $U(W)$ whose images span A.

Symmetrically to Lemma 3.1, we have:

Lemma 4.2. *There exist positive integers k and n and an integer ℓ such that every standard monomial $m = e_{j_1} \dots e_{j_k}$ with $j_1 \leqslant \dots \leqslant j_k \leqslant -n$ satisfies*

$$m = h + \sum c_t m_t,$$

where $h \in I$, the sum is finite, $c_t \in \mathbb{K}$, and the m_t are standard monomials so that for each t, we have $i \leqslant \ell$ for all letters e_i featuring in m_t, and either $|m_t| < k$ or $|m_t| = k$ and $i > -n$ for at least one of the letters featuring in m_t.

Proof. Consider the automorphism Φ of $U(W)$ defined by $\Phi(x_i) = -x_{-i}$ for $i \in \mathbb{Z}$. If we apply Lemma 3.1 to the ideal $\Phi(I)$ and the monomial $\pm\Phi(m)$, we obtain that $\Phi(m) = h + \sum c_t m_t$, where $h \in I$, the sum is finite, $c_t \in \mathbb{K}$, and the m_t are standard monomials so that for each t, we have $i \geqslant \ell$ for all letters e_i featuring in m_t, and either $|m_t| < k$ or $|m_t| = k$ and $i < n$ for at least one of the letters featuring in m_t. Applying Φ to both sides once again, we arrive at the result. □

Lemmata 3.1 and 4.2 allow us to construct our spanning set, which we define here.

Definition 4.3. For positive integers k, n, let $NS(k, n)$ be the set of standard monomials m which admit a factorisation $m = aub$, where a is a standard monomial of length $< k$ in e_{-n} and smaller letters, u is a standard monomial in $e_{1-n}, \dots e_0, e_1, e_2, \dots, e_{n-1}$, and b is a standard monomial of length $< k$ in e_n and bigger letters.

Lemma 4.4. *There exist positive integers k and n such that A is spanned by the image of $NS(k, n)$.*

Proof. As before, let k be the maximal length of monomials in g; that is $k = |g|$.

By Lemmata 3.1 and 4.2, there exist $n_1, n_2 \in \mathbb{Z}_{\geqslant 1}$ and $\ell_1, \ell_2 \in \mathbb{Z}$ such that for every standard monomial $m = e_{j_1} \dots e_{j_k}$, with $j_1 \leqslant \dots \leqslant j_k$, if $j_1 \geqslant n_1$, then

$$m = \sum c_s m_s + h,$$

where $h \in I$, the sum is finite, $c_s \in \mathbb{K}$, and the m_s are standard monomials such that $i \geqslant \ell_1$ for each e_i occurring in m_s, and $i < n_1$ for some e_i occurring in m_s; and if $j_k \leqslant -n_2$, then

$$m = \sum c_s m_s + h,$$

where $h \in I$, the sum is finite, $c_s \in \mathbb{K}$, and the m_s are standard monomials such that $i \leqslant \ell_2$ for each e_i occurring in m_s, and $i > -n_2$ for some e_i occurring in m_s.

Let $n = max\{n_1, n_2, |\ell_1|, |\ell_2|\}$. Repeatedly using the observations above to rewrite m modulo I, we obtain the result. □

For the rest of the section, let k, n be as given by Lemma 4.4, and let $NS = NS(k, n)$. We call the elements of NS *normal words*, and a representation of $m \in A$ as a linear combination of (images of) normal words a *normal form* for m, bearing in mind that this normal form may not be unique.

The growth of NS is polynomial, as we next show.

Lemma 4.5. *For any positive integer C, define a function*

$$\delta_C : \{\text{standard monomials in } U(W)\} \longrightarrow \mathbb{N}$$

by $\delta_C(e_{i_1} ... e_{i_k}) = |i_1| + ... + |i_k| + C$. For any $N \in \mathbb{N}$, let

$$p_C(N) = \#\{w \in NS \mid \delta_C(w) \leqslant N\}.$$

Then the function p_C has polynomial growth.

Proof. The growth of p_C does not depend on C, so without loss of generality let $C = 1$ and let $p = p_1$ and $\delta = \delta_1$. For a standard monomial m, we refer to $\delta(m)$ as the *absolute degree* of m. Note that $U(W)$ is *not* graded with respect to absolute degree.

Clearly $p(N) \leqslant q(N) r(N)^2$, where $q(N)$ is the number of standard monomials in

$$e_{1-n}, ..., e_0, e_1, ..., e_{n-1}$$

of absolute degree at most N, while $r(N)$ is the number of standard monomials in $e_n, e_{n+1}, ...$ of degree at most N and of length at most $k-1$.

Now, absolute degree, as a function on standard monomials, is always greater than or equal to length. Thus $q(N)$ does not exceed the number of standard monomials in $e_{1-n}, ..., e_{n-1}$ of length at most N, which is equal to $\binom{N+2n-1}{N} \leqslant cN^{2n-1}$, for some positive constant c which depends only on n and not on N.

On the other hand, in a monomial in $e_n, e_{n+1}, ...$ of absolute degree at most N and of length at most $k-1$, there are no more than N options for each letter and therefore $r(N) \leqslant N^{k-1}$ for all N. Hence $p(N) \leqslant cN^{2n+2k-3}$ and has polynomial growth. □

4.2. Choice of filtration

It remains to estimate the growth of A from the spanning set constructed in the previous subsection.

Since A is finitely generated, the growth of any finite filtration bounds the growth of A. The main result of this subsection is that there is a constant C such that the function δ_C induces a finite filtration of A, which by Lemma 4.5 will have polynomial growth.

To have a filtration $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ on $A = \bigcup A_i$ means to choose a map $\rho\colon A \to \mathbb{N}$, satisfying

$$\rho(uv) \leqslant \rho(u) + \rho(v), \quad \rho(u+v) \leqslant \max\{\rho(u), \rho(v)\} \quad \text{and} \quad \rho(\alpha u) = \rho(u), \tag{4.1}$$

for any $u, v \in A$ and $\alpha \in \mathbb{K}^*$. Suppose that for some C, the map $\delta_C\colon NS \to \mathbb{N}$ has the property that for any two normal words $m_1, m_2 \in NS$, we can find a normal form

$$\pi(m_1 m_2) = \sum c_i \pi(w_i),$$

where the $c_i \in \mathbb{K}$ and the w_i are normal words so that

$$\delta_C(w_i) \leqslant \delta_C(m_1) + \delta_C(m_2) \quad \text{for all } i. \tag{4.2}$$

We claim that this is enough to construct the required ρ. For, define

$$\rho(u) = \min_{\substack{\text{normal forms} \\ u = \sum c_j \pi(w_j)}} \Big\{ \max_j \delta_C(w_j) \Big\}.$$

Then ρ is easily seen to satisfy the required conditions (4.1).

So our goal is to show that there is some constant C so that δ_C satisfies (4.2).

Proposition 4.6. *Let* $\ell = \max\{|i| : e_i$ features in $g\}$. *Then* (4.2) *holds for* $C = 4k^2\ell$, *where we recall that* $k = |g|$.

Proof. It suffices to show that for any normal words $m_1, m_2 \in NS$, and for any normal form $\pi(m_1 m_2) = \sum c_i \pi(w_i)$ for $\pi(m_1 m_2)$, where $w_i \in NS$ and $c_i \in \mathbb{K}^*$, that

$$\delta_0(w_i) \leqslant \delta_0(m_1) + \delta_0(m_2) + C \quad \text{for all } i. \tag{4.3}$$

So we need to understand how δ_0 behaves on the words appearing in a normal form for $\pi(m_1 m_2)$.

Recall the definition of n from Lemma 4.4. Write

$$m_1 = a_1 u_1 b_1 \quad \text{and} \quad m_2 = a_2 u_2 b_2,$$

where u_1 and u_2 are standard monomials of any length on variables with indices strictly between $-n$ and n, a_1 and a_2 are standard monomials of length $<k$ on letters with indices $\leqslant -n$, while b_1, b_2 are standard monomials of length $<k$ on letters with indices $\geqslant n$. Now, normal words are standard monomials, and we first

use the commutation relations $e_ie_j = e_je_i + (j-i)e_{i+j}$ to write m_1m_2 as a linear combination of standard monomials, that is as a linear combination of words of the form

$$m_3 = a_3u_3b_3,$$

where a_3 is a standard monomial on letters with indices $\leqslant -n$, u_3 is a standard monomial on letters with indices strictly between $-n$ and n, and b_3 is a standard monomial on letters with indices $\geqslant n$. Let us call those letters with indexes $|i| \geqslant n$, *big letters*. Note that in the course of applying the commutation relations, the total number of big letters does not increase. Thus a_3 and b_3 have length $\leqslant 2k-2$, while u_3 may have any length.

Now we will use the reduction procedure from Lemma 3.1 to see how δ_0 changes when we get rid of big letters in b_3. If the length of b_3 is less than k we do not have to do anything. Otherwise let m be the monomial composed of the last k letters of b_3. According to Lemma 3.1, to get rid of one of the (big) letters in m, we find a sequence of derivations $D = \partial_{a_1} ... \partial_{a_k}$, with all the $a_i \geqslant 1$, and some $c \in \mathbb{K}^*$ such that

$$cD(g) = m + \sum c_s m'_s. \tag{4.4}$$

Let m' be some m'_s. Now m' is a standard monomial which falls into one of three cases:

I. $|m'| < |m|$, and m' is obtained from some monomial $\tilde{g}$ in g by applying D and then the commutation relations;

II. $|m'| = |m|$ and m' is obtained from some monomial $\tilde{g}$ in g (which necessarily has length k) by a non-permutational action of D;

III. $|m'| = |m|$ and m' is obtained from some monomial $\tilde{g}$ in g by a permutational action of D.

First, we note what happens to the number of big letters in each case. In case I, since m' is shorter than m, and all letters in m are big, m' contains fewer big letters than m. In case II, since the action is non-permutational, there is a letter in m' which was present in $\tilde{g}$. But monomials of g consist of letters which are not big, by definition of n. Thus the number of big letters in m' is smaller than in m. In case III, it may be that the number of big letters in m' is still equal to k, which is the number of (big) letters in m, but certainly there are no more than $k = |m'|$ big letters in m'. Further, by our choice of n, in case III all e_i occurring in m' have $i \geqslant 1$.

We now consider how δ_0 changes throughout this process. In situation III, as $m' \leqslant m$ and m' and m are made of letters $\geqslant e_1$, we have $\delta_0(m') = \deg(m') \leqslant \deg(m) = \delta_0(m)$. In cases I and II, $\delta_0(m')$ may be bigger than $\delta_0(m)$. Since applying the commutation relations does not increase δ_0, we may assume that the monomial m'

is a (possibly non standard) monomial obtained from applying D to a monomial $\tilde{g}$ of g.

Recall that $\bar{g}$ is the $\prec$-leading monomial of g, and m is obtained from the monomial $\bar{g}$ by applying D. Set $\deg(D)=a_1+\ldots+a_k$, so applying D increases the degrees of homogenous elements by $\deg(D)$. As $\delta_0(e_{a+b})\leqslant\delta_0(e_a)+\delta_0(e_b)$ and all the a_i are $\geqslant 1$, we have

$$\delta_0(m')\leqslant\delta_0(\tilde{g})+\deg D.$$

So

$$\begin{aligned}\delta_0(m')-\delta_0(m)&\leqslant\delta_0(\tilde{g})+\deg D-\delta_0(m)\\&=\delta_0(\tilde{g})+\deg D-\deg m\quad\text{as letters in } m \text{ are big}\\&\leqslant\delta_0(\tilde{g})+|\deg m-\deg D|\\&=\delta_0(\tilde{g})+|\deg\bar{g}|\\&\leqslant 2k\ell,\end{aligned}$$

by choice of ℓ, as $\tilde{g}$ and $\bar{g}$ have no more than k letters.

To summarize the discussion above: in cases I and II, we remove at least one big letter and increase δ_0 by no more than $2k\ell$. In case III, we do not remove big letters and do not increase δ_0. As proved in Lemma 3.1, after repeating this procedure finitely many times, we may write m_1m_2, modulo I, as a linear combination of normal words. However many times we repeat the procedure we remove a maximum of $k-1$ big letters from b_3, and thus we add a maximum of $2k(k-1)\ell$ to δ_0. Note that in applying this process to b_3, we never add any letters with indices $<-n$.

After we apply the procedure from Lemma 4.2 to a_3 at the other end of the word, we have found a normal form for m_1m_2 and have added maximum of another $2k(k-1)\ell$ to δ_0. In other words, we have written

$$\pi(m_1m_2)=\sum c_i\pi(w_i)$$

where the w_i satisfy (4.3), as required. □

The proof of Theorem 4.1 is now an easy combination of other results in this section.

Proof of Theorem 4.1. Let C be the constant given by Proposition 4.6. The discussion before that proposition shows that setting

$$A(N)=\operatorname{span}\{\pi(w): w\in NS, \pi_C(w)\leqslant N\}$$

defines a finite filtration on A. By Lemma 4.5, this filtration has polynomial growth, and so $\operatorname{GKdim}(A)<\infty$. □

Theorem 4.1 gives the second part of Theorem 1.2, dealing with $U(W)$.

Remark 4.7. Let $\mathbf{W}_1$ be the first *Cartan algebra*, which is isomorphic to the subalgebra of W spanned by $\{e_n : n \geqslant -1\}$. This is a simple graded Lie algebra of polynomial growth. Similar methods to those used in Theorem 4.1 show that $U(\mathbf{W}_1)$ has just infinite GK-dimension. To show this for all simple graded Lie algebras of polynomial growth is the subject of ongoing work.

5. Central quotients of the enveloping and symmetric algebras of *Vir*

In this section we first prove that all the central quotients $U(Vir)/(c-\varkappa)$ have just infinite GK-dimension, completing the proof of Theorem 1.2, and then consider the related Poisson algebras $S(Vir)/(c-\varkappa)$. Because the ideas of the proofs are similar to those in previous sections, we leave some details to the reader.

5.1. Central quotients of the enveloping algebra of *Vir*

Fix $\varkappa \in \mathbb{K}$ and let $R = U(Vir)/(c-\varkappa)$. Essentially the same argument as for the full Witt algebra works to show that R has just infinite GK-dimension; we give a sketch of the proof.

Note that R, like $U(W)$, has a basis of standard monomials in the e_i.

Definition 5.1. For positive integers k, n, let $NS(k,n) \subset R$ be the set of standard monomials m in the e_i which admit a factorisation $m = aub$, where a is a standard monomial of length $<k$ in e_{-n} and smaller letters, u is a standard monomial in $e_{1-n}, \dots e_0, e_1, e_2, \dots, e_{n-1}$, and b is a standard monomial of length $<k$ in e_n and bigger letters.

Lemma 5.2. *Let g be a nonzero element of R, let $I = (g)$, and let $A = R/I$. There exist positive integers k and n such that A is spanned by the image of $NS(k,n)$.*

Proof. The key point is that the reduction formulae in Lemmata 3.1 and 4.2 still hold. As before, let $\partial_a = [\_\_, e_a]$ as a linear operator on R and consider the effect of applying some $\partial_{a_1} \dots \partial_{a_k}$ to a standard monomial of length k. If $\varkappa \neq 0$ and some expression of the form $[e_{-a_j}, e_{a_j}]$ has been computed, we may obtain some standard monomials of length $<k$ in the result; but the leading term will have length k and will be given by the procedures in the previous section. Thus the proof of Lemma 4.4 goes through in this situation, almost without change. □

Theorem 5.3. *Let $I = (g)$ be a two-sided ideal in R generated by one nonzero element $g \in R$. Then $A = R/I$ has polynomial growth.*

Proof. We may define the functions δ_C just as with $U(W)$. The only part of the argument which is different for R is the proof of Proposition 4.6. When we compute

$$c\partial_{a_1}...\partial_{a_k}(g)=m+\sum_s c_s m'_s, \tag{5.1}$$

as in (4.4), consider some $m'=m'_s$ as before. In addition to cases I, II, III as in the proof of Proposition 4.6, we may have

I'. $|m'|<|m|$, and m' is obtained from some monomial $\tilde{g}$ in g by applying $\partial_{a_1}...\partial_{a_k}$ and then applying the relation

$$[e_{-i},e_i]=2ie_0+\frac{i-i^3}{12}\delta_{i+j,0}\varkappa.$$

As before, applying this relation does not increase δ_0, so we may assume that m' is a (possibly non standard) monomial obtained from applying D to a monomial $\tilde{g}$ of g. The argument of Proposition 4.6 goes through with only minor changes, and the result follows just as in the proof of Theorem 4.1. □

Theorem 5.3 completes the proof of Theorem 1.2 by establishing the part dealing with $U(Vir)$.

Remark 5.4. By the same method as in the proof of Theorem 5.3, one may show that the localised enveloping algebra $U(Vir)\otimes_{\mathbb{K}[c]}\mathbb{K}(c)$, considered as an algebra over $\mathbb{K}(c)$, has just infinite GK-dimension. It follows, using a similar argument to the proof of [KL00, Lemma 3.10], that $U(Vir)\otimes_{\mathbb{K}[c]}\mathbb{K}(c)$ has just infinite GK-dimension considered as a $\mathbb{K}$-algebra. We leave the details to the reader.

5.2. Central quotients of the symmetric algebra of *Vir*

In this subsection, let $\varkappa\in\mathbb{K}$ and let $R=S(W)/(c-\varkappa)$. Since $c-\varkappa$ is Poisson central, R is a Poisson algebra; in fact if we filter $U(Vir)$ by setting $|e_i|=1$ and $|c|=0$, then R is the associated graded ring of $U(Vir)/(c-\varkappa)$. Note that if we define $d_a=\{\_\_,x_a\}$ and $\partial_a=[\_\_,e_a]$ as before, then (3.2) still holds.

Similar arguments to those that have gone before prove:

Theorem 5.5. *Let g be a nonzero element of R and let $I=\{(g)\}$ be the Poisson ideal generated by g. Then $A=R/I$ has polynomial growth.*

Proof. The reduction process works as before: writing $g=\mathrm{gr}(g')$ and computing

$$h=c\partial_{a_1}...\partial_{a_k}(g')=m+\sum_s c_s m'_s$$

as in (5.1), by (3.2)

$$\mathrm{gr}(h) = \mathrm{gr}(m) + \sum_t c_t \,\mathrm{gr}(m'_t),$$

where the only m'_t surviving have length k. Thus as in Lemma 5.2 there are k and n so that A is spanned by the image of $\mathrm{gr}(NS(k,n))$.

Comparing $\delta_0(\mathrm{gr}(m'))$ with $\delta_0(\mathrm{gr}(m))$ as in the proof of Proposition 4.6 we see that only cases II and III occur. The conclusion of Proposition 4.6 still holds, and so as in the proof of Theorem 4.1 $\mathrm{GKdim}(A) < \infty$. □

Remark 5.6. Similarly, one may show that $S(\mathbf{W}_1)$ has just infinite GK-dimension. We omit the proof.

6. Applications

In this section we give several applications of Theorem 1.2. We first give a short proof that Verma modules for *Vir* are faithful over the appropriate central factor of $U(\mathit{Vir})$. (A more direct proof is an unpublished result of Nolan Wallach [WS13].) We next prove that $U(W_+)$, $U(W)$, and $U(\mathit{Vir})$ all satisfy the ascending chain condition on completely prime ideals. As a consequence, these algebras are *Hopfian*: they are not isomorphic to any proper quotient.

6.1. Annihilators of induced modules

Fix $\lambda, \varkappa \in \mathbb{K}$. Note that the Virasoro algebra *Vir* has a triangular decomposition: define $\mathfrak{n}_+ := \mathbb{K}(e_n : n \geqslant 1)$, $\mathfrak{h} := \mathbb{K}(c, e_0)$, and $\mathfrak{n}_- := \mathbb{K}(e_n : n \leqslant -1)$. Let $\mathfrak{b}_+ := \mathfrak{n}_+ \oplus \mathfrak{h}$. Let $\mathbb{K}_{\varkappa,\lambda}$ be the one-dimensional representation of $\mathfrak{b}_+$ where $\mathfrak{n}_+$ acts trivially, c acts as $\varkappa$, and e_0 acts as λ. Then define the *Verma module* $M_{\varkappa,\lambda}$ to be $U(\mathit{Vir}) \otimes_{U(\mathfrak{b}_+)} \mathbb{K}_{\varkappa,\lambda}$. It is immediate that

$$M_{\varkappa,\lambda} \cong U(\mathit{Vir})/U(\mathit{Vir})(c - \varkappa, e_0 - \lambda, e_n : n \geqslant 1)$$

and that $M_{\varkappa,\lambda}$ is non-positively graded, with $\dim (M_{\varkappa,\lambda})_{-n} = \mathcal{P}(n)$, the n'th partition number.

Verma modules are examples of the larger class we call, slightly imprecisely, *induced modules*. These are modules of the form $M = U(\mathit{Vir}) \otimes_{U(\mathfrak{b}_+)} M'$, where M' is a representation of $\mathfrak{b}_+$. Besides Verma modules, examples include *logarithmic representations*, where $\dim_{\mathbb{K}} M' < \infty$ and where $\mathfrak{n}_+$ acts trivially on M', c acts as a scalar, and e_0 acts as a non-semisimple matrix. These representations are important in logarithmic conformal field theory, see [GK96].

Whittaker modules [OW09] form another class of examples. Here let $\mathfrak{n}'=\mathfrak{n}\oplus\mathbb{K}c$ and let M'' be the one-dimensional $\mathfrak{n}'$ module where c acts as a scalar, and the e_n act trivially for $n\geqslant 3$. The module

$$M=U(Vir)\otimes_{U(\mathfrak{n}')}M''\cong U(Vir)\otimes_{U(\mathfrak{b}_+)}U(\mathfrak{b}_+)\otimes_{U(\mathfrak{n}')}M''$$

is a *Whittaker module.* If e_1, e_2 act nontrivially on M'', then M is simple by [OW09, Corollary 4.5]. All of these examples are annihilated by some $c-\varkappa$, where $\varkappa\in\mathbb{K}$, and so have *central character* $\varkappa$.

Using Theorem 1.2, we may immediately compute the annihilator of an induced module.

Theorem 6.1. *Let M' be a representation of $\mathfrak{b}_+$ with central character $\varkappa\in\mathbb{K}$. Let $M=U(Vir)\otimes_{U(\mathfrak{b}_+)}M'$. Then $\operatorname{Ann}_{U(Vir)}M=(c-\varkappa)$.*

Proof. Clearly $(c-\varkappa)M=0$.

Let $\mathcal{P}$ be the set of *negative partitions* $\underline{\lambda}=(\lambda_1,\dots,\lambda_k)$ where the λ_i are negative integers with $\lambda_1\leqslant\lambda_2\leqslant\dots\leqslant\lambda_k$. If $\underline{\lambda}=(\lambda_1,\dots,\lambda_k)\in\mathcal{P}$, let $e_{\underline{\lambda}}=e_{\lambda_1}\dots e_{\lambda_k}$. If $0\neq m\in M'$, it follows from the Poincaré-Birkhoff-Witt theorem that the elements $\{e_{\underline{\lambda}}\otimes m{:}\underline{\lambda}\in\mathcal{P}\}$ are linearly independent in M, and thus, as these elements are in bijection with partitions, that M has subexponential growth and infinite Gelfand-Kirillov dimension.

Let $K=\operatorname{Ann}_{U(Vir)}M$. If $K\supsetneq(c-\varkappa)$ then by Theorem 1.2 $U(Vir)/K$ has polynomial growth and thus, by [KL00, Proposition 5.1(d)], so does M. This contradiction shows that $K=(c-\varkappa)$. □

Corollary 6.2. *For any $\varkappa,\lambda\in\mathbb{K}$, the Verma module $M_{\varkappa,\lambda}$ is a faithful $U(Vir)/(c-\varkappa)$-module.*

Corollary 6.2 is an unpublished result of Nolan Wallach [WS13], and is independently due to Olivier Mathieu in unpublished work; see [CM07, footnote 2, p. 496]. We thank Rupert Wei Tze Yu for pointing out this reference to us.

It is known [FF84, Theorem 1.2] that for any $\varkappa$, the module $M_{\varkappa,\lambda}$ is simple for generic λ. Thus it follows immediately that $U(Vir)/(c-\varkappa)$ is primitive.

Corollary 6.3. *Let N be a logarithmic representation or a Whittaker module over Vir. Then $\operatorname{Ann}_{U(Vir)}(N)=(c-\varkappa)$ for some $\varkappa\in\mathbb{K}$.*

6.2. Completely prime ideals

In [PS17, Conjecture 1.3], it is conjectured that $U(W_+)$ satisfies the ascending chain condition on two-sided ideals. We cannot prove this, but we do show

Proposition 6.4. *The algebras $U(W_+)$ and $U(Vir)$ satisfy the ascending chain condition (ACC) on completely prime ideals.*

Proof. We first note that any ring R of finite or just infinite GK-dimension satisfies ACC on completely prime ideals. Letting P_0 be the first ideal in the chain, it is sufficient to show that R/P_0 has ACC on completely prime ideals. Thus we may replace R by R/P_0 and assume that R is a domain with GKdim $R<\infty$. Now if I is a nonzero ideal of R, then by [KL00, Proposition 3.15], GKdim $R/I\leqslant$GKdim $R-1$, so by induction the length of a chain of completely prime ideals is bounded by GKdim R. Thus by Theorem 1.2, $U(W_+)$ and $U(Vir)/(c-\varkappa)$ (for any $\varkappa\in\mathbb{K}$) have ACC on completely prime ideals.

In fact, note that if $f[x]\in\mathbb{K}[x]$ is irreducible, then $U(Vir)/(f(c))$ has just infinite GK-dimension and thus ACC on completely prime ideals. To see this, let $\mathbb{K}'$ be the extension field $\mathbb{K}[x]/f(x)$ of $\mathbb{K}$, and note that

$$U(Vir)/(f(c))=U_{\mathbb{K}}(Vir)/(f(c))\cong U_{\mathbb{K}'}(Vir)/(c-x).$$

This last has just infinite GK-dimension by Theorem 1.2.

We now consider an ascending chain $P_1\subseteq P_2\subseteq\ldots$ of completely prime ideals of $U(Vir)$. If $\bigcup P_n$ contains a nonzero element of $\mathbb{K}[c]$, then as the P_n are prime and c is central, some P_n contains an irreducible polynomial $f(c)\in\mathbb{K}[c]$. By the first part of the proof, therefore, the chain stabilizes.

So we may assume that each $P_n\cap\mathbb{K}[c]=0$. As P_n is prime, each $U(Vir)/P_n$ is $\mathbb{K}[c]$-torsionfree. Thus if $P_n\neq P_{n+1}$, then $(P_{n+1}/P_n)\otimes_{\mathbb{K}[c]}\mathbb{K}(c)\neq 0$ and so

$$P_n\otimes_{\mathbb{K}[c]}\mathbb{K}(c)\neq P_{n+1}\otimes_{\mathbb{K}[c]}\mathbb{K}(c).$$

Further, these ideals are completely prime as

$$U(Vir)\otimes_{\mathbb{K}[c]}\mathbb{K}(c)/P_n\otimes_{\mathbb{K}[c]}\mathbb{K}(c)\cong(U(Vir)/P_n)\otimes_{\mathbb{K}[c]}\mathbb{K}(c)$$

is a domain.

Thus it suffices to show that $U(Vir)\otimes_{\mathbb{K}[c]}\mathbb{K}(c)$ has ACC on completely prime ideals. By Remark 5.4, $U(Vir)\otimes_{\mathbb{K}[c]}\mathbb{K}(c)$ has just infinite GK-dimension. Thus by the first part of the proof, $U(Vir)\otimes_{\mathbb{K}[c]}\mathbb{K}(c)$ satisfies the ACC on completely prime ideals. □

6.3. The Hopfian and Bassian properties

To end the paper, we consider two ring-theoretic properties which are related to noetherianity. A ring R is *Hopfian* if R is not isomorphic to any proper quotient

R/J (equivalently, any epimorphism from $R \to R$ is an isomorphism). More strongly, R is *Bassian* if there is no injection of R into any proper quotient R/J. We thank Lance Small for introducing us to these concepts.

Proposition 6.5. *The algebras $U(W_+)$, $U(W)$, $U(\mathbf{W}_1)$, and $U(Vir)/(c-\varkappa)$ are Bassian and Hopfian, and $U(Vir)$ is Hopfian.*

That $U(W_+)$ is Hopfian is proved in [RS17, Remarks 2.2], and [RS17, Section 4] asks whether $U(W)$ is Bassian or Hopfian.

Proof. If R has just infinite GK-dimension, then $\mathrm{GKdim}\, R/J < \mathrm{GKdim}\, R$ for any proper ideal J of R, so R cannot inject in R/J. Thus the Bassian (and thus Hopfian) property for $U(W_+)$, $U(W)$, and $U(Vir)/(c-\varkappa)$ follows from Theorem 1.2. For $U(\mathbf{W}_1)$ it follows from Remark 4.7.

To show that $U(Vir)$ is Hopfian, let $R = U(Vir)$ and let f be a surjective endomorphism of R, with kernel J. As $R/J \cong \mathrm{Im}(f)$ is torsionfree as a module over $\mathbb{K}[c]$, the complex

$$0 \longrightarrow J \otimes_{\mathbb{K}[c]} \mathbb{K}(c) \longrightarrow R \otimes_{\mathbb{K}[c]} \mathbb{K}(c) \longrightarrow (R/J) \otimes_{\mathbb{K}[c]} \mathbb{K}(c) \longrightarrow 0$$

is exact. Now by Remark 5.4, we must have $J \otimes_{\mathbb{K}[c]} \mathbb{K}(c) = 0$. As R is $\mathbb{K}[c]$-torsionfree, $J = 0$. □

That $U(Vir)$ is Hopfian also follows from [RS17, Corollary 2.6] and Proposition 6.4. We do not know whether $U(Vir)$ is Bassian.

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