Sharp estimate on the inner distance in planar domains

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Abstract. We show that the inner distance inside a bounded planar domain is at most the one-dimensional Hausdorff measure of the boundary of the domain. We prove this sharp result by establishing an improved Painlevé length estimate for connected sets and by using the metric removability of totally disconnected sets, proven by Kalmykov, Kovalev, and Rajala. We also give a totally disconnected example showing that for general sets the Painlevé length bound $\varkappa(E) \leq \pi \mathcal{H}^1(E)$ is sharp.

1. Introduction

In this paper we continue the study of the internal distance for planar domains. For a set $A \subset \mathbb{R}^2$, we define the internal distance $d_A \colon A^2 \to [0, +\infty]$ as

$$(1.1) \qquad d_A(x,y) := \inf \left\{ \ell(\gamma) : \gamma \text{ is a curve in } A \text{ connecting } x \text{ to } y \right\},$$

where $\ell(\gamma)$ denotes the length of the curve γ. This is actually a distance when the set A is connected by rectifiable curves, otherwise d_A may take the value $+\infty$ (but the other axioms of a distance hold). Now fix a domain $\Omega \subset \mathbb{R}^2$. The internal distance for Ω is determined by how much the boundary $\partial\Omega$ blocks the curves γ. One result in this direction was proven in [6]: If the complement of the domain Ω is totally disconnected with finite $\mathcal{H}^1$-measure, then d_Ω is the Euclidean distance. (Here, $\mathcal{H}^1$ stands for the one-dimensional Hausdorff measure on $\mathbb{R}^2$, whose definition will be recalled at the beginning of Section 2.) In other words, totally disconnected

closed sets with finite $\mathcal{H}^1$-measure are (metrically) removable. The proof of this result used the estimate

$$(1.2) \qquad d_\Omega(x,y) \le |x-y| + \frac{\pi}{2}\,\mathcal{H}^1(\partial\Omega).$$

We improve (1.2) to the following sharp estimate:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a domain satisfying $\mathcal{H}^1(\partial\Omega) < \infty$. Then the estimate*

$$(1.3) \qquad d_\Omega(x,y) \le |x-y| + \mathcal{H}^1(E)$$

holds for every $x,y \in \Omega$, where $E \subset \partial\Omega$ is the union of all the connected components of $\partial\Omega$ with positive length. In the case when Ω is bounded, the above estimate can be improved to

$$(1.4) \qquad d_\Omega(x,y) \le \mathcal{H}^1(E).$$

The assumption $\mathcal{H}^1(\partial\Omega) < \infty$ in Theorem 1.1 is used for showing that the totally disconnected part of the boundary is removable. In view of the examples constructed in [5], it is at least necessary to assume that the Hausdorff dimension of $\partial\Omega$ is at most one. However, it is not clear if the assumption $\mathcal{H}^1(\partial\Omega) < \infty$ could be relaxed to $\partial\Omega$ having σ-finite $\mathcal{H}^1$-measure.

The sharpness of the estimate (1.3) in the unbounded case is seen simply by taking $\partial\Omega$ to be a line-segment. In the bounded case, the sharpness is seen for example by considering

$$\Omega = (0,1)^2 \setminus \bigcup_{i=1}^n \left(\left(\left\{ \frac{1}{2i} \right\} \times \left[0, 1 - \frac{1}{i} \right] \right) \cup \left(\left\{ \frac{1}{2i+1} \right\} \times \left[\frac{1}{i}, 1 \right] \right) \right)$$

for n larger and larger, and by scaling Ω.

As a consequence of Theorem 1.1, we obtain the following result:

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $\mathcal{H}^1(\partial\Omega) < \infty$. Let $x \in \Omega$ and $y \in \partial\Omega$ be given. Then for every $\varepsilon > 0$ there exists an injective Lipschitz curve $\gamma \colon [0,1] \to \mathbb{R}^2$ joining x to y such that $\gamma|_{[0,1)} \subset \Omega$ and $\ell(\gamma) \le \mathcal{H}^1(\partial\Omega) + \varepsilon$.*

The previous result can be proven by arguing as in the proof of Theorem 2.7, but replacing the estimate (1.2) with (1.4).

Remark 1.3. We point out that the curve γ in Theorem 1.2 can be chosen to be smooth in the open interval $(0,1)$, as follows from a standard approximation argument.

The paper is organized as follows. In Section 2 we recall, and prove, basic results in planar geometry; especially for planar domains whose boundary has finite length. In Section 3 we show an improved version of the Painlevé length estimate for connected sets and show the sharpness of the general Painlevé length estimate for disconnected sets. In the final Section 4 we prove our main theorem, Theorem 1.1.

2. Some auxiliary results

We collect in this section some standard results in planar geometry that will be needed in the remaining part of this paper. The one-dimensional Hausdorff measure $\mathcal{H}^1$ on $\mathbb{R}^2$ is the outer measure on $\mathbb{R}^2$ defined as

$$\mathcal{H}^1(A) := \sup_{\delta > 0} \inf \left\{ \sum_{i \in \mathbb{N}} \operatorname{diam}(A_i) \, \middle| \, A_i \subset \mathbb{R}^2 \text{ and } \operatorname{diam}(A_i) < \delta \text{ for all } i \in \mathbb{N}, A \subset \bigcup_{i \in \mathbb{N}} A_i \right\}$$

for every $A \subset \mathbb{R}^2$, where $\operatorname{diam}(A_i) := \sup \left\{ |x - y| : x, y \in A_i \right\}$ denotes the diameter of A_i. The one-dimensional Hausdorff content $\mathcal{H}^1_\infty$ on $\mathbb{R}^2$ is the outer measure on $\mathbb{R}^2$ defined as

$$\mathcal{H}^1_\infty(A) := \inf \left\{ \sum_{i \in \mathbb{N}} \operatorname{diam}(A_i) \, \middle| \, A_i \subset \mathbb{R}^2 \text{ for all } i \in \mathbb{N}, \ A \subset \bigcup_{i \in \mathbb{N}} A_i \right\} \quad \text{for every } A \subset \mathbb{R}^2.$$

An open, connected subset of $\mathbb{R}^2$ is referred to as a *(planar) domain*. Given any $x \in \mathbb{R}^2$ and $r > 0$, we shall denote by $B(x, r) := \left\{ y \in \mathbb{R}^2 : |y - x| < r \right\}$ the open ball of center x and radius r. More generally, given any nonempty subset A of $\mathbb{R}^2$, we shall denote by $B(A, r) := \left\{ y \in \mathbb{R}^2 : \operatorname{dist}(y, A) < r \right\}$ the open r-neighbourhood of A, where the quantity $\operatorname{dist}(y, A)$ is defined as $\operatorname{dist}(y, A) := \inf \left\{ |y - z| : z \in A \right\}$.

For our purposes, a curve in $\mathbb{R}^2$ is a continuous map $\gamma \colon I \to \mathbb{R}^2$, where I is an interval in the real line. For brevity, given $t \in I$ we shall always write γ_t in place of $\gamma(t)$. Moreover, we shall sometimes use the same notation γ to denote also the image $\gamma(I) \subset \mathbb{R}^2$ of the curve (whenever no ambiguity may occur). By *Jordan loop* we mean a closed simple curve $\sigma \colon [0, 1] \to \mathbb{R}^2$, namely, $\sigma|_{[0,1)}$ is injective and $\sigma_0 = \sigma_1$.

Lemma 2.1. *Let $C \subset \mathbb{R}^2$ be a connected set. Then it holds that*

$$\mathcal{H}^1(C) \geq |x - y| \quad \text{for every } x, y \in C.$$

In particular, we have that $\operatorname{diam}(C) \leq \mathcal{H}^1(C)$.

Proof. Fix $x \in C$ and consider the function $f_x \colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f_x(y) := |y - x|$ for every $y \in \mathbb{R}^2$. Observe that the function f_x is 1-Lipschitz, so that $\mathcal{H}^1(C) \geq$

$\mathcal{L}^1\big(f_x(C)\big)$, where $\mathcal{L}^1$ stands for the one-dimensional Lebesgue measure on $\mathbb{R}$. More-over, since C is connected, we know that $\big[0, |y-x|\big] = \big[f_x(x), f_x(y)\big] \subset f_x(C)$ holds for every $y \in C$. Consequently, we have that $|y-x| \leq \mathcal{L}^1\big(f_x(C)\big)$ for every $y \in C$. Therefore we conclude that

$$\mathcal{H}^1(C) \geq \mathcal{L}^1\big(f_x(C)\big) \geq |x-y| \quad \text{for every } x, y \in C,$$

as required. Taking the supremum over $x, y \in C$ we also get that $\mathrm{diam}(C) \leq \mathcal{H}^1(C)$. $\quad\square$

Lemma 2.2. *Let $\{G_i\}_{i \in I}$ be a family of Borel subsets of $\mathbb{R}^2$, for some $I \subset \mathbb{N}$. Suppose*

$$(2.1) \qquad \mathcal{H}^1(G_i \cap G_j \cap G_k) = 0 \quad \text{for every } i, j, k \in I \text{ with } i \neq j \neq k \neq i.$$

Then it holds that

$$(2.2) \qquad 2\,\mathcal{H}^1\bigg(\bigcup_{i \in I} G_i\bigg) \geq \sum_{i \in I} \mathcal{H}^1(G_i).$$

Proof. If $\mathcal{H}^1\big(\bigcup_{i \in I} G_i\big) = \infty$, then (2.2) is clearly satisfied, thus suppose $\mathcal{H}^1\big(\bigcup_{i \in I} G_i\big) < \infty$. Property (2.1) grants that $\mathcal{H}^1\big((G_i \cap G_j) \cap (G_{i'} \cap G_{j'})\big) = 0$ whenever $i, i', j, j' \in I$ satisfy $i < j$, $i' < j'$, and $(i, j) \neq (i', j')$, thus $\sum_{i < j} \mathcal{H}^1(G_i \cap G_j) = \mathcal{H}^1\big(\bigcup_{i < j} G_i \cap G_j\big)$. Therefore, by using the inclusion-exclusion principle (and (2.1) again) we obtain that

$$\mathcal{H}^1\bigg(\bigcup_{i \in I} G_i\bigg) = \sum_{i \in I} \mathcal{H}^1(G_i) - \sum_{i < j} \mathcal{H}^1(G_i \cap G_j) = \sum_{i \in I} \mathcal{H}^1(G_i) - \mathcal{H}^1\bigg(\bigcup_{i < j} G_i \cap G_j\bigg)$$

$$\geq \sum_{i \in I} \mathcal{H}^1(G_i) - \mathcal{H}^1\bigg(\bigcup_{i \in I} G_i\bigg),$$

whence the claimed inequality (2.2) is proven. $\quad\square$

The following result can be found in the proof of [7, Theorem VI.16.3, p. 168].

Lemma 2.3. *Let Ω be a domain in $\mathbb{R}^2$. Let F be some connected component of $\partial\Omega$. Denote by B the connected component of Ω^c that contains F. Then $\partial B = F$.*

A domain $\Omega \subset \mathbb{R}^2$ is said to be *locally connected along its boundary* provided for every point $x \in \partial\Omega$ and every radius $r > 0$ there exists $t \in (0, r)$ such that $\Omega \cap B(x, t)$ is contained in one connected component of $\Omega \cap B(x, r)$.

The following result has been stated and proved in [5, Corollary 3.3]:

Proposition 2.4. *Let $\Omega \subset \mathbb{R}^2$ be any domain such that $\mathbb{R}^2 \backslash \Omega$ is connected and not a singleton. Suppose that Ω is locally connected along its boundary and $\partial\Omega$ is bounded. Then $\partial\Omega$ is a Jordan loop.*

As an immediate consequence, we can obtain the following result:

Theorem 2.5. *Let Ω be a bounded domain in $\mathbb{R}^2$ with $\mathcal{H}^1(\partial\Omega) < +\infty$. Let U be a connected component of $\mathbb{R}^2 \backslash \overline{\Omega}$. Then ∂U is a Jordan loop (of finite length). Moreover, it holds that*:

i) *If U is bounded, then Ω lies in the unbounded connected component of $\mathbb{R}^2 \backslash \partial U$.*

ii) *If U is unbounded, then Ω lies in the bounded connected component of $\mathbb{R}^2 \backslash \partial U$.*

Proof. Consider a connected component U of $\mathbb{R}^2 \backslash \overline{\Omega}$. In view of Proposition 2.4, it is sufficient to show that U is locally connected along its boundary. Fix any $x \in \partial U$ and $r > 0$. We claim that:

(2.3) Only finitely many connected components of $U \cap B(x, 2r)$ intersect $B(x, r)$.

Call $\mathcal{F}$ the collection of the connected components of $U \cap B(x, 2r)$ intersecting $B(x, r)$. Suppose by contradiction $\mathcal{F}$ is not finite. This gives $U \backslash B(x, 2r) \neq \varnothing$. Fix any $n \in \mathbb{N}$, $n \geq 3$ with $n > \mathcal{H}^1(\partial\Omega)/r$. Choose distinct elements $\{E_i\}_{i \in \mathbb{Z}/n\mathbb{Z}}$ of $\mathcal{F}$. Call A the open annulus $B(x, 2r) \backslash \overline{B}(x, r)$. Since U is open and connected, we can find for any $i \in \mathbb{Z}/n\mathbb{Z}$ an injective curve $\gamma^i \colon [0, 1] \to U$ such that $\gamma_0^i \notin B(x, 2r)$ and $\gamma_1^i \in E_i \cap B(x, r)$. Define $s_i := \max\{t \in [0, 1) \colon \gamma_t^i \in \partial B(x, 2r)\}$ and $t_i := \min\{t \in (s_i, 1) \colon \gamma_t^i \in \partial B(x, r)\}$. Then the curves $\{\gamma^i|_{[s_i, t_i]}\}_{i \in \mathbb{Z}/n\mathbb{Z}}$ are pairwise disjoint and we have $G_i := \gamma^i((s_i, t_i)) \subset A \cap E_i$ for every $i \in \mathbb{Z}/n\mathbb{Z}$. Up to relabelling, we can assume that the points $\gamma_{s_0}^0, ..., \gamma_{s_{n-1}}^{n-1}$ are distributed along $\partial B(x, 2r)$ in a clockwise order. Given any $i \in \mathbb{Z}/n\mathbb{Z}$, we denote by F_i the connected component of $A \cap E_i$ containing G_i, while R_i stands for the connected component of $A \backslash (G_0 \cup ... \cup G_{n-1})$ whose boundary contains the set $G_i \cup G_{i+1}$. Observe that $A \cap \partial F_i \subset R_{i-1} \cup R_i$ for every $i \in \mathbb{Z}/n\mathbb{Z}$. Call $\Gamma_i^- := \partial F_i \cap R_{i-1}$ and $\Gamma_i^+ := \partial F_i \cap R_i$ for every $i \in \mathbb{Z}/n\mathbb{Z}$. Given that the sets $R_0, ..., R_{n-1}$ are pairwise disjoint, it holds that the family $\{\Gamma_i^- : i \in \mathbb{Z}/n\mathbb{Z}\} \cup \{\Gamma_i^+ : i \in \mathbb{Z}/n\mathbb{Z}\}$ satisfies the hypothesis of Lemma 2.2, as the only (possibly) non-empty intersections among its members are those ones of the form $\Gamma_i^- \cap \Gamma_{i-1}^+$ for $i \in \mathbb{Z}/n\mathbb{Z}$. Since $\Gamma_i^- \cup \Gamma_i^+ = A \cap \partial F_i \subset \partial E_i$ for every $i \in \mathbb{Z}/n\mathbb{Z}$, we have that

(2.4)
$$\mathcal{H}^1(\partial\Omega) \geq \mathcal{H}^1\left(\bigcup_{i \in \mathbb{Z}/n\mathbb{Z}} \partial E_i\right) \geq \mathcal{H}^1\left(\bigcup_{i \in \mathbb{Z}/n\mathbb{Z}} \Gamma_i^- \cup \Gamma_i^+\right) \geq \frac{1}{2} \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \mathcal{H}^1(\Gamma_i^-) + \mathcal{H}^1(\Gamma_i^+).$$
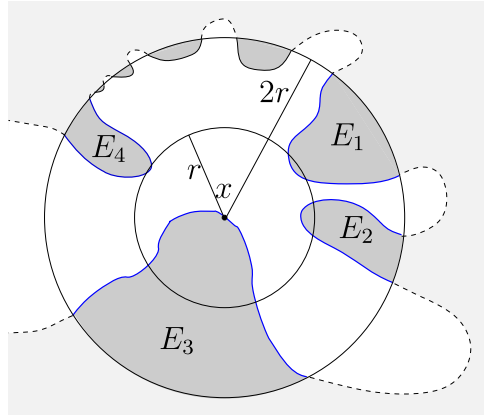
Figure 1. Proofs of Theorem 2.5 and Lemma 2.6 rely on the finiteness of the length of the boundary of Ω to deduce that there are only finitely many components intersecting both the small ball and the complement of the large ball.

(The last inequality follows from Lemma 2.2.) Moreover, it holds that $\Gamma_i^{\pm} \cap \partial B(x, \lambda) \neq \varnothing$ for every $i \in \mathbb{Z}/n\mathbb{Z}$ and $\lambda \in (r, 2r)$. Therefore, we have $\mathcal{H}^1(\Gamma_i^{\pm}) \geq r$ for all $i \in \mathbb{Z}/n\mathbb{Z}$, which (together with (2.4)) implies that $\mathcal{H}^1(\partial\Omega) \geq nr$. This contradicts our choice of n, thus the claim (2.3) is proven.

Since $\mathcal{F}$ has finite cardinality, we can find $t \in (0, r)$ so small that the element of $\mathcal{F}$ containing x in its boundary is the only one that intersects the ball $B(x, t)$. This forces $U \cap B(x, t)$ to be contained in one connected component of $U \cap B(x, 2r)$. Therefore U is locally connected along its boundary, as required. The proof of items i) and ii) follows by a standard topological argument. □

Lemma 2.6. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain satisfying $\mathcal{H}^1(\partial\Omega) < +\infty$. Fix any $x \in \partial\Omega$ and $\varepsilon > 0$. Then there exists a subdomain Ω' of Ω with $\mathcal{H}^1(\partial\Omega') \leq \varepsilon$ such that $x \in \partial\Omega'$ and $\mathrm{diam}(\Omega') \leq \varepsilon$. Moreover, we can further require that $\partial\Omega' \subset \partial\Omega \cup \partial B(x, r)$ for some $r > 0$.*

Proof. STEP 1. Given that $\mathcal{H}^1|_{\partial\Omega}$ is upper continuous, we can choose $r \in \big(0, \varepsilon/(4\pi)\big)$ such that $\mathcal{H}^1\big(\partial\Omega \cap B(x, r)\big) \leq \varepsilon/2$. If $\Omega \subset \overline{B}(x, r)$, then the set $\Omega' := \Omega$ does the job, thus let us assume that there exists a point $y \in \Omega \setminus \overline{B}(x, r)$. Let us denote by $\mathcal{F}$ the family of all connected components of $\Omega \cap B(x, r)$ that intersect $\partial B(x, r/2)$. Given any $V \in \mathcal{F}$, it holds:

(2.5)   There is a continuous curve $\alpha \colon [0, 1] \longrightarrow \mathbb{R}^2$ such that
$\alpha|_{(0,1]} \subset V$, $\alpha_0 \in \partial B(x, r) \cap \Omega$ and $\alpha_1 \in \partial B(x, r/2)$.

Indeed, we can find a continuous curve $\alpha' \colon [0,1] \to \Omega$ joining y to a point of $\partial B(x, r/2) \cap V$. Calling $t_0 := \max \left\{ t \in [0,1] \,\middle|\, \alpha'_t \in \partial B(x, r) \right\} < 1$, we see that the curve α obtained by restricting α' to $[t_0, 1]$ fulfills the requirements.

STEP 2. Given $V \in \mathcal{F}$, we call σ^V that Jordan loop for which $\sigma^V \subset \partial V$ and V is contained in the bounded connected component U^V of $\mathbb{R}^2 \setminus \sigma^V$ (cf. Theorem 2.5). We claim that

$$U^{V_1} \cap U^{V_2} = \varnothing \quad \text{if } V_1, V_2 \in \mathcal{F} \text{ are distinct.}$$

We prove it arguing by contradiction: suppose $U^{V_1} \cap U^{V_2} \neq \varnothing$ for some $V_1, V_2 \in \mathcal{F}$, $V_1 \neq V_2$. Possibly relabeling V_1 and V_2, we thus have that V_2 is contained in some bounded connected component $\mathcal{H}$ of $\mathbb{R}^2 \setminus \overline{V}_1$. Let α be a curve associated with V_2 as in (2.5). Since $\alpha_0 \in \Omega$, we can find $\delta \in (0, r)$ such that $B(\alpha_0, \delta) \subset \Omega$. Let us call $C := B(\alpha_0, \delta) \cap B(x, r)$ and $C' := B(\alpha_0, \delta) \setminus \overline{B}(x, r)$. Observe that $C \subset V_2$ (as C is a connected subset of $\Omega \cap B(x, r)$ intersecting V_2), while C' is contained in the unbounded connected component W of $\mathbb{R}^2 \setminus \overline{V}_1$. Also, the set $A := B(\alpha_0, \delta) \cap \partial B(x, r)$ has empty interior, whence $V_1 \cap B(\alpha_0, \delta) = \varnothing$. This grants that the point α_0 does not belong to the closure of V_1. Therefore, $C \cup C' \cup \{\alpha_0\}$ is a connected subset of $\mathbb{R}^2 \setminus \overline{V}_1$ that intersects both H and W, which leads to a contradiction.

Moreover, we claim that:

$$(2.6) \qquad \#(\partial V_1 \cap \partial V_2 \cap \partial V_3) \leq 2 \quad \text{if } V_1, V_2, V_3 \in \mathcal{F} \text{ are distinct.}$$

We know from (2.5) that the sets $U^{V_1}, U^{V_2}, U^{V_3}$ are pairwise disjoint. Suppose by contradiction that there exist at least three distinct points z_1, z_2, z_3 in $\partial V_1 \cap \partial V_2 \cap \partial V_3$. In particular, such points are forced to belong to $\sigma^{V_1} \cap \sigma^{V_2} \cap \sigma^{V_3}$. Fix some other points $a_j \in U^{V_j}$ for $j = 1, 2, 3$. We can build continuous curves $\gamma^{ij} \colon [0,1] \to \mathbb{R}^2$ for $i, j = 1, 2, 3$ such that the following properties hold:

i) γ^{ij} joins z_i to a_j for all $i, j = 1, 2, 3$.

ii) $\gamma^{ij}|_{(0,1)} \subset U^{V_j}$ for all $i, j = 1, 2, 3$.

iii) $\gamma^{1j}|_{(0,1)}, \gamma^{2j}|_{(0,1)}, \gamma^{3j}|_{(0,1)}$ are pairwise disjoint for all $j = 1, 2, 3$.

This would imply that the complete 3-by-3 bipartite graph $K_{3,3}$ is planar, thus leading to a contradiction (cf. [3, Chapter I, Theorem 17]). Therefore, the claim (2.6) is proven.

STEP 3. As a consequence, we can show that:

$$(2.7) \qquad \mathcal{F} \text{ is a finite family.}$$

In order to prove it, we argue by contradiction: suppose $\mathcal{F}$ is infinite, say $\mathcal{F} = (V_i)_{i \in \mathbb{N}}$. Thanks to (2.5) we know that $V_i \cap \partial B(x, \lambda) \neq \varnothing$ for any $i \in \mathbb{N}$ and $\lambda \in (r/2, r)$. Since the family $\mathcal{F}$ contains more than one element, we infer that also $\partial V_i \cap \partial B(x, \lambda) \neq \varnothing$ holds for any $i \in \mathbb{N}$ and $\lambda \in (r/2, r)$. In particular, we have that $\mathcal{H}^1(\partial V_i) \geq r/2$ for

every $i \in \mathbb{N}$. On the other hand, we know from (2.6) that the family $\{\partial V_i\}_{i \in \mathbb{N}}$ satisfies the hypothesis of Lemma 2.2, thus accordingly

$$2\,\mathcal{H}^1\bigg(\bigcup_{i \in \mathbb{N}} \partial V_i\bigg) \geq \sum_{i \in \mathbb{N}} \mathcal{H}^1(\partial V_i) \geq \sum_{i \in \mathbb{N}} \frac{r}{2} = +\infty.$$

This implies that $\bigcup_{i \in \mathbb{N}} \partial V_i$ has infinite $\mathcal{H}^1$-measure, which is in contradiction with the fact that $\bigcup_{i \in \mathbb{N}} \partial V_i \subset \partial\Omega \cup \partial B(x,r)$. Accordingly, property (2.7) is verified.

STEP 4. We also claim that:

$$(2.8) \qquad\qquad \Omega \cap B(x, r/2) \subset U := \bigcup_{V \in \mathcal{F}} V.$$

Indeed, fix any $z \in B(x, r/2) \cap \Omega$. Choose a continuous curve $\alpha \colon [0,1] \to \Omega$ such that $\alpha_0 = z$ and $\alpha_1 = y$. Call $t_0 := \min\big\{ t \in [0,1] \,\big|\, \alpha_t \in \partial B(x,r) \big\} > 0$. Then $\alpha|_{[0,t_0)}$ is a connected subset of $\Omega \cap B(x,r)$ that intersects $\partial B(x, r/2)$, whence it is contained in some element of $\mathcal{F}$. This shows that $z \in \bigcup_{V \in \mathcal{F}} V$, which yields the claim (2.8).

STEP 5. We can finally conclude the proof by combining (2.7) with (2.8): the latter ensures that $x \in \partial U$, thus the former implies that $x \in \partial\Omega'$ for some element $\Omega' \in \mathcal{F}$. Given that we have $\Omega' \subset B(x,r) \subset B(x, \varepsilon/2)$ (so that $\text{diam}(\Omega') \leq \varepsilon$ and

$$\mathcal{H}^1(\partial\Omega') \leq \mathcal{H}^1\big(\partial\Omega \cap B(x,r)\big) + \mathcal{H}^1\big(\partial B(x,r)\big) \leq \frac{\varepsilon}{2} + 2\,\pi\,r \leq \varepsilon,$$

the statement is achieved. $\quad\square$

Theorem 2.7. (Accessible points) *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $\mathcal{H}^1(\partial\Omega) < \infty$. Let $x \in \Omega$ and $y \in \partial\Omega$ be given. Then for every $\varepsilon > 0$ there exists an injective Lipschitz curve $\gamma \colon [0,1] \to \mathbb{R}^2$ joining x to y such that $\gamma|_{[0,1)} \subset \Omega$ and $\ell(\gamma) \leq |x-y| + \frac{\pi}{2}\,\mathcal{H}^1(\partial\Omega) + \varepsilon$.*

Proof. STEP 1. Fix $\varepsilon \in (0,1)$. Call $x_0 := x$ and $\Omega_0 := \Omega$. By applying Lemma 2.6 in a recursive way, we build a decreasing sequence $(\Omega_n)_{n \geq 1}$ of subdomains of $\Omega \setminus \{x_0\}$ satisfying the following properties:

i) $\mathcal{H}^1(\partial\Omega_n) < \varepsilon/(2^{n+1}\,\pi)$ for all $n \geq 1$.
ii) $y \in \partial\Omega_n$ for all $n \geq 1$.
iii) $\text{diam}(\Omega_n) \leq \varepsilon/2^{n+2}$ for all $n \geq 1$.
iv) $\text{dist}(\partial\Omega_n \cap \Omega, \partial\Omega_{n+1} \cap \Omega) > 0$ for all $n \geq 0$.

See Figure 2 for an illustration of the sequence of subdomains. For brevity, let us set $I_n := [1 - 2^{-n}, 1 - 2^{-n-1}]$ for all $n \in \mathbb{N}$. We claim that we can build a sequence
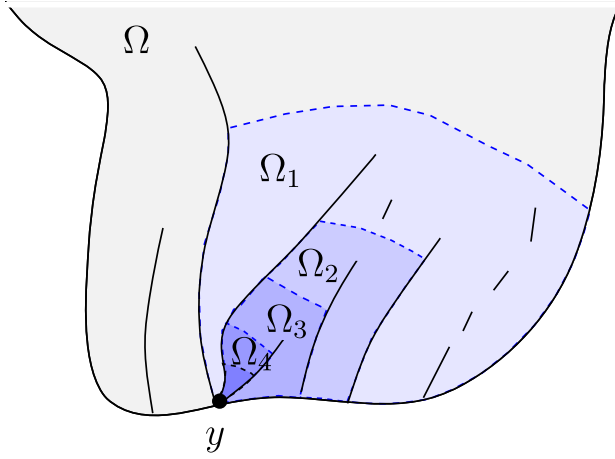
Figure 2. In the proof of Theorem 2.7 we use Lemma 2.6 to find a nested sequence of subdomains. The next step is to then connect points in subsequent subdomains by using the non-optimal inner distance estimate (1.2).

$(x_n)_{n\geq 1}\subset\Omega$ such that $x_n\in\Omega_n\setminus\overline{\Omega}_{n+1}$ for all $n\geq 1$, and a sequence of injective Lipschitz curves $(\alpha^n)_{n\geq 0}$ such that each $\alpha^n\colon I_n\to\Omega_n\setminus\overline{\Omega}_{n+2}$ joins x_n to x_{n+1} and satisfies

$$
(2.9) \qquad
\begin{aligned}
&\ell(\alpha^0)\leq |x-y|+\frac{\pi}{2}\,\mathcal{H}^1(\partial\Omega)+\frac{\varepsilon}{2},\\
&\ell(\alpha^n)\leq \frac{\varepsilon}{2^{n+1}}\quad\text{for every }n\geq 1.
\end{aligned}
$$

First, fix any sequence of points $(x'_n)_{n\geq 1}\subset\Omega$ that satisfies $x'_n\in\Omega_n$ for all $n\geq 1$. By using (1.2), we can find an injective curve $\widetilde{\alpha}^0\colon [0,1]\to\Omega_0$ joining x_0 to x'_1 such that $\ell(\widetilde{\alpha}^0)<|x_0-x'_1|+\frac{\pi}{2}\,\mathcal{H}^1(\partial\Omega)+\varepsilon/4$. By items ii) and iii) above, we have that $|x_0-x'_1|\leq |x-y|+|y-x'_1|\leq |x-y|+\varepsilon/4$, whence $\ell(\widetilde{\alpha}^0)\leq |x-y|+\frac{\pi}{2}\,\mathcal{H}^1(\partial\Omega)+\frac{\varepsilon}{2}$. Item iv) tells us that there exists $t_1\in(0,1]$ such that $\widetilde{\alpha}^0|_{[0,t_1]}$ lies in $\Omega_0\setminus\overline{\Omega}_2$ and $x_1:=\widetilde{\alpha}^0_{t_1}\in\Omega_1$. Therefore, the injective Lipschitz curve $\alpha^0\colon I_0\to\Omega_0$ obtained by reparametrizing $\widetilde{\alpha}^0|_{[0,t_1]}$ satisfies the first line of (2.9). Now suppose to have already defined $x_0,...,x_n$ and $\alpha^0,...,\alpha^{n-1}$ with the required properties. We can use again property (1.2) and item i) to find an injective curve $\widetilde{\alpha}^n\colon [0,1]\to\Omega_n$ joining x_n to x'_{n+1} such that $\ell(\widetilde{\alpha}^n)<|x_n-x'_{n+1}|+\varepsilon/2^{n+2}$. Since the points x_n,x'_{n+1} are in Ω_n, we infer from item iii) that $|x_n-x'_{n+1}|\leq\varepsilon/2^{n+2}$ and accordingly we have $\ell(\widetilde{\alpha}^n)\leq\varepsilon/2^{n+1}$. We can choose $t_{n+1}\in(0,1]$ so that $\widetilde{\alpha}^n|_{[0,t_{n+1}]}$ lies in $\Omega_n\setminus\overline{\Omega}_{n+2}$ and $x_{n+1}:=\widetilde{\alpha}^n_{t_{n+1}}\in\Omega_{n+1}$. Therefore, the injective Lipschitz curve $\alpha^n\colon I_n\to\Omega_n$ obtained by reparametrizing $\widetilde{\alpha}^n|_{[0,t_{n+1}]}$ satisfies the second line of (2.9). This proves our claim.

Consider the unique continuous curve $\alpha\colon [0,1)\to\Omega$ satisfying $\alpha|_{I_n}:=\alpha^n$ for all $n\in\mathbb{N}$. Items ii) and iii) grant that $|\alpha_t-y|\leq 2^{-n}$ holds for all $n\in\mathbb{N}$ and $t\in\bigcup_{k\geq n}I_k$.

This ensures that $\lim_{t\nearrow 1}\alpha_t=y$ whence α can be extended to a continuous curve $\alpha\colon[0,1]\to\mathbb{R}^2$ joining x to y. By using (2.9) we also deduce that

$$(2.10)\qquad \ell(\alpha)\le|x-y|+\frac{\pi}{2}\,\mathcal{H}^1(\partial\Omega)+\frac{\varepsilon}{2}+\sum_{n=1}^{\infty}\frac{\varepsilon}{2^{n+1}}=|x-y|+\frac{\pi}{2}\,\mathcal{H}^1(\partial\Omega)+\varepsilon.$$

STEP 2. Observe that the curve α might not be injective. We thus proceed as follows: we recursively build a sequence $(\gamma^n)_{n\ge1}$ of curves defined on $[0,1]$ such that

    a) γ^n is a constant-speed, $\ell(\alpha)$-Lipschitz and injective curve for all $n\ge1$,
    b) γ^n joins x to x_n for all $n\ge1$,
    c) the image of γ^n lies in $\alpha^0\cup...\cup\alpha^{n-1}$ for all $n\ge1$,
    d) calling $s_n:=\min\left\{t\in[0,1]\,\big|\,\gamma_t^n\in\partial\Omega_n\right\}$ it holds $\gamma^n|_{[0,s_n]}\subset\gamma^{n+1}$ for all $n\ge1$.

First, we take as $\gamma^1\colon[0,1]\to\Omega$ the constant-speed reparametrization of α^0. Now suppose to have already defined γ^n for some $n\ge1$. Let us define $t':=\max\{t\in I_n:\alpha_t^n\in\gamma^n\}$ and choose that $t''\in[0,1]$ for which $\gamma_{t''}^n=\alpha_{t'}^n$. Therefore, we call $\gamma^{n+1}\colon[0,1]\to\Omega$ the constant-speed reparametrization of the concatenation between $\gamma^n|_{[0,t'']}$ and $\alpha^n|_{[t',+\infty)\cap I_n}$. It follows from the very construction that γ^{n+1} satisfies items a), b), c) and d), as required.

The Ascoli-Arzelà theorem grants that (possibly passing to a not relabeled subsequence) the curves γ^n uniformly converge to some limit curve $\gamma\colon[0,1]\to\mathbb{R}^2$ with $\gamma_0=x$. By using item a) and the lower semicontinuity of the length functional, we deduce that $\ell(\gamma)\le\ell(\alpha)$. By item b) we know that $\gamma_1=\lim_n\gamma_1^n=\lim_n x_n=y$. Given $n\ge1$, we set $S_n:=\gamma^n\big([0,s_n]\big)$ and $\lambda_n:=\mathcal{H}^1(S_n)$. Item d) ensures that $S_n\subset\gamma^k$ for all $k\ge n$, thus $\ell(\gamma^k)\ge\lambda_n$. Thanks to item c) we also see that $\gamma^k\subset\gamma\cup\alpha^{k-1}\cup\alpha^k$, so that

$$\ell(\gamma^k)\le\ell(\gamma)+\ell(\alpha^{k-1})+\ell(\alpha^k)\le\ell(\gamma)+\frac{\varepsilon}{2^k}+\frac{\varepsilon}{2^{k+1}}\le\ell(\gamma)+\frac{1}{2^{n-1}}=:q_n.$$

Given that $\gamma^k\big([0,\lambda_n/\ell(\gamma^k)]\big)\subset S_n$ and $\lambda_n/\ell(\gamma^k)\ge\lambda_n/q_n$, we have $\gamma^k\big([0,\lambda_n/q_n]\big)\subset S_n$. Observe also that

$$(2.11)\quad d_{S_n}(\gamma_t^k,\gamma_s^k)=\ell(\gamma^k)\,|t-s|\ge\lambda_n\,|t-s|\quad\text{for every }k\ge n\text{ and }t,s\in[0,\lambda_n/q_n],$$

where the notation d_{S_n} stands for the internal distance for S_n defined in (1.1). Since $\gamma^n|_{[0,s_n]}\colon[0,s_n]\to S_n$ is a homeomorphism, we conclude from (2.11) by letting $k\to\infty$ that $d_{S_n}(\gamma_t,\gamma_s)\ge\lambda_n\,|t-s|$ for every $t,s\in[0,\lambda_n/q_n]$. In particular, one has that

$$(2.12)\qquad\qquad\gamma\text{ is injective on }[0,\lambda_n/q_n]\text{ for every }n\ge1.$$

Notice that $\alpha^0 \cup ... \cup \alpha^{n-2} \subset S_n$ for all $n \geq 2$ by construction, whence $\gamma^n \subset S_n \cup \alpha^{n-1}$ by item c) and accordingly $\ell(\gamma^n) \leq \lambda_n + \ell(\alpha^{n-1})$. This implies that

$$1 \geq \varliminf_{n \to \infty} \frac{\lambda_n}{q_n} = \frac{\varliminf_n \lambda_n}{\varliminf_n q_n} \geq \frac{\varliminf_n \ell(\gamma^n) - \varliminf_n \ell(\alpha^{n-1})}{\ell(\gamma)} \geq 1.$$

Therefore, we deduce that $[0, 1) = \bigcup_{n \geq 1} [0, \lambda_n / q_n]$, so that (2.12) grants that γ is injective on $[0, 1)$. Since $\gamma([0, \lambda_n / q_n]) \subset S_n \subset \alpha \setminus \{y\}$ for all $n \geq 1$, we see that $\gamma|_{[0,1)} \subset \alpha \setminus \{y\} \subset \Omega$. Finally, the curve γ is $\ell(\alpha)$-Lipschitz and $\ell(\alpha) \leq |x-y| + \frac{\pi}{2} \mathcal{H}^1(\partial\Omega) + \varepsilon$ by (2.10), whence $\ell(\gamma) \leq |x-y| + \frac{\pi}{2} \mathcal{H}^1(\partial\Omega) + \varepsilon$ as well. This completes the proof of the statement. $\square$

Lemma 2.8. *Let $K \subset \mathbb{R}^2$ be a compact, connected set. Let $\Omega \subset \mathbb{R}^2$ be an open set such that $K \setminus \overline{\Omega} \neq \varnothing$. Then for any connected component E of $K \cap \overline{\Omega}$ it holds that $E \cap \partial\Omega \neq \varnothing$.*

Proof. If $K \cap \overline{\Omega} = \varnothing$, then the statement is trivially verified (as the empty set admits no connected component). Then let us suppose that $K \cap \overline{\Omega} \neq \varnothing$. In this case, the statement readily follows from [1, Lemma 2.14], which we recall here for the reader's convenience:

(2.13)

Let X be a complete, connected metric space. Let $\varnothing \neq U \subset X$ be compact.

Let F be a connected component of U with $F \neq X$. Then $F \cap \partial U \neq \varnothing$.

To prove the statement, consider $X := K$ (with the induced Euclidean distance), $U := K \cap \overline{\Omega}$ and $F := E$. Given that $K \setminus \overline{\Omega} \neq \varnothing$, we have that $F \neq X$. Therefore, the result in (2.13) implies that $F \cap \partial_X U \neq \varnothing$, where $\partial_X U$ denotes the boundary of U in the metric space X. Notice that $\partial_X U = \partial_K(K \cap \overline{\Omega}) \subset \partial_{\mathbb{R}^2} \overline{\Omega} \subset \partial_{\mathbb{R}^2} \Omega$, thus we conclude that $E \cap \partial_{\mathbb{R}^2} \Omega \neq \varnothing$. $\square$

Remark 2.9. Let $\sigma \colon [0, 1] \to \mathbb{R}^2$ be a Jordan loop. Let us call U the bounded connected component of $\mathbb{R}^2 \setminus \sigma$. Fix any $t_1, t_2, t_3, t_4 \in [0, 1)$ with $t_1 < t_2 < t_3 < t_4$. Let us consider any two injective curves $\gamma, \gamma' \colon [0, 1] \to \mathbb{R}^2$ such that $\gamma_0 = \sigma_{t_1}$, $\gamma_1 = \sigma_{t_3}$, $\gamma'_0 = \sigma_{t_2}$, $\gamma'_1 = \sigma_{t_4}$, and $\gamma|_{(0,1)}, \gamma'|_{(0,1)} \subset U$. Then we claim that $\gamma[0,1] \cap \gamma'[0,1] \neq \varnothing$.

Indeed, calling V the bounded connected component of $\mathbb{R}^2 \setminus (\gamma[0,1] \cup \sigma[t_1, t_3])$, we have that $\gamma'_t \in V$ and $\gamma'_{1-t} \in \mathbb{R}^2 \setminus V$ for $t \in (0, 1/2)$ sufficiently small. Therefore, $\gamma'|_{(0,1)}$ must intersect the boundary of V. Given that $\gamma'(0, 1) \cap \sigma[t_1, t_3] = \varnothing$ by construction, we conclude that $\gamma'(0, 1) \cap \gamma(0, 1) \neq \varnothing$, as desired.

For the lack of an appropriate reference, we provide below a proof of the following fact:

Proposition 2.10. *Let $K \subset \mathbb{R}^2$ be a compact connected set such that $\mathcal{H}^1(K) < \infty$. Let us denote by C the convex hull of K. Then*

$$(2.14) \qquad\qquad \mathcal{H}^1(\partial C) \leq 2\, \mathcal{H}^1(K).$$

Proof. We subdivide the proof into several steps:

STEP 1. If $\mathring{C} = \varnothing$ then C is a segment, thus $\partial C = C = K$ and accordingly (2.14) is trivially verified. Now assume that $\mathring{C} \neq \varnothing$, so that (by convexity of C) we know that there exists a continuous curve $\gamma \colon [0,1] \to \mathbb{R}^2$, which is injective on $[0,1)$, such that $\gamma_0 = \gamma_1 \in K$ and $\gamma[0,1] = \partial C$. We can write $\gamma^{-1}(\mathbb{R}^2 \setminus K) = \bigcup_{i \in I}(a_i, b_i)$, where $I \subset \mathbb{N}$ and the sets (a_i, b_i) are pairwise disjoint subintervals of $(0,1)$. Since C is the convex hull of K, we have that the set $S_i := \gamma(a_i, b_i) \subset \partial C \setminus K$ is a segment for all $i \in I$.

STEP 2. Given any $i \in I$, let us call m_i the midpoint of S_i, while $v_i \in \mathbb{R}^2$ stands for the unit vector perpendicular to S_i such that $m_i + \mathbb{R}^{>0} v_i$ intersects C. Since m_i does not belong to the compact set $K \cup (\partial C \setminus S_i)$, we can find $\varepsilon_i > 0$ such that $m_i + [0, \varepsilon_i] v_i$ does not intersect $K \cup (\partial C \setminus S_i)$. Clearly, $m_i + (0, \varepsilon_i] v_i$ does not intersect S_i, whence $\big(m_i + (0, \varepsilon_i] v_i\big) \cap (K \cup \partial C) = \varnothing$. Then let us denote by A_i the connected component of $\mathbb{R}^2 \setminus (\partial C \cup K)$ containing $m_i + (0, \varepsilon_i] v_i$. Observe that A_i is open (as $\partial C \cup K$ is compact) and that $S_i \subset \partial A_i$. Moreover, we claim that

$$(2.15) \qquad\qquad A_i \cap A_j = \varnothing \quad \text{for every } i, j \in I \text{ with } i \neq j.$$

We argue by contradiction: suppose that $A_i \cap A_j \neq \varnothing$, thus necessarily $A_i = A_j$. By Theorem 2.7 we know that there exists an injective continuous curve $\sigma \colon [0,1] \to \mathbb{R}^2$ such that $\sigma_0 = m_i$, $\sigma_1 = m_j$ and $\sigma(0,1) \subset A_i$. Possibly interchanging i and j, we can assume that $b_i < a_j$. Choose $t \in (b_i, a_j)$ such that $\gamma_t \in K$. Since we are supposing that the set of indexes I has cardinality at least 2, we know that $K \cap \mathring{C} \neq \varnothing$ (otherwise K would be contained in ∂C and accordingly disconnected). Pick any $y \in K \cap \mathring{C}$. Either γ_0 or γ_t does not belong to the closure of the connected component of $\mathbb{R}^2 \setminus (\partial C \cup \sigma[0,1])$ containing y. Since K is connected, this forces K to intersect $\sigma(0,1)$, which leads to a contradiction. Therefore the claim (2.15) is proven.

STEP 3. Given any $i \in I$, we call R_i the strip $S_i + \mathbb{R}^{>0} v_i$. We also define $B_i := R_i \cap \partial A_i \cap \partial C$ and $C_i := (R_i \cap \partial A_i) \setminus \partial C$. Clearly, $B_i \cap C_i = \varnothing$ by definition. Moreover, as we are going to show, it follows from (2.15) that $B_i, C_i \subset K$. We claim that $\partial A_i \cap S_j = \varnothing$ if $j \in I \setminus \{i\}$. We argue by contradiction: suppose there exists $y \in \partial A_i \cap S_j$. Pick a sequence $(y_n)_n \subset A_i$ such that $y_n \to y$. Since the segment $[m_j, y]$ and the compact set $(\partial C \setminus S_j) \cup K$ are disjoint, there exists $\varepsilon > 0$ such that $B\big([m_j, y], \varepsilon\big) \cap \big((\partial C \setminus S_j) \cup K\big) = \varnothing$. Hence, the connected set $E := B\big([m_j, y], \varepsilon\big) \cap R_j$ does not intersect $\partial C \cup K$. Since $m_j + (0, \varepsilon_j] v_j \subset A_j$, we have that $E \cap A_j \neq \varnothing$ and accordingly $E \subset A_j$. Moreover, for

$n \in \mathbb{N}$ sufficiently big it holds that $y_n \in E$. This implies that $y_n \in A_i \cap A_j$, which contradicts (2.15). Therefore, we have proved that $\partial A_i \cap S_j = \varnothing$ whenever $j \in I$, $j \neq i$. This grants that $(R_i \cap \partial A_i) \cap \bigcup_{j \in I} S_j = \varnothing$. Since $\partial A_i \subset \partial C \cup K$, we conclude that $R_i \cap \partial A_i \subset K$ and thus $B_i, C_i \subset K$, as required.

Furthermore, we claim that

(2.16a) $$\#(B_i \cap B_j) \leq 2 \quad \text{for every } i,j \in I \text{ with } i \neq j,$$

(2.16b) $$\#(C_i \cap C_j \cap C_k) \leq 1 \quad \text{for every } i,j,k \in I \text{ with } i \neq j \neq k \neq i.$$

We argue by contradiction. In order to prove (2.16a), suppose to have three distinct points y_1, y_2, y_3 in $B_i \cap B_j$. The set $\partial C \setminus \{y_1, y_2, y_3\}$ is made of three arcs. By Theorem 2.7 we can find injective continuous curves $\sigma, \sigma' \colon [0,1] \to \mathbb{R}^2$ such that $\sigma_0 = m_i$, $\sigma_1 = y_2$, $\sigma(0,1) \subset A_i$, $\sigma_0' = m_j$, $\sigma_1' = y_1$ and $\sigma'(0,1) \subset A_j$. We now distinguish two cases:

i) S_i and S_j lie in the same arc of $\partial C \setminus \{y_1, y_2, y_3\}$. Then (up to relabeling y_1, y_2) we can assume that y_1, m_i, m_j, y_2 are distributed along ∂C in a clockwise order. Then Remark 2.9 grants that $\sigma[0,1] \cap \sigma'[0,1] \neq \varnothing$, thus contradicting (2.15).

ii) S_i and S_j lie in different arcs of $\partial C \setminus \{y_1, y_2, y_3\}$. Possibly relabeling y_1, y_2, y_3, we can assume that y_1, m_i, y_3, m_j, y_2 are distributed along ∂C in a clockwise order. Then Remark 2.9 grants that $\sigma[0,1] \cap \sigma'[0,1] \neq \varnothing$, again contradicting (2.15). Therefore (2.16a) is proven. In order to prove (2.16b), suppose that $C_i \cap C_j \cap C_k$ contains at least two distinct points z_1, z_2. Observe that $\partial A_i, \partial A_j, \partial A_k \subset \partial C \cup K$, as A_i, A_j, A_k are connected components of $\mathbb{R}^2 \setminus (\partial C \cup K)$. Being C a compact convex subset of $\mathbb{R}^2$, it holds that $\mathcal{H}^1(\partial C) < +\infty$ (since $\mathcal{H}^1(\partial C) \leq \pi \operatorname{diam}(C)$, as proven, e.g., in [2] or in [9, p. 257]). Given that $\mathcal{H}^1(K) < +\infty$ by assumption, we have that $\mathcal{H}^1(\partial A_i), \mathcal{H}^1(\partial A_j), \mathcal{H}^1(\partial A_k) < +\infty$ as well. Hence, by using Theorem 2.7 we can build an injective continuous curve $\sigma \colon [0,2] \to \mathbb{R}^2$ with $\sigma_0 = m_i$, $\sigma_1 = z_1$, $\sigma_2 = m_j$, $\sigma(0,1) \subset A_i$ and $\sigma(1,2) \subset A_j$. Possibly interchanging z_1 and z_2, we can assume that z_2 and m_k do not belong to the same connected component of $C \setminus \sigma(0,2)$. Hence (again by Theorem 2.7) we can pick an injective continuous curve $\sigma' \colon [0,1] \to \mathbb{R}^2$ such that $\sigma_0' = m_k$, $\sigma_1' = z_2$ and $\sigma'(0,1) \subset A_k$. This implies that $\sigma(0,2) \cap \sigma'(0,1) \neq \varnothing$, whence either $A_i \cap A_k \neq \varnothing$ or $A_j \cap A_k \neq \varnothing$. In both cases property (2.15) is violated, thus even the claim (2.16b) is proven.

STEP 4. Let $i \in I$ be fixed. For any $x \in S_i$ there is $t_x > 0$ such that $p_x := x + t_x v_i \in B_i \cup C_i$ and $\big(x + (0, t_x) v_i\big) \cap \partial A_i = \varnothing$. Call $\pi_i \colon \mathbb{R}^2 \to \mathbb{R}^2$ the orthogonal projection onto the line containing S_i, which is a 1-Lipschitz map. Then one has that

$$\mathcal{H}^1(S_i) = \mathcal{H}^1\big(\pi_i\{p_x : x \in S_i\}\big) \leq \mathcal{H}^1\big(\{p_x : x \in S_i\}\big) \leq \mathcal{H}^1(B_i) + \mathcal{H}^1(C_i).$$

Therefore, it holds that

(2.17)
$$\mathcal{H}^1(\partial C) = \mathcal{H}^1(\partial C \cap K) + \sum_{i \in I} \mathcal{H}^1(S_i) \le \mathcal{H}^1(\partial C \cap K) + \sum_{i \in I} \mathcal{H}^1(B_i) + \sum_{i \in I} \mathcal{H}^1(C_i).$$

We know from (2.16a) that $\mathcal{H}^1(B_i \cap B_j) = 0$ for all $i, j \in I$ with $i \ne j$. Since $B_i \subset \partial C \cap K$ holds for all $i \in I$, we infer that

(2.18)
$$\sum_{i \in I} \mathcal{H}^1(B_i) = \mathcal{H}^1\left(\bigcup_{i \in I} B_i\right) \le \mathcal{H}^1(\partial C \cap K).$$

Finally, (2.16b) grants that the family $\{C_i\}_{i \in I}$ satisfies the hypothesis of Lemma 2.2. Given that $C_i \subset \mathring{C} \cap K$ for all $i \in I$, we infer that

(2.19)
$$\sum_{i \in I} \mathcal{H}^1(C_i) \le 2\,\mathcal{H}^1\left(\bigcup_{i \in I} C_i\right) \le 2\,\mathcal{H}^1(\mathring{C} \cap K).$$

By plugging the estimates (2.18) and (2.19) into (2.17), we conclude that (2.14) is satisfied. This completes the proof of the statement. $\square$

Proposition 2.11. *Let $\Gamma \subset \mathbb{R}^2$ be a Borel set satisfying $\mathcal{H}^1(\Gamma) < +\infty$. Let $v \in \mathbb{R}^2 \setminus \{0\}$ and $x \in \mathbb{R}^2$ be given. Fix a point $y \in \mathbb{R}^2$ that does not belong to the line $x + \mathbb{R}v$. Then*
$$\mathcal{H}^1\big([x+tv, y] \cap \Gamma\big) = 0 \quad \text{for a.e. } t \in \mathbb{R}.$$

Proof. Notice that the elements of $\big\{[x+tv, y] \cap \Gamma\big\}_{t \in \mathbb{R}}$ are pairwise disjoint subsets of Γ. Given that $\mathcal{H}^1|_\Gamma$ is a finite Borel measure on $\mathbb{R}^2$, we conclude that $\mathcal{H}^1\big([x+tv, y] \cap \Gamma\big) = 0$ for all but countably many $t \in \mathbb{R}$, whence the statement follows. $\square$

Remark 2.12. Let Ω be an open, connected subset of $\mathbb{R}^2$. Let K be a compact subset of Ω. Then there exists an open, connected set $U \subset \mathbb{R}^2$ such that $K \subset U$ and $\overline{U} \subset \Omega$.

Indeed, the compactness of K ensures that we can find a finite family $B_1, ..., B_n$ of open balls such that $K \subset \bigcup_{i=1}^n B_i$ and $\bigcup_{i=1}^n \overline{B}_i \subset \Omega$. Given any $1 \le i < j \le n$, we can take a continuous curve γ^{ij} in Ω connecting a point of B_i to a point of B_j. Choose $\delta > 0$ so small that $\overline{B}(\gamma^{ij}, \delta) \subset \Omega$ for all $i < j$. Therefore, the set $U := \bigcup_{i=1}^n B_i \cup \bigcup_{i<j} B(\gamma^{ij}, \delta)$ does the job, as it is a domain containing K that is compactly contained in Ω.

Proposition 2.13. *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain with* $\mathcal{H}^1(\partial\Omega) < +\infty$. *Let* $K \subset \mathbb{R}^2$ *be a compact set such that* $K \subset \Omega$. *Let* $U \subset \mathbb{R}^2$ *be an open set satisfying* $\overline{\Omega} \subset U$. *Then there exists a bounded domain* $\Omega' \subset \mathbb{R}^2$ *with the following properties*:

    i) $K \subset \Omega' \subset \overline{\Omega'} \subset U$,

    ii) $\mathcal{H}^1(\partial\Omega') \leq \mathcal{H}^1(\partial\Omega)$,

    iii) $\partial\Omega'$ *is the union of finitely many pairwise disjoint analytic Jordan loops.*

Proof. We denote by $\{H_i\}_{i \in I}$ the bounded connected components of $\mathbb{R}^2 \setminus \overline{\Omega}$, while W stands for the unbounded one. We set $\sigma^i := \partial H_i$ for all $i \in I$ and $\sigma := \partial W$, which are Jordan loops by Theorem 2.5. Since $\{H_i\}_{i \in I}$ is an open covering of the compact set $\mathbb{R}^2 \setminus (U \cup W)$, we can select a finite subfamily F of I such that $\mathbb{R}^2 \setminus (U \cup W) \subset \bigcup_{i \in F} H_i$. Equivalently, we have that $(\mathbb{R}^2 \setminus W) \setminus \bigcup_{i \in F} H_i \subset U$.

One can now construct the Jordan loops for $\partial\Omega'$ by hand, but their existence is also easily seen as follows using conformal maps, for similar arguments see for instance [8, proof of Theorem 6.8]. For each H_i consider a conformal map φ_i mapping the unit disc $\mathbb{D}$ to H_i. Since H_i is Jordan, the map φ_i has a homeomorphic extension to the closed unit disc $\overline{\mathbb{D}}$. Now, by the maximum principle, for every $0 < r < 1$,

$$\mathcal{H}^1(\varphi_i(\{x : |x| = r\})) = r \int_0^{2\pi} |\varphi_i'(re^{it})| \, \mathrm{d}t \leq \mathcal{H}^1(\sigma^i).$$

By continuity of φ_i, for large enough $r_i \in (0,1)$ we have

$$H_i \setminus U \subset \varphi_i(\{x : |x| < r_i\}).$$

For the unbounded component W, we estimate using a conformal map φ from the unit disc to the complementary domain $\mathbb{R}^2 \setminus \overline{W}$. Similarly as for σ^i, we have $r \in (0,1)$ such that

$$K \cup \bigcup_{i \in F} \sigma^i \subset \varphi(\{x : |x| < r\}).$$

Recall that $\mathcal{H}^1(\varphi(\{x : |x| = r\})) \leq \mathcal{H}^1(\sigma)$. Now, we may simply define

$$\Omega' := \varphi(\{x : |x| < r\}) \setminus \bigcup_{i \in F} \varphi_i(\{x : |x| \leq r_i\})$$

to get the claimed bounded domain. Indeed, items i) and iii) are granted by the very construction of Ω', while ii) follows from $\mathcal{H}^1(\partial\Omega') \leq \mathcal{H}^1(\sigma) + \sum_{i \in F} \mathcal{H}^1(\sigma^i) \leq \mathcal{H}^1(\partial\Omega)$. $\quad\square$

3. Painlevé length estimates

Let us first recall the definition of Painlevé length.

Definition 3.1. The Painlevé length of a compact set $K \subset \mathbb{R}^2$, denoted $\varkappa(K)$, is the infimum of numbers ℓ with the following property: for every open set U containing K there exists an open set V such that $K \subset V \subset U$ and ∂V is a finite union of disjoint analytic Jordan loops of total length at most ℓ.

In [4, p. 25] the following estimate was stated. For a proof, we refer to [6, Proposition 4.3].

Proposition 3.2. *For every compact set $K \subset \mathbb{R}^2$ the following inequality holds*:

$$\varkappa(K) \leq \pi \, \mathcal{H}^1(K). \tag{3.1}$$

In [4] it was also noted that the estimate (3.1) is the best possible one for general compact sets, when the Hausdorff measure $\mathcal{H}^1$ is replaced by the (possibly smaller) Hausdorff content $\mathcal{H}^1_\infty$. In what follows we prove that the Painlevé length estimate (3.1) can be improved for connected sets.

Theorem 3.3. (Painlevé estimate for connected sets) *For every compact, connected set $K \subset \mathbb{R}^2$ the following inequality holds*:

$$\varkappa(K) \leq 2 \, \mathcal{H}^1(K). \tag{3.2}$$

Proof. The case $\mathcal{H}^1(K) = +\infty$ is trivial, thus let us suppose that $\mathcal{H}^1(K) < +\infty$. Fix an open neighbourhood $U \subset \mathbb{R}^2$ of K and any $\varepsilon > 0$. We aim to prove that there exists an open set $V \subset U$ containing K, whose boundary is a disjoint union of finitely many analytic Jordan loops, such that

$$\mathcal{H}^1(\partial V) < 2 \, \mathcal{H}^1(K) + \varepsilon. \tag{3.3}$$

Fix any positive radius $r < \text{dist}(K, \mathbb{R}^2 \setminus U)$. For any $x \in K$ we can choose $r_x > 0$ such that $r < r_x < \text{dist}(K, \mathbb{R}^2 \setminus U)$ and $\#\big(K \cap \partial B(x, r_x)\big) < +\infty$. We claim that:

(3.4) There exist $x_0, ..., x_n \in K$ and compact connected sets $x_i \in K_i \subset \overline{B}(x_i, r_{x_i})$ such that $K = K_0 \cup ... \cup K_n$ and $\mathcal{H}^1(K_i \cap K_j) = 0$ whenever $0 \leq i < j \leq n$.

In order to show the validity of the claim (3.4), we need the following property that can be readily obtained as a consequence of Lemmata 2.1 and 2.8:

FACT. If E is a compact connected subset of K and $x \in E$ satisfies $E \setminus \overline{B}(x, r_x) \neq \varnothing$, then the connected component F of $E \cap \overline{B}(x, r_x)$ containing x has the following properties:

i) $\mathcal{H}^1(F) \geq r_x > r$.

ii) $G := E \setminus \big(F \cap B(x, r_x)\big)$ is compact and has finitely many connected components (which accordingly are compact as well). More precisely, the number of connected components of G cannot exceed $\#\big(K \cap \partial B(x, r_x)\big)$.

iii) $\#(F \cap G) < +\infty$.

We now recursively apply the above FACT. First of all, fix any $x_0 \in K$. If $K \subset \overline{B}(x_0, r_{x_0})$ then we define $K_0 := K$. Otherwise, we call K_0 the connected component of $K \cap \overline{B}(x_0, r_{x_0})$ containing x_0. In the latter case, let us denote by $E_1, ..., E_k$ the connected components of the set $K \setminus \big(K_0 \cap B(x_0, r_{x_0})\big)$. Given any $i = 1, ..., k$, we pick a point $x_i \in E_i$ and proceed as before: if $E_i \subset \overline{B}(x_i, r_{x_i})$ then we set $K_i := E_i$; otherwise, we call K_i the connected component of $E_i \cap \overline{B}(x_i, r_{x_i})$ containing x_i and $E_{i,1}, ..., E_{i,k_i}$ the connected components of the set $E_i \setminus \big(K_i \cap B(x_i, r_{x_i})\big)$. We can repeat the same argument on each $E_{i,j}$, and so on. This iterated procedure must stop after finitely many passages thanks to item i) of FACT (recall that $\mathcal{H}^1(K) < +\infty$ and that the intersection $K_i \cap K_j$ has null $\mathcal{H}^1$-measure if $i \neq j$). Therefore the previous argument provides us with a family $K_0, ..., K_n$ as in claim (3.4).

Let us now denote by $\widetilde{C}_i$ the convex hull of K_i for every $i = 0, ..., n$. We know from Proposition 2.10 that $\mathcal{H}^1(\partial \widetilde{C}_i) \leq 2 \mathcal{H}^1(K_i)$. Moreover, recall that the open δ-neighbourhood $\widetilde{C}_i^\delta$ of $\widetilde{C}_i$ is convex for all $\delta > 0$ and satisfies $\mathcal{H}^1(\partial \widetilde{C}_i^\delta) \to \mathcal{H}^1(\partial \widetilde{C}_i)$ as $\delta \to 0$. Furthermore, given any $i = 0, ..., n$ and any Borel set $F \subset \mathbb{R}^2$ with $\mathcal{H}^1(F) < +\infty$, it holds that

$$\#(\partial \widetilde{C}_i^\delta \cap F) < +\infty \quad \text{for a.e. } \delta > 0.$$

By (3.4) we have that each K_i is contained in the convex set $\overline{B}(x_i, r_{x_i})$, whence accordingly the inclusions $\widetilde{C}_i \subset \overline{B}(x_i, r_{x_i}) \subset U$ hold for every $i = 0, ..., n$. Hence we can recursively choose $\delta_0, ..., \delta_n > 0$ so that (calling $C_i := \widetilde{C}_i^{\delta_i}$) the following properties are verified:

a) $C_i \subset U$ for every $i = 0, ..., n$,

b) $\mathcal{H}^1(\partial C_i) \leq 2 \mathcal{H}^1(K_i) + \varepsilon/(n+1)$ for every $i = 0, ..., n$.

Let us now define $V := C_0 \cup ... \cup C_n$. Then V is an open set such that $K \subset V \subset U$ and $\partial V \subset \partial C_0 \cup ... \cup \partial C_n$, thus it holds that

$$\mathcal{H}^1(\partial V) \leq \sum_{i=0}^n \mathcal{H}^1(\partial C_i) \overset{\text{b)}}{\leq} 2 \sum_{i=0}^n \mathcal{H}^1(K_i) + \varepsilon = 2 \mathcal{H}^1(K) + \varepsilon,$$

proving (3.3). The fact that the boundary of V can be supposed to be made of finitely many disjoint analytic Jordan loops is due to Proposition 2.13. This completes the proof. $\square$

The estimate (3.2) is easily seen to be sharp simply by taking K to be a line-segment. What is less trivial, is that also the estimate (3.1) is sharp for general compact sets. This is shown in the next example.

Example 3.4. We define a compact fractal set $K \subset \mathbb{R}^2$ with $\varkappa(K) = \pi \mathcal{H}^1(K)$ using an iteration procedure. We start with $K_1 = \overline{B}((0,0), 1)$ and continue by contracting and copying K_1 as follows. For each integers k and j with $k \geq 2$ and $1 \leq j \leq 2^k$ we define, using complex notation, a contractive similitude

$$f_{k,j}(x) := 2^{-k}x + (1 - 2^{-k})e^{j2^{1-k}\pi i}.$$

Using these functions we set

$$K_k := \bigcup_{(j_2, \ldots, j_k)} f_{2,j_2} \circ f_{3,j_3} \circ \ldots \circ f_{k,j_k}(K_1),$$

where the union runs over all $(k-1)$-tuples of indices with $1 \leq j_i \leq 2^k$, $i = 2, \ldots, k$. We call the balls in this union the construction balls of level k. Notice that $K_{k+1} \subset K_k$ for every $k \in \mathbb{N}$. Finally, we set

$$K := \bigcap_{k=1}^{\infty} K_k.$$

See Figure 3 for an illustration of the construction. For notational convenience we introduce the measure μ on K defined as the weak limit of measures $2(\mathcal{H}^2(K_k))^{-1}\mathcal{H}^2|_{K_k}$ as $k \to \infty$. That is, μ is the measure on K such that every construction ball has μ-measure equal to its diameter.

We claim that $\mathcal{H}^1(K) = 2$ and $\varkappa(K) = 2\pi$. Taking into account (3.1), it suffices to show that $\mathcal{H}^1(K) \leq 2$ and $\varkappa(K) \geq 2\pi = \pi\mu(K)$.

The inequality $\mathcal{H}^1(K) \leq 2$ follows directly by using the construction balls of level k as the cover for K in the definition of the Hausdorff measure and by letting $k \to \infty$. Thus, it only remains to show that $\varkappa(K) \geq \pi\mu(K)$.
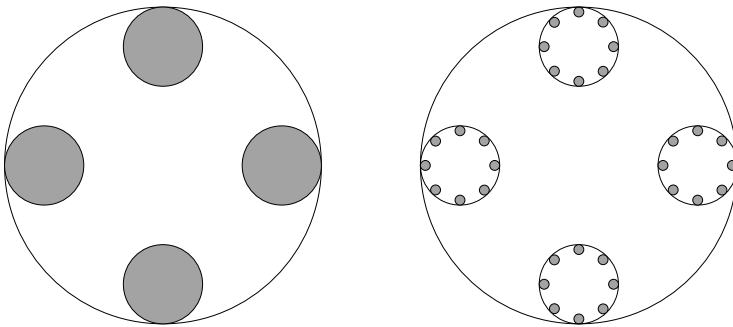

Figure 3. Here are approximations of the set K defined in Example 3.4. On the left is K_2 and on the right one step further refinement, the set K_3.

Let $\varepsilon > 0$. We show that

$$\varkappa(K) \geq (1-\varepsilon)\pi\mu(K).$$

By letting $U \subset \mathbb{R}^2$ be an open set containing K such that each connected component of U contains only one construction ball of level k, we may restrict ourselves to estimating $\varkappa(K \cap B)$ for a construction ball B of level k with k arbitrarily large. Let $V \subset U$ be open such that $K \cap B \subset V$. It suffices to show that for each connected component V' of V we have $\mathcal{H}^1(\partial V') \geq (1-\varepsilon)\pi\mu(V' \cap K)$. Since $\mathcal{H}^1(\partial V') \geq \mathcal{H}^1(\partial W)$ for $W = \mathrm{conv}(V' \cap K)$, it is enough to show that $\mathcal{H}^1(\partial W) \geq (1-\varepsilon)\pi\mu(V' \cap K)$.

Let k_0 be the smallest integer so that W intersects at least 2 of the level k_0 construction balls. By our assumption on U we have that $k_0 > k$. The set W is then contained in a level $k_0 - 1$ ball $B(x, r)$. We separate the rest of the proof into two cases:

i) W intersects exactly 2 level k_0 construction balls.

ii) W intersects at least 3 level k_0 construction balls.

Let us first consider the case i). Since the distance between two level k_0 construction balls is at least

$$(1 - 2^{1-k_0})\sin(2^{-k_0}\pi)r \geq (1 - 2^{2-k_0})2^{-k_0}\pi r,$$

we may assume that $\mu(V' \cap K) \geq 2^{-k_0}r$. Then, one of the construction balls contains a point of ∂W that has distance at least $\frac{3}{2}(1-2^{2-k_0})2^{-k_0}\pi r$ to the other construction ball. Thus, we may assume that $\mu(V' \cap K) \geq \frac{3}{2}2^{-k_0}r$. But then, there exist two points in ∂W with distance at least $2(1-2^{2-k_0})2^{-k_0}\pi r$, which then yields

$$\mathcal{H}^1(\partial W) \geq (1-2^{2-k_0})2\pi r 2^{2-k_0} \geq (1-2^{2-k_0})\pi\mu(V' \cap K).$$

Let us then consider the case ii). For each construction level k_0 ball B_i intersecting W there exists a point $x_i \in \partial W \cap B_i$, since none of the balls B_i is in the convex hull of the other balls. Let us then estimate $\mathcal{H}^1(\partial W)$ using the angle around the center x. If x_i and x_j are contained in adjacent construction balls, the boundary of W from x_i to x_j has length at least

$$(1 - 2^{1-k_0})\sin(\theta_{i,j})r \geq (1-2^{2-k_0})\theta_{i,j}\, r,$$

where $\theta_{i,j} := \measuredangle(x_i, x, x_j)$. See Figure 4 for an illustration for the estimate. If x_i and x_j are not contained in adjacent construction balls, the length of the boundary of W from x_i to x_j is at least $2^{1-k_0}\pi$. All in all, denoting by N the total number of the construction balls B_i intersecting W, we have

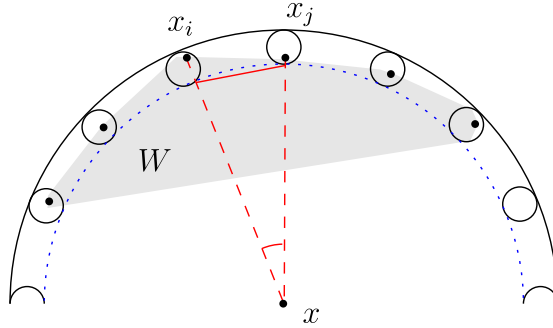$$\mathcal{H}^1(\partial W) \geq (1-2^{2-k_0})2\pi r N 2^{-k_0} \geq (1-2^{2-k_0})\pi\mu(V' \cap K).$$

Figure 4. In the case ii), the length of ∂W is estimated by summing up lengths of projections (red line) of parts connecting points in consecutive balls (here between x_i and x_j).

4. Proof of the main result

This section is entirely devoted to the proof of Theorem 1.1:

STEP 1. Let us denote by $\{E_i\}_{i=1}^N$ the connected components of $\partial\Omega$ with positive length, where $N\in\mathbb{N}\cup\{\infty\}$. We can clearly suppose without loss of generality that $N=\infty$. In the case in which Ω is bounded, we also assume that E_1 is the element containing the boundary of the unbounded connected component of $\mathbb{R}^2\setminus\overline{\Omega}$, which is connected as it is a Jordan loop by Theorem 2.5. Set $C:=\partial\Omega\setminus\bigcup_{i=1}^\infty E_i$. Notice that C can be a Cantor-type set, thus in particular it can have positive $\mathcal{H}^1$-measure. Lemma 2.1 grants that

$$\sum_{i=1}^\infty \operatorname{diam}(E_i) \le \sum_{i=1}^\infty \mathcal{H}^1(E_i) \le \mathcal{H}^1(\partial\Omega) < +\infty.$$

Consequently, we can relabel the sets $\{E_i\}_{i\ge2}$ so that $\operatorname{diam}(E_i)\ge\operatorname{diam}(E_j)$ if $2\le i\le j$.

Let us fix $\varepsilon>0$. For each $i\in\mathbb{N}$, we select a point $z_i\in E_i$. Observe that

$$\sum_{i=1}^\infty \mathcal{H}^1_\infty\big(B(z_i, 4\operatorname{diam}(E_i))\big) \le 8 \sum_{i=1}^\infty \operatorname{diam}(E_i) \le 8 \sum_{i=1}^\infty \mathcal{H}^1(E_i) \le 8\,\mathcal{H}^1(\partial\Omega) < +\infty.$$

By using the Borel-Cantelli lemma we deduce that $\mathcal{H}^1_\infty\big(\bigcap_{k=1}^\infty\bigcup_{i=k}^\infty B(z_i, 4\operatorname{diam}(E_i))\big)=0$. Since $\mathcal{H}^1\ll\mathcal{H}^1_\infty$ and the measure $\mathcal{H}^1|_{\partial\Omega}$ is continuous from above, we see that

$$\lim_{k\to\infty} \mathcal{H}^1\Big(\partial\Omega\cap\bigcup_{i=k}^\infty B\big(z_i, 4\operatorname{diam}(E_i)\big)\Big) = \mathcal{H}^1\Big(\partial\Omega\cap\bigcap_{k=1}^\infty \bigcup_{i=k}^\infty B\big(z_i, 4\operatorname{diam}(E_i)\big)\Big)$$
$$= 0.$$

Therefore, there exists $k \in \mathbb{N}$ such that

(4.1)
$$\mathcal{H}^1\Big(\partial\Omega \cap \bigcup\nolimits_{i=k}^{\infty} B\big(z_i, 4\,\mathrm{diam}(E_i)\big)\Big) < \frac{2\,\varepsilon}{5\,\pi},$$
$$\sum_{i=k}^{\infty} \mathrm{diam}(E_i) < \frac{\varepsilon}{10\,\pi}.$$

STEP 2. Choose any continuous curve $\gamma^1 \subset \Omega$ joining x to y. Theorem 3.3 grants that for any $i = 1, ..., k-1$ we can choose an open neighbourhood V_i of E_i in such a way that

$$\overline{V}_i \cap \overline{V}_j = \varnothing \quad \text{for every } 1 \leq i < j \leq k-1,$$
$$\overline{V}_i \cap \gamma^1 = \varnothing \quad \text{for every } i = 1, ..., k-1,$$
$$\mathcal{H}^1(\partial V_i) \leq 2\,\mathcal{H}^1(E_i) + \frac{2\,\varepsilon}{5\,(k-1)} \quad \text{for every } i = 1, ..., k-1.$$

We can also assume that the boundary of each set V_i consists of finitely many pairwise disjoint Jordan loops. Notice that x, y lie in the same connected component of $\mathbb{R}^2 \setminus \bigcup_{i=1}^{k-1} V_i$, thanks to the fact that the curve γ^1 does not intersect $\bigcup_{i=1}^{k-1} V_i$. We distinguish two cases:

• Ω is bounded. Let us call Ω' the (bounded) connected component of $\mathbb{R}^2 \setminus \overline{V}_1$ that contains γ^1 (thus also x, y). The boundary of Ω' is a Jordan loop $\sigma \colon [0,1] \to \mathbb{R}^2$ with $\ell(\sigma) \leq \mathcal{H}^1(\partial V_1) \leq 2\,\mathcal{H}^1(E_1) + \frac{2\,\varepsilon}{5(k-1)}$ (cf. Theorem 2.5). Possibly reparametrizing σ, we can suppose to have $0 < t_1 < t_2 < t_3 < t_4 < t_5 < 1$ such that

$$x, y \in (\sigma_0, \sigma_{t_3}) \subset \Omega', \quad x \in (\sigma_{t_1}, \sigma_{t_5}) \subset \Omega', \quad y \in (\sigma_{t_2}, \sigma_{t_4}) \subset \Omega'$$

and the segments $[\sigma_{t_1}, \sigma_{t_5}], [\sigma_{t_2}, \sigma_{t_4}]$ are perpendicular to $[x,y]$. It readily follows from Lemma 2.1 that $|x - \sigma_{t_1}| \leq \ell(\sigma|_{[0,t_1]})$, $|y - \sigma_{t_2}| \leq \ell(\sigma|_{[t_2,t_3]})$, $|y - \sigma_{t_4}| \leq \ell(\sigma|_{[t_3,t_4]})$ and $|x - \sigma_{t_5}| \leq \ell(\sigma|_{[t_5,0]})$. Calling $*$ the concatenation of curves, we thus see that

$$\ell\big([x, \sigma_{t_1}] * \sigma|_{[t_1,t_2]} * [\sigma_{t_2}, y]\big) + \ell\big([x, \sigma_{t_5}] * \sigma|_{[t_5,t_4]} * [\sigma_{t_4}, y]\big) \leq \ell(\sigma).$$

Then we define $s \colon [0,1] \to \mathbb{R}^2$ as the shortest curve between $[x, \sigma_{t_1}] * \sigma|_{[t_1,t_2]} * [\sigma_{t_2}, y]$ and $[x, \sigma_{t_5}] * \sigma|_{[t_5,t_4]} * [\sigma_{t_4}, y]$, so that $\ell(s) \leq \mathcal{H}^1(E_1) + \frac{\varepsilon}{5(k-1)}$. See Figure 5 for the curve s.

• Ω is unbounded. Then we define $s \colon [0,1] \to \mathbb{R}^2$ as $s_t := x + t(y-x)$ for all $t \in [0,1]$.

For the sake of simplicity, let us define the quantity $q > 0$ as

(4.2)
$$q := \begin{cases} 0 & \text{if } \Omega \text{ is bounded}, \\ |x-y| & \text{if } \Omega \text{ is unbounded}. \end{cases}$$
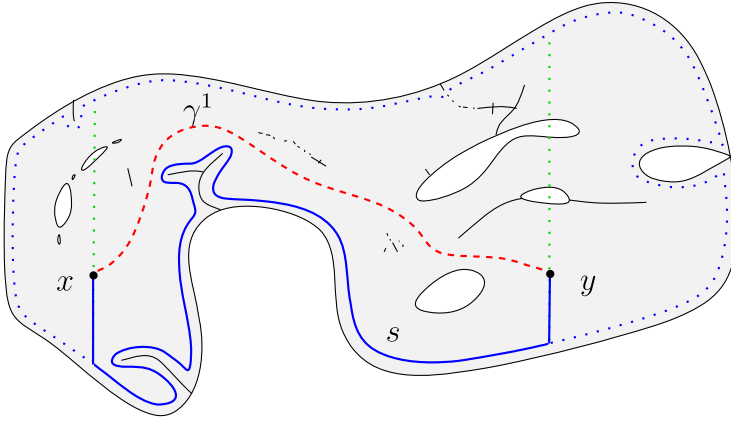
Figure 5. The first approximation s of the curve is obtained in the bounded case by going near the outer boundary. In the unbounded case, the initial curve is just the line-segment connecting the points.

We proceed in a recursive way: choose that $i_1 \in \{1, ..., k-1\}$ such that V_{i_1} is the first element of $\{V_i\}_{i=1}^{k-1}$ that is encountered by the curve s (note that $i_1 \geq 2$ if Ω is bounded). Put $a_1 := \min \{t \in (0,1) \,|\, s_t \in \partial V_{i_1}\}$. The connected component of ∂V_{i_1} containing s_{a_1} is the image of a Jordan loop σ^1. Now let us call $b_1 := \max \{t \in (a_1, 1) \,|\, s_t \in \sigma^1\}$. Observe that $s|_{(b_1,1)} \cap \partial V_{i_1} = \varnothing$. We can write the image of σ^1 as the union of two injective curves $\alpha^1, \widetilde{\alpha}^1$ joining s_{a_1} to s_{b_1}. Given that the length of σ^1 does not exceed $\mathcal{H}^1(\partial V_{i_1})$, which in turn is smaller than $2\,\mathcal{H}^1(E_{i_1}) + \frac{2\,\varepsilon}{5(k-1)}$, we can assume without loss of generality that the length of α^1 is smaller than $\mathcal{H}^1(E_{i_1}) + \frac{\varepsilon}{5(k-1)}$.

We can now argue in the same way starting from s_{b_1}. Take $i_2 \in \{1, ..., k-1\} \setminus \{i_1\}$ such that the first of the sets V_i that we meet while going from s_{b_1} to y is V_{i_2} (again, $i_2 \neq 1$ if Ω is bounded). We denote by a_2 the smallest $t \in (b_1, 1)$ for which $s_t \in \partial V_{i_2}$; the connected component of ∂V_{i_2} containing s_{a_2} is the image of a Jordan loop σ^2, and b_2 stands for the biggest $t \in (a_2, 1)$ such that $s_t \in \sigma^2$. Then we can find a curve α^2 in σ^2 joining s_{a_2} to s_{b_2}, which is shorter than $\mathcal{H}^1(E_{i_2}) + \frac{\varepsilon}{5(k-1)}$. By repeating this procedure finitely many times (see Figure 6), we obtain a curve γ^2 of the form

$$\gamma^2 := s|_{[0,a_1]} * \alpha^1 * s|_{[b_1,a_2]} * \alpha^2 * ... * s|_{[b_{\ell-1},a_\ell]} * \alpha^\ell * s|_{[b_\ell,1]}$$

for some $\ell \leq k-1$. Notice that γ^2 is contained in $\mathbb{R}^2 \setminus \bigcup_{i=1}^{k-1} E_i$ and connects x to y. By combining the previous estimates, we also deduce that

$$(4.3) \qquad\qquad \ell(\gamma^2) < q + \sum_{i=1}^{k-1} \mathcal{H}^1(E_i) + \frac{\varepsilon}{5}.$$
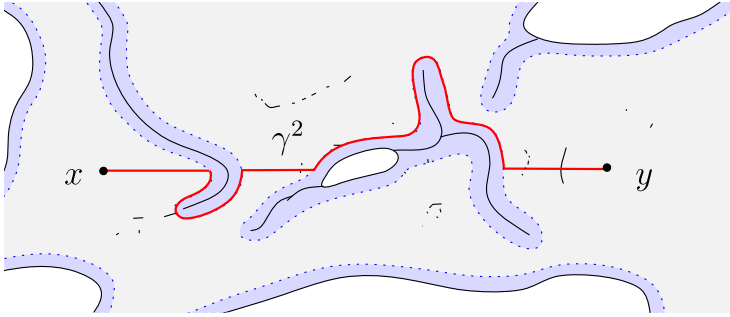
Figure 6. In the second approximation, the curve is constructed so that it avoids a finite number of the largest boundary components. Next, it is slightly perturbed so that it does not intersect the (remaining) boundary points in a positive $\mathcal{H}^1$-measure set.

STEP 3. In light of (4.3), we can choose some points $p_1, ..., p_{2h-1} \in \gamma^2 \setminus \{x, y\}$ having the following property: the curve $\gamma^3 := [x, p_1] * [p_1, p_2] * ... * [p_{2h-2}, p_{2h-1}] * [p_{2h-1}, y]$ is contained in $\mathbb{R}^2 \setminus \bigcup_{i=1}^{k-1} E_i$ and satisfies

$$\ell(\gamma^3) < q + \sum_{i=1}^{k-1} \mathcal{H}^1(E_i) + \frac{\varepsilon}{5}.$$

Now let us apply Proposition 2.11: we can find some points $q_1, q_3, ..., q_{2h-1}$ (sufficiently near to $p_1, p_3, ..., p_{2h-1}$, respectively) for which the following conditions are verified:

- The curve $\gamma^4 := [x, q_1] * [q_1, p_2] * ... * [p_{2h-2}, q_{2h-1}] * [q_{2h-1}, y]$ satisfies

$$(4.4) \qquad \ell(\gamma^4) < q + \sum_{i=1}^{k-1} \mathcal{H}^1(E_i) + \frac{\varepsilon}{5},$$

- γ^4 is contained in $\mathbb{R}^2 \setminus \bigcup_{i=1}^{k-1} E_i$,
- the set $\gamma^4 \cap \partial\Omega$ has null $\mathcal{H}^1$-measure.

By upper continuity of $\mathcal{H}^1|_{\partial\Omega}$, we can find $\delta > 0$ such that $B(\gamma^4, 2\delta) \subset \mathbb{R}^2 \setminus \bigcup_{i=1}^{k-1} E_i$ and

$$(4.5) \qquad \mathcal{H}^1\big(\partial\Omega \cap B(\gamma^4, 2\delta)\big) < \frac{2\varepsilon}{5\pi}.$$

Theorem 3.3 provides an open neighbourhood $U' \subset B(\gamma^4, \delta) \setminus \bigcup_{i=1}^{k-1} E_i$ of γ^4 such that

$$(4.6) \qquad \mathcal{H}^1(\partial U') \leq 2\,\ell(\gamma^4) + \frac{2\varepsilon}{5} \leq 2\,q + 2\sum_{i=1}^{k-1} \mathcal{H}^1(E_i) + \frac{4\varepsilon}{5},$$

where the second inequality stems from (4.4). Moreover, let us fix any index $m \geq k$ for which $\operatorname{diam}(E_m) < \operatorname{dist}(\gamma^4, \partial U')$. Since $i \mapsto \operatorname{diam}(E_i)$ is non-increasing for $i \geq k$, one has

$$\text{(4.7)} \qquad \operatorname{diam}(E_i) < \operatorname{dist}(\gamma^4, \partial U') \quad \text{for every } i \geq m.$$

Let us define

$$U := U' \cup \bigcup_{i=k}^{m} B\big(z_i, 2\operatorname{diam}(E_i)\big).$$

See Figure 7 for an illustration of the set U. Since $\overline{U} \subset B(\gamma^4, 2\delta) \cup \bigcup_{i=k}^{m} B\big(z_i, 4\operatorname{diam}(E_i)\big)$, we deduce from the first line of (4.1) and from (4.5) that

$$\text{(4.8)} \qquad \mathcal{H}^1(\partial\Omega \cap \overline{U}) < \frac{4\,\varepsilon}{5\,\pi}.$$

STEP 4. We claim that

$$\text{(4.9)} \qquad x, y \text{ belong to the same connected component of } U \setminus \partial\Omega.$$

We argue by contradiction: suppose that y does not belong to the connected component A of $U \setminus \partial\Omega$ containing x. Call B the connected component of $\mathbb{R}^2 \setminus A$ that contains y (notice that y lies in the interior of B). Call F the connected component of ∂A that is included in B. Hence, Lemma 2.3 yields $F = \partial B$. Given that γ^4 joins $x \notin B$ to $y \in B$, we deduce that $\gamma^4 \cap \partial B \neq \varnothing$. Choose any $z \in \gamma^4 \cap \partial B$. Observe that $\partial B \subset \partial A \subset \partial U \cup \partial\Omega$, thus the fact that $\gamma^4 \subset U$ gives $z \in \partial\Omega$. Call E the connected component of $\partial\Omega$ containing z. Given that $\gamma^4 \subset \mathbb{R}^2 \setminus \bigcup_{i=1}^{k-1} E_i$, we have that either
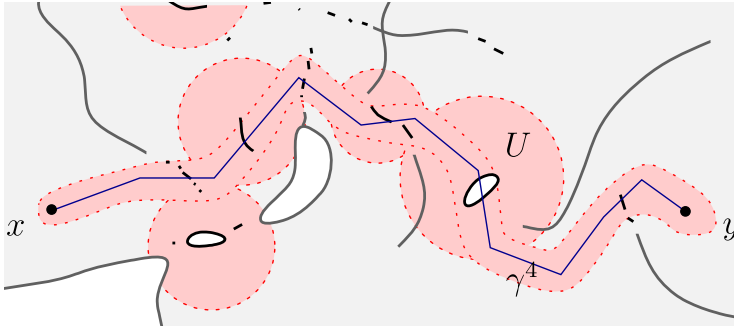

Figure 7. The final curve is found inside a set U that is obtained as the union of a neighbourhood of the curve γ^4 and suitable collection of balls. The neighbourhood of γ^4 allows us to avoid the largest pieces of the boundary, which the curve γ^4 avoided. The additional balls are added to the neighbourhood of γ^4 so that we can connect x to y inside U without crossing the remaining large boundary parts that γ^4 might originally intersect.

$E=E_i$ for some $i \geq k$ or $E \subset C$. To prove that $E \cap \partial U = \varnothing$ we distinguish the following three cases:

i) $E=E_i$ for some $i=k,...,m$. Then it holds $E \subset B(z_i, 2\operatorname{diam}(E_i)) \subset U$, whence accordingly $E \cap \partial U = \varnothing$.

ii) $E=E_i$ for some $i>m$. Since $\operatorname{diam}(E) < \operatorname{dist}(\gamma^4, \partial U')$ by (4.7) and $\gamma^4 \cap E \neq \varnothing$, we see that $E \subset U' \subset U$ and thus $E \cap \partial U = \varnothing$.

iii) $E \subset C$. Then E is a non-empty connected set with null diameter, namely a singleton, so that clearly $E \cap \partial U = \varnothing$.

Therefore, E is also a connected component of $\partial U \cup \partial \Omega$ and accordingly $\partial B \subset E$. The set $\mathbb{R}^2 \setminus \partial B$ cannot be connected, otherwise we would have $\mathring{B} = \mathbb{R}^2 \setminus \partial B$ and thus $B = \mathbb{R}^2$. Therefore, $\mathbb{R}^2 \setminus \partial B$ has at least two connected components: one coincides with $\mathring{B}$ (thus contains y), while another one contains the point x. Then any curve joining x to y must intersect $\partial B \subset \partial \Omega$, which is in contradiction with the assumption that Ω is connected. Consequently, the claim (4.9) is proven.

STEP 5. Thanks to (4.9), we can find a continuous curve $\gamma^5 \subset U \setminus \partial \Omega$ joining x to y. The Painlevé estimate (for general compact sets), namely Proposition 3.2, provides us with an open neighbourhood V of $\partial \Omega \cap \overline{U}$ such that $\overline{V} \cap \gamma^5 = \varnothing$ and

$$(4.10) \qquad \mathcal{H}^1(\partial V) \leq \pi \, \mathcal{H}^1(\partial \Omega \cap \overline{U}) < \frac{4\,\varepsilon}{5},$$

where the second inequality is a consequence of (4.8). Let us denote by W' the connected component of $U \setminus \overline{V}$ containing γ^5. Note that $\partial W' \subset \partial U' \cup \partial V \cup \bigcup_{i=k}^m \partial B(z_i, 2\operatorname{diam}(E_i))$. Therefore, by combining the estimates in (4.6), in (4.10) and in the second line of (4.1), we conclude that $\mathcal{H}^1(\partial W') < 2(q + \sum_{i=1}^{k-1} \mathcal{H}^1(E_i) + \varepsilon)$. Since $\gamma^5 \subset W' \subset \overline{W'} \subset \mathbb{R}^2 \setminus \partial \Omega$, we can apply Proposition 2.13 to obtain a bounded domain $W \subset \mathbb{R}^2$ with $\gamma^5 \subset W \subset \Omega$, whose boundary is the disjoint union of finitely many smooth Jordan loops and such that

$$(4.11) \qquad \mathcal{H}^1(\partial W) < 2\Big(q + \sum_{i=1}^{k-1} \mathcal{H}^1(E_i) + \varepsilon\Big).$$

We call λ the boundary of the unbounded connected component of $\mathbb{R}^2 \setminus \overline{W}$, while by $\{\lambda^j\}_{j \in J}$ (for some finite family of indices J) we denote the boundaries of the bounded connected components of $\mathbb{R}^2 \setminus \overline{W}$. Let us also define $\Lambda := \bigcup_{j \in J} \lambda^j$.

STEP 6. Call L_x and L_y the lines orthogonal to $[x,y]$ that pass through x and y, respectively. Take those points $u_1, u_2, u_3, u_4 \in \lambda$ such that $x \in [u_1, u_3] \subset L_x$, $y \in [u_2, u_4] \subset L_y$, and $(u_1, u_3) \cap \lambda, (u_2, u_4) \cap \lambda = \varnothing$. We can suppose that u_1, u_2 lie in the same connected component of $\mathbb{R}^2 \setminus \mathbb{R}(y-x)$ (thus u_3, u_4 are contained in the other one). By $\widehat{u_1 u_2}$ we mean the arc in λ joining u_1 to u_2 that does not contain any other point u_i, similarly for $\widehat{u_3 u_4}$ and so on. The set $\mathbb{R}(y-x) \setminus [x,y]$ is the union of

two half-lines; both of them intersect the curve λ, say at some points $u_5 \in \widehat{u_1 u_3}$ and $u_6 \in \widehat{u_2 u_4}$. By Lemma 2.1 we see that

$$(4.12) \qquad \begin{aligned} |x - u_1| &\leq \mathcal{H}^1(\widehat{u_1 u_5}), &\quad |y - u_2| &\leq \mathcal{H}^1(\widehat{u_2 u_6}), \\ |x - u_3| &\leq \mathcal{H}^1(\widehat{u_3 u_5}), &\quad |y - u_4| &\leq \mathcal{H}^1(\widehat{u_4 u_6}). \end{aligned}$$

Let us define the curves γ^6, γ^7 as

$$\gamma^6 := [x, u_1] * \widehat{u_1 u_2} * [u_2, y], \quad \gamma^7 := [x, u_3] * \widehat{u_3 u_4} * [u_4, y].$$

Therefore, (4.12) ensures that $\ell(\gamma^6) + \ell(\gamma^7) \leq \ell(\lambda)$, whence (possibly relabeling γ^6 and γ^7) it holds that $\ell(\gamma^6) \leq \ell(\lambda)/2$. Finally, take a curve $\gamma \subset \gamma^6 \cup \Lambda$ joining x to y such that $\gamma \cap \partial\Omega = \varnothing$ (thus $\gamma \subset \Omega$) and $\mathcal{H}^1(\gamma \cap \Lambda) \leq \mathcal{H}^1(\Lambda)/2$. Consequently, we deduce that $\ell(\gamma) \leq \ell(\gamma^6) + \mathcal{H}^1(\gamma \cap \Lambda) \leq \big(\ell(\lambda) + \mathcal{H}^1(\Lambda)\big)/2 = \mathcal{H}^1(\partial W)/2$. By recalling the inequality (4.11), we conclude that $\ell(\gamma) \leq q + \sum_{i=1}^{k-1} \mathcal{H}^1(E_i) + \varepsilon$. In view of (4.2), this explicitly means that

$$\ell(\gamma) \leq \begin{cases} \sum_{i=1}^{k-1} \mathcal{H}^1(E_i) + \varepsilon & \text{if } \Omega \text{ is bounded,} \\ |x - y| + \sum_{i=1}^{k-1} \mathcal{H}^1(E_i) + \varepsilon & \text{if } \Omega \text{ is unbounded.} \end{cases}$$

By arbitrariness of $\varepsilon > 0$, this completes the proof of Theorem 1.1.

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