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Hilbert's idea of a physical axiomatics: the analytical apparatus of quantum mechanics

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Abstract

We discuss the Hilbert program for the axiomatization of physics in the context of what Hilbert and von Neumann came to call the analytical apparatus and its conditions of reality. We suggest that the idea of a physical logic is the basis for a physical mathematics and we use quantum mechanics as a paradigm case for axiomatics in the sense of Hilbert. Finite probability theory requires finite derivations in the measurement theory of QM and we give a polynomial formulation of local complementation for the metric induced on the topology of the Hilbert space. The conclusion hints at a constructivist physics.

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1 Introduction (the consistency of physical theories)

Hilbert's idea of a physical axiomatics is introduced in his 6th problem in his 1900 list. It is the axiomatization of probability and mechanics, he says, that should concern the mathematician who wishes to secure the foundations of physics as rigorously as it is achieved in arithmetic and geometry. In his major work [14, vol. III, pp. 245–387], Kronecker, who had inspired Hilbert in more ways than one, referred to Kirchhoff's mechanism as a model of a scientific theory for its simplicity and completeness, attributes he claimed for his own general arithmetic. The same Kirchhoff furnished to Hilbert a radiation theory for his early work on foundations of physics [10, vol. III, pp. 217–257]. What we call now Kirchhoff's law on the equality between rates of emission and absorption of energy in thermal equilibrium is indeed a good example of a physical domain that should be investigated in view of the consistency of its axioms. One is reminded here that Hilbert had made of this question already in 1900 the sixth problem of his list "The mathematical treatment of the axioms of physics". Hilbert names probability theory and mechanics as the two privileged domains of such interpretations. The central problem in physical theories is still the consistency problem because a fundamental physical theory proceeds like geometry from general axioms to more specific ones and the extension from the first principles to the secondary ones must preserve consistency. Consistency is not a matter of feeling or experimentation, but of logic, Hilbert insists, and the extension of the theory of thermal radiation to elementary optics is possible only on the grounds of consistency.

The problem area under discussion is of no particular interest for our purposes, nor are Hilbert's contributions to relativity theory [10, vol. III, pp. 257–289] since they are mathematical elaborations and only partly foundationally illuminating—Hilbert had also worked on the foundations of the kinetic theory of gases and other occasional physical subjects. The work on (general) relativity theory in particular seems to have been inspired by the groundbreaking inquiries of Weyl, more than by Einstein's original work (see also [18]). Of greater interest to us is the paper written in collaboration with von Neumann and Nordheim "On the Foundations of Quantum Mechanics" [11].

In that paper, we find the clear exhortation to make explicit the concept of probability in order to extract the mathematical content from its mystical (philosophical) gangue, but the main themes are, in my view, associated with the notions of "analytical apparatus" (*analytischer Apparat*) and "conditions of reality" (*Realitätsbedingungen*). Which comes first, the analytical apparatus or conditions of reality, is a matter of foundational outlook and we will see how Hilbert conceived the so-called "physical axiomatics".

Probabilities and their relationships constitute the material we start from. First we have a mathematical probability theory which serves as the basic analytical apparatus for the physical theory; then follows a physical interpretation of the analytical structure and if the basis is fully determined, the analytical structure should be canonical. This is the axiomatic formulation already present in the Hilbertian foundations of geometry and the general argument leaves no doubt as to the permanence of the axiomatic ideal in Hilbert's work on the foundations of physics. What Hilbert seems to strive to is the conception of a categorical mathematical theory with a multiplicity of models; however, not all models would be isomorphic. Nonstandard models point rather to a complete first-order theory that generates a variety of interpretations, but the mathematical structure is generally not first-order. The dilemma of a physical axiomatics or of a "physical logic" opens up numerous avenues of research—see [4, 5] for other Hilbertian routes.

The analytical apparatus or the mathematical formalism is first conjectured and then tested through an interpretation in order to check its adequacy. The two components, analytical apparatus and its physical interpretation, must be sharply distinguished and that separation has the effect that the formalism is stable throughout the variations of its (physical) interpretations where some degree of freedom and arbitrariness cannot be eliminated. However, this is the price to pay for the axiomatization and vague concepts like probability will finally lose their fuzzy character. The conditions of reality for probability will prove to be intrinsically linked with the calculus of Hermitian operators and Hilbert's early theory of integral equations. Thus, the fact that a probability measure is real positive depends on the finiteness of the sum:

$$a_1x_1 + a_2x_2 + \cdots$$

for a linear function. Hilbert's result, which is a building-block of the Hilbert space formalism, was inspired by a similar result of Kronecker on linear forms. Kronecker's influence on Hilbert has also a conservative extension in the foundations of quantum mechanics.

Hilbert's ideas of the foundations of QM have been made to work by von Neumann [18, 19] in the Hilbert space formulation of quantum mechanics, which is the standard formulation of QM. We will explore in the following the continuation of Hilbert's programme in the hands of his followers. We start with a notion which is not found in Hilbert, but can be traced back to von Neumann's foundational work in QM.

2 Quantum mechanics

2.1 Hilbert space

The usual presentation of QM requires the analytical apparatus of Hilbert space as a linear vector space with complex coefficients; among all linear manifolds that constitute a Hilbert space, the closed ones or the subspaces are of special interest for physics (i.e., QM here), since notions like orthogonal vectors, orthogonal complements, projections, etc. can be defined on them. It is a well-known fact that not all linear manifolds are closed [9, p. 22] and that the set of all linear subsets of the infinite-dimensional Hilbert space is not orthocomplementable [12, p. 122]: it is this possibility which I want to exploit, keeping in mind that a Hilbert space is a metric and a topological space. The interesting fact about Hilbert space from a physical point of view is that it permits the definition of orthogonality:

$$(f,g) = 0$$

written $f \perp g$; the orthogonal complement of f, $f^{\perp\perp}$ obeys the Boolean rule $f^{\perp\perp} = f$ and $f^{\perp}$ forms a subspace of $\mathcal{H}$. For QM, it is important to notice that there is a bijection between subspaces and projections, that is, the linear operators E such that $EE^{\perp} = E$ for $E^{\perp}$ the adjoint of E defined by $(E^{\perp})^{\perp} = E$ (if $E^{\perp} = E$, then E is a self-adjoint or Hermitian operator). The spectral theorem states that there is a bijection between self-adjoint operators and spectral measures on (the Borel set of) the real line R^1, and the von Neumann *dogma* states that there is a bijection between self-adjoint operators and the observables of QM. (Von Neumann's dogma has been challenged in 1952 by Wick, Wightman, and Wigner who introduced superselection rules showing that there exist Hermitian operators that do not correspond to observables. However, Park and Margenau argue that there are observables, for example, the noncommuting x and z—components of spin which are not represented by Hermitian operators.) Let us look at the orthogonal complement: we have seen that $f^{\perp\perp} = f$; consequently, the orthogonal complement corresponds to the orthocomplement $(a^-)^- = a$ of a Boolean lattice, where $\leq$ corresponds to $\rightarrow$, a^- corresponds to $\neg a$, $a \cap b$ corresponds to $a \wedge b$, and $a \cup b$ corresponds to $a \vee b$. Orthocomplementation induces an involutive antiautomorphism $(a^{\perp})^{\perp}$ on the field vector space. It is such an antiautomorphism which yields Gleason's important theorem [6] stipulating that any probability measure $\mu(A)$ on the subspaces of $\mathcal{H}$ has the following form:

$$\mu(A) = Tr\big(WP_A\big),$$

where Tr means $TrX = \sum_r(\varphi_r, X\varphi_r)$ for any complete system of normalized orthogonal vectors, P_A denotes the orthogonal projection of A, and W is a Hermitian operator which satisfies

$$W > 0, \quad sTrW = 1, \quad W^2 \leq W.$$

Other spaces, like Banach spaces which lack the restriction of orthogonality, do not seem to be suited to the needs of QM.

The usual formulation of QM requires the analytical apparatus of the Hilbert space $\mathcal{H}$ as a complex vector space [12] with

$$\forall f, g \in \mathcal{H}\big((f+g) \in \mathcal{H}\big), \quad \forall f \in \mathcal{H}\ \forall \lambda \in C(\lambda f \in \mathcal{H})$$

for f and g and a complex coefficient λ with

$$1 \cdot f, \quad \theta + f = f, \quad \theta \cdot f = 0$$

for the null vector θ. The Hilbert space has also a scalar or interior product which is strictly positive. In particular, we have

$$(f, g+h) = (f,g) + (f,h), \quad (f, \lambda g) = \lambda(f,g), \quad (f,g) = (g,f)^*,$$

the complex conjugate with the norm:

$$\|f\|^2 \equiv (f,f) > 0, \quad \text{for } f \neq \theta.$$

The space $\mathcal{H}$ is separable (dense):

$$\forall f \in \mathcal{H},\ \forall \varepsilon > 0,\ \exists f_n \big\| f - f_n \big\| < \varepsilon, \quad \text{for } n = 1, 2, \ldots$$

and complete, that is, any Cauchy sequence:

$$\forall \varepsilon > 0,\ \exists N < i, j\big(\rho\big(x_1, x_j\big)\big) < \varepsilon_0$$

converges

$$\lim_{n\to\infty} \big\| f - f_n \big\| = 0, \quad \text{in } \mathcal{H}.$$

The analytical apparatus consists also of the following physical postulates or axioms:

(1) physical states are represented by state vectors (in $\mathcal{H}$);
(2) there is a bijection between observables and Hermitian operators—von Neumann's *dogma*;
(3) the evolution of the physical system is described by Schrödinger's equation;
(4) the probability to find a particle in a particular position is given by

$$\text{pr}(\underline{r}, t) = \psi^*(\underline{r}, t)\psi(\underline{r}, t) = \big|\psi(\underline{r}, t)\big|^2,$$

where $\underline{r}$ is the position vector and ψ^* is the complex conjugate of ψ:

$$\big(\psi^{\perp}\psi = |\psi|^2\big);$$

(5) the projection postulate which states that immediately after a measurement (i.e., an interaction), the superposition $\sum c_j \sigma_j \alpha_j$ is transformed or reduced to $\sigma_n \alpha_n$.

The fifth postulate for the wave packet reduction characterizes von Neumann's theory of measurement. For instance, the superposition of states $\sum \sigma_j \alpha_j$ is made up of the combined system—the observer and the observed system—and for von Neumann, a measurement projects the system Σ in a state $\sigma_n \alpha_n$ (we neglect the terms of the expansion here). Vectors $\sigma_n \alpha_n$ have a well-defined value since projections are in bijection with the subspaces of the Hilbert space, but the system is no more in a pure state, but in a mixture. Everett's multiverse theory or relative state theory supposes that the superposition is not reduced or projected in a determinate state, but ramifies after an interaction in a multitude of branches each corresponding to a component of the superposition: there would be as many worlds as there are components and the result of measurement would be valid on only one world among a (nondenumerable) infinity of universes. Here is the rub, more irritating than von Neumann's cut (*Schnitt*) between the observed system and the observer: the set of all values of the wave function ψ is C, the set of complex numbers, which has the cardinality $2^{\aleph_0}$; thus, the ramified ψ cannot be measured, for the set of all possible measurements certainly does not exceed $\aleph_0$ and there is no bijection between $\aleph_0$ and $2^{\aleph_0}$. The inconsistency is fatal in the view of Everett's idea that the formalism generates its own interpretation. If the ramification of ψ must have a probabilistic objective content, one is obliged to admit that it cannot emerge from the divergent ramification of nondenumerable probability values, a probability theory being at most σ-additive, that is denumerably convergent. Another example of an inconsistent probability theory of QM is the theory of consistent histories, first formulated by Griffiths [8] and adopted since by some important physicists, Gell-Mann and Hartle, among others. The theory can be considered as a variant of Everett's many-universe or multiverse interpretation with a historical component, since parallel universes can have different histories, that is, temporal sequences of quantum events. In order for a given history to be consistent, it is granted a weakened logical status which forbids, for instance, joining two incompatible events (e.g., spin states a and b of an electron) in a classical conjunction $a \wedge b$. These singular histories must preserve probability measures or σ-additivity for denumerable measures with the help of elementary logical notions such as *modus ponens*, conditional probabilities and counterfactuals, truth, and liability, but the main question is the consistency of consistent histories. Recent work by Goldstein and Page [7], Dowker and Kent [2] tends to show that Griffith's theory is inconsistent with its probabilistic assumptions about consistent histories. From a combinatorial point of view, denumerable or σ-additivity supposes that the decomposition of probability measures covers up inconsistent history subsequences (subsets) as well as consistent but irreconcilable subsequences in the density matrix of consistent histories; in other words, there is no bijection between the $\aleph_0$ sequences and the $2^{\aleph_0}$ subsequences (the power set of all histories), and standard probabilities are lost in the multiplicity of divergent histories (and subhistories). The lesson to be drawn here is perhaps that a paraconsistent logic that accommodates contradictions as well as tautologies can take care of a "quasi-consistency" for the "quasi-classicality" in a mixture of coherent histories in quantum systems and decoherent histories in classical (macroscopic) systems, as quantum decoherence theory seems to indicate, but the term "consistency histories" would nonetheless sound like a misnomer for a theory which makes room for too many divergent histories, as the universal ramification of the wave function would have it in Everett's multiverse interpretation.

2.2 Probabilities

Hilbert (followed in this by von Neumann) introduced the notion of analytical apparatus (*analytische Apparat*) drawn from the general structure of an axiomatic system in physics and he made no mystery of his intention to provide physics with the same kind of axiomatic foundations as geometry. Physical situations must be mirrored in an analytical apparatus, physical quantities are represented by mathematical constructs which are translated back into the language of physics in order to give real meaning to empirical statements. The analytical apparatus is not subjected to change, while its physical interpretation has a variable degree of freedom or arbitrariness. What this means is that the mathematical formalism of a physical theory is a syntactical structure which does not possess a canonical interpretation, the analytical apparatus does not generate a unique model. At the same time, axiomatization helps in clarifying a concept like probability which is thus rescued from its mystical state. It is noteworthy that another pair of renowned mathematicians, Hardy and Littlewood, expressed the same opinion at the same time: "Probability is not a notion of pure mathematics, but of philosophy or physics."

Probabilities had, long before quantum mechanics, been knocking at the door of physics, but Laplace had entitled his work *Essai Philosophique sur les Probabilités* (1814) after having called it *Théorie analytique des probabilités* (1812). Statistical mechanics can certainly count as a forerunner of QM as far as the statistical behavior of a large number of particles is an essential ingredient in the probability theory of quantum-mechanical systems, but even in the work of pioneers like Born and Pauli, probability has entered QM somehow through the backdoor and it seems that it is only reluctantly that Born, for example, has admitted the idea of probability. Later work by Kolmogorov on the axiomatic foundations of elementary probability theory or von Mises and Reichenbach on the frequentist interpretation of probability will achieve some measure of success, but it is the historical advent of a rigorous formalization of the notion of probability as it occurs in quantum physics which has not been sufficiently stressed.

If probability has evidently a multiple application in QM, it remains that it is mainly a mathematical notion. Von Neumann's work in 1927–1932 focuses on what is called the finiteness of the eigenvalue problem. The point here is that any calculation is finite and since we have only finite results, those must be the products of a finite calculation which is itself made possible only if the analytical apparatus contains the mathematical structures which enable such calculations. Such a formalism is the complex Hilbert space with

$$|\psi|^2 \in L^2(\mu),$$

where μ is a real positive measure on the functional space L^2 (i.e., the equivalence class of square-integrable functions). The integral

$$\int |\psi|^2 d\mu$$

is finite, which is equivalent to the fact that, in the theory of bounded quadratic forms, the sum:

$$K(x,x) = \sum_{p,q=1}^{\infty} k_{pq} x_p x_q$$

of all sequences $x_1, x_2, \ldots$ (of complex numbers) is finite in an orthonormal system of vectors. That mathematical fact, which Hilbert derived in the theory of integral equations in 1907, states that a linear expression:

$$k_1 x_1 + k_2 x_2 + \cdots$$

is a linear function, if and only if the sum of the squares of the coefficients in the linear expression $k_1, k_2, \ldots$ is *finite*. The theorem, inspired by Kronecker's result on linear forms (homogenous polynomials), is the very basis of the Hilbert space formulation of QM. Notice that on the probabilistic or statistical interpretation, the "acausal" interaction between an observed system and an observing system takes place in a given experimental situation and produces a univocal result of finite statistics for real or realized measurements.

In order for real measurements to have real positive probability values, the analytical apparatus must satisfy certain realizability conditions (*Realitätsbedingungen*) as Hilbert and von Neumann put it. For example, orthogonality for vectors, linearity, and hermiticity for functional operators and the finiteness of the eigenvalue problem for Hermitian operators, as in von Neumann's further work *Mathematische Grundlagen der Quantenmechanik*, are such constraints of realizability.

2.3 Logics

The requirements for realizability are not limited to additivity for Hermitian operators—Grete Hermann seems to have been the first one to criticize the requirement on philosophical grounds—but are strictures imposed by the analytical apparatus or the deductive structure of the theory in von Neumann's terminology. In their joint paper of 1936, Birkhoff and von Neumann attempt to define the "logical calculus" of quantum-mechanical propositions associated with projection operators alluded to in von Neumann [18, p. 134]. They are led to denote the orthogonal complement ($\perp$) as the "negative" of an experimental proposition in an orthocomplemented lattice satisfying

(1) $(a^\perp)^\perp = a$,
(2) $a \leq b$, if and only if $b^\perp \leq a^\perp$,
(3) $a \wedge a^\perp = 0$,
(4) $a \vee a^\perp = 1$.

The dual antiautomorphism of period two (or the involutory antiautomorphism of projective geometry) does not, however, uniquely determine complements in a continuous geometry and von Neumann came back to quantum logic in his paper "Quantum Logics" (1961) with the discussion of a continuous geometry without points and whose elements are all the linear subspaces of a given space (more general than a Hilbert space); von Neumann thought that the logic of quantum probabilities (frequencies) could be built upon such a geometry. But here, the probability measures must be infinite in order to be convergent and the probability statements that express those measures are required to have a finite meaning, as Reichenbach claimed for the verifiability theory of his probability theory. Von Neumann was dissatisfied with Hilbert space vector formalism,—but was unable to define a finite probability theory for his abstract projective geometry framework—the type II factor of a modular nonatomic lattice.

In that context, Birkhoff and von Neumann deny the distributive law of logic in favor of a weaker modular identity or orthomodularity:

$$a \leq b \longrightarrow a \vee b(b \wedge c) = (a \vee b) \wedge c,$$

weakened by Jauch and Piron [12] to

$$a \leq b, \quad \text{if and only if } a \text{ and } b \text{ are compatible}$$

(compatibility is an equivalence relation which is symmetric, but not transitive). The underlying logic here is the noncommutativity of operators P_1 and P_2:

$$P_1 P_2 \neq P_2 P_1$$

of which the uncertainty relations are "a direct intuitive explanation", as Heisenberg said.

The quantum logic of Jauch and Piron is another example of an impossibility proof for hidden variables as compatible propositions in the framework of essentially noncompatible quantum-mechanical propositions. Kochen and Specker devised rather a quantum logic for a partial Boolean algebra of commuting (or "commeasurable," as they say, [13, p. 64]) quantum-mechanical observables (or propositions) which are not embeddable in a commutative Boolean algebra—there is no 2-valued homomorphism h from the partial algebra A to the Boolean algebra B with the properties:

(1) $h(a) \llcorner h(a)h(b)$,
(2) $h(\mu a + b) = \mu\big(h(a)\big)\lambda\big(h(b)\big)$,
(3) $h(ab) = h(a)h(b)$,
(4) $h(1) = 1$,

where $\llcorner$ is the relation of commensurability, a, b are elements of A and μ, λ belong to a field of sets K (compare with the relation of compatibility defined in (2) and (4) above).

The fact that the 2-valued propositions form a commutative algebra which does not embed commeasurable quantum-mechanical propositions can be seen as a far-reaching consequence of Gleason's theorem on the measure of the closed subspaces of a Hilbert space [6].

2.4 Local complementation

Even in the case of set complementation (as in the theory of Hilbert spaces), we can have local complementation. Consider Hilbert space as a metric and a topological space; D is in this case the set of subspaces of the Hilbert space and E is obtained by local complementation; E is the "location" of the local observer. We will see that the Hilbert space can make room for a notion of local observer: the observer becomes the (local) complement of the observable, that is, the closed linear manifolds of the Hilbert space—of course the whole Hilbert space contains all bounded linear transformations (defined on open subsets) and is therefore not orthocomplementable. But here, we obtain nonorthocomplementability in a different way (remember that in a finite-dimensional space, every linear manifold is closed).

Theorem 2.1. *Hilbert space admits the observer through local negation (or complementation)—that is, we do not have orthocomplentation on the whole Hilbert space even in the finite-dimensional case.*

Proof. Let $\mathcal{H}$ be an n-dimensional Hilbert space and let $F^{\perp}$ be the set of closed linear manifolds $f^{\perp}$, $F^{\perp} = F^{-}$, the closure of all f. One can now define the relative complement F^{+} of F^{-} such that $\mathcal{H} - F^{-} = F^{+}$; F^{+} is then an open subset. From the topology, we pass to the metric of $\mathcal{H}$; for the metric of $\mathcal{H}$, a subset A of $\mathcal{H}$ is located (this notion of located subset has been introduced by Brouwer. E. Bishop has put it to use in his *Foundations of Constructive Analysis* [1]), if the distance

$$\forall x \in \mathcal{H}\big[\rho(x, A) \equiv \inf\big\{\rho(x, y) : y \in A\big\}\big]$$

from x to A exists. The metric complement $-A$ of a located subset A is the set:

$$-A \equiv \big\{x : x \in \mathcal{H},\ \rho(x, A) > 0\big\},$$

which is open, since

$$\forall x, y \in \mathcal{H}\big[\rho(x, A) \leq \rho(x, y) + \rho(y, A)\big].$$

Here the observer has a topological and metrical places as the local complement of the closed set of subspaces of $\mathcal{H}$. In order to further constructivize this result, we introduce the topological boundary operator b which is to be interpreted as the boundary between the observable (or observed) and the observer: we have the relations:

$$E = \neg D - b(E), \quad D = \neg E \cup b(D),$$

thus,

$$\neg D(\mathcal{H}) = E(\mathcal{H}) - b\big(E(\mathcal{H})\big).$$

The interior of E, that is, E°, is the complement of the closure of the complement of E and is thus open; we have also

$$E = E^{\circ}.$$

For any x, $D(\neg x)$ means that $x \in E$. So for some a, we have

$$E(a) = D(\neg a) - b\big(D(\neg a)\big).$$

However, the closure of D, that is, D^{-} implies that

$$B\big(D(\neg a)\big) = a^{-} \cap (D - a)^{-}.$$

Hence

$$A^{-} = a \cup b\big(D(\neg a)\big), \quad a^{-} \in E(\mathcal{H}) = a \in D(\mathcal{H}) \cup b\big(a \in D(\mathcal{H})\big),$$
$$a \in D(\mathcal{H}) = a^{-} \in E(\mathcal{H}) - b\big(E(\mathcal{H})\big),$$

which shows that E is disjoint from its boundary, that is, it is open and consequently, the whole Hilbert space $D(\mathcal{H}) \cup E(\mathcal{H})$ is not orthocomplementable, since local complementation excludes $(a^{-})^{-} = a$. (Orthocomplementation requires that $(a^{-})^{-} = a, a^{-} \cap a = \varnothing$ and $a \leq b \leftrightarrow b^{-} \leq a^{-}$.) □

Remark 2.2. The effect of abandoning orthocomplementation amounts to adopting an indefinite metric which may, in fact, be more convenient for some physical theories (e.g., quantum field theory).

2.5 The total Hilbert space

Gleason's theorem says that in a separable Hilbert space of dim ≥ 3, every measure on the closed subspaces has the following form:

$$\mu(A) = Tr\big(WP_A\big),$$

where the trace Tr means $TrX = \sum_R (\varphi_R, X_{\varphi_R})$ for any complete system of normalized orthogonal vectors φ_R, P_A denotes the orthogonal projection of A, and W is a Hermitian operator which satisfies

$$W > 0, \quad TrW = 1, \quad W^2 \leq W.$$

Since the sum for the linear span B over a countable set of orthogonal subspaces A_i:

$$\mu(B) = \sum \mu\big(A_i\big)$$

is finite, μ can be regarded as a real positive measure on the functional space L^2 as we saw above. Gleason's result states that "frame functions" defined on the unit sphere are regular, that is, there exists a self-adjoint operator T defined on the Hilbert space $\mathcal{H}$ such that the frame function f is

$$f(x) \equiv (Tx, x).$$

When the (real) Hilbert space is finite-dimensional, the frame functions are regular, if they are the restriction to the unit sphere of quadratic forms (homogenous polynomials of degree 2)—again in accordance with the Hilbert-Kronecker theorem on the finite sum of the squares of coefficients in a linear expression, but the total Hilbert space containing not only the subspaces (closed varieties), but all the linear varieties is infinite-dimensional and is not orthocomplementable. In view of the fact that complements in the total Hilbert space cannot be uniquely determined, a fact that von Neumann and Birkhoff had noticed, one can introduce a local or relative complement in the lattice of open subsets of $\mathcal{H}$ beyond the closed sequence of subspaces of $\mathcal{H}$. Topologically then, the local complement is an open subset of $\mathcal{H}$ and the topological boundary operator separates the space of the observed system from the space of the observing system, since points on the boundary are neither in A nor in $X - A$ for a given set and its complement in a topological space X. All linear varieties are closed in a finite-dimensional space $\mathcal{H}$ [9], and we have to "open up" that space; we need to locate finitely the relative complement, and a metric to that effect can be defined on the topology [3]. Brouwer has introduced the notion of located subset for subsequences [1]: a subsequence A of B, that is, $A \subset B$, is localized, if there exists a distance ρ (for points x and y) such that

$$\forall x \in \mathcal{H}\big[\rho(x, A) \equiv \inf\big\{p(x, y) : y \in A\big\}\big].$$

The metric local complement $-A$ of the subsequence A is

$$-A \equiv \big\{x : x \in \mathcal{H}, \rho(x, A) > 0\big\}$$

and is open, since

$$\forall x, y \in \mathcal{H}\big[\rho(x, A) \leq \rho(x, y) + \rho(y, A)\big].$$

The notion of local complement with its distance function constitutes the basis of a probability calculus which differs from the classical notions.

2.6 Finite derivation of the local complement

In accordance with Hilbert's result on finite sums for linear expressions, the local complement of our probability calculus is also embedded in a finite form. Instead of Kolmogorov's infinite probability space, we have a finite probability space as in Nelson [15]: a finite probability space is a finite set Ω and a (strictly positive) function pr on Ω such that for $\omega \in \Omega$

$$\sum \mathrm{pr}(\omega) = 1$$

and expectation is defined

$$Ex = \sum x(\omega)\,\mathrm{pr}(\omega)$$

for a random variable x; the probability of an event $A \subseteq \Omega$ is

$$\Pr A = \sum_{\omega \in A} \mathrm{pr}(\omega).$$

Nelson also defines the complementary event as $A^c = \Omega \backslash A$ for all $\omega \in \Omega - A$. This is the Boolean complement which we replace by our local complement $(\Omega - a) + b$ or $(1 - a) + b$. Putting $\bar{a}$ for $1-a$, we introduce polynomials in the following (binomial) form with decreasing powers:

$$\big(\bar{a}_o x + b_0 x\big)^n = \bar{a}_0^n x + n\bar{a}_0^{n-1} x b_0 x + \big[n(n-1)/2!\big]\bar{a}_0^{n-2} x b_0^2 x + \cdots + b_0^n x,$$

where the companion indeterminate x shares the same power expansion. By an easy calculation (on homogenous polynomials that are symmetric, that is, with a symmetric function $f(x,y) = f(y,x)$ of the coefficients),

$$\begin{aligned}
\big(\bar{a}_0 x + b_0 x\big)^n &= \bar{a}_0^n x + \sum_{k=1}^{n-1} (n-1/k-1)\bar{a}_0^{k-1} x + (n-1/k)\bar{a}_0^k x b_0^{n-k} x + b_0^n x \\
&= \sum_{k=1}^{n} (n/k-1)\bar{a}_0^k x b_0^{n-k} x + \sum_{k=0}^{n-1} (n-1/k)\bar{a}_0^k x b_0^{n-k} x \\
&= \sum_{k=0}^{n-1} (n-1/k)\bar{a}_0^{k+1} x b_0^{n-1-k} x + \sum_{k=0}^{n-1} (n-1/k)\bar{a}_0^k x b_0^{n-k} x \\
&= \bar{a}_0 \sum_{k=0}^{n-1} (n-1/k)\big(\bar{a}_0 - 1\big)^k b_0^{n-1-k} x + \sum_{k=0}^{n-1} (n-1/k)\bar{a}_0^k x\big(b_0 - 1\big)^{n-1-k} c \\
&= \big(\bar{a}_1 x + b_1 x\big)\big(\bar{a}_1 x + b_1 x - 1\big)^{n-1},
\end{aligned}$$

and continuing by descent and omitting the x's, we have

$$\begin{aligned}
&\big(\bar{a}_2 + b_2\big)\big(\bar{a}_2 + b_2 - 2\big)^{n-2} \\
&\qquad\vdots \\
&\big(\bar{a}_{n-2} + b_{n-2} + \bar{a}_{n-2} + b_{n-2} - (n-2)\big)^{n-(n-2)} \\
&\big(\bar{a}_{n-1} + b_{n-1} + \bar{a}_{n-1} + b_{n-1} - (n-1)\big)^{n-(n-1)} \\
&\big(\bar{a}_n + b_n\big)\big(\bar{a}_n + b_n\big)^{n-n}.
\end{aligned}$$

Applying descent again on $(\bar{a}_n + b_n)$, we obtain

$$\big(\bar{a}_0 + b_0\big),$$

or reinstating the x's

$$\big(\bar{a}_0 x + b_0 x\big).$$

Remembering that

$$(\bar{a}x + bx)^n_{k<n} = \sum_{k+m=n} \big(k + m/k\big)\bar{a}^k b^m x^n,$$

we have

$$(\bar{a}x + bx)^{n+m=n}_{k<n} = \prod_{k+m=n} (k, m) = 2^n,$$

or more explicitly,

$$\sum_{i=0}^{m+n} c_1 x^{m+n-1} = \bar{a}_0 x \cdot b_0 x \prod_{i=1}^{m+n} \big(1 + c_i x\big) = 2^n,$$

where the product is over the coefficients (with indeterminates) of convolution of the two polynomials (monomials) a_0 and b_0. The descent that we have applied here is the arithmetic finite descent from a given n to the first ordinal (0 or 1). The finite descent (or derivation) is applied to a probability calculus, but it could be applied also to a propositional calculus as in Kochen and Specker [13]. The interesting difference is that the calculus is no more classical nor Boolean, but intuitionistic, since the local complement corresponds to intuitionistic implication:

$$a \longrightarrow b = \mathrm{In}\big((X - a) \cup b\big),$$

and the algebra of propositions (or events) is not even a partial Boolean algebra, but a Brouwerian lattice, that is a partially ordered set with two binary operations (meet and join) and a relative "pseudocomplement":

$$a \longrightarrow b = a \cap c \leq b$$

for c the greatest element different from a. The Brouwerian lattice is isomorphic to a Heyting algebra, which is the algebraic structure corresponding to the intuitionistic logic of propositions. The open subsets of a topological space also determine a Brouwerian lattice.

Kolmogorov's axiomatization of the probability calculus is based on a triple $\langle \Omega, \Sigma, \mu \rangle$ for μ a probability measure on the σ-algebra Σ of subsets or events A of a probability space Ω:

(1) $A \in \Omega$;
(2) $\forall A \in \Omega \rightarrow \bigcup_{j=1}^{x} A_i \in \Omega$;
(3) $A' = \Omega - A$ for A' the complement of A

with $0 \leq \mu(A) \leq 1$ for $A \in \Omega$ and $\mu(0) = 0$, $\mu(\Omega) = 1$; countable or σ-additivity means that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

for $A_i \cap A_j = \varnothing$, if $i \neq j$. Properties of the Boolean complementation of probabilities are summarized as follows:

$$\begin{aligned} &\big(A'\big)' = A, \quad A \cup A' = \Omega, \quad A \cap A' = \varnothing, \\ &(A \cup B)' = A' \cap B', \quad (A \cap B)^{-} = \big(A' \cup B'\big). \end{aligned}$$

For local complementation, we have $(A')' \neq A$ for

$$C = A \Longrightarrow B = \mathrm{In}\big((X - a) \cup B\big),$$

where In is the set of interior points, and A, B, C are open subsets of a topological space X (C is here the largest open subset distinct from A). This relative "pseudo-complement" is the main distinctive feature of a Brouwerian lattice. We see that probabilities according to the local complement do not satisfy the Boolean equality or duality and make it possible to adjoin an intermediary or included third, that is the open subset B here. The expectation value:

$$\mathrm{Exp}(A) \int A \, d\mu$$

for a dispersion ΔA is given by

$$\mathrm{Var}(A) = \mathrm{Exp}\big[A - \mathrm{Exp}(A)\big]^2,$$

and it is easy to see that in order to take into account the local complement, we must have

$$\Delta A^2 = \mathrm{Exp}\left[A - \mathrm{Exp}(A) + \mathrm{Exp}(\neg A)\right]^2,$$

where $\neg A$ is the local complement of the space of events. For noninteractive systems and dispersion-free states, the local complement has a negligible effect on the statistics, but in quantum interactive systems (where a measurement is *some kind* of interaction between an observed system and an observing system), the statistical *weight* of the local complement cannot be ignored, although it is confined and *indeterminate*. The indeterminacy has something to do with the undeterminacy or uncertainty relations, but only indirectly in that the local complement acquires a determinate value upon measurement and only within actual measurement results as a relative complementation of probabilities.

3 Conclusion

It is possible to reconstruct the EPR argument for "elements-of-reality" without Bell's inequalities by appealing to indirect measurements or *reductio ad absurdum* arguments. The contextuality and nonlocality appear as features of a realist interpretation incompatible with QM to the extent that undefined values for observables of QM become definite for "elements-of-reality" in the EPR reconstruction. The simple case of the spin angular momentum will suffice for our argument.

The fundamental relationship for the x, y, z components of spin along the $x-$, $y-$, $z-$ axes is (S being the spin observable and squaring)

$$S_x^2 + S_y^2 + S_z^2 = S^2.$$

Direct measurement of the z component excludes attributing definite values to the other components, but realism specifically supposes that there are *independent* "elements-of-reality" that can be subjected to *indirect* measurements

Léon Rosenfeld, a harsh defender of the Copenhagen interpretation, summarizes the instrumentalist view:

> a phenomenon is therefore a process (endowed with the characteristic quantal wholeness) involving a definite type of interaction between the system and the apparatus [17, p. 82].

Thus, the probability calculus must be inherent to the quantum-mechanical measurement process. The tacit assumption of a classical probability structure must be questioned and a better adjustment of the analytical apparatus and its physical interpretation remains a lasting problem for foundational research. The completeness of quantum mechanics, despite Bell's pronouncement, is such a problem, may it be of a quantum-logical or mathematical nature. The topological solution suggested here has instrumentalist overtones, but it aims essentially at explaining quantum mechanics as the physics of "local" experiments. Although the metaphysics of wholeness or nonseparability is not totally dispelled by such an attempt, it might provide the sceptic with some good reasons not to despair about the so-called incompleteness of quantum mechanics in his search for reality. As M. Redhead [16, p. 45] puts it, on Bohr's complementarity interpretation, the value of an observable Q, when the state of the system is not an eigenstate of Q, is undefined or "meaningless" and one cannot impugn such an interpretation by denying a locality principle which says that a previously undefined value for an observable cannot be defined by measurements performed "at a distance". On Redhead's reckoning, the charge of incompleteness cannot be levelled against Bohr's view, unless staunch realism and nonconstructive *reductio ad absurdum* arguments are invoked, but if the undeterminacy or uncertainty principle has given rise to a noncommutative geometry and analysis (A. Connes), Bohr's Complementarity Principle could yield on a par a nonclassical logic and probability calculus. And this militates for a proportionate anti-realist or, as I prefer to say, a constructivist (instrumentalist) interpretation of QM.

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