Journal of Generalized Lie Theory and Applications Vol. 3 (2009), No. 2, 95–111

Deforming $\mathcal{K}(1)$ superalgebra modules of symbols

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Abstract

We study nontrivial deformations of the natural action of the Lie superalgebra $\mathcal{K}(1)$ of contact vector fields on the $(1,1)$-dimensional superspace $\mathbb{R}^{1|1}$ on the space of symbols $\widetilde{\mathcal{S}}_\delta^n = \bigoplus_{k=0}^n \mathfrak{F}_{\delta-\frac{k}{2}}$. We calculate obstructions for integrability of infinitesimal multiparameter deformations and determine the complete local commutative algebra corresponding to the miniversal deformation.

1 Introduction

We consider the superspace $\mathbb{R}^{1|1}$ equipped with the contact 1-form $\alpha = dx + \theta d\theta$, where θ is the odd variable, the Lie superalgebra $\mathcal{K}(1)$ of contact polynomial vector fields on $\mathbb{R}^{1|1}$ (also called superconformal Lie algebra see [16]), and the $\mathcal{K}(1)$-module of symbols $\widetilde{\mathcal{S}}_\delta^n = \bigoplus_{k=0}^n \mathfrak{F}_{\delta-\frac{k}{2}}$, where $\mathfrak{F}_{\delta-\frac{k}{2}}$ is the module of the weighted densities on $\mathbb{R}^{1|1}$. As Lie superalgebra $\mathcal{K}(1)$ is rigid like the Lie algebra of Virasoro [13], so one tries deformations of its modules. We will use the framework of Fialowski [1, 3] and Fialowski-Fuchs [2] (see also [10]) and consider (multiparameter) deformations over complete local commutative algebras related to this deformation. The first step of any approach to the deformation theory consists in the determination of infinitesimal deformations. According to Nijenhuis-Richardson [4], infinitesimal deformations of the action of a Lie algebra on some module are classified by the first cohomology space of the Lie algebra with values in the module of endomorphisms of that module. In our case,

$$\mathrm{H}^1_{\mathrm{diff}}\left(\mathcal{K}(1);\mathrm{End}_{\mathrm{diff}}(\widetilde{\mathcal{S}}_\delta^n)\right) = \oplus_{\lambda,k}\,\mathrm{H}^1_{\mathrm{diff}}\left(\mathcal{K}(1);\mathfrak{D}_{\lambda,\lambda+k}\right)$$

where $\mathfrak{D}_{\lambda,\mu}$ is the superspace of linear differential operators from the superspace of weighted densities $\mathfrak{F}_\lambda$ to $\mathfrak{F}_\mu$, and hereafter $2(\delta-\lambda), 2(\delta-\mu) \in \{0,1,\dots,n\}$.

While the obstructions for integrability of this infinitesimal deformations belong to the second cohomology space,

$$\mathrm{H}^2_{\mathrm{diff}}\left(\mathcal{K}(1);\mathrm{End}_{\mathrm{diff}}(\widetilde{\mathcal{S}}_\delta^n)\right) = \oplus_{\lambda,k}\,\mathrm{H}^2_{\mathrm{diff}}\left(\mathcal{K}(1);\mathfrak{D}_{\lambda,\lambda+k}\right)$$

The odd first space $\mathrm{H}^1_{\mathrm{diff}}\left(\mathcal{K}(1);\mathfrak{D}_{\lambda,\lambda+k}\right)_1$ was calculated in [9]: our task, therefore, is to calculate the even first space $\mathrm{H}^1_{\mathrm{diff}}\left(\mathcal{K}(1);\mathfrak{D}_{\lambda,\lambda+k}\right)_0$ and the obstructions.

We shall give concrete explicit examples of the deformed action.

2 Definitions and notations

2.1 The Lie superalgebra of contact vector fields on $\mathbb{R}^{1|1}$

Let $\mathbb{R}^{1|1}$ be the superspace with coordinates (x,θ), where θ is the odd variables ($\theta^2=0$). We consider the superspace $\mathbb{R}^{1|1}[x,\theta]$ of superpolynomial functions on $\mathbb{R}^{1|1}$:

$$\mathbb{R}^{1|1}[x,\theta]=\big\{F=f_0+f_1\theta:\ f_0 \text{ and } f_1 \text{ are in } \mathbb{R}[x]\big\}$$

where $\mathbb{R}[x]$ is the space of polynomial functions on $\mathbb{R}$. The superspace $\mathbb{R}^{1|1}[x,\theta]$ has a structure of superalgebra given by the contact bracket

$$\{F,G\}=FG'-F'G+\frac{1}{2}(-1)^{p(F)+1}\overline{\eta}(F)\cdot\overline{\eta}(G) \tag{2.1}$$

where $\eta=\frac{\partial}{\partial\theta}+\theta\frac{\partial}{\partial x}$, $\overline{\eta}=\frac{\partial}{\partial\theta}-\theta\frac{\partial}{\partial x}$ and $p(F)$ is the parity of F. Note that $\eta\circ\eta=\frac{\partial}{\partial x}$, so η is sometimes called a "square root" of $\frac{\partial}{\partial x}$.

Any contact structure on $\mathbb{R}^{1|1}$ can be defined by the following 1-form:

$$\alpha=dx+\theta d\theta$$

Let $\mathrm{Vect}_P(\mathbb{R}^{1|1})$ be the superspace of superpolynomial vector fields on $\mathbb{R}^{1|1}$:

$$\mathrm{Vect}_p(\mathbb{R}^{1|1})=\big\{F_0\partial_x+F_1\partial \mid F_i\in\mathbb{R}^{1|n}[x,\theta]\big\}$$

where ∂ stands for $\frac{\partial}{\partial\theta}$ and ∂_x stands for $\frac{\partial}{\partial x}$, and consider the superspace $\mathcal{K}(1)$ of contact polynomial vector fields on $\mathbb{R}^{1|1}$ defined by

$$\mathcal{K}(1)=\big\{v\in\mathrm{Vect}_p(\mathbb{R}^{1|1}):v\alpha=F\alpha,\ \text{for some } F\in\mathbb{R}^{1|1}[x,\theta]\big\}$$

where $v\alpha$ is the Lie derivative of α along the vector fields v. Any contact superpolynomial vector field on $\mathbb{R}^{1|1}$ can be given by the following explicit form:

$$v_F=F\partial_x+\frac{1}{2}(-1)^{p(F)+1}\overline{\eta}(F)\overline{\eta},\quad \text{where } F\in\mathbb{R}^{1|1}[x,\theta]$$

2.2 The space of polynomial weighted densities on $\mathbb{R}^{1|1}$

Recall the definition of the $\mathrm{Vect}_p(\mathbb{R})$-module of polynomial weighted densities on $\mathbb{R}$, where $\mathrm{Vect}_p(\mathbb{R})$ is the Lie algebra of polynomial vector fields on $\mathbb{R}$. Consider the 1-parameter action of $\mathrm{Vect}_p(\mathbb{R})$ on the space of polynomial functions $\mathbb{R}[x]$ given by

$$L^{\lambda}_{X\partial_x}(f)=Xf'+\lambda X'f$$

where $\lambda\in\mathbb{R}$. Denote by $\mathcal{F}_\lambda$ the $\mathrm{Vect}_p(\mathbb{R})$-module structure on $\mathbb{R}[x]$ defined by this action. Geometrically, $\mathcal{F}_\lambda$ is the space of polynomial weighted densities of weight λ on $\mathbb{R}$, i.e.,

$$\mathcal{F}_\lambda=\big\{f(x)(dx)^\lambda|f\in\mathbb{R}[x]\big\} \tag{2.2}$$

Now, in supersetting, we have an analogous definition of weighted densities (see [9]) with dx replaced by $\alpha=dx+\theta d\theta$. Consider the 1-parameter action of $\mathcal{K}(1)$ on $\mathbb{R}[x,\theta]$ given by the rule

$$\mathfrak{L}^{\lambda}_{v_F}(G)=\mathfrak{L}_{v_F}(G)+\lambda F'\cdot G \tag{2.3}$$

where $F, G \in \mathbb{R}[x,\theta]$ and $F' = \partial_x F$ or, in components,

$$\mathfrak{L}_{v_F}^{\lambda}(G) = L_{a\partial_x}^{\lambda}(g_0) + \frac{1}{2}bg_1 + \left(L_{a\partial_x}^{\lambda+\frac{1}{2}}(g_1) + \lambda g_0 b' + \frac{1}{2}g_0' b\right)\theta \tag{2.4}$$

where $F = a + b\theta$, $G = g_0 + g_1\theta$. In particular, we have

$$\mathfrak{L}_{v_a}^{\lambda}(g_0) = L_{a\partial_x}^{\lambda}(g_0), \quad \mathfrak{L}_{v_a}^{\lambda}(g_1\theta) = \theta L_{a\partial_x}^{\lambda+\frac{1}{2}}(g_1)$$
$$\mathfrak{L}_{v_{b\theta}}^{\lambda}(g_0) = \left(\lambda g_0 b' + \frac{1}{2}g_0' b\right)\theta, \quad \mathfrak{L}_{v_{b\theta}}^{\lambda}(g_1\theta) = \frac{1}{2}bg_1$$

We denote this $\mathcal{K}(1)$-module by $\mathfrak{F}_\lambda$, the space of all polynomial weighted densities on $\mathbb{R}^{1|1}$ of weight λ:

$$\mathfrak{F}_\lambda = \left\{\phi = f(x,\theta)\alpha^\lambda \mid f(x,\theta) \in \mathbb{R}[x,\theta]\right\} \tag{2.5}$$

Let $\mathfrak{D}_{\lambda,\mu} := \mathrm{Hom}_{\mathrm{diff}}(\mathfrak{F}_\lambda, \mathfrak{F}_\mu)$ be the $\mathcal{K}(1)$-module of linear differntial superoperators, the $\mathcal{K}(1)$-action on this superspace is given by

$$\mathfrak{L}_{v_F}^{\lambda,\mu}(A) = \mathfrak{L}_{v_F}^{\mu} \circ A - (-1)^{p(A)p(F)} A \circ \mathfrak{L}_{v_F}^{\lambda} \tag{2.6}$$

Obviously,

(1) The adjoint $\mathcal{K}(1)$-module is isomorphic to $\mathfrak{F}_{-1}$.
(2) As a $\mathrm{Vect}_p(\mathbb{R})$-module, $\mathfrak{F}_\lambda \simeq \mathcal{F}_\lambda \oplus \Pi(\mathcal{F}_{\lambda+\frac{1}{2}})$, where $\mathcal{F}_\lambda$ is the $\mathrm{Vect}_p(\mathbb{R})$-module of polynomial weighted densities of weight λ and Π is the functor of the change of parity.

Proposition 2.1. *As a* $\mathrm{Vect}_p(\mathbb{R})$*-module, we have for the homogeneous relative parity components,*

$$(\mathfrak{D}_{\lambda,\mu})_0 \simeq \mathcal{D}_{\lambda,\mu} \oplus \mathcal{D}_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}, \quad (\mathfrak{D}_{\lambda,\mu})_1 \simeq \Pi\big(\mathcal{D}_{\lambda+\frac{1}{2},\mu} \oplus \mathcal{D}_{\lambda,\mu+\frac{1}{2}}\big)$$

2.3 The supertransvectants: Explicit formula

Definition 2.2 (see [12])**.** The supertransvectants are the bilinear $\mathfrak{osp}(1|2)$-invariant maps

$$\mathfrak{J}_k^{\alpha,\beta} : \mathfrak{F}_\alpha \otimes \mathfrak{F}_\beta \longrightarrow \mathfrak{F}_{\alpha+\beta+k}$$

where $k \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$. These operators were calculated in [11] (see also [19]), let us give their explicit formula.

One has

$$\mathfrak{J}_k^{\alpha,\beta}(f,g) = \sum_{i+j=2k} J_{i,j}^k \overline{\eta}^i(f)\overline{\eta}^j(g) \tag{2.7}$$

where the numeric coefficients are

$$J_{i,j}^k = (-1)^{\left(\left[\frac{j+1}{2}\right]+j(i+p(f))\right)} \frac{\begin{pmatrix} [k] \\ \left[\frac{2j+1+(-1)^{2k}}{4}\right] \end{pmatrix} \begin{pmatrix} 2\alpha + [k-\frac{1}{2}] \\ \left[\frac{2j+1-(-1)^{2k}}{4}\right] \end{pmatrix}}{\begin{pmatrix} 2\beta + \left[\frac{j-1}{2}\right] \\ \left[\frac{j+1}{2}\right] \end{pmatrix}} \tag{2.8}$$

where $[a]$ denotes the integer part of $a \in \mathbb{R}$, as above, the binomial coefficients $\binom{a}{b}$ are well defined if b is integer. It can be deduced directly that those operators are, indeed, $\mathfrak{osp}(1|2)$-invariant.

2.4 The first cohomology space $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(1), \mathfrak{D}_{\lambda,\mu})$

Let us first recall some fundamental concepts from cohomology theory (see [10]). Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra acting on a super vector space $V = V_0 \oplus V_1$. The space $\mathrm{Hom}(\mathfrak{g}, V)$ is $(\mathbb{Z}/2\mathbb{Z})$-graded via

$$\mathrm{Hom}(\mathfrak{g}, V)_b = \oplus_{a\in(\mathbb{Z}/2\mathbb{Z})} \mathrm{Hom}(\mathfrak{g}_a, V_{a+b}), \quad b \in \mathbb{Z}/2\mathbb{Z} \tag{2.9}$$

Let

$$Z^1(\mathfrak{g}, V) = \left\{\gamma \in \mathrm{Hom}(\mathfrak{g}, V); \gamma([g,h]) = (-1)^{p(g)p(\gamma)} g\cdot\gamma(h) - (-1)^{p(h)(p(g)+p(\gamma))} h\cdot\gamma(g),\ \forall g,h \in \mathfrak{g}\right\}$$

be the space of 1-cocycles for the Chevalley-Eilenberg differential. According to the $\mathbb{Z}/2\mathbb{Z}$-grading (2.9), any 1-cocycle $\gamma \in Z^1(\mathfrak{g}; V)$ is broken to $(\gamma', \gamma'') \in \mathrm{Hom}(\mathfrak{g}_0, V) \oplus \mathrm{Hom}(\mathfrak{g}_1, V)$.

The first cohomology space $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(1), \mathfrak{D}_{\lambda,\mu})$ inherits the $(\mathbb{Z}/2\mathbb{Z})$-grading from (2.9) and it decomposes into odd and even parts as follows:

$$\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(1), \mathfrak{D}_{\lambda,\mu}) = \mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(1), \mathfrak{D}_{\lambda,\mu})_0 \oplus \mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(1), \mathfrak{D}_{\lambda,\mu})_1$$

The odd first space $\mathrm{H}^1_{\mathrm{diff}}\left(\mathcal{K}(1); \mathfrak{D}_{\lambda,\lambda+k}\right)_1$ was calculated in [9]; we calculate, here, the even first space $\mathrm{H}^1_{\mathrm{diff}}\left(\mathcal{K}(1); \mathfrak{D}_{\lambda,\lambda+k}\right)_0$.

Lemma 2.3 (see [9])**.** *The 1-cocycle γ is a coboundary over $\mathcal{K}(1)$ if and only if γ' is a coboundary over* $\mathrm{Vect}_p(\mathbb{R})$.

The following theorem recalls the result.

Theorem 2.4. *(1) The space $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(1), \mathfrak{D}_{\lambda,\mu})_0$ is isomorphic to the following:*

$$\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(1), \mathfrak{D}_{\lambda,\mu})_0 \simeq \begin{cases} \mathbb{R} & \text{if } \mu-\lambda = 0 \\ \mathbb{R} & \text{if } \mu-\lambda = 2 \text{ for } \lambda \neq -1 \\ \mathbb{R} & \text{if } \mu-\lambda = 3 \text{ for } \lambda = 0 \text{ or } \lambda = \dfrac{-5}{2} \\ \mathbb{R} & \text{if } \mu-\lambda = 4 \text{ for } \lambda = \dfrac{-7\pm\sqrt{33}}{4} \\ 0 & \text{otherwise} \end{cases}$$

The space $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(1), \mathfrak{D}_{\lambda,\mu})_0$ is generated by the cohomology classes of the 1-cocycles:

- *For $\lambda = \mu$, the generator can be chosen as follows:*

$$\gamma_{\lambda,\lambda}(v_G)(F\alpha^\lambda) = G'F\alpha^\lambda$$

where, here and below, $F, G \in \mathbb{R}^{1|1}[x,\theta]$.

- *For $\mu-\lambda = 2$ and $\lambda \neq -1$, the generator can be chosen as follows:*

$$\gamma_{\lambda,\lambda+2}(v_G)(F\alpha^\lambda) = (2\lambda G^3 F + 3(-1)^{p(G)}\overline{\eta}(G'')\overline{\eta}(F))\alpha^{\lambda+2}$$

- *For $\mu-\lambda = 3$ and $\lambda = 0$, the generator can be chosen as follows:*

$$\gamma_{0,3}(v_G)(F\alpha^0) = \left(G^4F - (-1)^{p(G)}\overline{\eta}(G^3)\overline{\eta}(F) + G^3F'' + (-1)^{p(G)}\frac{3}{2}\overline{\eta}(G'')\overline{\eta}(F')\right)\alpha^3$$

- *For $\mu-\lambda=3$ and $\lambda=\frac{-5}{2}$, the generator can be chosen as follows:*

$$\gamma_{\frac{-5}{2},\frac{1}{2}}(v_G)(F\alpha^{\frac{-5}{2}})=\left(G^4F-(-1)^{p(G)}\overline{\eta}(G^3)\overline{\eta}(F)+G^3F'-(-1)^{p(G)}\frac{3}{8}\overline{\eta}(G'')\overline{\eta}(F')\right)a^{\frac{1}{2}}$$

- *For $\mu-\lambda=4$ and $\lambda=\frac{-7\pm\sqrt{33}}{4}$, the generator can be chosen as follows:*

$$\gamma_{\lambda,\lambda+4}(v_G)(F\alpha^{\lambda})=\Big(G^5F+(-1)^{p(G)}\frac{5}{2\lambda}\overline{\eta}(G^4)\overline{\eta}(F)-\frac{5}{\lambda}G^4F' -(-1)^{p(G)}\frac{20}{\lambda(2\lambda+1)}\overline{\eta}(G^3)\overline{\eta}(F')\Big)\alpha^{\lambda+4}$$

(2) The space $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(1),\mathfrak{D}_{\lambda,\mu})_1$ is isomorphic to the following:

$$\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(1),\mathfrak{D}_{\lambda,\mu})_1\simeq\begin{cases}\mathbb{R}^2 & \text{if } \lambda=0,\ \mu=\frac{1}{2}\\ \mathbb{R} & \text{if } \mu=\lambda+\frac{3}{2}\\ \mathbb{R} & \text{if } \mu=\lambda+\frac{5}{2} \text{ for all } \lambda\\ 0 & \text{otherwise}\end{cases}\tag{2.10}$$

The space $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(1),\mathfrak{D}_{\lambda,\mu})_1$ is generated by the cohomology classes of the 1-cocycles:

- *For $\lambda=0$ and $\mu=\frac{1}{2}$, the generators can be chosen as follows:*

$$\gamma_{0,\frac{1}{2}}(v_G)(F)=\eta(G')F\alpha^{\frac{1}{2}},\quad \widetilde{\gamma}_{0,\frac{1}{2}}(v_G)(F)=(-1)^{p(F)}\eta(G'F)\alpha^{\frac{1}{2}}$$

- *For $\lambda=-\frac{1}{2}$ and $\mu-\lambda=\frac{3}{2}$, the generator can be chosen as follows:*

$$\gamma_{-\frac{1}{2},1}(v_G)\big(F\alpha^{-\frac{1}{2}}\big)=\Big(\frac{3}{2}\big(\eta(G'')+(-1)^{p(G)}\eta(G'')\big)F-\frac{1}{2}\big(\eta(G)-(-1)^{p(G)}\eta(G)\big)F'' +(-1)^{p(F)}\Big(\eta(G')F'+\frac{1}{2}\big(G''+(-1)^{p(G)}G''\big)\eta(F)\Big) +(-1)^{p(G)+1}\eta(G'')\big(F+(-1)^{p(F)}F\big)\Big)\alpha^1$$

- *For $\lambda\neq-\frac{1}{2}$ and $\mu-\lambda=\frac{3}{2}$, the generator can be chosen as follows:*

$$\gamma_{\lambda,\lambda+\frac{3}{2}}(v_G)(F\alpha^{\lambda})=\big(\overline{\eta}(G'')F\big)\,\alpha^{\lambda+\frac{3}{2}}$$

- *For $\mu-\lambda=\frac{5}{2}$, the generator can be chosen as follows:*

$$\gamma_{\lambda,\lambda+\frac{5}{2}}(v_G)(F\alpha_1^{\lambda})=(2\lambda G^3F+3(-1)^{p(G)}\overline{\eta}(G'')\overline{\eta}(F))\alpha^{\lambda+\frac{5}{2}}$$

Proof. The odd cohomology $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(1), \mathfrak{D}_{\lambda,\mu})_1$ was calculated in [9].

Now, we are interested in the even cohomology. The adjoint $\mathcal{K}(1)$-module is $\mathrm{Vect}_p(\mathbb{R})$-isomorphic to $\mathrm{Vect}_p(\mathbb{R}) \oplus \Pi(\mathcal{F}_{-\frac{1}{2}})$, so the even 1-cocycle γ_0 decomposes into two components: $\gamma_0 = (\gamma_{00}, \gamma_{11})$, where

$$\gamma_{00} : \mathrm{Vect}_p(\mathbb{R}) \longrightarrow (\mathfrak{D}_{\lambda,\mu})_0$$
$$\gamma_{11} : \mathcal{F}_{-\frac{1}{2}} \longrightarrow (\mathfrak{D}_{\lambda,\mu})_1$$

- For $\lambda = \mu$, a straightest computation shows that $\gamma_{\lambda,\lambda}$ is prolongation of $c_{\lambda,\lambda}(X, F) = X'F$ calculated by Feigen and Fuchs in [2].
- For $\mu - \lambda \geq 2$, we have $(\mathfrak{D}_{\lambda,\mu})_0 = \mathcal{D}_{\lambda,\mu} \oplus \mathcal{D}_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}$. Then the component γ_{00} of γ is broken on $(\gamma_{000}, \gamma_{001})$, where

$$\gamma_{000} : \mathrm{Vect}_p(\mathbb{R}) \longrightarrow \mathcal{D}_{\lambda,\mu}$$
$$\gamma_{001} : \mathrm{Vect}_p(\mathbb{R}) \longrightarrow \mathcal{D}_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}$$

If the component γ_{000} is a differential operator with degree ≥ 2, then it vanish on $\mathfrak{sl}(2)$, thus γ_0 is a supertransvectant by the following lemma.

Lemma 2.5 (see [8, Theorem 3.1])**.** *Up to coboundary, any even 1-cocycle* $\gamma \in Z^1(\mathcal{K}(1), \mathfrak{D}_{\lambda,\mu})$ *vanishing on* $\mathfrak{sl}(2)$ *is* $\mathfrak{osp}(1|2)$*-invariant. That is, if* $\gamma(X_1) = \gamma(X_x) = \gamma(X_{x^2}) = 0$*, then the restriction of* γ *to* $\mathfrak{osp}(1|2)$ *is trivial.*

As the adjoint $\mathcal{K}(1)$-module is isomorphic to $\mathfrak{F}_{-1}$, the 1-cocycle $\gamma : \mathcal{K}(1) \to \mathfrak{D}_{\lambda,\mu}$ can be looked as a differential operator $\gamma : \mathfrak{F}_{-1} \otimes \mathfrak{F}_\lambda \to \mathfrak{F}_\mu$. We consider the supertransvectants $\mathfrak{J}_k^{-1,\lambda}$ as $k = \mu - \lambda$. If $\mu - \lambda \geq 2$, we look for those which are nontrivial 1-cocycles. In this way we can deduce $\gamma_{\lambda,\lambda+2}$, $\gamma_{0,3}$, $\gamma_{-\frac{1}{2},\frac{5}{2}}$, and $\gamma_{a,a+4}$, where $a = \frac{-7\pm\sqrt{33}}{4}$. □

3 Deformation theory and cohomology

Deformation theory of Lie algebra homomorphisms was first considered for one-parameter deformations [2, 4, 14]. Recently, deformations of Lie (super)algebras with multiparameters were intensively studied (e.g., [1, 2, 3, 5, 6, 17, 18]). Here we give an outline of this theory.

3.1 Infinitesimal deformations

Let $\rho_0 : \mathfrak{g} \to \mathrm{End}(V)$ be an action of a Lie superalgebra $\mathfrak{g}$ on a vector superspace V. When studying deformations of the $\mathfrak{g}$-action ρ_0, one usually starts with *infinitesimal* deformations:

$$\rho = \rho_0 + t\gamma \tag{3.1}$$

where $\gamma : \mathfrak{g} \to \mathrm{End}(V)$ is a linear map and t is a formal parameter. The homomorphism condition

$$[\rho(x), \rho(y)] = \rho([x, y]) \tag{3.2}$$

where $x, y \in \mathfrak{g}$, is satisfied in order 1 in t if and only if γ is a 1-cocycle. That is, the map γ satisfies

$$\gamma[x, y] - (-1)^{p(x)p(\gamma)}[\rho_0(x), \gamma(y)] + (-1)^{p(y)(p(x)+p(\gamma))}[\rho_0(y), \gamma(x)] = 0$$

If $\dim \mathrm{H}^1(\mathfrak{g}; \mathrm{End}(V)) = m$, then one can choose 1-cocycles $\gamma_1, \dots, \gamma_m$ as a basis of $\mathrm{H}^1(\mathfrak{g}; \mathrm{End(V)})$ and consider the following infinitesimal deformation:

$$\rho = \rho_0 + \sum_{i=1}^{m} t_i \gamma_i \tag{3.3}$$

where $t_1, \dots, t_m$ are independent formal parameters with t_i and γ_i are the same parity, i.e., $p(t_i) = p(\gamma_i)$.

To study deformations of $\mathcal{K}(1)$-action on $\widetilde{\mathcal{S}}^n_\delta$, we must consider the space $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(1), \mathrm{End}(\widetilde{\mathcal{S}}^n_\delta))$. Any infinitesimal deformation of the $\mathcal{K}(1)$-module $\widetilde{\mathcal{S}}^n_\delta$ is then of the form

$$\widetilde{\mathfrak{L}}_{v_F} = \mathfrak{L}_{v_F} + \mathfrak{L}^{(1)}_{v_F} \tag{3.4}$$

where $\mathfrak{L}_{v_F}$ is the Lie derivative of $\widetilde{\mathcal{S}}^n_\delta$ along the vector field v_F defined by (2.3), and

$$\begin{aligned} \mathfrak{L}^{(1)}_{v_F} =& \sum_{\lambda} \sum_{k=0,3,4,5} t_{\lambda,\lambda+\frac{k}{2}} \gamma_{\lambda,\lambda+\frac{k}{2}}(v_F) + t_{0,3}\gamma_{0,3}(v_F) + t_{\frac{-5}{2},\frac{1}{2}}\gamma_{\frac{-5}{2},\frac{1}{2}}(v_F) \\ &+ \sum_{i=1,2} t_{a_i,a_i+4}\gamma_{a_i,a_i+4}(v_F) + \widetilde{t}_{0,\frac{1}{2}}\widetilde{\gamma}_{0,\frac{1}{2}}(v_F) + t_{0,\frac{1}{2}}\gamma_{0,\frac{1}{2}}(v_F) + t_{-\frac{1}{2},0}\gamma_{-\frac{1}{2},0}(v_F) \end{aligned} \tag{3.5}$$

where $a_1 = \frac{-7-\sqrt{33}}{4}$ and $a_2 = \frac{-7+\sqrt{33}}{4}$.

Let us emphasize that we restrict our study to the deformation (3.4) for generic values of λ.

3.2 Integrability conditions

Consider the supercommutative associative superalgebra $\mathbb{C}[[t_1, \dots, t_m]]$ with unity and consider the problem of integrability of infinitesimal deformations. Starting with the infinitesimal deformation (3.3), we look for a formal series

$$\rho = \rho_0 + \sum_{i=1}^{m} t_i \gamma_i + \sum_{i,j} t_i t_j \rho^{(2)}_{ij} + \cdots \tag{3.6}$$

where the highest-order terms $\rho^{(2)}_{ij}, \rho^{(3)}_{ijk}, \dots$ are linear maps from $\mathfrak{g}$ to End(V) with $p(\rho^{(2)}_{ij}) = p(t_i t_j),\ p(\rho^{(3)}_{ijk}) = p(t_i t_j t_k), \dots$ such that the map

$$\rho : \mathfrak{g} \to \mathrm{End(V)} \otimes \mathbb{C}[[t_1, \dots, t_m]] \tag{3.7}$$

satisfies the homomorphism condition (3.2) at any order in $t_1, \dots, t_m$.

However, quite often the above problem has no solution. Following [1] and [5], we must impose extra algebraic relations on the parameters $t_1, \dots, t_m$ in order to get the full deformation. Let $\mathcal{R}$ be an ideal in $\mathbb{C}[[t_1, \dots, t_m]]$ generated by some set of relations, the quotient

$$\mathcal{A} = \mathbb{C}[[t_1, \dots, t_m]]/\mathcal{R} \tag{3.8}$$

is a supercommutative associative superalgebra with unity, and one can speak about deformations with base $\mathcal{A}$ (see [2] for details). The map (3.7) sends $\mathfrak{g}$ to $\mathrm{End}(V) \otimes \mathcal{A}$.

3.3 Equivalence and the first cohomology

The notion of equivalence of deformations over commutative associative algebras has been considered in [1].

Definition 3.1. Two deformations ρ and ρ' with the same base $\mathcal{A}$ are called equivalent if there exists a formal inner automorphism Ψ of the associative superalgebra $\mathrm{End}(V)\otimes\mathcal{A}$ such that

$$\Psi\circ\rho=\rho' \text{ and } \Psi(\mathbb{I})=\mathbb{I}$$

where $\mathbb{I}$ is the unity of the superalgebra $\mathrm{End}(V)\otimes\mathcal{A}$.

As a consequence, two infinitesimal deformations $\rho_1=\rho_0+t\gamma_1$ and $\rho_2=\rho_0+t\gamma_2$ are equivalent if and only if $\gamma_1-\gamma_2$ is a coboundary:

$$(\gamma_1-\gamma_2)(x)=(-1)^{p(x)p(A_1)}[\rho_0(x),A_1]=\delta A_1(x)$$

where $A_1\in\mathrm{End}(V)$ and δ stands for the cohomological Chevalley-Eilenberg coboundary for cochains on $\mathfrak{g}$ with values in $\mathrm{End}(V)$ [10, 4].

So the first cohomology space $\mathrm{H}^1(\mathfrak{g};\mathrm{End}(V))$ determines and classifies infinitesimal deformations up to equivalence.

4 Computing the second-order Maurer-Cartan equation

Any infinitesimal deformation of the $\mathcal{K}(1)$-module $\widetilde{\mathcal{S}}^n_\delta$ can be integrated to a formal deformation, such deformation is then of the form

$$\widetilde{\mathfrak{L}}_{v_F}=\mathfrak{L}_{v_F}+\mathfrak{L}^{(1)}_{v_F}+\mathfrak{L}^{(2)}_{v_F}+\cdots \tag{4.1}$$

where

$$\mathfrak{L}^{(2)}_{v_F}=\sum_{i,j}t_it_j\rho^{(2)}_{ij},\quad \mathfrak{L}^{(3)}_{v_F}=\sum_{i,j,k}t_it_jt_k\rho^{(3)}_{ijk},\ldots$$

By setting

$$\varphi_t=\rho-\rho_0,\quad \mathcal{L}^{(1)}=\sum_{i=1}^{m}t_i\gamma_i,\quad \mathcal{L}^{(2)}=\sum_{i,j}t_it_j\rho^{(2)}_{ij},\ldots$$

we can rewrite the relation (3.2) as follows:

$$[\varphi_t(G),\rho_0(H)]+[\rho_0(G),\varphi_t(H)]-\varphi_t([G,H])+\sum_{i,j>0}[\mathcal{L}^{(i)}(G),\mathcal{L}^{(j)}(H)]=0 \tag{4.2}$$

The first three terms give $(\delta\varphi_t)(G,H)$. The relation (4.2) becomes now equivalent to

$$\delta\varphi_t(G,H)+\sum_{i,j>0}[\mathcal{L}^{(i)}(G),\mathcal{L}^{(j)}(H)]=0 \tag{4.3}$$

Definition 4.1. Let $\gamma_1, \gamma_2 : \mathfrak{g} \to \mathrm{End}(V)$ be two arbitrary linear maps, we denote by $[[\cdot,\cdot]]$ the *cup-product* defined by

$$\begin{aligned}&[[\gamma_1,\gamma_2]] : \mathfrak{g}\otimes\mathfrak{g} \longrightarrow \mathrm{End}\\ &[[\gamma_1,\gamma_2]](G,H) = (-1)^{|G||\gamma_2|}\gamma_1(G)\circ\gamma_2(H) - (-1)^{|H|(|G|+|\gamma_2|)}\gamma_1(H)\circ\gamma_2(G)\end{aligned} \tag{4.4}$$

where $|\cdot|$ denotes the parity.

Expanding (4.3) in power series in $t_1,\dots,t_m$, we obtain the following equation for $\mathcal{L}^{(s)}$:

$$\delta\mathcal{L}^{(s)}(G,H) + \sum_{i+j=s}[\mathcal{L}^{(i)}(H),\mathcal{L}^{(j)}(G)] = 0 \tag{4.5}$$

The first nontrivial relation is

$$\delta\mathcal{L}^{(2)} = -\frac{1}{2}\left[\left[\sum_{\lambda}\sum_{j\in\{0,3,4,5\}} t_{\lambda,\lambda+\frac{j}{2}}\gamma_{\lambda,\lambda+\frac{j}{2}}, \sum_{\lambda}\sum_{j\in\{0,3,4,5\}} t_{\lambda,\lambda+\frac{j}{2}}\gamma_{\lambda,\lambda+\frac{j}{2}}\right]\right] \tag{4.6}$$

Therefore, it is easy to check that for any two 1-cocycles γ_1 and $\gamma_2 \in Z^1(\mathfrak{g},\mathrm{End}(V))$, the bilinear map $[[\gamma_1,\gamma_2]]$ is a 2-cocycle. The first nontrivial relation (4.6) is precisely the condition for this 2-cocycle to be a coboundary. Moreover, if one of the 1-cocycles γ_1 or γ_2 is a coboundary, then $[[\gamma_1,\gamma_2]]$ is a 2-coboundary. We, therefore, naturally deduce that the operation (4.4) defines a bilinear map:

$$\mathrm{H}^1(\mathfrak{g};\mathrm{End}(V))\otimes\mathrm{H}^1(\mathfrak{g};\mathrm{End}(V)) \longrightarrow \mathrm{H}^2(\mathfrak{g};\mathrm{End}(V)) \tag{4.7}$$

All the potential obstructions are in the image of $\mathrm{H}^1(\mathfrak{g};\mathrm{End}(V))$ under the cup-product in $\mathrm{H}^2(\mathfrak{g};\mathrm{End}(V))$.

The bilinear map (4.7) can be decomposed in homogeneous components as follows:

$$\mathrm{H}^1(\mathfrak{g};\mathrm{End}(V))_i\otimes\mathrm{H}^1(\mathfrak{g};\mathrm{End}(V))_j \longrightarrow \mathrm{H}^2(\mathfrak{g};\mathrm{End}(V))_{i+j} \tag{4.8}$$

where $i,j\in\mathbb{Z}/2\mathbb{Z}$.

4.1 Cup-products of the nontrivial 1-cocycles

Let us consider the 2-cochains

$$B_{\lambda,\lambda+k}(G,H) = \sum_{j\in\{0,\frac{1}{2},1,\dots,k\}} t_{\lambda+j,\lambda+k}t_{\lambda,\lambda+j}[[\gamma_{\lambda+j,\lambda+k},\gamma_{\lambda,\lambda+j}]](G,H) \tag{4.9}$$

then it is easy to see that

$$B_{\lambda,\lambda+k} \in \mathrm{Z}^2(\mathcal{K}(1),\mathfrak{D}_{\lambda,\mu}) \tag{4.10}$$

and we we compute successively the 2-cocycles $B_{\lambda,\lambda+k}(G,H)$ for $G = g_0+\theta g_1$ and $H = h_0+\theta h_1$, two contact vectors, and $F = f_0+\theta f_1 \in \mathfrak{F}_\lambda$. For generic values of λ, we have the following:

✓ For $k=0$, let

$$B_{\lambda,\lambda}(G,H)=t^2_{\lambda,\lambda}[[\gamma_{\lambda,\lambda},\gamma_{\lambda,\lambda}]]:\mathcal{K}(1)\times\mathcal{K}(1)\longrightarrow\mathfrak{D}_{\lambda,\lambda}$$

and

$$B_{\lambda,\lambda}(G,H)=0$$

✓ For $k=\frac{3}{2}$, let

$$\begin{aligned}B_{\lambda,\lambda+\frac{3}{2}}(G,H)=(&t_{\lambda,\lambda+\frac{3}{2}}t_{\lambda,\lambda}[[\gamma_{\lambda,\lambda+\frac{3}{2}},\gamma_{\lambda,\lambda}]]+t_{\lambda+\frac{3}{2},\lambda+\frac{3}{2}}t_{\lambda,\lambda+\frac{3}{2}}[[\gamma_{\lambda+\frac{3}{2},\lambda+\frac{3}{2}},\gamma_{\lambda,\lambda+\frac{3}{2}}]])(G,H)\\&:\mathcal{K}(1)\times\mathcal{K}(1)\longrightarrow\mathfrak{D}_{\lambda,\lambda+\frac{3}{2}}\end{aligned}$$

and

$$\begin{aligned}B_{\lambda,\lambda+\frac{3}{2}}(G,H)(F)=&\big(t_{\lambda,\lambda+\frac{3}{2}}t_{\lambda,\lambda}-t_{\lambda+\frac{3}{2},\lambda+\frac{3}{2}}t_{\lambda,\lambda+\frac{3}{2}}\big)\\&\times\big(\big(h_0^3g_0'-h_0'g_0^3\big)f_0+\big(g_0'h_1''-g_1''h_0'\big)\big(f_0+\theta f_1\big)+\theta\big(g_1'h_1'\big)f_0\big)\end{aligned}$$

✓ For $k=2$, let

$$\begin{aligned}B_{\lambda,\lambda+2}(G,H)=(&t_{\lambda,\lambda+2}t_{\lambda,\lambda}[[\gamma_{\lambda,\lambda+2},\gamma_{\lambda,\lambda}]]+t_{\lambda+2,\lambda+2}t_{\lambda,\lambda+2}[[\gamma_{\lambda+2,\lambda+2},\gamma_{\lambda,\lambda+2}]])\,(G,H)\\&:\mathcal{K}(1)\times\mathcal{K}(1)\longrightarrow\mathfrak{D}_{\lambda,\lambda+2}\end{aligned}$$

and

$$\begin{aligned}B_{\lambda,\lambda+2}(G,H)(F)=&(t_{\lambda,\lambda+2}t_{\lambda,\lambda}-t_{\lambda+2,\lambda+2}t_{\lambda,\lambda+2})\big(2\lambda(h_0^3g_0'-g_0^3h_0')f_0\\&+2\lambda(g_0'h_1''-g_1''h_0')f_1+\theta((2\lambda+7)(g_0^3h_0'-h_0^3g_0')f_1\\&+2\lambda(h_1^3g_0'-g_1^3h_0')f_0\big)-\theta t_{\lambda,\lambda+2}t_{\lambda,\lambda}\big(2\lambda+3)(g_0^3h_1'-h_0^3g_1')f_0\\&-3(h_1''g_0''-h_0''g_1'')f_0+3(g_1''h_1'+g_1'h_1'')f_0\big)\\&+\theta t_{\lambda+2,\lambda+2}t_{\lambda,\lambda+2}\big(-2\lambda(g_0'h_1^3-h_0'g_1^3)f_0+3(g_0'h_1''-h_0'g_1'')f_0'\\&-3(g_1'h_1''+h_1'g_1'')f_1\big)\end{aligned}$$

✓ For $k=\frac{5}{2}$, let

$$\begin{aligned}B_{\lambda,\lambda+\frac{5}{2}}=\big(&t_{\lambda,\lambda+\frac{5}{2}}t_{\lambda,\lambda}\big[[\gamma_{\lambda,\lambda+\frac{5}{2}},\gamma_{\lambda,\lambda}]\big]+t_{\lambda+\frac{5}{2},\lambda+\frac{5}{2}}t_{\lambda,\lambda+\frac{5}{2}}\big[[\gamma_{\lambda+\frac{5}{2},\lambda+\frac{5}{2}},\gamma_{\lambda,\lambda+\frac{5}{2}}]\big]\big)(G,H)\\&:\mathcal{K}(1)\times\mathcal{K}(1)\longrightarrow\mathfrak{D}_{\lambda,\lambda+\frac{5}{2}}\end{aligned}$$

and

$$\begin{aligned}B_{\lambda,\lambda+\frac{5}{2}}(G,H)(F)=&(t_{\lambda+\frac{5}{2},\lambda+\frac{5}{2}}t_{\lambda,\lambda+\frac{5}{2}}-t_{\lambda+\frac{5}{2},\lambda+\frac{5}{2}}t_{\lambda,\lambda+\frac{5}{2}}\big)\big(\big(g_0^3h_0'-h_0^3g_0'\big)f_1\\&+3\big(g_1''h_0'-g_0'h_1''\big)f_0'-\theta\big(-4\big(g_0^3h_0'-h_0^3g_0'\big)f_0'+2\lambda\big(g_0^4h_0'-h_0^4g_0'\big)f_0\\&-(2\lambda+1)\big(g_1^3h_0'-h_1^3g_0'\big)f_1+3\big(g_1''h_1'+g_1'h_1''\big)f_0'+3\big(g_1''h_0''-h_1''g_0''\big)f_1\big)\big)\\&+t_{\lambda+\frac{5}{2},\lambda+\frac{5}{2}}t_{\lambda,\lambda+\frac{5}{2}}\big(3\big(g_1''h_0''-h_1''g_1''\big)f_0+\theta\big(-4\big(g_0^3h_0''-g_0''h_0^3\big)f_0\\&+6g_1''h_1''f_0-(1+2\lambda)\big(g_1^3h_1'+h_1^3g_1'\big)f_0+3\big(g_1''h_0'-h_1''g_0'\big)f_1'\\&-4\big(g_1'h_0^3-h_1'g_0^3\big)f_0+\theta\big(\big(g_1'h_1^3+g_1^3h_1'\big)f_0-4\big(g_1'h_0^3-h_1'g_0^3\big)f_1\big)\big)\end{aligned}$$

✓ For $k=3$, let

$$B_{\lambda,\lambda+3}(G,H)=t_{\lambda+\frac{3}{2},\lambda+3}t_{\lambda,\lambda+\frac{3}{2}}[[\gamma_{\lambda+\frac{3}{2},\lambda+3},\gamma_{\lambda,\lambda+\frac{3}{2}}]](G,H):\mathcal{K}(1)\times\mathcal{K}(1)\longrightarrow\mathfrak{D}_{\lambda,\lambda+3}$$

and

$$B_{\lambda,\lambda+3}(G,H)(F)=-2t_{\lambda+\frac{3}{2},\lambda+3}t_{\lambda,\lambda+\frac{3}{2}}\big(g_1''h_1''f_0+\theta(g_1''h_1''f_1-g_0^3h_1''f_0+h_0^3g_1''f_0)\big)$$

✓ For $k=\frac{7}{2}$, let

$$\begin{aligned}B_{\lambda,\lambda+\frac{7}{2}}(G,H)=&t_{\lambda+\frac{3}{2},\lambda+\frac{7}{2}}t_{\lambda,\lambda+\frac{3}{2}}[[\gamma_{\lambda+\frac{3}{2},\lambda+\frac{7}{2}},\gamma_{\lambda,\lambda+\frac{3}{2}}]](G,H)\\&+t_{\lambda+2,\lambda+\frac{7}{2}}t_{\lambda,\lambda+2}[[\gamma_{\lambda+2,\lambda+\frac{7}{2}},\gamma_{\lambda,\lambda+2}]](G,H):\mathcal{K}(1)\times\mathcal{K}(1)\longrightarrow\mathfrak{D}_{\lambda,\lambda+4}\end{aligned}$$

and

$$\begin{aligned}B_{\lambda,\lambda+\frac{7}{2}}(G,H)(F)=&(t_{\lambda+\frac{3}{2},\lambda+\frac{7}{2}}t_{\lambda,\lambda+\frac{3}{2}}-t_{\lambda+2,\lambda+\frac{7}{2}}t_{\lambda,\lambda+2})\big(6g_1''h_1''f_1-2\lambda(g_0^3h_1''-h_0^3g_1'')f_0\\&+\theta\big(-6g_1''h_1''f_0'-2(\lambda+3)(g_1''h_1^3+g_1^3h_1'')f_0-2(\lambda+3)(g_0^3h_1''-h_0^3g_1'')f_1\big)\big)\end{aligned}$$

✓ For $k=4$, let

$$\begin{aligned}B_{\lambda,\lambda+4}(G,H)=&(t_{\lambda+\frac{3}{2},\lambda+4}t_{\lambda,\lambda+\frac{3}{2}}[[\gamma_{\lambda+\frac{3}{2},\lambda+4},\gamma_{\lambda,\lambda+\frac{3}{2}}]]+t_{\lambda+\frac{5}{2},\lambda+4}t_{\lambda,\lambda+\frac{5}{2}}[[\gamma_{\lambda+\frac{5}{2},\lambda+4},\gamma_{\lambda,\lambda+\frac{5}{2}}]]\\&+t_{\lambda+2,\lambda+4}t_{\lambda,\lambda+2}[[\gamma_{\lambda+2,\lambda+4},\gamma_{\lambda,\lambda+2}]])(G,H):\mathcal{K}(1)\times\mathcal{K}(1)\longrightarrow\mathfrak{D}_{\lambda,\lambda+4}\end{aligned}$$

and

$$\begin{aligned}B_{\lambda,\lambda+4}(G,H)(F)=&\left(t_{\lambda+\frac{3}{2},\lambda+4}t_{\lambda,\lambda+\frac{3}{2}}+t_{\lambda+\frac{5}{2},\lambda+4}t_{\lambda,\lambda+\frac{5}{2}}+\frac{1}{3}t_{\lambda+2,\lambda+4}t_{\lambda,\lambda+2}\right)\\&\times\big(-2\lambda(g_1''h_1^3+g_1^3h_1'')f_0+6g_1''h_1''f_0'+4(g_0^3h_1''-h_0^3g_1'')f_1\\&+\theta(-(2\lambda+1)(g_1''h_1^3+g_1^3h_1'')f_1+6g_1''h_1''f_1'+2\lambda(g_0^4h_1''-h_0^4g_1'')f_0\\&-7(g_0^3h_1''-h_0^3g_1'')f_0'+2\lambda(g_0^3h_1^3-h_0^3g_1^3)f_0)\big)\end{aligned}$$

✓ For $k=\frac{9}{2}$, let

$$\begin{aligned}B_{\lambda,\lambda+\frac{9}{2}}(G,H)=&t_{\lambda+2,\lambda+\frac{9}{2}}t_{\lambda,\lambda+2}\big[\big[\gamma_{\lambda+2,\lambda+\frac{9}{2}},\gamma_{\lambda,\lambda+2}\big]\big](G,H)\\&+t_{\lambda+\frac{5}{2},\lambda+\frac{9}{2}}t_{\lambda,\lambda+\frac{5}{2}}\big[\big[\gamma_{\lambda+\frac{5}{2},\lambda+\frac{9}{2}},\gamma_{\lambda,\lambda+\frac{5}{2}}\big]\big](G,H):\mathcal{K}(1)\times\mathcal{K}(1)\longrightarrow\mathfrak{D}_{\lambda,\lambda+\frac{9}{2}}\end{aligned}$$

and

$$\begin{aligned}B_{\lambda,\lambda+\frac{9}{2}}(G,H)(F)=&(t_{\lambda+2,\lambda+\frac{9}{2}}t_{\lambda,\lambda+2}-t_{\lambda+\frac{7}{2},\lambda+\frac{9}{2}}t_{\lambda,\lambda+\frac{5}{2}})\big(6(\lambda+2)(g_0^3h_1''-g_1''h_0^3)f_0'\\&-4\lambda(\lambda+4)(g_0^3h_1^3-g_1^3h_0^3)f_0+2\lambda(g_0^4h_1''-h_0^4g_1'')f_0\\&+3(2\lambda+1)(g_1^3h_1''+g_1''h_1^3)f_1-18g_1''h_1''f_1'\\&+\theta\big(-4\lambda(\lambda+4)(h_0^4g_0^3-g_0^4h_0^3)f_0-3(g_0^4h_1''-g_1''h_0^4)f_1\\&+(12-(2\lambda+5)(2\lambda+3))(g_1^3h_0^3-h_1^3g_0^3)f_1+3\lambda(2\lambda+7)(g_0^3h_1''-g_1''h_0^3)f_1'\\&+4\lambda(2\lambda+5)g_1^3h_1^3f_0-6(\lambda+2)(g_1^3h_1''+g_1''h_1^3)f_0-6\lambda(h_1^4g_1''-g_1^4h_1'')f_0\\&+18g_1''h_1''f_0''\big)\end{aligned}$$

✓ For $k=5$, let

$$B_{\lambda,\lambda+5}(G,H)=t_{\lambda+\frac{5}{2},\lambda+5}t_{\lambda,\lambda+\frac{5}{2}}[[\gamma_{\lambda+\frac{5}{2},\lambda+5},\gamma_{\lambda,\lambda+\frac{5}{2}}]](G,H):\mathcal{K}(1)\times\mathcal{K}(1)\longrightarrow\mathfrak{D}_{\lambda,\lambda+5}$$

and

$$\begin{aligned}B_{\lambda,\lambda+5}(G,H)(F)=\;&t_{\lambda+\frac{5}{2},\lambda+5}t_{\lambda,\lambda+\frac{5}{2}}\\&\times\Bigg(\frac{-1}{2\lambda+5}(h_0^3g_0^4-h_0^4g_0^3)f_0-\frac{2\lambda+3}{\lambda(2\lambda+5)}\lambda(g_0^3h_1^3-g_1^3h_0^3)f_1\\&+\frac{3}{2\lambda(2\lambda+5)}(h_1''g_0^4-g_1''h_0^4)f_1+\frac{3}{\lambda(2\lambda+5)}(h_1''g_0^3-g_1''h_0^3)f_1'\\&+\frac{3}{\lambda(2\lambda+5)}(g_1''h_1^4+h_1''g_1^4)f_0+\frac{6(2\lambda+1)}{2\lambda(2\lambda+5)}(g_1''h_1^3+h_1''g_1^3)f_0'\\&-\frac{9}{\lambda(2\lambda+5)}g_1''h_1''f_0''-2g_1^3h_1^3f_0+\theta(\frac{-(2\lambda+1)}{2\lambda(2\lambda+5)}(h_0^4g_0^3-h_0^3g_0^4)f_1\\&+\frac{4}{2\lambda+5}(g_0^3h_1^4-g_1^4h_0^3)f_0+\frac{6}{2\lambda+5}(g_0^3h_1^3-g_1^3h_0^3)f_0'\\&+\frac{3(4\lambda+1)}{2\lambda(2\lambda+5)}(h_1''g_0^4-g_1''h_0^4)f_0'-\frac{4\lambda+11}{2\lambda+5}(g_0^4h_1^3-h_0^4g_1^3)f_0\\&-\frac{12}{\lambda(2\lambda+5)}(g_0^3h_1''-h_0^3g_1'')f_0''+\frac{3}{\lambda(2\lambda+5)}(g_0^5h_1''-h_0^5g_1'')f_0\\&+\frac{3(2\lambda+1)}{2\lambda(2\lambda+5)}(g_1''h_1^4-h_1''g_1^4)f_1-\frac{6(1+\lambda)}{\lambda(2\lambda+5)}(g_1''h_1^3+h_1''g_1^3)f_1'\\&-\frac{9}{\lambda(2\lambda+5)}g_1''h_1''f_1''-\frac{2(\lambda+3)(2\lambda+1)}{\lambda(2\lambda+5)}g_1^3h_1^3f_1)\Bigg)\end{aligned}$$

Proposition 4.2. *(a) Each of the 2-cocycles*

$$B_{\lambda,\lambda+\frac{3}{2}}\ \textit{for}\ \lambda\neq-\frac{1}{2},\quad B_{\lambda,\lambda+2},\ B_{\lambda,\lambda+\frac{5}{2}}\ \textit{for}\ \lambda\neq-1$$

defines a nontrivial cohomology class. Moreover, these classes are linearly independant.

(b) Each of the 2-cocycles $B_{\lambda,\lambda+3}$, $B_{\lambda,\lambda+\frac{7}{2}}$, $B_{\lambda,\lambda+4}$, $B_{\lambda,\lambda+\frac{9}{2}}$, and $B_{\lambda,\lambda+5}$ is a coboundary.

Proof. A 2-cocycle $B_{\lambda,\lambda+k}$ for $k\in\{\frac{3}{2},2,\frac{5}{2},3,\frac{7}{2},4,\frac{9}{2},5\}$ is a coboundary if and only if it satisfies

$$B_{\lambda,\lambda+k}(G,H)(F)=\delta b_{\lambda,\lambda+k}(G,H)(F)\tag{4.11}$$

where $b_{\lambda,\lambda+k}:\mathcal{K}(1)\to\mathfrak{D}_{\lambda,\lambda+k}$ and

$$\begin{aligned}\delta b_{\lambda,\lambda+k}(G,H)(F)=\;&b_{\lambda,\lambda+k}[G,H](F)-(-1)^{|G||b_{\lambda,\lambda+k}|}\mathcal{L}_G^{\lambda,\lambda+k}\circ(b_{\lambda,\lambda+k})(H)(F)\\&+(-1)^{|G|(|H|+|b_{\lambda,\lambda+k}|)}\mathcal{L}_H^{\lambda,\lambda+k}\circ(b_{\lambda,\lambda+k})(G)(F)\end{aligned}$$

□

For $k\in\{\frac{3}{2},2,\frac{5}{2}\}$, a direct computation shows that those $B_{\lambda,\lambda+k}$ are nontrivial 2-cocycles. For $k\in\{3,\frac{7}{2},4,\frac{9}{2},5\}$, we need the following lemma.

Lemma 4.3. *Let*

$$b : \mathcal{K}(1) \longrightarrow \mathfrak{D}_{\lambda,\mu}$$

be a 1-cochain. If

$$\delta b_{|\mathfrak{osp}(1|2)\times\mathcal{K}(1)} = 0$$

then, for $\lambda \neq \mu$ *or* $\lambda \neq \frac{1-k}{2}$ *or* $\mu \neq \frac{k}{2}$ *where* k *is an integer,* b *is a supertransvectant.*

Proof. The condition $\delta b_{|\mathfrak{osp}(1|2)\times\mathcal{K}(1)} = 0$ implies that b is a 1-cocycle on $\mathfrak{osp}(1|2)$. From the result of [8, Theorem 3.1] the space $\mathrm{H}^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\mu}) = 0$ if $\lambda \neq \mu$ or $\lambda \neq \frac{1-k}{2}$ or $\mu \neq \frac{k}{2}$, where k is an integer. For such values of λ and μ, the condition of 1-cocycle

$$\delta b(X, Y) = b([X, Y]) - (-1)^{p(X)p(b)} X \cdot b(Y) + (-1)^{p(Y)(p(X)+p(b))} Y.b(X) = 0$$

is equivalent to the condition of $\mathfrak{osp}(1|2)$-invariance. Then b is a supertransvectant. □

Remark that the cup-products for $k \in \{3, \frac{7}{2}, 4, \frac{9}{2}, 5\}$ are $\mathfrak{osp}(1|2)$-invariant, then, by Lemma 4.3, they are supertransvectant boundaries. A simple computation shows that

$$B_{\lambda,\lambda+3} = \rho(\lambda, t_\lambda)\delta\mathfrak{J}_4^{-1,\lambda}$$

where

$$\rho(\lambda, t_\lambda) = T_1 \frac{2\lambda + 1}{3(2\lambda + 5)}$$

$$T_1 = -2t_{\lambda+\frac{3}{2},\lambda+3} t_{\lambda,\lambda+\frac{3}{2}}$$

Also one can check that

$$B_{\lambda,\lambda+\frac{7}{2}} = \psi(\lambda, t_\lambda)\delta\mathfrak{J}_{\frac{9}{2}}^{-1,\lambda}$$

where

$$\psi(\lambda, t_\lambda) = T_2 \frac{2\lambda(2\lambda + 1)}{2\lambda + 3}$$

$$T_2 = (t_{\lambda+\frac{3}{2},\lambda+\frac{7}{2}} t_{\lambda,\lambda+\frac{3}{2}} - t_{\lambda+2,\lambda+\frac{7}{2}} t_{\lambda,\lambda+2})$$

Also

$$B_{\lambda,\lambda+4} = \alpha(\lambda, t_\lambda)\delta\mathfrak{J}_5^{-1,\lambda}$$

where

$$\alpha(\lambda, t_\lambda) = T_3 \frac{-3(\lambda + 1)(2\lambda + 1)}{5(2\lambda + 4)(2\lambda^2 + 7\lambda + 2)}$$

$$T_3 = t_{\lambda+\frac{3}{2},\lambda+4} t_{\lambda,\lambda+\frac{3}{2}} + t_{\lambda+\frac{5}{2},\lambda+4} t_{\lambda,\lambda+\frac{5}{2}} + \frac{1}{3} t_{\lambda+2,\lambda+4} t_{\lambda,\lambda+2}$$

Also one can check that

$$B_{\lambda,\lambda+\frac{9}{2}} = \nu(\lambda, t_\lambda)\delta\mathfrak{J}_{\frac{11}{2}}^{-1,\lambda}$$

where

$$\nu(\lambda, t_\lambda) = -T_4\frac{5(\lambda+4)}{\lambda(\lambda+1)(2\lambda+1)}$$
$$T_4 = \left(t_{\lambda+2,\lambda+\frac{9}{2}}t_{\lambda,\lambda+2} - t_{\lambda+\frac{5}{2},\lambda+\frac{9}{2}}t_{\lambda,\lambda+\frac{5}{2}}\right)$$

Finally

$$B_{\lambda,\lambda+5} = \zeta(\lambda, t_\lambda)\delta\mathfrak{J}_{6}^{-1,\lambda}$$

where

$$\zeta(\lambda, t_\lambda) = T_5\frac{(2\lambda+5)(\lambda+1)(2\lambda+1)}{10(\lambda^2+24\lambda+8)}$$
$$T_5 = \left(t_{\lambda+\frac{5}{2},\lambda+5}t_{\lambda,\lambda+\frac{5}{2}}\right)$$

5 Integrability conditions

In this section, we obtain the necessary second-order integrability conditions for the infinitesimal deformation (3.4).

Theorem 5.1. *The following conditions are necessary for integrability for the deformation (3.4):*

1) For $2(\delta-\lambda) \in \{3, \dots, n\}$ *and* $\lambda \neq -\frac{1}{2}$*,*

$$t_{\lambda,\lambda+\frac{3}{2}}t_{\lambda,\lambda} - t_{\lambda+\frac{3}{2},\lambda+\frac{3}{2}}t_{\lambda,\lambda+\frac{3}{2}} = 0$$

2) For $2(\delta-\lambda) \in \{4, \dots, n\}$*,*

$$t_{\lambda,\lambda+2}t_{\lambda,\lambda} = t_{\lambda+2,\lambda+2}t_{\lambda,\lambda+2} = 0$$

3) For $2(\delta-\lambda) \in \{5, \dots, n\}$ *and* $\lambda \neq -1$*,*

$$t_{\lambda,\lambda+\frac{5}{2}}t_{\lambda,\lambda} = t_{\lambda+\frac{5}{2},\lambda+\frac{5}{2}}t_{\lambda,\lambda+\frac{5}{2}} = 0$$

Proof. If we take account of the Proposition 4.2, we deduce the integrability conditions (1), (2) and (3) and we have

$$\mathfrak{L}^{(2)} = -\left(\sum_\lambda \rho(\lambda, t_\lambda)\mathfrak{J}_4^{-1,\lambda} + \sum_\lambda \psi(\lambda, t_\lambda)\mathfrak{J}_{\frac{9}{2}}^{-1,\lambda} + \sum_\lambda \alpha(\lambda, t_\lambda)\mathfrak{J}_5^{-1,\lambda} + \sum_\lambda \nu(\lambda, t_\lambda)\mathfrak{J}_{\frac{11}{2}}^{-1,\lambda} + \sum_\lambda \zeta(\lambda, t_\lambda)\mathfrak{J}_6^{-1,\lambda}\right)$$

□

6 An open problem

It seems to be an interesting open problem to compute the full cohomology ring $\mathrm{H}^*_{\mathrm{diff}}(\mathcal{K}(1); \mathfrak{D}_{\lambda,\lambda+k})$. The only complete result here concerns the first cohomology space. Proposition 4.2 provides a lower bound for the dimension of the second cohomology space. We formulate the following.

Conjecture 6.1. *The space of second cohomology of $\mathcal{K}(1)$ with coefficients in the superspace $\mathfrak{D}_{\lambda,\mu}$ has the following structure:*

$$\mathrm{H}^2_{\mathrm{diff}}(\mathcal{K}(1), \mathfrak{D}_{\lambda,\mu}) \simeq \begin{cases} \mathbb{R} & \text{if } \mu-\lambda=\frac{3}{2} \text{ and } \lambda\neq-\frac{1}{2} \\ \mathbb{R} & \text{if } \mu-\lambda=2 \text{ for all } \lambda \\ \mathbb{R} & \text{if } \mu-\lambda=\frac{5}{2} \text{ and } \lambda\neq-1 \\ 0 & \text{otherwise} \end{cases}$$

7 Examples

We study deformations of $\mathcal{K}(1)$-modules $\widetilde{\mathcal{S}}^n_{\lambda+n}$ for any $n\in\mathbb{N}$ and for arbitrary generic $\lambda\in\mathbb{R}$.

Example 7.1. Let us consider the $\mathcal{K}(1)$-modules $\widetilde{\mathcal{S}}^0_\lambda$ and $\widetilde{\mathcal{S}}^1_{\lambda+1}$.

Proposition 7.2. *Every deformation of $\mathcal{K}(1)$-modules $\widetilde{\mathcal{S}}^0_\lambda$ and $\widetilde{\mathcal{S}}^1_{\lambda+1}$ is equivalent to infinitesimal one.*

Proof. Let us consider the $\mathcal{K}(1)$-module $\widetilde{\mathcal{S}}^0_\lambda$. Any infinitesimal deformation is given by

$$\widetilde{\mathcal{L}}_{v_F} = \mathcal{L}_{v_F} + \mathcal{L}^{(1)}_{v_F} \tag{7.1}$$

where $\mathcal{L}_{v_F}$ is the Lie derivative of $\widetilde{\mathcal{S}}^0_\lambda$ along the vector field v_F defined by (2.3), and

$$\mathcal{L}^{(1)}_{v_F} = t_{\lambda,\lambda}\gamma_{\lambda,\lambda} \tag{7.2}$$

$$\partial(\mathcal{L}^{(2)}_{v_F}) = t^2_{\lambda,\lambda}[[\gamma_{\lambda,\lambda}, \gamma_{\lambda,\lambda}]] \tag{7.3}$$

but, by a direct computation, we show that $[[\gamma_{\lambda,\lambda}, \gamma_{\lambda,\lambda}]] = 0$ for all λ, then $\partial(\mathcal{L}^{(2)}_{v_F}) = 0$ and as a consequence $\mathcal{L}^{(2)}_{v_F} = 0$.

Now consider the $\mathcal{K}(1)$-module $\widetilde{\mathcal{S}}^1_{\lambda+1}$. Any infinitesimal deformation is given by

$$\widetilde{\mathcal{L}}_{v_F} = \mathcal{L}_{v_F} + \mathcal{L}^{(1)}_{v_F} \tag{7.4}$$

where $\mathcal{L}_{v_F}$ is the Lie derivative of $\widetilde{\mathcal{S}}^1_{\lambda+1}$ along the vector field v_F defined by (2.3), and

$$\mathcal{L}^{(1)}_{v_F} = \sum_{j\in\{\frac{1}{2},1\}} t_{\lambda+j,\lambda+j}\gamma_{\lambda+j,\lambda+j} \tag{7.5}$$

By the same arguments, we can that show in this case $\mathcal{L}^{(2)} = 0$, then the deformation is infinitesimal. □

Example 7.3. Consider the $\mathcal{K}(1)$-module $\widetilde{\mathcal{S}}^3_{\lambda+3}$. In this case,

$$\widetilde{\mathcal{S}}^3_{\lambda+3} = \sum_{k=0}^{3} \mathfrak{F}_{(\lambda+3)-\frac{k}{2}}$$

For $\lambda \neq -2$, the deformation of this $\mathcal{K}(1)$-module is of degree 1 given by

$$\widetilde{\mathcal{L}}_{v_F} = \mathcal{L}_{v_F} + \mathcal{L}^{(1)}_{v_F} \tag{7.6}$$

where $\mathcal{L}_{v_F}$ is the Lie derivative of $\widetilde{\mathcal{S}}^3_{\lambda+3}$ along the vector field v_F defined by (2.3), $\mathcal{L}^{(1)}_{v_F}$ is defined as

$$\mathcal{L}^{(1)}_{v_F} = \sum_{j\in\{\frac{3}{2},2,\frac{5}{2},3\}} t_{\lambda+j,\lambda+j}\gamma_{\lambda+j,\lambda+j} + t_{\lambda+\frac{3}{2},\lambda+3}\gamma_{\lambda+\frac{3}{2},\lambda+3}$$
$$\partial(\mathfrak{L}^{(2)}) = t_{\lambda+3,\lambda+3}t_{\lambda+\frac{3}{2},\lambda+3}[[\gamma_{\lambda+3,\lambda+3}, \gamma_{\lambda+\frac{3}{2},\lambda+3}]]$$

The condition of integrability is

$$t_{\lambda+3,\lambda+3}t_{\lambda+\frac{3}{2},\lambda+3} = 0 \tag{7.7}$$

where $\lambda + \frac{3}{2} \neq -\frac{1}{2}$, i.e., $\lambda \neq -2$.

In this case, we have $\mathfrak{L}^{(2)} = 0$, then this condition is necessary and sufficient for integrability of the deformation (7.6).

Let, in this case (i.e., $\lambda \neq -2$), $\mathcal{A}$ be the supercommutative associative superalgebra defined by the quotient of $\mathbb{C}[[t_{\lambda+3,\lambda+3}, t_{\lambda+\frac{3}{2},\lambda+3}]]$ by the ideal $\mathcal{R}$ generated by equation (7.7). Then we speak about a deformation with base $\mathcal{A}$.

For $\lambda = -2$, one has $\partial(\mathfrak{L}^{(2)}) = 0$, then the deformation of this $\mathcal{K}(1)$-module is equivalent to infinitesimal one.

Acknowledgement

We are grateful to Claude Roger for his constant support.

References

[1] A. Fialowski. An example of formal deformations of Lie algebras. In *"Proceedings of NATO Conference on Deformation Theory of Algebras and Applications (Il Ciocco, Italy, 1986)"*. Kluwer, Dordrecht, 1988, 375–401.

[2] A. Fialowski and D. B. Fuchs. Construction of miniversal deformations of Lie algebras. J. Func. Anal. **161** (1999), 76–110.

[3] A. Fialowski. Deformations of Lie algebras. Mat. Sbornyik USSR, **127** (169), (1985), 476–482; English translation: Math. USSR-Sb. **55** (1986), 467–473.

[4] A. Nijenhuis and R. W. Richardson. Deformations of homomorphisms of Lie groups and Lie algebras. Bull. Amer. Math. Soc. **73** (1967), 175–179.

[5] B. Agrebaoui, F. Ammar, P. Lecomte, and V. Ovsienko. Multi-parameter deformations of the module of symbols of differential operators. Int. Math. Res. Notices. **16** (2002), 847–869.

[6] B. Agrebaoui, N. B. Fraj, M. B. Ammar, and V. Ovsienko. Deformations of modules of differential forms. Nonlinear Math. Phys. **10** (2003), 148-156.

[7] M. Ben Ammar and M. Boujelbene. $\mathfrak{sl}(2)$-trivial deformations of $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$-modules of symbols. SIGMA Symmetry Integrability Geom. Methods Appl. **4** (2008), Paper 065, 19.

[8] I. Basdouri and M. Ben Ammar. Cohomology of $\mathfrak{osp}(1|2)$ acting on Linear Differential Operators on the supercercle $S^{1|1}$. Lett. Math. Phys. **81** (2007), 239–251.

[9] I. Basdouri, M. Ben Ammar, N. Ben Fraj, M. Boujelben, and K. Kamoun. Cohomology of the Lie superalgebra of contact vector fields on $\mathbb{R}^{1|1}$ and deformations of the superspace of symbols. Preprint arXiv:math/0702645v1.

[10] D. B. Fuchs. *Cohomology of infinite-dimensional Lie algebras.* Plenum Publ., New York, 1986.

[11] F. Gieres and S. Theisen. Superconformally covariant operators and super W-algebras. J. Math. Phys. **34** (1993), 5964–5985.

[12] H. Gargoubi and V. Ovsienko. Supertransvectants and symplectic geometry. Int. Math. Res. Not. IMRN. **2008** (2008), article ID rnn021, 19 pages.

[13] L. Guieu and C. Roger. *L'algèbre et le groupe de Virasoro.* Publication of CRM, Montréal, 2007, ISBN 2-921120-44-5.

[14] R. W. Richardson. Deformations of subalgebras of Lie algebras. J. Diff. Geom. **3** (1969), 289–308.

[15] S. Bouarroud and V. Ovsienko. Three cocycles on $\mathrm{Diff}(S^1)$ generalizing the Schwarzsian derivative. Int. Math. Res. Notices **1** (1998), 25–39.

[16] V. G. Kac and W. Van De Leur. On classification of superconformal algebras. In *"Strings 88"*. S. J. Gates, C. R. Preitschopf, and W. Siegel, Eds. World Scientific, Singapore, 1989, 77–106.

[17] V. Ovsienko and C. Roger. Deforming the Lie algebra of vector fields on S^1 inside the Lie algebra of pseudodifferential operators on S^1. AMS Transl. Ser. 2 (Adv. Math. Sci.) **194** (1999), 211–227.

[18] V. Ovsienko and C. Roger. Deforming the Lie algebra of vector fields on S^1 inside the Poisson algebra on $\dot{T}^*S^1$. Comm. Math. Phys. **198** (1998) 97–110.

[19] W.-J. Huang. Superconformal covariantization of superdifferential operator on $(1|1)$ superspace and classical $N = 2W$ superalgebras. J. Math. Phys. **35** (1994), 2570–2582.

Received April 16, 2008
Revised December 05, 2008