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Research Article

Dynamical Yang-Baxter Maps Associated with Homogeneous Pre-Systems⋆

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Dedicated to Professor Susumu Okubo on the occasion of his eightieth birthday

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Abstract We construct dynamical Yang-Baxter maps, which are set-theoretical solutions to a version of the quantum dynamical Yang-Baxter equation, by means of homogeneous pre-systems, that is, ternary systems encoded in the reductive homogeneous space satisfying suitable conditions. Moreover, a characterization of these dynamical Yang-Baxter maps is presented.

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1 Introduction

The quantum dynamical Yang-Baxter equation (QDYBE for short) [9,10], a generalization of the quantum Yang-Baxter equation (QYBE for short) [2,3,40,41], has been studied extensively in recent years (see [7] and the references therein). Dynamical Yang-Baxter maps [31,32,34] are set-theoretical solutions to a version of the QDYBE.

Let H and X be nonempty sets with a map $(\cdot) : H \times X \ni (\lambda, x) \mapsto \lambda \cdot x \in H$. A map $R(\lambda) : X \times X \to X \times X$ $(\lambda \in H)$ is a dynamical Yang-Baxter map associated with H, X and $(\cdot)$, if and only if, for every $\lambda \in H$, $R(\lambda)$ satisfies the following equation on $X \times X \times X$:

$$R_{23}(\lambda)R_{13}\Big(\lambda \cdot X^{(2)}\Big)R_{12}(\lambda) = R_{12}\Big(\lambda \cdot X^{(3)}\Big)R_{13}(\lambda)R_{23}\Big(\lambda \cdot X^{(1)}\Big). \tag{1.1}$$

Here $R_{12}(\lambda)$, $R_{12}(\lambda \cdot X^{(3)})$, $R_{23}(\lambda \cdot X^{(1)})$, and others are the maps from $X \times X \times X$ to itself defined as follows: for $u, v, w \in X$,

$$\begin{aligned} R_{12}(\lambda)(u,v,w) &= \big(R(\lambda)(u,v), w\big), \\ R_{12}\Big(\lambda \cdot X^{(3)}\Big)(u,v,w) &= R_{12}(\lambda \cdot w)(u,v,w), \\ R_{23}\Big(\lambda \cdot X^{(1)}\Big)(u,v,w) &= \big(u, R(\lambda \cdot u)(v,w)\big). \end{aligned}$$

Set-theoretical solutions to the QYBE [6], also known as Yang-Baxter maps [39], are dynamical Yang-Baxter maps constant for the parameter λ of any set H; indeed, the dynamical Yang-Baxter map is a generalization of the set-theoretical solution to the QYBE.

This dynamical Yang-Baxter map yields a bialgebroid [4]. Every dynamical Yang-Baxter map with some conditions gives birth to an (H, X)-bialgebroid [35], a generalization of the quantum group [5,11], through the Faddeev-Reshetikhin-Takhtajan construction [8].

It is worth pointing out that a ternary system (Definition 1(3)) can produce the dynamical Yang-Baxter map [32]. Each triple (L, M, π) consisting of a left quasigroup $L = (L, \cdot)$ (Definition 1(1)), a ternary system M satisfying (2.2) and (2.3), and a (set-theoretic) bijection $\pi : L \to M$ gives a dynamical Yang-Baxter map $R(\lambda)$ associated with L, L and $(\cdot)$ (see Section 2 for more details).

Homogeneous systems [18,19,20,21,22,23] are algebraic features of the reductive homogeneous space [24,28] satisfying suitable conditions. Let A be a group with its subgroup K. We assume that a subset G of the group A satisfies the following:

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(1) the group A is uniquely factorized as $A = GK$,
(2) $G^{-1} = G$,
(3) $kGk^{-1} = G$, for all $k \in K$.

Let $p : A \to G$ denote the canonical projection with respect to the factorization $A = GK$, and $(\cdot)$ the binary operation on G defined by $x \cdot y = p(xy)$ $(x, y \in G)$. Because the map $L_x : G \ni y \mapsto x \cdot y \in G$ is bijective, we define the ternary operation η on G by

$$\eta(x, y, z) = L_x\Big(\big(L_x\big)^{-1}(y) \cdot \big(L_x\big)^{-1}(z)\Big), \quad x, y, z \in G.$$

This ternary system $G = (G, \eta)$ is a homogeneous system [23, Proposition 1] (see also Definition 7), and every homogeneous system is constructed in such a way.

If G is a connected and second countable C^∞-manifold with a C^∞-map $\eta : G \times G \times G \to G$, then the homogeneous system $G = (G, \eta)$ is isomorphic to a reductive homogeneous space A/K for a connected Lie group A with its closed subgroup K [19, Theorem 1]. The homogeneous system is a ternary system, an algebraic structure, encoded in the reductive homogeneous space (for ternary systems in differential geometry and mathematical physics, see [13, 14, 15, 29]).

It is natural to relate this homogeneous system to the dynamical Yang-Baxter map through the ternary system.

The aim of this paper is to produce the dynamical Yang-Baxter maps by means of homogeneous pre-systems, which generalize the homogeneous system. Furthermore, we characterize such dynamical Yang-Baxter maps.

The organization of this paper is as follows.

Section 2 contains a brief summary of the dynamical Yang-Baxter map. We focus on its construction by means of the ternary system. This construction yields a category $\mathcal{A}$ concerning the ternary systems, which is equivalent to a category $\mathcal{D}$ consisting of the dynamical Yang-Baxter maps.

Section 3 presents the notion of a homogeneous pre-system, together with examples.

In Section 4, our main results are stated and proved. Every homogeneous pre-system satisfying (4.1) can produce a dynamical Yang-Baxter map via the ternary system. More precisely, we construct a category $\mathcal{H}$, isomorphic to the category $\mathcal{A}$, by means of the homogeneous pre-systems with (4.1). Because the category $\mathcal{A}$ is equivalent to the category $\mathcal{D}$, each object of $\mathcal{H}$ gives a dynamical Yang-Baxter map; in particular, we demonstrate dynamical Yang-Baxter maps provided by a certain left quasigroup and the examples in Section 3.

The last section, Section 5, deals with a relation between the homogeneous pre-system satisfying (4.1) and the left quasigroup with (5.1), which is due to the work in [32, Section 6]. We introduce a category $\mathcal{B}$ concerning the left quasigroups satisfying (5.1) and an essentially surjective functor $J : \mathcal{B} \to \mathcal{H}$ to construct the dynamical Yang-Baxter maps by means of quasigroups of reflection [17, 27].

Our viewpoint sheds some light on the relation between geometry and the dynamical Yang-Baxter map.

2 Dynamical Yang-Baxter maps

In this section, we briefly summarize without proofs the relevant material in [32] on the construction of the dynamical Yang-Baxter map.

Definition 1. (1) $(L, \cdot)$ is a left quasigroup (resp. right quasigroup [38, Section I.4.3]), if and only if L is a nonempty set, together with a binary operation $(\cdot)$ on L having the property that, for all $u, w \in L$, there uniquely exists $v \in L$ such that $u \cdot v = w$ (resp. $v \cdot u = w$). For the simplicity, one uses the notation uv instead of $u \cdot v$ $(u, v \in L)$.
(2) A quasigroup $(Q, \cdot)$ is a left and right quasigroup (see [30, Definition I.1.1] and [38, Section I.2]).
(3) A ternary system (M, μ) is a pair of a nonempty set M and a ternary operation $\mu : M \times M \times M \to M$.

By this definition, the left quasigroup $L = (L, \cdot)$ has another binary operation $\backslash_L$ called a left division [38, Section I.2.2]. For $u, w \in L$, we denote by $u \backslash_L w$ the unique element $v \in L$ satisfying $uv = w$,

$$u \backslash_L w = v \Longleftrightarrow uv = w. \tag{2.1}$$

The binary operation on the quasigroup is not always associative.

Example 2. We define the binary operation $(*)$ on the set $Q = \{1, 2, 3, 4, 5\}$ of five elements by Table 1. Here $1 * 2 = 3$. This $Q = (Q, *)$ is a quasigroup, because each element in Q appears once and only once in each row and in each column of Table 1 [30, Theorem I.1.3]. The binary operation $(*)$ is not associative, since $(1*2)*3 \neq 1*(2*3)$. This quasigroup Q is due to Nobusawa [27, Section 6, type 1]. However, the order of the binary operation $(*)$ in Table 1 is reversed.

$*$	1	2	3	4	5
1	1	3	5	2	4
2	5	2	4	1	3
3	4	1	3	5	2
4	3	5	2	4	1
5	2	4	1	3	5

Table 1: Binary operation $(*)$ on Q.

Each ternary system $M = (M, \mu)$ satisfying

$$\mu\big(a, \mu(a,b,c), \mu\big(\mu(a,b,c), c, d\big)\big) = \mu\big(a, b, \mu(b,c,d)\big), \tag{2.2}$$

$$\mu\big(\mu\big(a,b,\mu(b,c,d)\big), \mu(b,c,d), d\big) = \mu\big(\mu(a,b,c), c, d\big), \tag{2.3}$$

for any $a,b,c,d \in M$, can provide a dynamical Yang-Baxter map [32, Theorem 3.2]. Let $L = (L, \cdot)$ be a left quasigroup isomorphic to M as sets, and $\pi : L \to M$ a (set-theoretic) bijection. For $\lambda, u \in L$, we define the maps $\xi_\lambda^{(L,M,\pi)}(u) : L \to L$ and $\eta_\lambda^{(L,M,\pi)}(u) : L \to L$ as follows: for $v \in L$,

$$\xi_\lambda^{(L,M,\pi)}(u)(v) = \lambda \backslash_L \pi^{-1}\Big(\mu\Big(\pi(\lambda), \pi(\lambda u), \pi\big((\lambda u)v\big)\Big)\Big), \tag{2.4}$$

$$\eta_\lambda^{(L,M,\pi)}(u)(v) = \big(\lambda \xi_\lambda^{(L,M,\pi)}(v)(u)\big) \backslash_L \big((\lambda v)u\big). \tag{2.5}$$

Let $R^{(L,M,\pi)}(\lambda)$ $(\lambda \in L)$ denote the map from $L \times L$ to itself defined by

$$R^{(L,M,\pi)}(\lambda)(u,v) = \big(\eta_\lambda^{(L,M,\pi)}(v)(u), \xi_\lambda^{(L,M,\pi)}(u)(v)\big), \quad u, v \in L. \tag{2.6}$$

Theorem 3. *The map $R^{(L,M,\pi)}(\lambda)$ (2.6) is a dynamical Yang-Baxter map (1.1) associated with L, L and $(\cdot)$.*

We now introduce two categories $\mathcal{A}$ and $\mathcal{D}$ concerning a special class of the dynamical Yang-Baxter maps, which play a central role in this article.

The first task is to explain the category $\mathcal{A}$ (cf. the category $\mathcal{A}_2$ in [32, Section 6]). We follow the notation of [16, Chapter XI]. Let $L = (L, \cdot)$ be a left quasigroup, $M = (M, \mu)$ a ternary system satisfying (2.2) and

$$\mu(a,b,b) = a, \quad \forall a, b \in M, \tag{2.7}$$

$$\mu\big(\mu(a,b,c), c, d\big) = \mu(a,b,d), \quad \forall a,b,c,d \in M, \tag{2.8}$$

and $\pi : L \to M$ a bijection. The object of $\mathcal{A}$ is, by definition, a triple (L, M, π).

The morphism $f : (L, (M, \mu), \pi) \to (L', (M', \mu'), \pi')$ of $\mathcal{A}$ is a homomorphism $f : L \to L'$ of left quasigroups such that $h := \pi' \circ f \circ \pi^{-1} : M \to M'$ is a homomorphism of ternary systems; that is, the map $f : L \to L'$ satisfies

$$\begin{gathered} f\big(a \cdot_L b\big) = f(a) \cdot_{L'} f(b), \quad \forall a, b \in L, \\ h\big(\mu(a,b,c)\big) = \mu'\big(h(a), h(b), h(c)\big), \quad \forall a,b,c \in M. \end{gathered} \tag{2.9}$$

The identity id, the source $s(f)$ and the target $b(f)$ of a morphism $f : (L, M, \pi) \to (L', M', \pi')$, and the composition $g \circ f$ for morphisms $f : (L, M, \pi) \to (L', M', \pi')$ and $g : (L', M', \pi') \to (L'', M'', \pi'')$ are defined as follows: for an object $(L, M, \pi) \in \mathcal{A}$,

$$\begin{gathered} \mathrm{id}_{(L,M,\pi)} = \mathrm{id}_L, \\ s\big(f : (L,M,\pi) \longrightarrow \big(L', M', \pi'\big)\big) = (L, M, \pi), \\ b\big(f : (L,M,\pi) \longrightarrow \big(L', M', \pi'\big)\big) = \big(L', M', \pi'\big), \end{gathered}$$

the composition $g \circ f$ is the usual one of the maps $f : L \to L'$ and $g : L' \to L''$.

Proposition 4. *$\mathcal{A}$ is a category.*

The next task is to describe the category $\mathcal{D}$, which is exactly the category $\mathcal{D}_2$ in [32, Section 6]. The object of this category $\mathcal{D}$ is a pair (L, R) of a left quasigroup $L = (L, \cdot)$ and a dynamical Yang-Baxter map $R(\lambda) : L \times L \to L \times L$ $(\lambda \in L)$ satisfying

$$\xi_\lambda(u)\big((\lambda u)\backslash_L(\lambda u)\big) = \lambda\backslash_L\lambda, \quad \forall\lambda, u \in L,$$
$$\big(\lambda\xi_\lambda(u)(v)\big)\eta_\lambda(v)(u) = (\lambda u)v, \quad \forall\lambda, u, v \in L,$$
$$\big(\lambda\xi_\lambda(u)(v)\big)\xi_{\lambda\xi_\lambda(u)(v)}\big(\eta_\lambda(v)(u)\big)(w) = \lambda\xi_\lambda(u)\big((\lambda u)\backslash_L\big(\big((\lambda u)v\big)w\big)\big), \quad \forall\lambda, u, v, w \in L.$$

Here, $(\eta_\lambda(v)(u), \xi_\lambda(u)(v)) := R(\lambda)(u, v)$ $(\lambda, u, v \in L)$.

The morphism $f : (L, R) \to (L', R')$ of $\mathcal{D}$ is a homomorphism $f : L \to L'$ of left quasigroups satisfying

$$R'\big(f(\lambda)\big) \circ f \times f = f \times f \circ R(\lambda), \quad \forall\lambda \in L.$$

Proposition 5. *$\mathcal{D}$ is a category; the definitions of the identity, the source, the target and the composition are similar to those of the category $\mathcal{A}$.*

We can construct functors $S : \mathcal{A} \to \mathcal{D}$ and $T : \mathcal{D} \to \mathcal{A}$, which establish an equivalence of the categories $\mathcal{A}$ and $\mathcal{D}$ (cf. [32, Proposition 6.15]): for $(L, M, \pi) \in \mathcal{A}$, set $S(L, M, \pi) = (L, R^{(L,M,\pi)})$. Here, $R^{(L,M,\pi)}(\lambda)$ is defined by (2.4), (2.5) and (2.6); for a morphism f of $\mathcal{A}$, write $S(f) = f$; for $(L, R) \in \mathcal{D}$, $T(L, R)$ denotes the triple $(L, (M, \mu), \mathrm{id}_L)$, where $M = L$ as sets and $\mu(a, b, c) = a\xi_a(a\backslash_L b)(b\backslash_L c)$ $(a, b, c \in M(= L))$; for a morphism f of $\mathcal{D}$, set $T(f) = f$.

These functors S and T satisfy $ST = \mathrm{id}_{\mathcal{D}}$, and $\theta(L, M, \pi) := \mathrm{id}_L$ $((L, M, \pi) \in \mathcal{A})$ gives a natural isomorphism $\theta : TS \to \mathrm{id}_{\mathcal{A}}$. Thus, the following theorem holds.

Theorem 6. *$S : \mathcal{A} \to \mathcal{D}$ is an equivalence of categories.*

3 Homogeneous pre-systems

This section is devoted to introducing homogeneous pre-systems.

Definition 7. (1) A ternary system $G = (G, \eta)$ (Definition 1(3)) is a homogeneous pre-system if and only if the ternary operation η satisfies

$$\eta(x, y, x) = y, \quad \forall x, y \in G, \tag{3.1}$$
$$\eta\big(x, y, \eta(u, v, w)\big) = \eta\big(\eta(x, y, u), \eta(x, y, v), \eta(x, y, w)\big), \tag{3.2}$$

for all $x, y, u, v, w \in G$.

(2) A homogeneous system $G = (G, \eta)$ [18] is a homogeneous pre-system satisfying

$$\eta(x, x, y) = y, \quad \forall x, y \in G,$$
$$\eta\big(x, y, \eta(y, x, z)\big) = z, \quad \forall x, y, z \in G. \tag{3.3}$$

We explain two examples in this section: one homogeneous pre-system and one homogeneous system, which imply dynamical Yang-Baxter maps in the next section.

Let G be an abelian group. We define the ternary operation η on G by

$$\eta(x, y, z) = x + y - z, \quad x, y, z \in G. \tag{3.4}$$

A trivial verification shows that $G = (G, \eta)$ is a homogeneous pre-system, which is not always a homogeneous system because of (3.3) (cf. [18, Remark 4]).

Another example is a homogeneous system on an arbitrary group G [18, Example in Section 1]. We define the ternary operation η on the group G by

$$\eta(x, y, z) = yx^{-1}z, \quad x, y, z \in G. \tag{3.5}$$

It is clear that this $G = (G, \eta)$ is a homogeneous system.

Remark 8. The homogeneous system (G, η) (3.5) is equivalent to the notion of a torsor [25,33,36], also known as the principal homogeneous space, up to the choice of the unit element. Hence, the principal homogeneous space provides a homogeneous system.

4 A relation between dynamical Yang-Baxter maps and homogeneous pre-systems

In this section, we construct dynamical Yang-Baxter maps (2.6) by means of homogeneous pre-systems $G = (G, \eta)$ satisfying

$$\eta(x, y, z) = \eta\big(w, \eta(x, y, w), z\big), \quad \forall x, y, z, w \in G. \tag{4.1}$$

In fact, we present a category $\mathcal{H}$ concerning the homogeneous pre-systems with (4.1); this $\mathcal{H}$ is isomorphic to the category $\mathcal{A}$ in Section 2, and, on account of Theorem 6, every object of $\mathcal{H}$ consequently gives a dynamical Yang-Baxter map.

Let $L = (L, \cdot)$ be a left quasigroup, $G = (G, \eta)$ a homogeneous pre-system satisfying (4.1) and $\pi : L \to G$ a (set-theoretic) bijection. The object of $\mathcal{H}$ is a triple (L, G, π).

The morphism $f : (L, (G, \eta), \pi) \to (L', (G', \eta'), \pi')$ of $\mathcal{H}$ is a homomorphism $f : L \to L'$ of left quasigroups such that $h := \pi' \circ f \circ \pi^{-1} : G \to G'$ is a homomorphism of ternary systems; that is, the map $f : L \to L'$ satisfies (2.9) and

$$h\big(\eta(x, y, z)\big) = \eta'\big(h(x), h(y), h(z)\big), \quad \forall x, y, z \in G.$$

Proposition 9. *$\mathcal{H}$ is a category; the definitions of the identity, the source, the target and the composition are similar to those of the category $\mathcal{A}$.*

In order to prove that the category $\mathcal{H}$ is isomorphic to the category $\mathcal{A}$, we construct functors $F : \mathcal{A} \to \mathcal{H}$ and $F' : \mathcal{H} \to \mathcal{A}$.

We first introduce the functor $F : \mathcal{A} \to \mathcal{H}$. Let $(L, (M, \mu), \pi) \in \mathcal{A}$. Define the ternary system $G = (G, \eta)$ by $G = M$ as sets and

$$\eta(x, y, z) = \mu(y, x, z), \quad x, y, z \in G(= M). \tag{4.2}$$

Proposition 10. *$(L, G, \pi) \in \mathcal{H}$.*

Proof. We need only show that G is a homogeneous pre-system satisfying (4.1).

An easy computation shows (3.1) and (4.1).

By virtue of (4.2), the left-hand side of (3.2) is $\mu(y, x, \mu(v, u, w))$, and, with the aid of (2.2), (2.7) and (2.8),

$$\begin{aligned}\mu\big(y, x, \mu(v, u, w)\big) &= \mu\big(\mu(y, x, v), v, \mu(v, u, w)\big)\\ &= \mu\Big(\mu(y, x, v), \mu\big(\mu(y, x, v), v, u\big), \mu\Big(\mu\big(\mu(y, x, v), v, u\big), u, w\Big)\Big)\\ &= \mu\big(\mu(y, x, v), \mu(y, x, u), \mu(y, x, w)\big),\end{aligned}$$

which is the right-hand side of (3.2). This proves the proposition. □

By setting $F(L, (M, \mu), \pi) = (L, G, \pi)$ and $F(f) = f$ for a morphism f of $\mathcal{A}$, the following proposition holds.

Proposition 11. *$F : \mathcal{A} \to \mathcal{H}$ is a functor.*

The next task is to construct a functor $F' : \mathcal{H} \to \mathcal{A}$. Let $(L, (G, \eta), \pi) \in \mathcal{H}$. We define the ternary system $M_G = (M_G, \mu)$ by $M_G = G$ as sets and

$$\mu(a, b, c) = \eta(b, a, c), \quad a, b, c \in M_G(= G). \tag{4.3}$$

Proposition 12. *$(L, M_G, \pi) \in \mathcal{A}$.*

Proof. It suffices to prove that M_G satisfies (2.2), (2.7) and (2.8).

A trivial verification shows (2.7) and (2.8).

Due to (4.1) and (4.3), the left-hand side of (2.2) is

$$\mu\big(a, \mu(a, b, c), \mu\big(\mu(a, b, c), c, d\big)\big) = \eta\big(\eta(b, a, c), a, \eta(b, a, d)\big).$$

From (3.1) and (3.2),

$$\eta\big(\eta(b, a, c), a, \eta(b, a, d)\big) = \eta\big(\eta(b, a, c), \eta(b, a, b), \eta(b, a, d)\big) = \eta\big(b, a, \eta(c, b, d)\big),$$

which is exactly the right-hand side of (2.2). □

By setting $F'(L,(G,\eta),\pi) = (L, M_G, \pi)$ and $F'(f) = f$ for a morphism f of $\mathcal{H}$, the following proposition holds.

Proposition 13. *$F' : \mathcal{H} \to \mathcal{A}$ is a functor.*

Since the functors F and F' satisfy $F'F = \mathrm{id}_{\mathcal{A}}$ and $FF' = \mathrm{id}_{\mathcal{H}}$, the following theorem holds.

Theorem 14. *The categories $\mathcal{A}$ and $\mathcal{H}$ are isomorphic.*

By taking account of Theorem 6, we have the following corollary.

Corollary 15. *Each object of $\mathcal{H}$ provides a dynamical Yang-Baxter map.*

The proof of the following proposition is straightforward.

Proposition 16. *The ternary operations* (3.4) *and* (3.5) *satisfy* (4.1).

As a consequence of Corollary 15 and Proposition 16, the homogeneous pre-system G (3.4) and the homogeneous system G (3.5) imply dynamical Yang-Baxter maps. Let $L = (G, \cdot)$ denote the left quasigroup whose binary operation $(\cdot)$ is defined by

$$u \cdot v = v, \quad u, v \in L, \tag{4.4}$$

and let $\pi : L(= G) \to G$ be the identity map on G. The corresponding dynamical Yang-Baxter maps are as follows: if G is a homogeneous pre-system (3.4), then

$$R^{(L,M_G,\pi)}(\lambda)(u,v) = (v, \lambda + u - v), \quad \lambda, u, v \in L(= G),$$

and if G is a homogeneous system (3.5), then

$$R^{(L,M_G,\pi)}(\lambda)(u,v) = \big(v, \lambda u^{-1} v\big), \quad \lambda, u, v \in L(= G).$$

5 A relation between homogeneous pre-systems and left quasigroups

Because of the work in [32, Section 6] and the fact that the categories $\mathcal{A}$ and $\mathcal{H}$ are isomorphic, every homogeneous pre-system G (Definition 7(1)) in the object $(L, G, \pi) \in \mathcal{H}$ is a left quasigroup (Definition 1(1)) whose binary operation gives the ternary operation of G. This last section demonstrates it by constructing a category $\mathcal{B}$ concerning the left quasigroups with (5.1) and an essentially surjective functor $J : \mathcal{B} \to \mathcal{H}$ (see [32, Proposition 6.17]). The functors $J : \mathcal{B} \to \mathcal{H}$, $S : \mathcal{A} \to \mathcal{D}$ in Section 2, and $F' : \mathcal{H} \to \mathcal{A}$ in Section 4, together with quasigroups of reflection [17,27], provide examples of the dynamical Yang-Baxter map.

The first task is to introduce a category $\mathcal{B}$. Let $L_1, L_2 = (L_2, *)$ be left quasigroups. We assume that the left quasigroup L_2 satisfies

$$(a * c)\backslash_{L_2}\big((a * b) * c\big) = (a' * c)\backslash_{L_2}\big((a' * b) * c\big), \quad \forall a, a', b, c \in L_2. \tag{5.1}$$

Here the symbol $\backslash_{L_2}$ is the left division (2.1) of L_2. Let $\pi : L_1 \to L_2$ be a (set-theoretic) bijection. An object of $\mathcal{B}$ is such a triple (L_1, L_2, π).

A morphism $f : (L_1, L_2, \pi) \to (L_1', L_2', \pi')$ is a homomorphism $f : L_1 \to L_1'$ of left quasigroups such that $\pi' \circ f \circ \pi^{-1} : L_2 \to L_2'$ is also a homomorphism of left quasigroups.

Proposition 17. *$\mathcal{B}$ is a category; the definitions of the identity, the source, the target and the composition are similar to those of the category $\mathcal{A}$.*

The next task is to construct a functor $J : \mathcal{B} \to \mathcal{H}$. Let $(L_1, (L_2, *), \pi) \in \mathcal{B}$. We define the ternary system $G_{L_2} = (G_{L_2}, \eta_{L_2})$ by $G_{L_2} = L_2$ as sets and

$$\eta_{L_2}(x, y, z) = z * \big(x \backslash_{L_2} y\big), \quad x, y, z \in G_{L_2}\big(= L_2\big). \tag{5.2}$$

Proposition 18. $(L_1, G_{L_2}, \pi) \in \mathcal{H}$.

Proof. It suffices to prove that G_{L_2} is a homogeneous pre-system satisfying (4.1).

We give a proof only for (3.2) because the rest of the proof is straightforward. Let $x, y, u, v, w \in G(= L_2)$. From (5.2) we have

$$\begin{aligned}\eta_{L_2}\big(x, y, \eta_{L_2}(u, v, w)\big) &= \big(w * \big(u \backslash_{L_2} v\big)\big) * \big(x \backslash_{L_2} y\big)\\ &= \big(w * \big(x \backslash_{L_2} y\big)\big) * \big(\big(w * \big(x \backslash_{L_2} y\big)\big) \backslash_{L_2} \big(\big(w * \big(u \backslash_{L_2} v\big)\big) * \big(x \backslash_{L_2} y\big)\big)\big).\end{aligned} \tag{5.3}$$

With the aid of (5.1), the right-hand side of (5.3) is

$$\begin{aligned}&\big(w * \big(x \backslash_{L_2} y\big)\big) * \big(\big(w * \big(x \backslash_{L_2} y\big)\big) \backslash_{L_2} \big(\big(w * \big(u \backslash_{L_2} v\big)\big) * \big(x \backslash_{L_2} y\big)\big)\big)\\ &\quad= \big(w * \big(x \backslash_{L_2} y\big)\big) * \big(\big(u * \big(x \backslash_{L_2} y\big)\big) \backslash_{L_2} \big(\big(u * \big(u \backslash_{L_2} v\big)\big) * \big(x \backslash_{L_2} y\big)\big)\big)\\ &\quad= \big(w * \big(x \backslash_{L_2} y\big)\big) * \big(\big(u * \big(x \backslash_{L_2} y\big)\big) \backslash_{L_2} \big(v * \big(x \backslash_{L_2} y\big)\big)\big),\end{aligned}$$

which is exactly $\eta_{L_2}(\eta_{L_2}(x, y, u), \eta_{L_2}(x, y, v), \eta_{L_2}(x, y, w))$. This is the desired conclusion. □

Let $f : (L_1, L_2, \pi) \to (L_1', L_2', \pi')$ be a morphism of the category $\mathcal{B}$. The map $f : L_1 \to L_1'$ is a homomorphism of left quasigroups. Moreover, $h := \pi' \circ f \circ \pi^{-1} : L_2 \to L_2'$ is a homomorphism of ternary systems from G_{L_2} to $G_{L_2'}$, because h is a homomorphism of left quasigroups. As a result, $f : (L_1, L_2, \pi) \to (L_1', L_2', \pi')$ is a morphism of the category $\mathcal{H}$.

We set $J(L_1, L_2, \pi) = (L_1, G_{L_2}, \pi)$ for $(L_1, L_2, \pi) \in \mathcal{B}$ and $J(f) = f$ for a morphism f of $\mathcal{B}$.

Proposition 19. *$J : \mathcal{B} \to \mathcal{H}$ is a functor.*

This functor J is essentially surjective. In fact, for any $(L, G, \pi) \in \mathcal{H}$, we can construct a left quasigroup L_2 such that $(L, L_2, \pi) \in \mathcal{B}$ and $J(L, L_2, \pi) = (L, G, \pi)$. We fix any element $\lambda_0 \in G$. Set $L_2 = G$ as sets and

$$a * b = \eta\big(\lambda_0, b, a\big), \quad a, b \in L_2(= G). \tag{5.4}$$

Due to (3.1) and (4.1), L_2 is a left quasigroup; its left division is defined by

$$a \backslash_{L_2} c = \eta\big(a, c, \lambda_0\big), \quad a, c \in L_2. \tag{5.5}$$

Proposition 20. $(L, L_2, \pi) \in \mathcal{B}$.

Proof. We need only show (5.1). Let $a, a', b, c \in L_2(= G)$. With the aid of (5.4) and (5.5) we have

$$(a * c) \backslash_{L_2} \big((a * b) * c\big) = \eta\big(\eta\big(\lambda_0, c, a\big), \eta\big(\lambda_0, c, \eta\big(\lambda_0, b, a\big)\big), \lambda_0\big). \tag{5.6}$$

From (3.2) and (4.1),

$$\begin{aligned}\eta\big(\lambda_0, c, \eta\big(\lambda_0, b, a\big)\big) &= \eta\big(\lambda_0, c, \eta\big(a', \eta\big(\lambda_0, b, a'\big), a\big)\big)\\ &= \eta\big(\eta\big(\lambda_0, c, a'\big), \eta\big(\lambda_0, c, \eta\big(\lambda_0, b, a'\big)\big), \eta\big(\lambda_0, c, a\big)\big).\end{aligned}$$

By taking into account (4.1) again, the right-hand side of (5.6) is

$$\eta\big(\eta\big(\lambda_0, c, a\big), \eta\big(\lambda_0, c, \eta\big(\lambda_0, b, a\big)\big), \lambda_0\big) = \eta\big(\eta\big(\lambda_0, c, a'\big), \eta\big(\lambda_0, c, \eta\big(\lambda_0, b, a'\big)\big), \lambda_0\big),$$

which is exactly the right-hand side of (5.1) by virtue of (5.4) and (5.5). This proves the proposition. □

It is immediate that $J(L, L_2, \pi) = (L, G, \pi)$, and consequently, the following holds.

Proposition 21. *The functor $J : \mathcal{B} \to \mathcal{H}$ is essentially surjective.*

Corollary 22. *The functor $SF'J : \mathcal{B} \to \mathcal{D}$ is essentially surjective.*

The final task of this section is to construct dynamical Yang-Baxter maps by means of the functor $SF'J : \mathcal{B} \to \mathcal{D}$ and quasigroups of reflection; see [17, Section 1].

Definition 23. A pair $(G, *)$ of a nonempty set G and a binary operation $(*)$ on G is called a quasigroup of reflection if and only if $(G, *)$ is a left quasigroup (Definition 1(1)) satisfying

$$x * x = x, \quad \forall x \in G,$$

$$(x * y) * y = x, \quad \forall x, y \in G, \tag{5.7}$$

$$(x * y) * z = (x * z) * (y * z), \quad \forall x, y, z \in G. \tag{5.8}$$

It follows from (5.7) that $(G, *)$ is a quasigroup (Definition 1(2)).

Remark 24. (1) The above definition is slightly different from that in [17] (see also [26, II.1.1] and [27, Section 1]); the order of the binary operation $(*)$ on G is reversed.

(2) The identity (5.8) is called a right distributive law [30, Section V.2].

(3) The quasigroup of reflection gives an involutory quandle [1, 12, 37] by reversing the order of the binary operation in Definition 23.

A straightforward computation shows that Nobusawa's quasigroup $(Q, *)$ in Example 2 is a quasigroup of reflection.

Let $(G, *)$ be a quasigroup of reflection, and L a left quasigroup isomorphic to G as sets. We denote by π a set-theoretic bijection from L to G. Because (5.8) immediately induces (5.1),

Proposition 25. $(L, G, \pi) \in \mathcal{B}$.

The quasigroup $G = (G, *)$ of reflection hence produces the dynamical Yang-Baxter map $R(\lambda)$ defined by $(L, R) = SF'J(L, G, \pi) \in \mathcal{D}$.

For example, let $L = (G, \cdot)$ denote the left quasigroup (4.4) and $\pi : L(= G) \to G$ the identity map on G. The above dynamical Yang-Baxter map $R(\lambda)$ induced by $(L, G, \pi) \in \mathcal{B}$ is

$$R(\lambda)(u, v) = \big(v, v * \big(u \backslash_G \lambda\big)\big), \quad \lambda, u, v \in L(= G). \tag{5.9}$$

For Nobusawa's quasigroup $Q = (Q, *)$, the corresponding dynamical Yang-Baxter map (5.9) is really dependent on the parameter λ; in fact,

$$R(1)(1, 1) = (1, 1), \quad R(2)(1, 1) = (1, 2).$$

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References

[1] N. Andruskiewitsch and M. Graña, *From racks to pointed Hopf algebras*, Adv. Math., 178 (2003), 177–243.
[2] R. J. Baxter, *Partition function of the eight-vertex lattice model*, Ann. Physics, 70 (1972), 193–228.
[3] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, London, 1982.
[4] T. Brzeziński and G. Militaru, *Bialgebroids, $\times_A$-bialgebras and duality*, J. Algebra, 251 (2002), 279–294.
[5] V. G. Drinfel'd, *Quantum groups*, in Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), Providence, RI, USA, 1987, American Mathematical Society, 798–820.
[6] V. G. Drinfel'd, *On some unsolved problems in quantum group theory*, in Quantum Groups (Leningrad, 1990), vol. 1510 of Lecture Notes in Mathematics, Springer, Berlin, 1992, 1–8.
[7] P. Etingof and F. Latour, *The Dynamical Yang-Baxter Equation, Representation Theory, and Quantum Integrable Systems*, vol. 29 of Oxford Lecture Series in Mathematics and Its Applications, Oxford University Press, Oxford, 2005.
[8] L. D. Faddeev, N. Y. Reshetikhin, and L. A. Takhtajan, *Quantization of Lie groups and Lie algebras*, in Algebraic Analysis, M. Kashiwara and T. Kawai, eds., vol. 1, Academic Press, Boston, MA, USA, 1988, 129–139.
[9] G. Felder, *Conformal field theory and integrable systems associated to elliptic curves*, in Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), Basel, 1995, Birkhäuser, 1247–1255.
[10] J.-L. Gervais and A. Neveu, *Novel triangle relation and absence of tachyons in Liouville string field theory*, Nuclear Phys. B, 238 (1984), 125–141.
[11] M. Jimbo, *A q-difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation*, Lett. Math. Phys., 10 (1985), 63–69.
[12] D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra, 23 (1982), 37–65.
[13] N. Kamiya, *On radicals of triple systems*, in Groups, Rings, Lie and Hopf Algebras (St. John's, NF, 2001), vol. 555 of Math. Appl., Kluwer Academic Publishers, Dordrecht, 2003, 75–83.
[14] N. Kamiya, *On Lie algebras and triple systems*, in Noncommutative Structures in Mathematics and Physics, S. Caenepeel, J. Fuchs, S. Gutt, C. Schweigert, A. Stolin, and F. Van Oystayen, eds., Koninklijke Vlaamse Academie van Belgie voor Wetenschappen en Kunsten, Brussels, 2010, 201–208.
[15] N. Kamiya and S. Okubo, *On triple systems and Yang-Baxter equations*, Int. J. Math. Game Theory Algebra, 8 (1998), 81–88.
[16] C. Kassel, *Quantum Groups*, vol. 155 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1995.
[17] M. Kikkawa, *On some quasigroups of algebraic models of symmetric spaces*, Mem. Fac. Lit. Sci. Shimane Univ. Natur. Sci., 6 (1973), 9–13.
[18] M. Kikkawa, *On the left translations of homogeneous loops*, Mem. Fac. Lit. Sci. Shimane Univ. Natur. Sci., 10 (1976), 19–25.
[19] M. Kikkawa, *On homogeneous systems. I*, Mem. Fac. Lit. Sci. Shimane Univ. Natur. Sci., 11 (1977), 9–17.
[20] M. Kikkawa, *On homogeneous systems. II*, Mem. Fac. Sci. Shimane Univ., 12 (1978), 5–13.
[21] M. Kikkawa, *On homogeneous systems. III*, Mem. Fac. Sci. Shimane Univ., 14 (1980), 41–46.
[22] M. Kikkawa, *On homogeneous systems. IV*, Mem. Fac. Sci. Shimane Univ., 15 (1981), 1–7.
[23] M. Kikkawa, *On homogeneous systems. V*, Mem. Fac. Sci. Shimane Univ., 17 (1983), 9–13.
[24] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry. Vol. II*, Interscience Tracts in Pure and Applied Mathematics, Interscience Publishers John Wiley & Sons, New York, 1969.

[25] M. Kontsevich, *Operads and motives in deformation quantization*, Lett. Math. Phys., 48 (1999), 35–72.
[26] O. Loos, *Symmetric Spaces. I: General Theory*, W. A. Benjamin, New York, 1969.
[27] N. Nobusawa, *On symmetric structure of a finite set*, Osaka J. Math., 11 (1974), 569–575.
[28] K. Nomizu, *Invariant affine connections on homogeneous spaces*, Amer. J. Math., 76 (1954), 33–65.
[29] S. Okubo and N. Kamiya, *Quasi-classical Lie superalgebras and Lie supertriple systems*, Comm. Algebra, 30 (2002), 3825–3850.
[30] H. O. Pflugfelder, *Quasigroups and Loops: Introduction*, vol. 7 of Sigma Series in Pure Mathematics, Heldermann Verlag, Berlin, 1990.
[31] Y. Shibukawa, *Dynamical Yang-Baxter maps*, Int. Math. Res. Not., 36 (2005), 2199–2221.
[32] Y. Shibukawa, *Dynamical Yang-Baxter maps with an invariance condition*, Publ. Res. Inst. Math. Sci., 43 (2007), 1157–1182.
[33] Y. Shibukawa, *Dynamical braided monoids and dynamical Yang-Baxter maps*, in Quantum Groups and Quantum Topology (RIMS Kokyuroku 1714), A. Masuoka, ed., Res. Inst. Math. Sci. Kyoto Univ., Kyoto, Japan, 2010, 80–89.
[34] Y. Shibukawa, *Survey on dynamical Yang-Baxter maps*, in Noncommutative Structures in Mathematics and Physics, S. Caenepeel, J. Fuchs, S. Gutt, C. Schweigert, A. Stolin, and F. Van Oystayen, eds., Koninklijke Vlaamse Academie van Belgie voor Wetenschappen en Kunsten, Brussels, Belgium, 2010, 239–244.
[35] Y. Shibukawa and M. Takeuchi, *FRT construction for dynamical Yang-Baxter maps*, J. Algebra, 323 (2010), 1698–1728.
[36] Z. Škoda, *Quantum heaps, cops and heapy categories*, Math. Commun., 12 (2007), 1–9.
[37] J. D. H. Smith, *Quasigroups and quandles*, Discrete Math., 109 (1992), 277–282.
[38] J. D. H. Smith and A. B. Romanowska, *Post-Modern Algebra*, Wiley Series in Pure and Applied Mathematics, John Wiley & Sons, New York, 1999.
[39] A. Veselov, *Yang-Baxter maps: dynamical point of view*, in Combinatorial Aspect of Integrable Systems, vol. 17 of Mathematical Society of Japan Memoirs, Mathematical Society of Japan, Tokyo, 2007, 145–167.
[40] C. N. Yang, *Some exact results for the many-body problem in one dimension with repulsive delta-function interaction*, Phys. Rev. Lett., 19 (1967), 1312–1315.
[41] C. N. Yang, *S matrix for the one-dimensional N-body problem with repulsive or attractive δ-function*, Phys. Rev., 168 (1968), 1920–1923.