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Equivariant L^2-Euler characteristics of G-CW-complexes

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Abstract. We show that if X is a cocompact G-CW-complex such that each isotropy subgroup G_σ is $L^{(2)}$-good over an arbitrary commutative ring k, then X satisfies some fixed-point formula which is an $L^{(2)}$-analogue of Brown's formula in 1982. Using this result we present a fixed point formula for a cocompact proper G-CW-complex which relates the equivariant $L^{(2)}$-Euler characteristic of a fixed point CW-complex X^s and the Euler characteristic of X/G. As corollaries, we prove Atiyah's theorem in 1976, Akita's formula in 1999 and a result of Chatterji-Mislin in 2009. We also show that if X is a free G-CW-complex such that $C_*(X)$ is chain homotopy equivalent to a chain complex of finitely generated projective $\mathbb{Z}\pi_1(X)$-modules of finite length and X satisfies some fixed-point formula over $\mathbb{Q}$ or $\mathbb{C}$ which is an $L^{(2)}$-analogue of Brown's formula, then $\chi(X/G)=\chi^{(2)}(X)$. As an application, we prove that the weak Bass conjecture holds for any finitely presented group G satisfying the following condition: for any finitely dominated CW-complex Y with $\pi_1(Y)=G$, $\widetilde{Y}$ satisfies some fixed-point formula over $\mathbb{Q}$ or $\mathbb{C}$ which is an $L^{(2)}$-analogue of Brown's formula.

1. Introduction

Let G be a discrete group and X a G-CW-complex. Then for each $s\in G$, a fixed point set X^s is a $C_G(s)$-CW-complex, where $C_G(s)$ is the centralizer of s in G.

In 1999, Akita presented a formula expressing the Euler characteristic of the orbit space of a proper cocompact G-CW-complex in terms of equivariant Euler characteristics [1, Theorem 1]. More precisely, he asserted that if X is a cocompact proper G-CW-complex, then

$$\chi(X/G)=\sum_{[s]\in\mathcal{F}(G)}\chi_{C_G(s)}(X^s),$$

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where $\mathcal{F}(G)$ is a set of representatives for the conjugacy classes of G of finite order. If G is finite, this result implies the following well-known result [1] and [13]:

$$\chi(X/G)=\frac{1}{|G|}\sum_{s\in G}\chi(X^s)$$

When G is virtually torsion-free and X is a proper cocompact G-CW-complex such that X^s is nonempty and $\mathbb{Q}$-acyclic for every $s\in G$ of finite order, the result also implies the following, which is a special case of Brown's formula [1], [6] and [7]:

$$\sum_i(-1)^i\dim_{\mathbb{Q}}H_i(G,\mathbb{Q})=\sum_{[s]\in\mathcal{F}(G)}\chi_{C_G(s)}(X^s).$$

On the other hand, Brown conjectured the following formula for a group of type FP over $\mathbb{Q}$ under suitable finiteness conditions for a group G, and gave some partial affirmative answers in many cases, including groups with cocompact $\underline{E}G$ [6]:

$$\text{For each } s\in G,\quad E(G,\mathbb{Q})(s)=\begin{cases} e(C_G(s)) & \text{if } s \text{ has finite order,}\\ 0 & \text{if } s \text{ has infinite order,}\end{cases}$$

where $E(G,\mathbb{Q}):=\mathrm{HS}_{\mathbb{Q}G}(P_*)$ is the complete equivariant Euler characteristic of G which is the Hattori-Stallings rank of an alternating sum of finitely generated projective $\mathbb{Q}G$-modules, and $e(C_G(s))$ is the Euler characteristic of $C_G(s)$ in the sense of [3] or [9] (see Section 2 for more details). In order to give a positive answer to his conjecture, Brown considered more general situation as in [6, Theorem 3.1].

In 2000, Chatterji and Mislin [8] conjectured that if G is a group of type FP over $\mathbb{C}$ such that the centralizer of every element of finite order in G has finite L^2-Betti numbers, then

$$E(G,\mathbb{C})(s)=\chi^{(2)}(C_G(s))$$

for every $s\in G$. This amounts to putting Brown's formula within the framework of L^2-homology. They gave a class of groups which satisfy their conjecture. Thus we may naturally ask the following questions which are L^2-analogues of [6, Theorem 3.1] (Theorem 7 and Proposition 4 explain the reason why we call these questions L^2-analogues of [6, Theorem 3.1]).

Question 1. Let k be an arbitrary commutative ring and X a G-CW-complex of type FP over k such that $\chi^{(2)}_{C_G(s)}(X^s)$ is defined for each element s of finite order in G. What kind of G-CW-complexes X do satisfy the following equation?

$$\text{For each } s\in G,\quad E_G(X,k)(s)=\begin{cases} \chi^{(2)}_{C_G(s)}(X^s) & \text{if } s \text{ has finite order,}\\ 0 & \text{if } s \text{ has infinite order,}\end{cases}$$

where $E_G(X,k)$ is the complete equivariant Euler characteristic of X.

Question 2. Let k be an arbitrary commutative ring and X a G-CW-complex of type FP over k such that $\chi^{(2)}(X^s)$ is defined for each element s of finite order in G. What kind of G-CW-complexes X do satisfy the following equation?

$$\text{For each } s\in G, \quad E_G(X,k)(s)=\begin{cases}\chi^{(2)}(X^s) & \text{if } s \text{ has finite order,}\\ 0 & \text{if } s \text{ has infinite order,}\end{cases}$$

where $E_G(X,k)$ is the complete equivariant Euler characteristic of X.

In Theorem 10, we show that if X is a cocompact G-CW-complex such that each isotropy subgroup G_σ is $L^{(2)}$-good over an arbitrary commutative ring k, then X satisfies the equation in Question 1. Using this result we present, in Theorem 11, a fixed point formula for a cocompact proper G-CW-complex which relates the equivariant $L^{(2)}$-Euler characteristic of a fixed point CW-complex X^s and the Euler characteristic of X/G. As corollaries, we prove Akita's formula in Corollary 13, a result of Chatterji-Mislin in Corollary 14 and Atiyah's theorem in Corollary 12.

We also show in Theorem 15 that if X is a free G-CW-complex such that $C_*(X)$ is chain homotopy equivalent to complex of finitely generated projective $\mathbb{Z}\pi_1(Y)$-modules of finite length and X satisfies the equation over $\mathbb{Q}$ or $\mathbb{C}$ in Question 2, then $\chi(X/G)=\chi^{(2)}(X)$. As an application, we prove that the weak Bass conjecture holds for any finitely presented group G satisfying the following conditions: for any finitely dominated CW-complex Y with $\pi_1(Y)=G$, $\widetilde{Y}$ satisfies the equation over $\mathbb{Q}$ or $\mathbb{C}$ in Question 2.

2. Preliminaries

Throughout this paper, let G be an arbitrary discrete group and $\mathbb{Z}G$ its group ring. We denote by k an arbitrary commutative ring and by R an arbitrary ring. All CW-complexes we consider are connected admissible ones [6] and [7]. For a CW-complex X, $\widetilde{X}$ will denote the universal covering of X.

In this section, we introduce terminologies and notations, and review many well known facts about various topics, which we will use throughout the paper. For more details, we recommend each reference.

1. ([6]) A chain complex of R-modules $\mathcal{P}=(P_i)$ is said to be of finite type if each P_i is finitely generated projective. If, in addition, $P_i=0$ for sufficiently large i, then $\mathcal{P}=(P_i)$ is said to be finite. A chain complex of R-modules $\mathcal{C}$ is said to be of type FP_∞ or FP over R if there is a weak equivalence $f:\mathcal{P}\to\mathcal{C}$, i.e., $f_*:H_*(\mathcal{P})\to H_*(\mathcal{C})$ is an isomorphism, with $\mathcal{P}$ finite type or finite, respectively. For a G-CW-complex X, and (G,X) or X is said to be of type FP_∞ or FP over k if $C_*(X,k)$ is of type FP_∞ or FP over kG, respectively, where $C_*(X,k)$ is a the cellular chain complex of

kG-modules. In case X is a point, $C_*(X,k)$ is k, with trivial G-action, concentrated in dimension 0. Thus a point with G-action is of type FP_∞ or FP over k if and only if k is a kG-module of type FP_∞ or FP over k if and only if k admits a finitely type or finite projective resolution, in which case we say that G is of type FP_∞ or FP over k, respectively. A group G is said to be of type VFP over k if some subgroup of finite index is of type FP over k.

2. ([5], [6] and [7]) A group G is said to be of finite homological type if (i) $\mathrm{vcd}G<\infty$ and (ii) for every $\mathbb{Z}G$-module M which is finitely generated as an abelian group, $H_i(G,M)$ is finitely generated for all i. The groups of type VFP over $\mathbb{Z}$ are examples of groups of finite homological type. For a torsion-free group G of finite homological type, the Euler characteristic $\chi(G)$ is defined by $\chi(G):=\sum_i(-1)^i\mathrm{rk}_{\mathbb{Z}}(H_i(G))$. For a group G of finite homological type, the Euler characteristic $\chi(G)$ is defined by $\chi(G):=\chi(G')/|G:G'|$, where G' is a torsion-free subgroup of finite index. For an admissible G-CW-complex such that (i) every G_σ is of finite homological type and (ii) X has finitely many cells mod G, i.e., X is cocompact, the equivariant Euler characteristic of X is defined by $\chi_G(X):=\sum_{\sigma\in\mathcal{E}}(-1)^{\dim\sigma}\chi(G_\sigma)$, where $\mathcal{E}$ is a set of representatives for the cells of X mod G.

3. ([12]) For a G-space X, its p-th L^2-Betti number is defined by

$$b(X):=\dim_{\mathcal{N}(G)}(H_p^G(X,\mathcal{N}(G))),$$

where $H_p^G(X,\mathcal{N}(G))$ is the homology of the $\mathcal{N}(G)$-chain complex $\mathcal{N}(G)\otimes_{\mathbb{Z}G} C_*^{\mathrm{sing}}(X)$. The L^2-Euler characteristic of a G-space X is defined by $\chi^{(2)}(X):=\sum_{p\geq 0}(-1)^p b_p^{(2)}(X)$ provided that $h^{(2)}(X):=\sum_{p\geq 0} b_p^{(2)}(X)<\infty$. A G-space X is called L^2-finite if $h^{(2)}(X)<\infty$. Thus the condition of being L^2-finite ensures that $\chi^{(2)}(X)<\infty$. For any discrete group G its p-th L^2-Betti number by $b_p^{(2)}(G):=b_p^{(2)}(EG)$, where EG is the classifying space for free G-action. The L^2-Euler characteristic of G is defined by $\chi^{(2)}(G):=\chi^{(2)}(EG)$ provided that $h^{(2)}(G):=h^{(2)}(EG)<\infty$.

4. ([4], [6] and [7]) Let F be a finitely generated free R-module, and $\alpha: F\to F$ an endomorphism. The Hattori-Stallings rank of α is defined by $\mathrm{HS}_R(\alpha)=\sum\overline{\alpha}_{ii}$, where $[\alpha_{ij}]$ is the matrix of α relative to a basis of F, and $\overline{\alpha}_{ii}\in T(R)=R/[R,R]$. The Hattori-Stallings rank of F is defined by $\mathrm{HS}_R(F):=\mathrm{HS}_R(1_F)$. For a finitely generated projective R-module and $\alpha: P\to P$ an endomorphism, define $\mathrm{HS}_R(\alpha):=\mathrm{HS}_R(i\alpha\pi)$, where $i: P\to F$, $\pi: F\to P$, and $\pi i=1_P$. The Hattori-Stallings rank of P is defined by $\mathrm{HS}_R(P):=\mathrm{HS}_R(1_P)$. For a group ring kG and a finitely generated projective kG-module P, it can be seen that $\mathrm{HS}_{kG}(P)=\sum_{[s]\in[G]}\mathrm{HS}_{kG}(P)(s)\cdot[s]\in\bigoplus_{[G]}k$, where $[G]$ is a set of representatives for the conjugacy classes of elements of G. For a G-CW-complex X of type FP, the complete Euler characteristic $E_G(X,k)$ is defined by $E_G(X,k):=\sum_{i=1}^n(-1)^i\mathrm{HS}_{kG}(P_i)$, where P_* is a finite chain complex of projective kG-modules weak equivalent to $C_*(X,k)$. Then $E_G(X,k)$ is

a finite linear combination of the conjugacy classes $[s]$ of elements of G. Denote by $E_G(X,k)(s)$ the coefficient of the conjugacy class $[s]$ of an element $s{\in}G$. The equivariant Euler characteristic of X over k is defined by $e_G(X,k){:=}E_G(X,k)(1)$. For a group G of type FP over k, the complete Euler characteristic $E(G,k)$ is defined by $E(G,k){:=}E_G(\mathrm{point},k){=}\sum_{i=1}^{n}(-1)^i\mathrm{HS}_{kG}(P_i)$, where $0{\to}P_n{\to}P_{n-1}{\to}\ldots{\to}P_0{\to}k{\to}0$ is a finite projective resolution of k over kG. For a group G of type FP over k, define $e(G,k)$ by $E(G,k)(1)$, which is the same as the Euler characteristic of G in the sense of [3] or [9]. If k is the integer ring $\mathbb{Z}$, then we will suppress k from the notations above. The weak Bass conjecture (for $\mathbb{Z}G$) predicts that $\sum_{[s]\in[G]}\mathrm{HS}_{\mathbb{Z}G}(P)(s){=}\mathrm{HS}_{\mathbb{Z}G}(P)(1)$. It is known that it suffices to consider only finitely presented groups in the weak Bass conjecture, and the weak Bass conjecture holds for a finitely presented group G if and only if $\chi^{(2)}(\widetilde{Y}){=}\chi(Y)$ for any finitely dominated CW-complex Y with $\pi_1(Y){=}G$.

3. Main results

Definition 3. For a G-CW-complex X, define the *equivariant L^2-Euler characteristic* $\chi_G^{(2)}(X)$ of X by $\chi_G^{(2)}(X){:=}\sum_{\sigma\in\mathcal{E}}(-1)^{\dim\sigma}\chi^{(2)}(G_\sigma)$ if it is defined, where $\mathcal{E}$ is a set of representatives for the cells of X mod G.

Proposition 4. *Let X be a G-CW-complex such that G_σ is amenable for each $\sigma{\in}\mathcal{E}$, where $\mathcal{E}$ is a set of representatives for the cells of X mod G. If $\chi_G^{(2)}(X)$ and $\chi^{(2)}(X)$ are defined, then $\chi_G^{(2)}(X){=}\chi^{(2)}(X){<}\infty$.*

Proof. Note that for any amenable group H, $b_p^{(2)}(H){=}0$ for all $p{\geq}1$ and thereby $\chi^{(2)}(H){=}\frac{1}{|H|}$, where $\frac{1}{|H|}$ is defined to be zero if H is infinite [12, Theorems 6.54 and 6.73]. Since G_σ is amenable for any $\sigma{\in}\mathcal{E}$, it follows from the Generalized Euler-Poincarè formula [12, Theorem 6.80 (1)] that

$$\chi_G^{(2)}(X)=\sum_{\sigma\in\mathcal{E}}(-1)^{\dim\sigma}\chi^{(2)}(G_\sigma)=\sum_{\sigma\in\mathcal{E}}(-1)^{\dim\sigma}\frac{1}{|G_\sigma|}=\chi^{(2)}(X). \quad \square$$

Lemma 5. (Cf. [12, Exercise 6.20]) *Let X be a contractible G-CW-complex with $\chi^{(2)}(X){<}\infty$. If each isotropy subgroup G_σ is finite or satisfies $b_p^{(2)}(G_\sigma){=}0$ for $p{\geq}0$, then $\chi^{(2)}(X){=}\chi^{(2)}(G)$.*

Proof. Note that $EG{\times}X$ is a model for EG. From [12, Theorem 6.54] it follows that $b_p^{(2)}(X){=}b_p^{(2)}(G)$ for $p{\geq}0$. Hence $\chi^{(2)}(X){=}\chi^{(2)}(G)$. $\square$

Let X be a finite dimensional virtually free G-CW-complex. Suppose that a G-CW-complex X is of finite homological type in the sense of [6], i.e., X satisfies that $H_i(X/G')$ is finitely generated for each subgroup G' of finite index which acts freely on X. Then a homologically defined equivariant Euler characteristic $\overline{\chi}_G(X)$ is defined by $\overline{\chi}_G(X){:=}\chi(X/G')/|G{:}G'|$, where $\chi(X/G'){:=}\sum_i(-1)^i\mathrm{rk}_{\mathbb{Z}}(H_i(X/G'))$. Brown denote this by $\chi_G(X)$ in [6]. It is known that this is well-defined, i.e., it is independent of the choice of G' [6].

Lemma 6. *Let X be a finite dimensional virtually free G-CW-complex of finite homological type. Then $\overline{\chi}_G(X){=}\chi_G(X)$.*

Proof. Let G' be a subgroup of finite index which acts freely on X. Since G' acts freely on X, it is known that $H_*^{G'}(X){\cong}H_*(X/G')$, where $H_*^{G'}(X)$ is the equivariant homology of X [7, Proposition VII.7.8]. Thus we see that $\chi(X/G'){=}\widetilde{\chi}_{G'}(X)$, where $\widetilde{\chi}_{G'}(X){:=}\sum_i(-1)^i\mathrm{rk}_{\mathbb{Z}}(H_i^{G'}(X))$. Since G' acts freely on X, it is known that $\chi_{G'}(X){=}\widetilde{\chi}_{G'}(X)$, [7, Proposition IX.7.3 (c)]. Thus we conclude from [7, Proposition IX.7.3 (b′)] that

$$\overline{\chi}_G(X)=\chi(X/G')/|G:G'|=\chi_{G'}(X)/|G:G'|=\chi_G(X). \quad \square$$

In [6, Remark 2.5], Brown mentioned that if X is a finite dimensional virtually free G-CW-complex of finite homological type and G has a subgroup of finite index which satisfies the weak Bass conjecture, then $\overline{\chi}_G(X){=}e_G(X)$. Using Lemma 6, we have the following result.

Theorem 7. *Let X be a finite dimensional virtually free G-CW-complex of type FP_∞ over $\mathbb{Z}$. If G has a subgroup of finite index which satisfies the weak Bass conjecture, then $\chi_G(X){=}e_G(X)$.*

Proof. Let G' be a subgroup of finite index in G which satisfies the weak Bass conjecture. We may assume that G' acts freely on X. Then it follows from the argument of the proof of [6, Proposition 2.4] that (G', X) is of type FP over $\mathbb{Z}$ and $H_i(X/G')$ is finitely generated for all i. Note from [7, IX.(1.1)] and [7, IX.(4.3)] that

$$\begin{aligned}\chi(X/G') &= \sum_i(-1)^i\,\mathrm{rk}_{\mathbb{Z}}(H_i(X/G')) = \sum_i(-1)^i\,\mathrm{rk}_{\mathbb{Z}}(C_i(X/G'))\\ &= \sum_i(-1)^i\,\mathrm{rk}_{\mathbb{Z}}(C_i(X)\otimes_{\mathbb{Z}G'}\mathbb{Z}) = \sum_i(-1)^i\,\varepsilon(C_i(X)) = \sum_t E_{G'}(X)(t).\end{aligned}$$

Since the weak Bass conjecture holds for G', it follows that $\sum_{[t]\in[G]\setminus[1]}\mathrm{HS}_{\mathbb{Z}G'}(P)(t){=}0$ and so $\chi(X/G'){=}e_{G'}(X)$. Hence by Lemma 6 and [7, Proposition IX.4.1], we see that

$$\chi_G(X)=\overline{\chi}_G(X)=\chi(X/G')/|G:G'|=e_{G'}(X)/|G:G'|=e_G(X). \quad \square$$

Definition 8. A group G is said to be $L^{(2)}$-*good* over k if the following are satisfied:

(a) Up to conjugacy G has only finitely many elements of finite order.

(b) For each $s \in G$ of finite order, $C_G(s)$ is of type FP over k.

(c) $E(G,k)(s) = \begin{cases} \chi^{(2)}(C_G(s)) & \text{if } s \text{ has finite order,} \\ 0 & \text{if } s \text{ has infinite order.} \end{cases}$

Example 9. Note that if a group H is of type FP over $\mathbb{Q}$ or $\mathbb{C}$, then $\chi^{(2)}(H) = e(H) < \infty$ (cf. [8, Lemma 2.1]). Thus every good group over $\mathbb{Q}$ or $\mathbb{C}$ in the sense of [6] is $L^{(2)}$-good over $\mathbb{Q}$ or $\mathbb{C}$. It is known from [8, Theorems 4.2 and 4.4] and [10] that there are many groups satisfying the condition (c) over $\mathbb{C}$ in Definition 8. Note also that if a group G admits a cocompact $\underline{E}G$ which is the classifying space for proper G-action, then G has only finitely many elements of finite order up to conjugacy (cf. [11]). Thus if a group G admits a cocompact $\underline{E}G$, then G is $L^{(2)}$-good over $\mathbb{C}$.

Theorem 10. *Let X be a cocompact G-CW-complex such that each isotropy subgroup G_σ is $L^{(2)}$-good over k. Then the following hold*:

(a) *Up to conjugacy G has only finitely many elements of finite order which have fixed points in X.*

(b) *For each $s \in G$ of finite order, $C_G(s)$-CW-complex X^s is of type FP over k.*

(c) *For each $s \in G$,* $E_G(X,k)(s) = \begin{cases} \chi^{(2)}_{C_G(s)}(X^s) & \textit{if } s \textit{ has finite order,} \\ 0 & \textit{if } s \textit{ has infinite order,} \end{cases}$

Proof. It can be proved by the same argument of the proof of [6, Theorem 3.1]. □

Theorem 11. *Let X be a cocompact proper G-CW-complex. Then*

$$\chi(X/G) = \sum_{[s]\in[G]} \chi^{(2)}_{C_G(s)}(X^s) = \sum_{[s]\in\mathcal{F}(G)} \chi^{(2)}_{C_G(s)}(X^s).$$

Proof. Since X is cocompact and proper, it follows that each $C_i(X,\mathbb{Q})$ is a finitely generated projective $\mathbb{Q}G$-module (cf. [1, Lemma 6]) and

$$H_*(G, C_*(X,\mathbb{Q})) \cong H^G_*(X,\mathbb{Q}) \cong H_*(X/G,\mathbb{Q})$$

(cf. [1] and [7, Exercise VII.7.2]). It is well-known that $\dim_{\mathbb{Q}}(\mathbb{Q}\otimes_{\mathbb{Q}G} P) = \sum_{s\in[G]} \mathrm{HS}_{\mathbb{Q}}(P)(s)$ for any finitely generated projective $\mathbb{Q}G$-module P (cf. [7, IX.4.(4.3)]). Note that every finite group is $L^{(2)}$-good over $\mathbb{Q}$. Thus it follows from Theorem 10 that

$$\begin{aligned}\chi(X/G) &= \sum_{i\geq 0}(-1)^i \dim_{\mathbb{Q}} H_i(G, C_*(X,\mathbb{Q})) = \sum_{i\geq 0}(-1)^i \dim_{\mathbb{Q}} (\mathbb{Q}\otimes_{\mathbb{Q}G} C_i(X,\mathbb{Q})) \\ &= \sum_{i\geq 0}(-1)^i \sum_{s\in[G]} \mathrm{HS}_{\mathbb{Q}}(C_i(X,\mathbb{Q}))(s) = \sum_{s\in[G]} E_G(X,\mathbb{Q})(s) \\ &= \sum_{s\in[G]} \chi^{(2)}_{C_G(s)}(X^s) = \sum_{[s]\in\mathcal{F}(G)} \chi^{(2)}_{C_G(s)}(X^s). \quad \square\end{aligned}$$

Using Theorem 11 we have the following corollary, which is well-known as Atiyah's theorem [2] and [4].

Corollary 12. *For a finite CW-complex Y, $\chi(Y)=\chi^{(2)}(\widetilde{Y})$.*

Proof. Since $\widetilde{Y}$ is a free $\pi_1(Y)$-CW-complex, the result follows immediately from Theorem 11. □

The following corollary is the formula of Atika appeared in [1, Theorem 1].

Corollary 13. *Let X be a cocompact proper G-CW-complex. Then*

$$\chi(X/G) = \sum_{[s]\in\mathcal{F}(G)} \chi_{C_G(s)}(X^s).$$

Proof. Note that for any finite group H, $\chi^{(2)}(H)=\chi(H)$. Thus it is clear that if Y is a cocompact proper G-complex, then $\chi^{(2)}_G(Y)=\chi_G(Y)$. Hence the result follows from Theorem 11. □

Using Theorem 11 we have the following corollary, which also follows from [8, Corollary 4.5].

Corollary 14. *Let G be a group which admits a cocompact G-CW-model for $\underline{E}G$. Then*

$$\chi^{(2)}(G) = \sum_{[s]\in[G]} \chi^{(2)}(C_G(s)) = \sum_{[s]\in\mathcal{F}(G)} \chi^{(2)}(C_G(s)).$$

Proof. Since G admits a cocompact G-CW-model for $\underline{E}G$, it follows that $\chi^{(2)}(G)$ is defined. Let X be a cocompact G-CW model for $\underline{E}G$. From Lemma 5 it follows that $\chi^{(2)}(X)=\chi^{(2)}(G)$. Let $s\in G$ be an element of finite order. Since X is a cocompact G-CW model for $\underline{E}G$, it follows that X^s is a contractible $C_G(s)$-CW-complex. Put $H=C_G(s)$. Since H_σ is a subgroup of G_σ, it is finite for every $\sigma\in X^s$. Thus X^s is a contractible proper $C_G(s)$-CW-complex. Hence $\chi^{(2)}(X^s)=\chi^{(2)}_{C_G(s)}(X^s)=\chi^{(2)}(C_G(s))$ by Lemmas 4 and 5. The required result now follows from Theorem 11. □

Theorem 15. *Let X be a free G-CW-complex and k a commutative ring which is $\mathbb{Q}$ or $\mathbb{C}$. If $C_*(X)$ is chain homotopy equivalent to a chain complex of finitely generated projective $\mathbb{Z}G$-modules of finite length, then*

$$\chi(X/G) = \sum_{[s]\in[G]} E_G(X,k)(s).$$

Moreover, if X satisfies the equation over $\mathbb{Q}$ or $\mathbb{C}$ in Question 2, then

$$\chi(X/G) = \chi^{(2)}(X).$$

Proof. Let Q_* be a chain complex of finitely generated projective $\mathbb{Z}G$-modules of finite length which is chain homotopy equivalent to $C_*(X)$. Define $P_* := Q_* \otimes \mathbb{Q}$. Since G acts freely on X, it follows that $C_*(X/G) \cong C_*(X) \otimes_{\mathbb{Z}G} \mathbb{Z}$. Thus we have

$$\begin{aligned}\chi(X/G) &= \sum_{i\geq 0}(-1)^i \dim_{\mathbb{Q}} H_i(X/G,\mathbb{Q}) = \sum_{i\geq 0}(-1)^i \dim_{\mathbb{Q}} H_i(C_*(X)\otimes_{\mathbb{Z}G}\mathbb{Q})\\ &= \sum_{i\geq 0}(-1)^i \dim_{\mathbb{Q}} H_i(Q_*\otimes_{\mathbb{Z}G}\mathbb{Q}) = \sum_{i\geq 0}(-1)^i \dim_{\mathbb{Q}} H_i(P_*\otimes_{\mathbb{Q}G}\mathbb{Q})\\ &= \sum_{i\geq 0}(-1)^i \dim_{\mathbb{Q}} (P_i\otimes_{\mathbb{Q}G}\mathbb{Q}) = \sum_{i\geq 0}(-1)^i \sum_{[s]\in[G]} \mathrm{HS}(P_i,\mathbb{Q})(s)\\ &= \sum_{[s]\in[G]} E_G(X,\mathbb{Q})(s)\end{aligned}$$

Moreover, if X satisfies the equation over $\mathbb{Q}$ in Question 2. Note that $\chi^{(2)}(X)$ is defined. Since G acts freely on X, we have

$$\chi(X/G) = \sum_{[s]\in\mathcal{F}(G)} \chi^{(2)}(X^s) = \chi^{(2)}(X).$$

By the same argument in the above, we conclude that the result also holds for the case of $k=\mathbb{C}$. □

Corollary 16. *Let Y be a finitely dominated CW-complex with $\pi_1(Y)=G$. If $\widetilde{Y}$ satisfies the equation over $\mathbb{Q}$ or $\mathbb{C}$ in Question 2, then $\chi(Y)=\chi^{(2)}(\widetilde{Y})$.*

Proof. From the well-known result of Wall ([7, Remark after Proposition VIII.6.4]) it follows that $C_*(\widetilde{Y})$ is chain homotopy equivalent to a chain complex of finitely generated projective $\mathbb{Z}\pi_1(Y)$-modules of finite length. Hence the result follows immediately from Theorem 15. □

Corollary 17. *Let G be a finitely presented group. Suppose that for any finitely dominated CW-complex Y with $\pi_1(Y)=G$, $\widetilde{Y}$ satisfies the equation over $\mathbb{Q}$ or $\mathbb{C}$ in Question 2. Then the weak Bass conjecture holds for G.*

Proof. This follows from [4, Lemma 8.1] and Corollary 16. □

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