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Matched pairs of generalized Lie algebras and cocycle twists

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Abstract

We introduce the conception of matched pairs of (H,β)-Lie algebras, and construct an (H,β)-Lie algebra through them. We prove that the cocycle twist of a matched pair of (H,β)-Lie algebras can also be matched.

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1 Introduction and preliminaries

A generalized Lie algebra in the comodule category of a cotriangular Hopf algebra which included Lie superalgebras and Lie color algebras as special cases has been studied by many authors (see [1, 2] and the references therein).

On the other hand, there is a general theory of matched pairs of Lie algebras which was introduced and studied by Majid in [4]. It says that we can construct a Lie algebra through a matched pair of Lie algebras.

In this note, we introduce the conception of matched pairs of (H,β)-Lie algebras, and construct an (H,β)-Lie algebra through them. Furthermore, we prove that the cocycle twist of a matched pair of (H,β)-Lie algebras can also be matched.

We now fix some notation. Let H be a Hopf algebra, and write the comultiplication $\Delta : H \to H \otimes H$ as $\Delta(h) = \sum h_1 \otimes h_2$. When V is a left H-comodule with coaction $\rho : V \to H \otimes V$, we write $\rho(v) = \sum v_{(-1)} \otimes v_{(0)}$. We frequently omit the summation sign in the following context.

A pair (H,β) is called a cotriangular Hopf algebra if H is a Hopf algebra and $\beta : H \otimes H \to k$ is a convolution-invertible bilinear map satisfying, for all $h, g, l \in H$,

(CT1) $\beta(h_1, g_1)g_2h_2 = \beta(h_2, g_2)h_1g_1$;
(CT2) $\beta(h, gl) = \beta(h_1, g)\beta(h_2, l)$;
(CT3) $\beta(hg, l) = \beta(g, l_1)\beta(h, l_2)$;
(CT4) $\beta(h_1, g_1)\beta(g_2, h_2) = \varepsilon(g)\varepsilon(h)$.

A map satisfying (CT2)–(CT4) is called a skew-symmetric bicharacter. We always assume H is (co)commutative if necessary.

A convolution invertible map $\sigma : H \otimes H \to k$ is called a left cocycle if, for $h, g, l \in H$,

$$\sigma(h_1, g_1)\sigma(h_2g_2, l) = \sigma(g_1, l_1)\sigma(h, g_2l_2)$$

and a right cocycle if

$$\sigma(h_1g_1, l)\sigma(h_2, g_2) = \sigma(h, g_1l_1)\sigma(g_2, l_2)$$

Definition 1.1 (see [1])**.** Let (H, β) be a cotriangular Hopf algebra. An (H, β)-Lie algebra is a left H-comodule $\mathcal{L}$ together with Lie bracket $[\cdot, \cdot] : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$ that is an H-comodule morphism satisfying, for all $a, b, c \in \mathcal{L}$,

(1) β-anticommutativity:

$$[a, b] = -\beta(a_{(-1)}, b_{(-1)})[b_{(0)}, a_{(0)}]$$

(2) β-Jacobi identity:

$$[[a, b], c] + \beta(a_{(-1)}, b_{(-1)}c_{(-1)})[[b_{(0)}, c_{(0)}], a_{(0)}] + \beta(a_{(-1)}b_{(-1)}, c_{(-1)})[[c_{(0)}, a_{(0)}], b_{(0)}] = 0$$

When $H = k\mathbb{Z}_2$, $\beta(x, y) = (-1)^{xy}$ for all $x, y \in \mathbb{Z}_2$, this is exactly Lie superalgebra. When $H = kG$, where G is an Abelian group with a bicharacter $\beta : G \times G \to k^*$ such that $\beta(h, g) = \beta(g, h)^{-1}$ for all $h, g \in G$, this is exactly Lie color algebra studied in [6].

Example 1.1. Let A be a left H-comodule algebra. Define $[\ ,\ ]_\beta$ to be $[a, b]_\beta := ab - \sum \beta(a_{(-1)}, b_{(-1)})b_{(0)}a_{(0)}$. Then $(A, [\ ,\ ]_\beta)$ is an (H, β)-Lie algebra and is denoted by A_β

Definition 1.2. Let (H, β) be a cotriangular Hopf algebra. An (H, β)-Lie coalgebra is a left H-comodule $\mathcal{A}$ together with Lie cobracket $\delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ that is an H-comodule morphism satisfying

(1) β-anticocommutativity:

$$\delta(a) = -\beta(a_{[1](-1)}, a_{[2](-1)})a_{[2](0)} \otimes a_{[1](0)}$$

(2) β-co-Jacobi identity:

$$\begin{aligned} a_{[1][1]} \otimes a_{[1][2]} \otimes a_{[2]} &+ \beta(a_{[1][1](-1)}a_{[1][2](-1)}, a_{[2](-1)})a_{[2]} \otimes a_{[1][1]} \otimes a_{[1][2]} \\ &+ \beta(a_{[1][1](-1)}, a_{[1][2](-1)}a_{[2](-1)})a_{[1][2]} \otimes a_{[2]} \otimes a_{[1][1]} = 0 \end{aligned}$$

where we use the notion $\delta(a) = \sum a_{[1]} \otimes a_{[2]}$ for all $a \in \mathcal{A}$.

Example 1.2. Let C be a left H-comodule coalgebra. Define $\delta_\beta : C \to C \otimes C$ to be

$$\delta_\beta(c) = \sum c_1 \otimes c_2 - \beta(c_{1(-1)}, c_{2(-1)})c_{2(0)} \otimes c_{1(0)}$$

Then (C, δ_β) is an (H, β)-Lie coalgebra and is denoted by (C_β, δ_β).

Proposition 1.1. *Let H be a Hopf algebra with a skew-symmetric bicharacter $\beta : H \otimes H \to k$, and suppose $\sigma : H \otimes H \to k$ is a left cocycle.*

(a) Define H_σ to be H as a coalgebra, with multiplication defined to be

$$h \cdot_\sigma l := \sigma^{-1}(h_1, l_1)h_2l_2\sigma(h_3, l_3)$$

Then H (with a suitable antipode) is a Hopf algebra.

(b) *Define the map* $\beta_\sigma : H_\sigma \otimes H_\sigma \to k$ *by, for all* $h, l \in H$,

$$\beta_\sigma(h,l) := \sigma^{-1}(l_1, h_1)\beta(h_2, l_2)\sigma(h_3, l_3)$$

If (H,β) *is cotriangular, then* (H_σ, β_σ) *is also cotriangular.*

(c) *If* A *is a left* H*-comodule algebra, define* A^σ *to be* A *as a vector space and* H_σ*-comodule, with multiplication given by*

$$a \cdot^\sigma b := \sigma(a_{(-1)}, b_{(-1)})a_{(0)}b_{(0)}$$

Then A^σ *is an* H_σ*-comodule algebra.*

Definition 1.3. An (H,β)-Lie bialgebra $\mathcal{H}$ is a vector space equipped simultaneously with an (H,β)-Lie algebra structure $(\mathcal{H}, [\ ,\ ])$ and an (H,β)-Lie coalgebra $(\mathcal{H}, \delta)$ structure such that the following compatibility condition is satisfied:

$$\begin{aligned}\delta([a,b]) = [a, b_{[1]}] \otimes b_{[2]} + \beta(a_{(-1)}, b_{[1](-1)})b_{[1](0)} \otimes [a_{(0)}, b_{[2]}] + a_{[1]} \otimes [a_{[2]}, b] \\ + \beta(a_{[2](-1)}, b_{(-1)})[a_{[1]}, b_{(0)}] \otimes a_{[2](0)}\end{aligned} \tag{LB}$$

and we denoted it by $(\mathcal{H}, [\ ,\ ], \delta)$.

2 Matched pair of (H,β)-Lie algebras

Let $\mathcal{A}, \mathcal{H}$ be both (H,β)-Lie algebras, and for $a, b \in \mathcal{A}$, $h, g \in \mathcal{H}$, denote maps $\triangleright : \mathcal{H} \otimes \mathcal{A} \to \mathcal{A}$, $\triangleleft : \mathcal{H} \otimes \mathcal{A} \to \mathcal{H}$ by $\triangleright(h \otimes a) = h \triangleright a$, $\triangleleft(h \otimes a) = h \triangleleft a$. If $\mathcal{H}$ is an (H,β)-Lie algebra and the map $\triangleright : \mathcal{H} \otimes \mathcal{A} \to \mathcal{A}$ satisfies

$$[h,g] \triangleright a = h \triangleright g \triangleright a - \beta(h_{(-1)}, g_{(-1)})g_{(0)} \triangleright h_{(0)} \triangleright a$$

then $\mathcal{A}$ is called a left $\mathcal{H}$-module. Note that when considering (H,β)-Lie algebras, all action maps must be H-comodule maps. So, for $h \in \mathcal{H}, a \in \mathcal{A}$, we have

$$\rho(h \triangleright a) = \sum h_{(-1)}a_{(-1)} \otimes h_{(0)} \triangleright a_{(0)}$$

Also, if $\mathcal{A}$ is an $\mathcal{H}$-module Lie algebra, then

$$h \triangleright [a,b] = [h \triangleright a, b] + \beta(h_{(-1)}, a_{(-1)})[a_{(0)}, h_{(0)} \triangleright b]$$

and if $\mathcal{A}$ is an $\mathcal{H}$-module Lie coalgebra, then

$$\delta(h \triangleright a) = h \triangleright a_{[1]} \otimes a_{[2]} + \beta(h_{(-1)}, a_{[1](-1)})a_{[1](0)} \otimes h_{(0)} \triangleright a_{[2]}$$

Definition 2.1. Assume that $\mathcal{A}$ and $\mathcal{H}$ are (H,β)-Lie algebras. If $\mathcal{A}$ is a left $\mathcal{H}$-module, $\mathcal{H}$ is a right $\mathcal{A}$-module, and the following (BB1) and (BB2) hold, then $(\mathcal{A}, \mathcal{H})$ is called a *matched pair of* (H,β)*-Lie algebras*:

(BB1) $h \triangleright [a,b] = [h \triangleright a, b] + \beta(h_{(-1)}, a_{(-1)})[a_{(0)}, h_{(0)} \triangleright b] + (h \triangleleft a) \triangleright b - \beta(a_{(-1)}, b_{(-1)})(h \triangleleft b_{(0)}) \triangleright a_{(0)}$;

(BB2) $[h,g] \triangleleft a = [h, g \triangleleft a] + \beta(g_{(-1)}, a_{(-1)})[h \triangleleft a_{(0)}, g_{(0)}] + h \triangleleft (g \triangleright a) - \beta(h_{(-1)}, g_{(-1)})g_{(0)} \triangleleft (h_{(0)} \triangleright a)$.

Theorem 2.1. *If $(\mathcal{A},\mathcal{H})$ is a matched pair of (H,β)-Lie algebras, then the double cross sum $\mathcal{A}\bowtie\mathcal{H}$ forms an (H,β)-Lie algebra which equals $\mathcal{A}\oplus\mathcal{H}$ as a linear space, but with Lie bracket*

$$[a\oplus h, b\oplus g] = ([a,b]+h\triangleright b-\beta(g_{(-1)},a_{(-1)})g_{(0)}\triangleright a_{(0)})\oplus([h,g]+h\triangleleft b-\beta(g_{(-1)},a_{(-1)})g_{(0)}\triangleleft a_{(0)})$$

Proof. We show that the β-Jacobi identity holds for $\mathcal{A}\bowtie\mathcal{H}$. By definition, $[h,a]=h\triangleright a+h\triangleleft a$, $[a,h]=-\beta(a_{(-1)},h_{(-1)})a_{(0)}\triangleright h_{(0)}-\beta(a_{(-1)},h_{(-1)})a_{(0)}\triangleleft h_{(0)}$. So, for $h,g\in\mathcal{H}, a\in\mathcal{A}$, $[[h,g],a]=[h,g]\triangleright a+[h,g]\triangleleft a$, and for the second item of β-Jacobi identity,

$$\begin{aligned}
&\beta(h_{(-1)}g_{(-1)},a_{(-1)})[[a_{(0)},h_{(0)}],g_{(0)}]\\
&\quad=\beta(h_{(-2)}g_{(-2)},a_{(-2)})\beta(a_{(-1)},h_{(-1)})\beta((h_{(0)}\triangleright a_{(0)})_{(-1)},g_{(-1)})\\
&\qquad\times g_{(0)}\triangleright(h_{(0)}\triangleright a_{(0)})_{(0)}-\beta(h_{(-2)}g_{(-2)},a_{(-2)})\beta(a_{(-1)},h_{(-1)})\\
&\qquad\times[h_{(0)}\triangleleft a_{(0)},g_{(0)}]+\beta(h_{(-2)}g_{(-2)},a_{(-2)})\beta(h_{(-1)},a_{(-1)})\\
&\qquad\times\beta((h_{(0)}\triangleright a_{(0)})_{(-1)},g_{(-1)})g_{(0)}\triangleleft(h_{(0)}\triangleright a_{(0)})_{(0)}
\end{aligned}$$

The right-hand sides are as follows:

$$\begin{aligned}
\text{1st RHS}&=\beta(h_{(-3)}g_{(-2)},a_{(-3)})\beta(a_{(-2)},h_{(-2)})\beta(h_{(-1)}a_{(-1)},g_{(-1)})g_{(0)}\triangleright h_{(0)}\triangleright a_{(0)}\\
&=\beta(h_{(-3)},a_{(-3)})\beta(g_{(-3)},a_{(-4)})\beta(a_{(-2)},h_{(-2)})\\
&\quad\times\beta(h_{(-1)},g_{(-1)})\beta(a_{(-1)},g_{(-2)})g_{(0)}\triangleright h_{(0)}\triangleright a_{(0)}\\
&=\beta(g_{(-3)},a_{(-2)})\beta(h_{(-1)},g_{(-1)})\beta(a_{(-1)},g_{(-2)})g_{(0)}\triangleright h_{(0)}\triangleright a_{(0)}\\
&=\beta(g_{(-2)},a_{(-2)})\beta(h_{(-1)},g_{(-3)})\beta(a_{(-1)},g_{(-1)})g_{(0)}\triangleright h_{(0)}\triangleright a_{(0)}\\
&=\beta(h_{(-1)},g_{(-1)})g_{(0)}\triangleright h_{(0)}\triangleright a
\end{aligned}$$

where we use the fact that $\triangleright:\mathcal{H}\otimes\mathcal{A}\to\mathcal{A}$ is a left H-comodule map in the first equality, (CT2) and (CT3) for β in second equality, the cocommutative of H in the fourth equality, and (CT4) for β in the fifth equality. Similarly,

$$\text{3rd RHS}=\beta(h_{(-1)},g_{(-1)})g_{(0)}\triangleleft(h_{(0)}\triangleright a)$$

It is easy to see that

$$\begin{aligned}
\text{2nd RHS}&=-\beta(h_{(-2)},a_{(-2)})\beta(g_{(-2)},a_{(-3)})\beta(a_{(-1)},h_{(-1)})[h_{(0)}\triangleleft a_{(0)},g_{(0)}]\\
&=-\beta(g_{(-1)},a_{(-1)})[h\triangleleft a_{(0)},g_{(0)}]
\end{aligned}$$

As for the third item of β-Jacobi identity,

$$\begin{aligned}
&\beta(h_{(-1)},g_{(-1)}a_{(-1)})[[g_{(0)},a_{(0)}],h_{(0)}]\\
&\quad=-\beta(h_{(-2)},g_{(-1)}a_{(-1)})\beta((g_{(0)}\triangleright a_{(0)})_{(-1)},h_{(-1)})h_{(0)}\triangleright(g_{(0)}\triangleright a_{(0)})_{(0)}\\
&\qquad-\beta(h_{(-2)},g_{(-1)}a_{(-1)})\beta((g_{(0)}\triangleright a_{(0)})_{(-1)},h_{(-1)})h_{(0)}\triangleleft(g_{(0)}\triangleright a_{(0)})_{(0)}\\
&\qquad+\beta(h_{(-1)},g_{(-1)}a_{(-1)})[g_{(0)}\triangleleft a_{(0)},h_{(0)}]
\end{aligned}$$

The right-hand sides are as follows:

$$\begin{aligned}
\text{1st RHS} &= -\beta(h_{(-2)}, g_{(-2)}a_{(-2)})\beta(g_{(-1)}a_{(-1)}, h_{(-1)})h_{(0)} \triangleright (g_{(0)} \triangleright a_{(0)}) \\
&= -\beta(h_{(-4)}, g_{(-2)})\beta(h_{(-3)}, a_{(-2)})\beta(g_{(-1)}, h_{(-2)}) \\
&\quad \times \beta(a_{(-1)}, h_{(-1)})h_{(0)} \triangleright (g_{(0)} \triangleright a_{(0)}) \\
&= -\beta(h_{(-4)}, g_{(-2)})\beta(h_{(-2)}, a_{(-2)})\beta(g_{(-1)}, h_{(-3)}) \\
&\quad \times \beta(a_{(-1)}, h_{(-1)})h_{(0)} \triangleright (g_{(0)} \triangleright a_{(0)}) \\
&= -\beta(h_{(-2)}, g_{(-2)})\beta(g_{(-1)}, h_{(-1)})h_{(0)} \triangleright (g_{(0)} \triangleright a_{(0)}) \\
&= -h \triangleright (g \triangleright a)
\end{aligned}$$

Similarly, 2nd RHS $= -h \triangleleft (g \triangleright a)$. We now see

$$\begin{aligned}
\text{3rd RHS} &= -\beta(h_{(-2)}, g_{(-1)}a_{(-1)})\beta((g_{(0)} \triangleleft a_{(0)})_{(-1)}, h_{(-1)})[h_{(0)}, (g_{(0)} \triangleleft a_{(0)})_{(0)}] \\
&= -\beta(h_{(-2)}, g_{(-2)}a_{(-2)})\beta(g_{(-1)}a_{(-1)}, h_{(-1)})[h_{(0)}, g_{(0)} \triangleleft a_{(0)}] \\
&= -\beta(h_{(-4)}, g_{(-2)})\beta(h_{(-3)}, a_{(-2)})\beta(g_{(-1)}, h_{(-1)}) \\
&\quad \times \beta(a_{(-1)}, h_{(-2)})[h_{(0)}, g_{(0)} \triangleleft a_{(0)}] \\
&= -\beta(h_{(-2)}, g_{(-2)})\beta(h_{(-3)}, a_{(-2)})\beta(g_{(-1)}, h_{(-1)}) \\
&\quad \times \beta(a_{(-1)}, h_{(-4)})[h_{(0)}, g_{(0)} \triangleleft a_{(0)}] \\
&= -\beta(h_{(-2)}, a_{(-2)})\beta(a_{(-1)}, h_{(-1)})[h_{(0)}, g_{(0)} \triangleleft a_{(0)}] \\
&= -[h, g \triangleleft a]
\end{aligned}$$

Now by (BB2) and the fact that $\mathcal{A}$ is a left $\mathcal{H}$-module, we have that the sum of three item equals zero. The other cases can be checked similarly. □

Proposition 2.1. *Assume that $\mathcal{A}$ and $\mathcal{H}$ are (H,β)-Lie bialgebras and $(\mathcal{A},\mathcal{H})$ is a matched pair of (H,β)-Lie algebras, i.e., $\mathcal{A}$ is a left $\mathcal{H}$-module Lie coalgebra and $\mathcal{H}$ is a right $\mathcal{A}$-module (H,β)-Lie coalgebra. If*

$$\begin{aligned}
&(\mathrm{id}_{\mathcal{H}} \otimes \triangleleft)(\delta_{\mathcal{H}} \otimes \mathrm{id}_{\mathcal{A}}) + (\triangleright \otimes \mathrm{id}_{\mathcal{A}})(\mathrm{id}_{\mathcal{H}} \otimes \delta_{\mathcal{A}}) = 0 \\
&\sum h_{[1]} \otimes h_{[2]} \triangleright a + \sum h \triangleleft a_{[1]} \otimes a_{[2]} = 0 \qquad \text{(BB3)}
\end{aligned}$$

then $\mathcal{A} \bowtie \mathcal{H}$ becomes an (H,β)-Lie bialgebra.

Proof. The Lie algebra structure is as in Theorem 2.1. The Lie cobracket is the one inherited from $\mathcal{A}$ and $\mathcal{H}$. $\mathcal{A}$ and $\mathcal{H}$ are also (H,β)-Lie sub-bialgebras of $\mathcal{A} \bowtie \mathcal{H}$. So we only check equation (LB) on $\mathcal{A} \otimes \mathcal{H}$. For $h \in \mathcal{H}, a \in \mathcal{A}$, $\delta[h,a] = \delta(h \triangleright a) + \delta(h \triangleleft a)$, and by the ad-action on tenor product,

$$\begin{aligned}
h \triangleright \delta(a) + \delta(h) \triangleleft a = {}& h \triangleright a_{[1]} \otimes a_{[2]}(1) + h \triangleleft a_{[1]} \otimes a_{[2]}(2) + \beta(h_{(-1)}, a_{[1](-1)}) \\
&\times a_{[1](0)} \otimes h_{(0)} \triangleright a_{[2]}(3) + \beta(h_{(-1)}, a_{[1](-1)})a_{[1](0)} \otimes h_{(0)} \triangleleft a_{[2]}(4) \\
&+ h_{[1]} \otimes h_{[2]} \triangleleft a(5) + h_{[1]} \otimes h_{[2]} \triangleright a(6) + \beta(h_{[2](-1)}, a_{(-1)}) \\
&\times h_{[1]} \triangleleft a_{(0)} \otimes h_{[2](0)}(7) + \beta(h_{[2](-1)}, a_{(-1)})h_{[1]} \triangleright a_{(0)} \otimes h_{[2](0)}(8)
\end{aligned}$$

By (BB3), (2) + (6) = 0, (4) + (8) = 0. For the remaining four terms, $\delta(h \triangleright a) = (1) + (3)$ and $\delta(h \triangleleft a) = (5) + (7)$. □

3 Cocycle twists of matched pairs of (H, β)-Lie algebras

The cocycle twist of an (H, β)-Lie algebra was introduced in [1]. If $\mathcal{A}$ is a left $\mathcal{H}$-module $\triangleright : \mathcal{H} \otimes \mathcal{A} \to \mathcal{A}$, then we also have that $\mathcal{A}^\sigma$ is a left $\mathcal{H}^\sigma$-module by $\triangleright^\sigma : \mathcal{H} \otimes \mathcal{A} \to \mathcal{A}$,

$$h \triangleright^\sigma a = \sigma(h_{(-1)}, a_{(-1)}) h_{(0)} \triangleright a_{(0)}$$

(see [1, Propostion 4.7]). Similarly, we get a right $\mathcal{A}^\sigma$-module $\triangleleft^\sigma : \mathcal{H} \otimes \mathcal{A} \to \mathcal{H}$ by

$$h \triangleleft^\sigma a = \sigma(h_{(-1)}, a_{(-1)}) h_{(0)} \triangleleft a_{(0)}$$

We now prove that the cocycle twist of a matched pair of (H, β)-Lie algebras can also be matched.

Theorem 3.1. *If $(\mathcal{A}, \mathcal{H})$ is a matched pair of (H, β)-Lie algebras, then $(\mathcal{A}^\sigma, \mathcal{H}^\sigma)$ is a matched pair of (H, β_σ)-Lie algebras, so their double cross sum $\mathcal{A}^\sigma \bowtie \mathcal{H}^\sigma$ forms an (H, β_σ)-Lie algebra.*

Proof. Note that the brackets in $\mathcal{A}^\sigma \bowtie \mathcal{H}^\sigma$ are given by

$$[a, b]^\sigma = \sigma(a_{(-1)}, b_{(-1)})[a_{(0)}, b_{(0)}]$$
$$[h, a] = \sigma(h_{(-1)}, a_{(-1)}) h_{(0)} \triangleright a_{(0)} + \sigma(h_{(-1)}, a_{(-1)}) h_{(0)} \triangleleft a_{(0)}$$
$$[a, h] = -\sigma(h_{(-2)}, a_{(-2)})\beta(h_{(-1)}, a_{(-1)}) h_{(0)} \triangleright a_{(0)} - \sigma(h_{(-2)}, a_{(-2)})\beta(h_{(-1)}, a_{(-1)}) h_{(0)} \triangleleft a_{(0)}$$

We check that the matched pair conditions (BB1) and (BB2) are valid on $(\mathcal{A}^\sigma, \mathcal{H}^\sigma)$. We want to see that

$$\begin{aligned} h \triangleright^\sigma [a, b]^\sigma = {} & [h \triangleright^\sigma a, b]^\sigma + \beta_\sigma(h_{(-1)}, a_{(-1)})[a_{(0)}, h_{(0)} \triangleright^\sigma b]^\sigma + (h \triangleleft^\sigma a) \triangleright^\sigma b \\ & - \beta_\sigma(a_{(-1)}, b_{(-1)})(h \triangleleft^\sigma b_{(0)}) \triangleright^\sigma a_{(0)} \end{aligned}$$

In fact,

$$\begin{aligned} h \triangleright^\sigma [a, b]^\sigma &= \sigma(h_{(-1)}, [a_{(0)}, b_{(0)}]_{(-1)})\sigma(a_{(-1)}, b_{(-1)}) h_{(0)} \triangleright [a_{(0)}, b_{(0)}]_{(0)} \\ &= \sigma(h_{(-1)}, a_{(-1)} b_{(-1)})\sigma(a_{(-2)}, b_{(-2)}) h_{(0)} \triangleright [a_{(0)}, b_{(0)}] \end{aligned}$$

and

$$\begin{aligned} [h \triangleright^\sigma a, b]^\sigma &= \sigma(h_{(-1)}, a_{(-1)})\sigma((h_{(0)} \triangleright a_{(0)})_{(-1)}, b_{(-1)})[(h_{(0)} \triangleright a_{(0)})_{(0)}, b_{(0)}] \\ &= \sigma(h_{(-2)}, a_{(-2)})\sigma(h_{(-1)} a_{(-1)}, b_{(-1)})[h_{(0)} \triangleright a_{(0)}, b_{(0)}] \end{aligned}$$

Similarly, we get

$$(h \triangleleft^\sigma a) \triangleright^\sigma b = \sigma(h_{(-2)}, a_{(-2)})\sigma(h_{(-1)} a_{(-1)}, b_{(-1)})(h_{(0)} \triangleleft a_{(0)}) \triangleright b_{(0)}$$

Also

$$\begin{aligned} & \beta_\sigma(h_{(-1)}, a_{(-1)})[a_{(0)}, h_{(0)} \triangleright^\sigma b]^\sigma \\ & \quad = \sigma(h_{(-3)}, a_{(-3)})\beta(h_{(-2)}, a_{(-2)})\sigma(a_{(-1)}, h_{(-1)})[a_{(0)}, h_{(0)} \triangleright^\sigma b]^\sigma \\ & \quad = \sigma(h_{(-5)}, a_{(-4)})\beta(h_{(-4)}, a_{(-3)})\sigma(a_{(-2)}, h_{(-3)}) \\ & \qquad \times \sigma(h_{(-2)}, b_{(-2)})\sigma(a_{(-1)}, h_{(-1)} b_{(-1)})[a_{(0)}, h_{(0)} \triangleright b_{(0)}] \\ & \quad = \sigma(h_{(-5)}, a_{(-5)})\beta(h_{(-4)}, a_{(-4)})\sigma^{-1}(a_{(-3)}, h_{(-3)}) \\ & \qquad \times \sigma(a_{(-2)}, h_{(-2)})\sigma(a_{(-1)} h_{(-1)}, b_{(-1)})[a_{(0)}, h_{(0)} \triangleright b_{(0)}] \\ & \quad = \sigma(h_{(-3)}, a_{(-3)})\beta(h_{(-2)}, a_{(-2)})\sigma(h_{(-1)} a_{(-1)}, b_{(-1)})[a_{(0)}, h_{(0)} \triangleright b_{(0)}] \\ & \quad = \sigma(h_{(-3)}, a_{(-3)})\sigma(h_{(-2)} a_{(-2)}, b_{(-2)})\beta(h_{(-1)}, a_{(-1)})[a_{(0)}, h_{(0)} \triangleright b_{(0)}] \end{aligned}$$

Similarly, we get

$$\begin{aligned}&\beta_\sigma(a_{(-1)}, b_{(-1)})(h \triangleleft^\sigma b_{(0)}) \triangleright^\sigma a_{(0)}\\&\qquad = \sigma(h_{(-3)}, a_{(-3)})\sigma(h_{(-2)}a_{(-2)}b_{(-2)})\beta(h_{(-1)}, a_{(-1)})(h_{(0)} \triangleleft b_{(0)}) \triangleright a_{(0)}\end{aligned}$$

Now, by the cocycle condition of σ and (BB1) on $(\mathcal{A}, \mathcal{H})$, we get the result. Similar argument can be performed for (BB2) on $(\mathcal{A}^\sigma, \mathcal{H}^\sigma)$. □

We now give the relationship between the (H, β)-Lie algebras $\mathcal{A}^\sigma \bowtie \mathcal{H}^\sigma$ and $(\mathcal{A} \bowtie \mathcal{H})^\sigma$ by the following theorem; the proof can easily be seen from their construction, so we omit it.

Theorem 3.2. *If $(\mathcal{A}, \mathcal{H})$ is a matched pair of (H, β)-Lie algebras, then $\mathcal{A}^\sigma \bowtie \mathcal{H}^\sigma \cong (\mathcal{A} \bowtie \mathcal{H})^\sigma$.*

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