Ark. Mat., **57** (2019), 107–120
DOI: 10.4310/ARKIV.2019.v57.n1.a6

The asymptotic zero-counting measure of iterated derivaties of a class of meromorphic functions

Christian Hägg

Abstract. We give an explicit formula for the logarithmic potential of the asymptotic zero-counting measure of the sequence $\{(d^n/dz^n)\,(R(z)\exp T(z))\}_{n=1}^{\infty}$. Here, $R(z)$ is a rational function with at least two poles, all of which are distinct, and $T(z)$ is a polynomial. This is an extension of a recent measure-theoretic refinement of Pólya's Shire theorem for rational functions.

1. Introduction

Consider a meromorphic function f, and let S denote its set of poles. Pólya proved in 1922 that the zeros of the iterated derivatives f', f'', f''', ... of such a function asymptotically accumulate along the boundaries of the Voronoi diagram associated with S. This classical result is called Pólya's Shire theorem (see [3] and [4]). In a recent paper by Rikard Bögvad and this author (see [2]), a measure-theoretic refinement of Pólya's Shire theorem was given for the special case that $f=P/Q$, where P and Q are polynomials with $\gcd(P,Q)=1$, and $P\not\equiv 0$.

In this paper, we generalize the main result of the aforementioned paper (see Theorem 1 of [2]) to the situation when $f=(P/Q)e^T$, where P and Q are defined as previously, and T is a nonconstant polynomial. Furthermore, we assume that Q is monic and has at least two zeros, all of which are distinct. Under these conditions, it follows from Hadamard's factorization theorem (see [5]) that the class of such functions is equivalent to the class of meromorphic functions that are quotients of two entire functions of finite order, each with a finite number of zeros. For convenience, we denote $p:=\deg P$, $q:=\deg Q$ and $t:=\deg T$ throughout this paper, and additionally set $P=\sum_{k=0}^{p} b_k z^k$, $Q=\sum_{k=0}^{q} c_k z^k$ and $T=\sum_{k=0}^{t} d_k z^k$.

Before we state the main result of this paper in Theorem 1.1 below, we remind the reader that if $\widetilde{P}(z)$ is a polynomial of degree $d\geq 1$, then its *zero-counting measure* μ is a probability measure that assigns mass $1/d$ to each zero of $\widetilde{P}(z)$, accounting for multiplicity (see [1]).

Theorem 1.1. *Let* $f:=(P/Q)e^T$, *where* P, Q *and* T *are polynomials with* $\gcd(P,Q)=1$, $P\not\equiv 0$, $\deg Q\geq 2$ *and* $\deg T\geq 1$. *Furthermore, assume that* Q *is monic, and that all of its zeros* $z_1,\dots,z_q$ *are distinct. Then*
(i) the zero-counting measures μ_n *of the sequence* $\{f^{(n)}\}_{n=1}^{\infty}$ *converge to a measure* μ_S *with mass* $(q-1)/(q-1+t)$.
(ii) The logarithmic potentials $\mathcal{L}_{\mu_n}(z)$ *of* μ_n *diverge as* $n\to\infty$.
(iii) The shifted logarithmic potentials $\tilde{\mathcal{L}}_{\mu_n}(z):=\mathcal{L}_{\mu_n}(z)-(\log n!)/(n(q+t-1)+p)$ *of* μ_n *converge in* L^1_{loc} *to the distribution* $\Psi(z)$, *where*

$$\Psi(z)=\frac{1}{q+t-1}\left(\max_{i=1,\dots,q}\left\{\log|z-z_i|^{-1}\right\}+\log|Q|-\log\left(|d_t|\,t\right)\right). \tag{1}$$

(iv) The measure μ_S *is given by* $(2\pi)^{-1}\Delta\Psi(z)$.

In the terminology of Theorem 1.1, it is intuitive to refer to $\Psi(z)$ as the shifted logarithmic potential of μ_S. Additionally, note that the formula used to reconstruct the measure μ_S in (iv) is identical to the formula used in the reconstruction of a measure from its associated logarithmic potential (see [1]). Furthermore, note that if $t=0$, it follows from Theorem 1 of [2] that the logarithmic potential of the asymptotic zero-counting measure μ of the sequence $\{(P/Q)^{(n)}\}_{n=1}^{\infty}$ is given by

$$\mathcal{L}_{\mu}(z)=\frac{1}{q-1}\left(\max_{i=1,\dots,q}\left\{\log|z-z_i|^{-1}\right\}+\log|Q|\right).$$

Thus, there are strong similarities with Theorem 1.1 above. An illustration of Theorem 1.1 is given in Figure 1.

The connection between the probability measures μ_n and the measure μ_S with mass $(q-1)/(q+t-1)<1$ (which we will detail later in Proposition 4.2) may seem surprising, as it implies that a mass of $t/(q+t-1)$ disappears as $n\to\infty$. This mass discrepancy appears to arise due to the “bubbles” in Figure 1, whose structure does not appear to converge on the Voronoi diagram of $Z(Q)$ (compare this to Figure 1 in [2], where $t=0$, and no such structures seem to arise). Numerical experiments indicate that these “bubbles” expand toward ∞ asymptotically.

The author is indebted to Rikard Bögvad for discussions, ideas, corrections and comments, and to Boris Shapiro for additional corrections and suggestions.

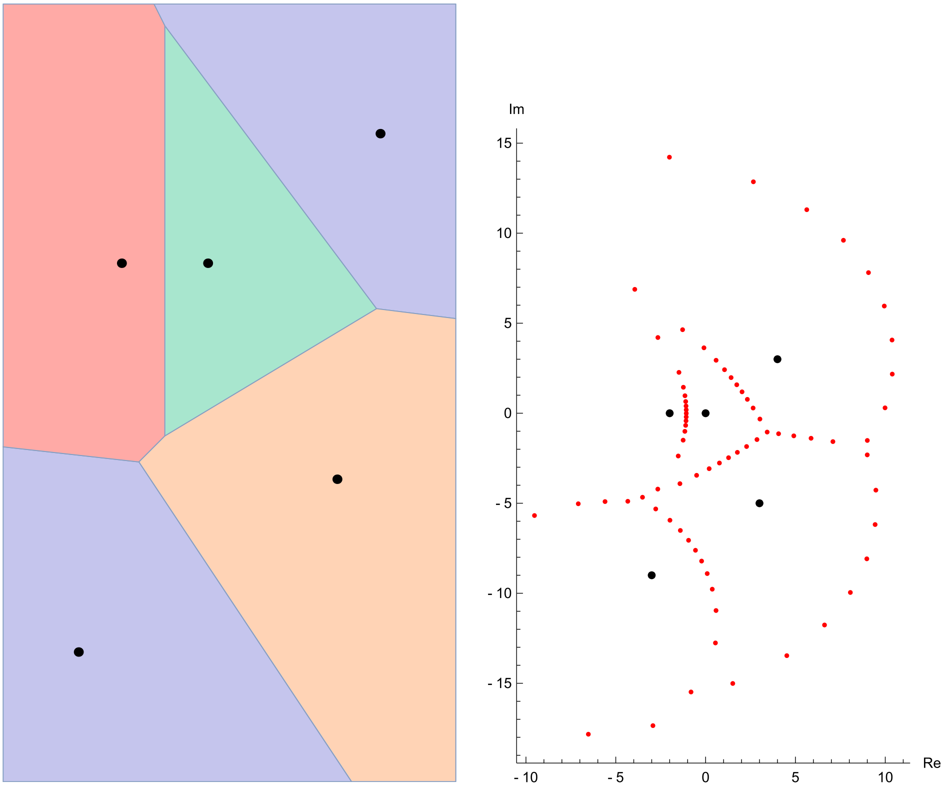

Figure 1. The Voronoi diagram generated by the zeros of the polynomial $Q{=}z(z{+}2)(z{-}4{-}3i)(z{-}3{+}5i)(z{+}3{+}9i)$ (left), and the 75 zeros of $\left((1/Q)e^T\right)^{(15)}$ (small dots), where $T{=}z{+}1$ (right). Note that $56/75{\approx}4/5{=}(q{-}1)/(q{+}t{-}1)$ of the zeros approximately appear to be supported on the Voronoi diagram, in accordance with Theorem 1.1.

2. Voronoi diagrams

Consider a set of q distinct points $S{=}\{z_1,\dots,z_q\}{\subset}\mathbb{C}$. The Voronoi diagram associated with S, denoted by Vor_S, is a partitioning of $\mathbb{C}$ into q distinct cells $V_1,\dots,V_q$, where any interior point α_i in V_i is closest to z_i of all points in S. The boundary between two adjacent cells V_i and V_j consists of a segment of the line $|z-z_i|=|z-z_j|$.

Based on the aforementioned definition of Voronoi diagrams, it is natural to stratify the complex plane using the function

$$\Phi(z) := \min_{i=1,\dots,q} \{|z-z_i|\}.$$

Thus, the (closed) cell V_i that contains the point $z_i \in S$ is equal to the set

$$V_i = \{z : \Phi(z) = |z - z_i|\}.$$

Similarly, the boundary V_{ij} between two cells V_i and V_j is given by

$$V_{ij} = \{z : \Phi(z) = |z - z_i| = |z - z_j|\}.$$

Together, these boundaries form the $1-$*skeleton* of $\mathrm{Vor}_{\mathrm{S}}$, which we denote by $\mathrm{Vor}_{\mathrm{S}}^{\mathrm{B}}$ (where B means boundary). Finally, the vertices of $\mathrm{Vor}_{\mathrm{S}}$ are points z such that at least three distances $|z - z_i|$, $i = 1, \ldots, q$, coincide with $\Phi(z)$.

3. Uniform convergence of the shifted logarithmic potentials

We return to the function $f = (P/Q)e^T$, defined as in Theorem 1.1. By Pólya's Shire theorem, the zeros of the iterated derivatives f', f'', f''', ... of f tend to accumulate along $\mathrm{Vor}_{\mathrm{S}}^{\mathrm{B}}$, where $S = \{z_1, \ldots, z_q\}$ is the set of zeros of Q. Simple computations show that

$$f^{(n)} = \frac{P_n}{Q^{n+1}} e^T, \tag{2}$$

where P_n is a polynomial such that P_n and Q are relatively prime. Clearly, the zeros of $f^{(n)}$ are the zeros of the polynomial P_n, so it is of interest to investigate the structure of P_n. It follows trivially from (2) that

$$P_n = (QT' - nQ')P_{n-1} + QP'_{n-1}, \; n \geq 1, \tag{3}$$

where $P_0 := P$.

Example 3.1. If $f = e^z/(z(z-1))$, it follows that $P_0 = 1$, $P_1 = z^2 - 3z + 1$, $P_2 = z^4 - 6z^3 + 13z^2 - 8z + 2$, and $P_3 = z^6 - 9z^5 + 36z^4 - 73z^3 + 63z^2 - 30z + 6$.

To proceed, we make use of the assumptions that $t = \deg T \geq 1$ and $P \not\equiv 0$ in Theorem 1.1. In this situation, we see from equation (3) that the $QT'P_{n-1}$ term dominates the degree of P_n. Thus, it follows that

$$\deg P_n = \deg (QT'P_{n-1}) = n(q + t - 1) + p. \tag{4}$$

In addition to the degree of P_n, we will soon make use of the coefficient A_n of the highest-power term of P_n. To determine it explicitly, let $\alpha_1, \ldots, \alpha_{\deg P_n}$ be the zeros of P_n, and let $P_n = A_n \prod_{k=1}^{\deg P_n} (z - \alpha_k)$. Now note that $A_0 = b_p = (d_t\, t)^0\, b_p$.

Since A_n depends only on the $QT'P_{n-1}$ term in (3) when $t\geq 1$, $A_n = d_t\, t\, A_{n-1}$, for all $n\geq 1$, and thus,

$$A_n = (d_t\, t)^n \cdot b_p, \quad n\geq 0, \quad t\geq 1. \tag{5}$$

According to Pólya's Shire theorem, $\left|f^{(n)}/n!\right|^{1/n}$ converges pointwise a.e. in any open Voronoi cell to $\max\left\{|z-z_i|^{-1},\ i=1,\dots,q\right\}$. To make use of this, we see from equation (2) that

$$\begin{aligned}\log\left(\left|\frac{f^{(n)}}{n!}\right|^{\frac{1}{n}}\right) = \frac{\log|A_n|}{n} + \frac{\deg P_n}{n}\cdot\frac{\log\left|\prod_{k=1}^{\deg P_n}(z-\alpha_k)\right|}{\deg P_n} + \\ + \frac{\log\left|e^T\right|}{n} - \frac{\log|n!|}{n} - \frac{(n+1)\log|Q|}{n}.\end{aligned} \tag{6}$$

Note that the term $\left(\log\left|\prod_{k=1}^{\deg P_n}(z-\alpha_k)\right|\right)/\deg P_n$ is the logarithmic potential $\mathcal{L}_{\mu_n}(z)$ of the zero-counting measure μ_n of P_n/A_n. Passing to the limit in n in equation (6) and making use of (4) and (5), we see that

$$\lim_{n\to\infty}\frac{\log|A_n|}{n} = \log(|d_t|\, t), \tag{7}$$

$$\lim_{n\to\infty}\frac{\deg P_n}{n} = q+t-1, \tag{8}$$

$$\lim_{n\to\infty}\frac{\log\left|e^T\right|}{n} = 0, \tag{9}$$

$$\lim_{n\to\infty} -\frac{\log n!}{n} = -\infty, \tag{10}$$

and

$$\lim_{n\to\infty} -\frac{(n+1)\log|Q|}{n} = -\log|Q|. \tag{11}$$

Thus, since the left-hand side of (6) converges to

$$\log\left(\max_{i=1,\dots,q}\left\{\frac{1}{|z-z_i|}\right\}\right) = \max_{i=1,\dots,q}\left\{\log|z-z_i|^{-1}\right\} \tag{12}$$

inside open Voronoi cells, which is finite outside of S, it follows from (7)-(12) that $\lim_{n\to\infty}\mathcal{L}_{\mu_n}(z)=\infty$. This proves part (ii) of Theorem 1.1.

Although the logarithmic potential of the asymptotic zero-counting measure μ_S diverges as $n\to\infty$, the shifted logarithmic potential $\tilde{\mathcal{L}}_{\mu_n}(z)$ of μ_n (defined as in part (iii) of Theorem 1.1) can be used to rewrite equation (6) as

(13)

$$\tilde{\mathcal{L}}_{\mu_n}(z) = \frac{n}{n(q+t-1)+p}\left(\log\left(\left|\frac{f^{(n)}}{n!}\right|^{\frac{1}{n}}\right) + \frac{(n+1)\log|Q|}{n} - \frac{\log|A_n|}{n} - \frac{\log|e^T|}{n}\right),$$

or, as we will find use for later, by using the expression for $f^{(n)}$ in (2),

$$\tilde{\mathcal{L}}_{\mu_n}(z) = \frac{1}{n(q+t-1)+p}\left(\log\left|\frac{P_n}{A_n}\right| - \log n!\right). \tag{14}$$

By letting $n\to\infty$ in (13), we obtain the equation (1), where the right-hand side has converged pointwise (in any open Voronoi cell V_i^o) to a *continuous subharmonic function* defined in the whole complex plane, as we will see in Lemma 4.1 in the next section. More generally, we have the following proposition, the proof of which is analogous to that of Proposition 4.5 in [2], and is omitted for brevity.

Proposition 3.2. *Let $\mathcal{L}_{\mu_n}(z) = (\log|P_n| - \log|A_n|)\,/\deg P_n$ be the logarithmic potential of the zero-counting measure μ_n of P_n/A_n. Furthermore, let $\tilde{\mathcal{L}}_{\mu_n}(z) = \mathcal{L}_{\mu_n}(z) - (\log n!)/(n(q+t-1)+p)$. Then for any z in the interior of the Voronoi cell V_i^o, we have pointwise convergence*

(15)

$$\lim_{n\to\infty}\tilde{\mathcal{L}}_{\mu_n}(z) = \frac{1}{q+t-1}\left(\max_{i=1,\dots,q}\left\{\log|z-z_i|^{-1}\right\} + \log|Q| - \log(|d_t|\,t)\right) =: \Psi(z).$$

The convergence is uniform on compact subsets of V_i^o.

4. The subharmonic function $\Psi(z)$

The two results in this section describe properties of the asymptotic zero-counting measure of P_n/A_n. Their proofs are analogous to those of Lemma 2.1 and Proposition 2.2 in [2], respectively.

Lemma 4.1. *The function $\Psi(z)$, defined in $\mathbb{C}$, is a continuous subharmonic function, and is harmonic in the interior of any cell V_i.*

Since $\Psi(z)$ is subharmonic, $\Delta\Psi(z) = 4(\partial^2\Psi(z)/\partial\bar{z}\partial z)$ is a positive measure with support on Vor_S^B.

The following proposition provides the definition and some properties of what will turn out to be the asymptotic zero-counting measure.

Proposition 4.2. *For each pair i,j, define a measure with support on the line l_{ij}: $|z-z_i|=|z-z_j|$ as*

$$\delta_{ij} = \frac{1}{4(q+t-1)} \frac{|z_i-z_j|}{|(z-z_i)(z-z_j)|}\, ds,$$

where ds is the Euclidean length measure in the complex plane. Then
1. $(\partial^2\Psi/\partial\bar{z}\partial z)$ *is the sum of all δ_{ij}, each restricted to V_{ij}.*
2. $\mu_S := 2\pi^{-1}(\partial^2\Psi/\partial\bar{z}\partial z)$ *has mass $(q-1)/(q+t-1)$.*

5. Proof of the main theorem

Uniform convergence a.e. as in Proposition 3.2 does not by itself imply convergence of the logarithmic potentials in L^1_{loc}, though it tells us that there is only one possible limit, since a function in L^1_{loc} is determined by its behavior a.e. We will prove the L^1_{loc}-convergence directly, with the main difficulty being the unboundedness of the zeros of P_n as $n\to\infty$. To deal with this problem, we give rough bounds of the growth of the zeros of P_n in Lemma 5.2 below.

5.1. Growth of zeros

Consider a fixed meromorphic function $f(z):=(P/Q)e^T$ as in Theorem 1.1. Lemma 5.1 below shows that if the statement of the theorem holds for $f(z)$, it also holds for $\hat{f}(z):=f(\tau z+a)$, $\tau\in\mathbb{R}_+$, $a\in\mathbb{C}$, i.e. the statement of the theorem is invariant under scaling and translation. For convenience, let $\hat{\mu}_n$ be the zero-counting measure of $\hat{f}^{(n)}$ (or, technically, of the polynomial $\prod_k(z-\widehat{\alpha}_k)$, where the product is taken over all zeros $\widehat{\alpha}_1, \widehat{\alpha}_2, \ldots$ of $\hat{f}^{(n)}$), and let $\tilde{\mathcal{L}}_{\hat{\mu}_n}(z)$ be its shifted logarithmic potential.

Lemma 5.1. *Assume that $\tilde{\mathcal{L}}_{\mu_n}(z)\to\Psi(z)$ in L^1_{loc}, where $\Psi(z)$ is the shifted logarithmic potential given by (15) of the asymptotic zero-counting measure* $\lim_{n\to\infty}\mu_n$. *Then $\tilde{\mathcal{L}}_{\hat{\mu}_n}(z)\to\widehat{\Psi}(z)$ in L^1_{loc}, where $\widehat{\Psi}(z)$ is the shifted logarithmic potential of* $\lim_{n\to\infty}\hat{\mu}_n$.

Proof. First note that Theorem 1.1, and, in particular, the L^1_{loc}-convergence to $\Psi(z)$ in part (iii) of the theorem, are not actually dependent on the fact that the polynomial $Q(z)$ is monic. For general $Q(z)$, equation (1) needs to be adjusted to

$$\Psi(z) = \frac{1}{q+t-1}\left(\max_{i=1,\ldots,q}\left\{\log|z-z_i|^{-1}\right\} + \log|Q| - \log\left(|c_q|\,|d_t|\,t\right)\right). \tag{16}$$

We see from (2) that

$$\hat{f}^{(n)}(z) = \frac{\widehat{P}_n(z)}{(Q(\tau z+a))^{n+1}} e^{T(\tau z+a)}, \tag{17}$$

for some polynomial $\widehat{P}_n(z) := \hat{A}_n \prod_{k=1}^{n(q+t-1)+p}(z-\widehat{\alpha}_k)$. Similarly,

$$\begin{aligned}\hat{f}^{(n)}(z) &= (f(\tau z+a))^{(n)} = \tau^n f^{(n)}(\tau z+a)\\ &= \tau^n \left(\frac{P_n(\tau z+a)}{(Q(\tau z+a))^{n+1}} e^{T(\tau z+a)} \right).\end{aligned} \tag{18}$$

By comparing equations (17) and (18), we see that

$$\widehat{P}_n(z) = \tau^n P_n(\tau z+a). \tag{19}$$

Consequently, by using the definitions of $\widehat{P}_n(z)$ and $P_n(z)$ in (19), it follows that

$$\hat{A}_n \prod_{k=1}^{n(q+t-1)+p} (z-\widehat{\alpha}_k) = \tau^{n(q+t)+p} A_n \prod_{k=1}^{n(q+t-1)+p} \left(z - \frac{\alpha_k - a}{\tau} \right),$$

and thus,

$$\hat{A}_n = \tau^{n(q+t)+p} A_n. \tag{20}$$

As a result, by using (19) and (20) in (14),

$$\tilde{\mathcal{L}}_{\hat{\mu}_n}(z) = \frac{1}{n(q+t-1)+p} \left(\log \left| \frac{\widehat{P}_n(z)}{\hat{A}_n} \right| - \log n! \right) = \tilde{\mathcal{L}}_{\mu_n}(\tau z+a) - \log \tau. \tag{21}$$

As a result of (21) and the assumption of the lemma, $\tilde{\mathcal{L}}_{\hat{\mu}_n}(z) \to \Psi(\tau z+a) - \log\tau$ in L^1_{loc}.

To see that $\widehat{\Psi}(z) := \Psi(\tau z+a) - \log\tau$ is the correct shifted logarithmic potential of $\lim_{n\to\infty} \hat{\mu}_n$ (rather than some other L^1_{loc}-function), we also need to prove that it satisfies equation (16). To do this, define $\hat{c}_k$ and $\hat{d}_k$ as the coefficients of z^k in $Q(\tau z+a)$ and $T(\tau z+a)$, respectively. Then, by using the definition of $Q(z)$, we see that

$$Q(\tau z+a) = \sum_{k=0}^{q} \hat{c}_k z^k = \sum_{k=0}^{q} c_k (\tau z+a)^k,$$

so $\hat{c}_q = \tau^q$, and similarly, $\hat{d}_t = \tau^t d_t$.

Furthermore, for each zero z_i of $Q(z)$, $\hat{z}_i:=(z_i-a)/\tau$ is a zero of $Q(\tau z+a)$. Consequently, by using this bijective correspondence between z_i and $\hat{z}_i$, we get

$$\max_{i=1,\dots,q}\left\{\log|z-\hat{z}_i|^{-1}\right\}=\max_{i=1,\dots,q}\left\{\log\left|z+\frac{a-z_i}{\tau}\right|^{-1}\right\}\\=\max_{i=1,\dots,q}\left\{\log|\tau z+a-z_i|^{-1}\right\}+\log\tau. \tag{22}$$

Finally, by using (22) in the right-hand side of (16) for $\hat{f}(z)$, we see that

$$\frac{1}{q+t-1}\left(\max_{i=1,\dots,q}\left\{\log|z-\hat{z}_i|^{-1}\right\}+\log|Q(\tau z+a)|-\log\left(|\hat{c}_q||\hat{d}_t|t\right)\right)$$
$$=\frac{1}{q+t-1}\left(\max_{i=1,\dots,q}\left\{\log|\tau z+a-z_i|^{-1}\right\}+\log\tau+\log|Q(\tau z+a)|-\log\left(\tau^{q+t}|d_t|t\right)\right)$$
$$=\Psi(\tau z+a)-\log\tau=\widehat{\Psi}(z). \quad\square$$

Next, let $D_\rho(b)$ denote the open disk with center b and of radius ρ. By choosing b as one of the poles of $f(z)$, and by letting ρ be sufficiently small, it follows from Pólya's Shire theorem that $D_\rho(b)$ contains no zeros of $f^{(n)}(z)$ for all large enough n. More precisely, after scaling and translation, we may assume that the following holds due to Lemma 5.1:

(*) The closed disk $\overline{D}_2(0)$ contains exactly one pole $z_i=0$ (so that $Q(0)=0$).

It follows from (*), by Proposition 3.2, that there is a positive number N such that $z\in\overline{D}_1(0)\subset V_i^o\Longrightarrow P_n(z)\neq 0$, if $n\geq N$. Equivalently, if $n\geq N$ and $P_n(z)=0$, then $|z|>1$.

Before we give bounds for the growth of the zeros of P_n, we define some additional notation for convenience. For $K\subset\mathbb{C}$, let

$$|z_{K,n}|:=\prod_{z\in K:P_n(z)=0}|z|,$$

where zeros are taken with multiplicities; note that if there are no zeros of $P_n(z)$ in K, then $|z_{K,n}|=1$. Furthermore, let $\mathcal{D}_\rho:=D_\rho(0)=\{z:|z|<\rho\}$, for $\rho>0$, and set $m_n:=\deg P_n=n(q+t-1)+p$.

Lemma 5.2. *Assume* (*). *Then there are real numbers* C_1, C_2, *and* N *such that* $C_1\leq(1/m_n)\log(|z_{\mathcal{D}_\rho^c,n}|/n!)\leq C_2$ *for all* $n\geq N$.

Proof. Since $Q(0)=0$ by (*), it follows from the assumptions in Theorem 1.1 that $P(0)\neq0$, and $Q'(0)\neq0$. Consequently, the recurrence relation (3) yields that

$$P_n(0)=-nQ'(0)P_{n-1}(0),\ \text{for all}\ n\geq1. \tag{23}$$

Because $P_0(0)=P(0)$, the solution of (23) is

$$P_n(0)=n!(-Q'(0))^n P(0), \text{ for all } n\geq 0. \tag{24}$$

Thus, it follows from (24) that

$$\lim_{n\to\infty}\frac{1}{m_n}\log\left|\frac{P_n(0)}{n!}\right| = \lim_{n\to\infty}\left(\frac{n\log|Q'(0)|}{n(q+t-1)+p}+\frac{\log|P(0)|}{n(q+t-1)+p}\right) = \frac{\log|Q'(0)|}{q+t-1}. \tag{25}$$

Next, we consider the situation in which $0<\rho\leq 1$. Since $|P_n(0)|=|A_n||z_{\mathcal{D}_\rho,n}||z_{\mathcal{D}_\rho^c,n}|$, we obtain the equation

$$\frac{1}{m_n}\log\left|\frac{P_n(0)}{n!}\right| = \frac{1}{m_n}\left(\log|A_n|+\log|z_{\mathcal{D}_\rho,n}|+\log|z_{\mathcal{D}_\rho^c,n}|-\log n!\right). \tag{26}$$

Because $\lim_{n\to\infty}(1/m_n)\log|A_n|=(\log|d_t|t)/(q+t-1)$ by (7), $(1/m_n)\log|z_{\mathcal{D}_\rho,n}|=0$ for all $n\geq N$ due to Proposition 3.2, and the fact that the left-hand side (and thus also the right-hand side) of equation (26) converges due to the limit in (25), it follows that

$$\lim_{n\to\infty}\frac{1}{m_n}\log\left(\frac{|z_{\mathcal{D}_\rho^c,n}|}{n!}\right) = \frac{\log|Q'(0)|-\log(|d_t|t)}{q+t-1} =: C.$$

Hence, for any fixed $\varepsilon>0$, we can choose $C_1=C-\varepsilon$ and $C_2=C+\varepsilon$. Consequently, there exists a number $N=N(\varepsilon)$ such that the lemma follows in this case.

We proceed with the case $\rho>1$. In this situation, we see from (26) that

$$\lim_{n\to\infty}\frac{1}{m_n}\left(\log|z_{\mathcal{D}_\rho,n}|+\log|z_{\mathcal{D}_\rho^c,n}|-\log n!\right)=C. \tag{27}$$

Assume that $\lim_{n\to\infty}(1/m_n)\log(|z_{\mathcal{D}_\rho^c,n}|/n!)=\infty$, for some subsequence of n. In order for (27) to be valid, we must have that $\lim_{n\to\infty}(1/m_n)\log|z_{\mathcal{D}_\rho,n}|=-\infty$ over the same subsequence. Furthermore, note that there exists a number N' such that all the zeros of R_n in $\mathcal{D}_\rho$ are contained in the annulus $\{z:1\leq|z|<\rho\}$ for all $n\geq N'$. Hence, $\rho>1\Longrightarrow|z_{\mathcal{D}_\rho,n}|\geq 1\Longrightarrow(1/m_n)\log|z_{\mathcal{D}_\rho,n}|\geq 0$ for all large enough n, resulting in a contradiction. Thus, there exists a number C_2 such that

$$\frac{1}{m_n}\log\left(\frac{|z_{\mathcal{D}_\rho^c,n}|}{n!}\right)\leq C_2, \tag{28}$$

for all $n\geq N'$.

Next, assume that $\lim_{n\to\infty}(1/m_n)\log(|z_{\mathcal{D}_\rho^c,n}|/n!)=-\infty$ for some subsequence of n. Then by (27), it follows that $\lim_{n\to\infty}(1/m_n)\log|z_{\mathcal{D}_\rho,n}|=\infty$ over the same subsequence. Since the number of zeros of P_n in $\mathcal{D}_\rho$ is at most $m_n=n(q+t-1)+p$

for any fixed n, it follows that $|z_{\mathcal{D}_\rho,n}|<\rho^{m_n}$, or equivalently, $(1/m_n)\log|z_{\mathcal{D}_\rho,n}|<(1/m_n)\log(\rho^{m_n})=\log\rho$, for all $n\geq 1$. This is another contradiction. Consequently, there exist numbers C_1 and N'' such that

$$C_1\leq\frac{1}{m_n}\log\left(\frac{|z_{\mathcal{D}_\rho^c,n}|}{n!}\right), \tag{29}$$

for all $n\geq N''$. Thus, by choosing $N=\max\{N',N''\}$, the lemma follows from (28) and (29) in this case. □

5.2. L^1_{loc}-convergence of the logarithmic potentials

Recall that we have previously proven (ii) of Theorem 1.1 in Section 2. Note that if we prove (iii) of the theorem, then the whole theorem follows, since parts (i) and (iv) are immediate consequences of (iii).

To proceed, fix a number $0<\varepsilon<1$. Recall that $\mathcal{D}_\rho$ is the disk of fixed radius $\rho>0$ centered at the origin, and let $U\subset\mathcal{D}_\rho$ be the set of points on $\mathcal{D}_\rho$ that are at least a distance ε away from $\mathrm{Vor}^{\mathrm{B}}_{\mathrm{S}}$. To prove that the convergence of $\tilde{\mathcal{L}}_{\mu_n}(z)$ to $\Psi(z)$ is L^1_{loc}, we must show that, for arbitrary ρ,

$$I_1:=\int_{\mathcal{D}_\rho}\left|\tilde{\mathcal{L}}_{\mu_n}(z)-\Psi(z)\right|\,d\lambda=O(\varepsilon),$$

(that is, an ε can be chosen so that I_1 is arbitrarily close to 0) where λ is Lebesgue measure on $\mathbb{C}$. It is appropriate to split the integral I_1 into two integrals and deal with each one separately:

$$I_1=\int_U\left|\tilde{\mathcal{L}}_{\mu_n}(z)-\Psi(z)\right|\,d\lambda+\int_{\mathcal{D}_\rho\setminus U}\left|\tilde{\mathcal{L}}_{\mu_n}(z)-\Psi(z)\right|\,d\lambda=:I_2+I_3.$$

Since U is the union of q compact subsets of $\mathcal{D}_\rho\setminus\mathrm{Vor}^{\mathrm{B}}_{\mathrm{S}}$, it follows from the uniform convergence in Proposition 3.2 that there exists a number N such that $n\geq N$ implies that $\left|\tilde{\mathcal{L}}_{\mu_n}(z)-\Psi(z)\right|\leq\varepsilon$ if $z\in U$. Hence

$$I_2=\int_U\left|\tilde{\mathcal{L}}_{\mu_n}(z)-\Psi(z)\right|\,d\lambda\leq\pi\rho^2\varepsilon=O(\varepsilon). \tag{30}$$

The integral I_3 is appropriately bounded by the triangle inequality:

$$I_3=\int_{\mathcal{D}_\rho\setminus U}\left|\tilde{\mathcal{L}}_{\mu_n}(z)-\Psi(z)\right|\,d\lambda\leq\int_{\mathcal{D}_\rho\setminus U}\left|\tilde{\mathcal{L}}_{\mu_n}(z)\right|\,d\lambda+\int_{\mathcal{D}_\rho\setminus U}|\Psi(z)|\,d\lambda=:I_5+I_4.$$

If $M_1:=\max\{\Psi(z),\ z\in\mathcal{D}_\rho\}$, the last integral satisfies

$$I_4\leq M_1\lambda(\mathcal{D}_\rho\setminus U)\leq 2\ell\varepsilon M_1, \tag{31}$$

where ℓ denotes the length of $\mathrm{Vor}_{\mathrm{S}}^{\mathrm{B}} \cap \mathcal{D}_\rho$. Thus, $I_4 = O(\varepsilon)$.

To deal with the last integral I_5, we write

$$\tilde{\mathcal{L}}_{\mu_n}(z) = \frac{1}{m_n}\left(\sum_{k=1}^{m_n} \log|z-\alpha_k| - \log n!\right) = \tilde{\mathcal{L}^o}_{\mu_n}(z) + \tilde{\mathcal{L}^i}_{\mu_n}(z),$$

where

$$\tilde{\mathcal{L}^o}_{\mu_n}(z) := (1/m_n)\left(\sum_{|\alpha_k|\geq \rho+1} \log|z-\alpha_k| - \log n!\right)$$

and

$$\tilde{\mathcal{L}^i}_{\mu_n}(z) := (1/m_n)\left(\sum_{|\alpha_k|<\rho+1} \log|z-\alpha_k|\right).$$

Thus, by using the triangle inequality again,

$$I_5 \leq \int_{\mathcal{D}_\rho \setminus U} \left|\tilde{\mathcal{L}^o}_{\mu_n}(z)\right| \, d\lambda + \int_{\mathcal{D}_\rho \setminus U} \left|\tilde{\mathcal{L}^i}_{\mu_n}(z)\right| \, d\lambda =: I_6 + I_7.$$

Consequently, for such ρ,

$$0 \leq \log|z-\alpha_k| \leq \log(\rho+|\alpha_k|) \leq \log(\rho+1) + \log|\alpha_k|, \qquad \text{if } |z|<\rho,\ |\alpha_k| \geq \rho+1,$$

so it follows that

$$\begin{aligned} I_6 &= \int_{\mathcal{D}_\rho \setminus U} \left|\tilde{\mathcal{L}^o}_{\mu_n}(z)\right| \, d\lambda \\ &\leq \frac{1}{m_n} \int_{\mathcal{D}_\rho \setminus U} \left| \sum_{|\alpha_k|\geq \rho+1} (\log(\rho+1) + \log|\alpha_k|) - \log n! \right| \, d\lambda \\ &\leq \int_{\mathcal{D}_\rho \setminus U} \left| \log(\rho+1) + \frac{1}{m_n} \log\left(\frac{|z_{\mathcal{D}^c_{\rho+1},n}|}{n!}\right) \right| \, d\lambda \\ &\leq (\log(\rho+1) + \max\{|C_1|, |C_2|\})\, \lambda(\mathcal{D}_\rho \setminus U) = O(\varepsilon), \end{aligned} \tag{32}$$

where the last inequality holds for all sufficiently large n due to Lemma 5.2. (Also note that the inequality $\log(\rho+|\alpha_k|) \leq \log(\rho+1) + \log|\alpha_k|$ corrects a minor mistake in [2], where the corresponding, incorrect inequality was $\log(\rho+|\alpha_k|) \leq \log\rho + \log|\alpha_k|$.)

Finally, if in addition to $|z|<\rho$ and $|\alpha_k|<\rho+1$, we also have $|z-\alpha_k|>\varepsilon$, then $|\log|z-\alpha_k||<\max\{-\log\varepsilon, \log(2\rho+1)\}$. This leads to the inequalities

$$\begin{aligned}&\int_{\mathcal{D}_\rho\setminus U}|\log|z-\alpha_k||\,d\lambda\\ &<\int_{|z-\alpha_k|\le\varepsilon}|\log|z-\alpha_k||\,d\lambda+\max\{-\log\varepsilon,\log(2\rho+1)\}\lambda(\mathcal{D}_\rho\setminus U)\\ &\le 2\pi(1/2-\log\varepsilon)(\varepsilon^2/2)+\max\{-\log\varepsilon,\log(2\rho+1)\}\varepsilon=o(1).\end{aligned} \tag{33}$$

Consequently, from (33),

$$\begin{aligned}I_7&=\int_{\mathcal{D}_\rho\setminus U}\left|\tilde{\mathcal{L}}^i{}_{\mu_n}(z)\right|\,d\lambda\\ &\le\frac{1}{m_n}\sum_{|\alpha_k|<\rho+1}\left(\int_{\mathcal{D}_\rho\setminus U}|\log|z-\alpha_k||\,d\lambda\right)=o(1),\end{aligned} \tag{34}$$

where the inequality in (34) follows because the sum has at most m_n terms.

As a result, part (iii) of Theorem 1.1 (except for the statement of Proposition 5.3 below, which needs to be dealt with separately) follows from the fact that the upper bounds in (30), (31), (32), and (34) go to 0 when ε goes to 0.

Proposition 5.3. *$\Psi(z)=L(z)-D$, where $L(z):=\int_{\mathbb{C}}\log|z-\zeta|\,d\mu_S(\zeta)$ is the logarithmic potential of μ_S and $D:=(\log(|d_t|t))/(q+t-1)$.*

Proof. We will first prove that $L(z):=\int_{\mathbb{C}}\log|z-\zeta|\,d\mu_S(\zeta)$ is well-defined as a L^1_{loc}–function. Let $l_{ij}=\{z\colon |z-z_i|=|z-z_j|\}$, and use the notation of Proposition 4.2. Then, for a compact set $K\subset\mathbb{C}$,

$$\int_K|L(z)|d\lambda(z)\le\sum_{i,j}\int_{l_{ij}}\left(\int_K|\log|z-\zeta||d\lambda(z)\right)d\delta_{ij}(\zeta).$$

Now fix a line l_{ij}. An affine change of coordinates transforms l_{ij} into the real axis, and then δ_{ij} is given by $(\pi(1+t^2))^{-1}\,dt$. Hence it suffices to prove that

$$\int_{\mathbb{R}}\left(\int_K\frac{|\log|z-t||}{1+t^2}d\lambda(z)\right)dt$$

is finite. This is clear, since for large $|t|$, the integrand is approximately $\lambda(K)\log|t|/t^2$.

Secondly, we will prove that $L(z)$ has the property that

$$\lim_{|z|\to\infty}\left(L(z)-\frac{q-1}{q+t-1}\log|z|\right)=0. \tag{35}$$

Since $\Psi(z)$, by inspection from (15), has the property that

$$\lim_{|z|\to\infty}\left(\Psi(z)-\frac{q-1}{q+t-1}\log|z|\right)=\frac{\log(|d_t|t)}{q+t-1}=D,$$

it will follow that $\Psi(z)-L(z)$ is bounded. However, $\Psi(z)$ and $L(z)$ have by definition the same Laplacian, and hence $\Psi(z)-L(z)$ is harmonic. By Harnack's theorem, this implies that $\Psi(z)-L(z)$ is constant, and hence by taking the limit as $|z|\to\infty$, this difference is equal to $-D$.

Now to prove (35) as above, using that the total mass of μ_s is $(q-1)/(q+t-1)$, we observe that

$$\left|L(z)-\frac{q-1}{q+t-1}\log|z|\right|\leq\sum_{i,j}\int_{l_{ij}}\left|\log\left|1-\frac{\zeta}{z}\right|\right|d\delta_{ij}(\zeta).$$

Thus, after another affine transformation, it is enough to consider

$$\int_{\mathbb{R}}\frac{\left|\log\left|1-\frac{t}{z}\right|\right|}{1+t^2}\,dt,$$

which is easily seen to have the limit 0 as $|z|\to\infty$. $\square$

References

1. Bergkvist, T. and Rullgård, H., On polynomial eigenfunctions for a class of differential operators, *Math. Res. Lett.* **9** (2002), 153–171.
2. Bögvad, R. and Hägg, C., A refinement for rational functions of Pólya's method to construct Voronoi diagrams, *J. Math. Anal. Appl.* **452** (2017), 312–334.
3. Hayman, W. K., *Meromorphic functions*, Clarendon Press, Oxford, 1964.
4. Pólya, G., Über die Nullstellen sukzessiver Derivierten, *Math. Z.* **12** (1922).
5. Titchmarsh, E. C., *The theory of functions*, 2nd ed., Oxford University Press, Oxford, 1939.

Christian Hägg
Department of Mathematics
Stockholm University
Sweden
hagg@math.su.se

Received October 4, 2017