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On jets, extensions and characteristic classes I

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Abstract

In this paper, we give general definitions of non-commutative jets in the local and global situation using square zero extensions and derivations. We study the functors $\mathrm{Exan}_k(A, I)$, where A is any k-algebra, and I is any left and right A-module and use this to construct affine non-commutative jets. We also study the Kodaira-Spencer class $\mathrm{KS}(\mathcal{L})$ and relate it to the Atiyah class.

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1 Introduction

In this paper, we give general definitions of non-commutative jets in the local and global situation using square zero extensions and derivations. We study the functors $\mathrm{Exan}_k(A, I)$, where A is any k-algebra, and I is any left and right A-module and use this to construct affine non-commutative jets. In the final section of the paper, we define and prove basic properties of the Kodaira-Spencer class $\mathrm{KS}(\mathcal{L})$ and relate it to the Atiyah class.

2 Jets, liftings, and small extensions

We give an elementary discussion of structural properties of square zero extensions of arbitrary associative unital k-algebras. We introduce for any k-algebra A and any left and right A-module I the set $\mathrm{Exan}_k(A, I)$ of isomorphism classes of square zero extensions of A by I and show it is a left and right module over the center $C(A)$ of A. This structure generalize the structure as left $C(A)$-module introduced in [3]. We also give an explicit construction of $\mathrm{Exan}_k(A, I)$ in terms of cocycles. Finally, we give a direct construction of non-commutative jets and generalized Atiyah sequences using derivations and square zero extensions.

Let in the following k be a fixed base field, and let

$$0 \longrightarrow I \xrightarrow{i} B \xrightarrow{p} A \longrightarrow 0$$

be an exact sequence of associative unital k-algebras with $i(I)^2 = 0$. Assume s is a map of k-vector spaces with the following properties:

$$s(1) = 1,$$

and

$$p \circ s = \mathrm{id}.$$

Such a section always exists since B and A are vector spaces over the field k. Note: s gives the ideal I a left and right A-action.

Lemma 2.1. *There is an isomorphism:*

$$B \cong I \oplus A$$

of k-vector spaces.

Proof. Define the following maps of vector spaces: $\phi : B \to I \oplus A$ by $\phi(x) = (x - sp(x), p(x))$ and $\psi : I \oplus A \to B$ by $\psi(u, x) = u + s(x)$. It follows that $\psi \circ \phi = \mathrm{id}$ and $\phi \circ \psi = \mathrm{id}$ and the claim of the proposition follows. □

Define the following element:

$$\tilde{C} : A \times A \longrightarrow I,$$

by

$$\tilde{C}(x \times y) = s(x)s(y) - s(xy).$$

It follows that $\tilde{C} = 0$ if and only if s is a ring homomorphism.

Lemma 2.2. *The map $\tilde{C}$ gives rise to an element $C \in \mathrm{Hom}_k(A \otimes_k A, I)$.*

Proof. We easily see that $\tilde{C}(x+y, z) = \tilde{C}(x, z) + \tilde{C}(y, x)$ and $\tilde{C}(x, y+z) = \tilde{C}(x, y) + \tilde{C}(x, z)$ for all $x, y, z \in A$. Moreover, for any $a \in k$, it follows that

$$\tilde{C}(ax, y) = \tilde{C}(x, ay) = a\tilde{C}(x, y).$$

Hence we get a well-defined element $C \in \mathrm{Hom}_k(A \otimes_k A, I)$ as claimed. □

Define the following product on $I \oplus A$:

$$(u, x) \times (v, y) = \big(uy + xv + C(x, y), xy\big). \tag{2.1}$$

We let $I \oplus^C A$ denote the abelian group $I \oplus A$ with product defined by (2.1).

Proposition 2.3. *The natural isomorphism:*

$$B \cong I \oplus A$$

of vector spaces is a unital ring isomorphism if and only if the following holds:

$$xC(y, z) - C(xy, z) + C(x, yz) - C(x, y)z = 0$$

for all $x, y, z \in A$.

Proof. We have defined two isomorphisms of vector spaces ϕ, ψ:

$$\phi(x) = \big(x - sp(x), p(x)\big),$$

and

$$\psi(u, x) = u + s(x).$$

We define a product on the direct sum $I \oplus A$ using ϕ and ψ:

$$(u, x) \times (v, y) = \phi\big(\psi(u, x)\psi(v, y)\big) = \phi\big(\big(u + s(x)\big)\big(v + s(y)\big)\big)$$

$$\begin{aligned}&= \phi\big(uv + us(y) + s(x)v + s(x)s(y)\big)\\ &= \big(us(y) + s(x)v + s(x)s(y) - s(xy), xy\big)\\ &= \big(uy + xv + C(x,y), xy\big).\end{aligned}$$

Here, we define

$$uy = us(y),$$

and

$$xv = s(x)v.$$

One checks that

$$\phi(1) = \big(1 - sp(1), 1\big) = (0,1) = \mathbf{1},$$

and

$$\mathbf{1}(u,x) = (u,x)\mathbf{1} = (u,x)$$

for all $(u,x) \in I \oplus A$. It follows that the morphism ϕ is unital. Since $C(x+y,z) = C(x,z) + C(y,z)$ and $C(x,y+z) = C(x,y) + C(x,z)$ the following holds:

$$(u,x)\big((v,y) + (w,z)\big) = (u,x)(v,y) + (u,x)(w,z),$$

and

$$\big((v,y) + (w,z)\big)(u,x) = (v,y)(u,x) + (w,z)(u,x).$$

Hence, the multiplication is distributive over addition. Hence for an arbitrary section s of p of vector spaces mapping the identity to the identity, it follows the multiplication defined above always has a left and right unit and is distributive. We check when the multiplication is associative:

$$\big((u,x)(v,y)\big)(w,z) = \big(uyz + xvz + xyw + C(x,y)z + C(xy,z), xyz\big).$$

Also,

$$(u,x)\big((v,y)(w,z)\big) = \big(uyz + xvz + xyw + xC(y,z) + C(x,yz), xyz\big).$$

It follows that the multiplication is associative if and only if the following equation holds for the element C:

$$xC(y,z) - C(xy,z) + C(x,yz) - C(x,y)z = 0$$

for all $x,y,z \in A$. The claim follows. □

Let

$$xC(y,z) - C(xy,z) + C(x,yz) - C(x,y)z = 0 \tag{2.2}$$

be the *cocycle condition.*

Definition 2.4. Let $\text{exan}_k(A, I)$ be the set of elements $C \in \text{Hom}_k(A \otimes_k A, I)$ satisfying the cocycle condition (2.2).

Proposition 2.5. *Equation (2.2) holds for all $x, y, z \in A$:*

Proof. We get,

$$\begin{aligned}
&xC(y, z) = s(x)s(y)s(z) - s(x)s(yz), \\
&C(xy, z) = s(xy)s(z) - s(xyz), \\
&C(x, yz) = s(x)s(yz) - s(xyz),
\end{aligned}$$

and

$$C(x, y)z = s(x)s(y)s(z) - s(xy)s(z).$$

We get

$$\begin{aligned}
&xC(y, z) - C(xy, z) + C(x, yz) - C(x, y)z \\
&\quad = s(x)s(y)s(z) - s(x)s(yz) - s(xy)s(z) + s(xyz) \\
&\quad\quad + s(x)s(yz) - s(xyz) - s(x)s(y)s(z) + s(xy)s(z) \\
&\quad = 0,
\end{aligned}$$

and the claim follows. □

Corollary 2.6. *The morphism $\phi : B \to I \oplus^C A$ is an isomorphism of unital associative k-algebras.*

Proof. This follows from Proposition 2.5 and Proposition 2.3. □

Hence, there is always a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow = & & \downarrow \cong & & \downarrow = & & \\
0 & \longrightarrow & I & \xrightarrow{i} & I \oplus^C A & \xrightarrow{p} & A & \longrightarrow & 0
\end{array}$$

where the middle vertical morphism is an isomorphism associative unital k-algebras.

Define the following left and right A-action on the ideal I:

$$xu = s(x)u, \quad ux = us(x),$$

where s is the section of p and $x \in A$, $u \in I$. Recall $I^2 = 0$.

Proposition 2.7. *The actions defined above give the ideal I a left and right A-module structure. The structure is independent of choice of section s.*

Proof. One checks that for any $x, y \in A$ and $u, v \in I$, the following holds:

$$(x + y)u = xu + yu, \quad x(u + v) = xu + xv, \quad 1u = 1.$$

Also,

$$(xy)u - x(yu) = s(xy)u - s(x)s(y)u = \big(s(xy) - s(x)s(y)\big)u = 0,$$

since $I^2 = 0$. It follows that $(xy)u = x(yu)$, hence I is a left A-module. A similar argument prove I is a right A-module. Assume t is another section of p. It follows that

$$s(x)u - t(x)u = \big(s(x) - t(x)\big)u = 0,$$

since $I^2 = 0$. It follows that $s(x)u = t(x)u$. Similarly, $us(x) = ut(x)$ hence s and t induce the same structure of A-module on I and the proposition is proved. $\square$

We have proved the following theorem: let A be any associative unital k-algebra and let I be a left and right A-module. Let $C : A \otimes_k A \to I$ be a morphism satisfying the cocycle condition (2.2).

Theorem 2.8. *The exact sequence:*

$$0 \longrightarrow I \longrightarrow I \oplus^C A \longrightarrow A \longrightarrow 0$$

is a square zero extension of A with the module I. Moreover, any square zero extension of A with I arise this way for some morphism $C \in \mathrm{Hom}_k(A \otimes_k A, I)$ satisfying equation (2.2).

Proof. The proof follows from the discussion above. $\square$

Let

$$0 \longrightarrow I \longrightarrow E \longrightarrow A \longrightarrow 0$$

with $i : I \to E$ and $p : E \to A$ and

$$0 \longrightarrow J \longrightarrow F \longrightarrow B \longrightarrow 0$$

with $j : J \to F$ and $q : F \to B$ be square zero extensions of associative k-algebras A, B with left and right modules I, J. This means the sequences are exact and the following holds $i(I)^2 = j(J)^2 = 0$. A triple (w, u, v) of maps of k-vector spaces giving rise to a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \xrightarrow{i} & E & \xrightarrow{p} & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \\
0 & \longrightarrow & J & \xrightarrow{j} & F & \xrightarrow{q} & B & \longrightarrow & 0
\end{array}$$

is a morphism of extensions if u and v are maps of k-algebras and w is a map of left and right modules. This means

$$w(x + y) = w(x) + w(y), \quad w(ax) = v(a)w(x), \quad w(xa) = w(x)v(a)$$

for all $x, y \in I$ and $a \in A$.

We say two square zero extensions:

$$0 \longrightarrow I \longrightarrow E \longrightarrow A \longrightarrow 0$$

and

$$0 \longrightarrow I \longrightarrow F \longrightarrow A \longrightarrow 0$$

are *equivalent* if there is an isomorphism $\phi : E \to F$ of k-algebras making all diagrams commute.

Definition 2.9. Let $\mathrm{Exan}_k(A,I)$ denote the set of all isomorphism classes of square zero extensions of A by I.

Theorem 2.10. *Let $C(A)$ be the center of A. The set $\mathrm{exan}_k(A,I)$ is a left and right module over $C(A)$. Moreover, there is a bijection:*

$$\mathrm{Exan}_k(A,I) \cong \mathrm{exan}_k(A,I)$$

of sets.

Proof. We first prove that $\mathrm{exan}_k(A,I)$ is a left and right $C(A)$-module. Let $C, D \in \mathrm{exan}_k(A,I)$. This means $C, D \in \mathrm{Hom}_k(A \otimes_k A, I)$ are elements satisfying the cocycle condition (2.2). let $a, b \in C(A) \subseteq A$ be elements. Define aC, Ca as follows:

$$(aC)(x,y) = aC(x,y),$$

and

$$(Ca)(x,y) = C(x,y)a.$$

We see

$$\begin{aligned} &x(aC)(y,x) - (aC)(xy,z) + (aC)(x,yz) - (aC)(x,y)z \\ &\quad = a\big(xC(y,z) - C(xy,z) + C(x,yz) - C(x,y)z\big) = a(0) = 0, \end{aligned}$$

hence $aC \in \mathrm{exan}_k(A,I)$. Similarly, one proves $Ca \in \mathrm{exan}_k(A,I)$ hence we have defined a left and right action of $C(A)$ on the set $\mathrm{exan}_k(A,I)$. Given $C, D \in \mathrm{exan}_k(A,I)$ define

$$(C+D)(x,y) = C(x,y) + D(x,y).$$

One checks that $C + D \in \mathrm{exan}_k(A,I)$ hence $\mathrm{exan}_k(A,I)$ has an addition operation. One checks the following hold:

$$\begin{aligned} &a(C+D) = aC + aD, \quad (C+D)a = Ca + Da, \\ &(a+b)C = aC + bC, \quad C(a+b) = Ca + Cb, \\ &a(bC) = (ab)C, \quad C(ab) = (Ca)b, \quad 1C = C1 = C, \end{aligned}$$

hence the set $\mathrm{exan}_k(A,I)$ is a left and right $C(A)$-module. Define the following map: let $[B] = [I \oplus^C A] \in \mathrm{Exan}_k(A,I)$ be an equivalence class of a square zero extension. Define

$$\phi : \mathrm{Exan}_k(A,I) \longrightarrow \mathrm{exan}_k(A,I)$$

by

$$\phi[B] = \phi\big[I \oplus^C A\big] = C.$$

We prove this gives a well-defined map of sets. Assume $[I \oplus^C A]$ and $[I \oplus^D A]$ are two elements in $\mathrm{Exan}_k(A,I)$. Note: we use brackets to denote isomorphism classes of extensions. The two extensions are equivalent if and only if there is an isomorphism:

$$f : I \oplus^C A \longrightarrow I \oplus^D A$$

of k-algebras such that all diagrams are commutative. This means that

$$f(u,x) = (u,x)$$

for all $(u,x) \in I \oplus^C A$. We get

$$f\big((u,x)(v,y)\big) = f(u,x)f(v,y).$$

This gives the equality:

$$\big(uy + xv + C(x,y), xy\big) = \big(uy + xv + D(x,y), xy\big)$$

for all $(u,x),(v,y) \in I \oplus^C A$. Hence, $\phi[I \oplus^C A] = C = D = \phi[I \oplus^D A]$, and the map ϕ is well defined. It is clearly an injective map. It is surjective by Theorem 2.8 and the claim of the theorem follows. □

Theorem 2.10 shows that there is a structure of left and right $C(A)$-module on the set of equivalence classes of extensions $\mathrm{Exan}_k(A,I)$. The structure as left $C(A)$-module agrees with the one defined in [3].

Let $\phi \in \mathrm{Hom}_k(A,I)$. Let $C^\phi \in \mathrm{Hom}_k(A \otimes_k A, I)$ be defined by

$$C^\phi(x,y) = x\phi(y) - \phi(xy) + \phi(x)y.$$

One checks that $C^\phi \in \mathrm{exan}_k(A,I)$ for all $\phi \in \mathrm{Hom}_k(A,I)$.

Definition 2.11. Let $\mathrm{exan}_k^{inn}(A,I)$ be the subset of $\mathrm{exan}_k(A,I)$ of maps C^ϕ for $\phi \in \mathrm{Hom}_k(A,I)$.

Lemma 2.12. *The set* $\mathrm{exan}_k^{inn}(A,I) \subseteq \mathrm{exan}_k(A,I)$ *is a left and right sub* $C(A)$*-module.*

Proof. The proof is left to the reader as an exercise. □

Definition 2.13. Let $\mathrm{Exan}_k^{inn}(A,I) \subseteq \mathrm{Exan}_k(A,I)$ be the image of $\mathrm{exan}_k^{inn}(A,I)$ under the bijection $\mathrm{exan}_k(A,I) \cong \mathrm{Exan}_k(A,I)$.

It follows that $\mathrm{Exan}_k^{inn}(A,I) \subseteq \mathrm{Exan}_k(A,I)$ is a left and right sub $C(A)$-module.

Recall the definition of the *Hochschild complex* as follows.

Definition 2.14. Let A be an associative k-algebra, and let I be a left and right A-module. Let $C^p(A,I) = \mathrm{Hom}_k(A^{\otimes p}, I)$. Let $d^p : C^p(A,I) \to C^{p+1}(A,I)$ be defined as follows:

$$\begin{aligned} d^p(\phi)\big(a_1 \otimes \cdots \otimes a_{p+1}\big) = a_1\phi\big(a_2 \otimes \cdots \otimes a_{p+1}\big) \\ + \sum_{1 \le i \le p} (-1)^i \phi\big(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{p+1}\big) \\ + (-1)^{p+1}\phi\big(a_1 \otimes \cdots \otimes a_p\big)a_{p+1}. \end{aligned}$$

We let $\mathrm{HH}^i(A,I)$ denote the i'th cohomology of this complex. It is the *i'th Hochschild cohomology* of A with values in I.

Proposition 2.15. *There is an exact sequence:*

$$0 \longrightarrow \mathrm{Exan}_k^{inn}(A,I) \longrightarrow \mathrm{Exan}_k(A,I) \longrightarrow \mathrm{HH}^2(A,I) \longrightarrow 0$$

of left and right $C(A)$*-modules.*

Proof. The proof is left to the reader as an exercise. □

Example 2.16. Characteristic classes of L-connections.

Let A be a commutative k-algebra and let $\alpha : L \to \mathrm{Der}_k(A)$ be a Lie-Rinehart algebra. Let W be a left A-module with an L-connection $\nabla : L \to \mathrm{End}_k(W)$. In [6], we define a characteristic class $c_1(E) \in \mathrm{H}^2(L|_U, \mathcal{O}_U)$ when W is of finite presentation, $U \subseteq \mathrm{Spec}(A)$ is the open set, where W is locally free, and $\mathrm{H}^2(L|_U, \mathcal{O}_U)$ is the Lie-Rinehart cohomology of $L|_U$ with values in $\mathcal{O}_U$. If L is locally free, it follows that $\mathrm{H}^2(L, A) \cong \mathrm{Ext}^2_{U(L)}(A, A)$, where $U(L)$ is the generalized universal enveloping algebra of L. There is an obvious structure of left and right $U(L)$-module on $\mathrm{End}_k(A)$ and an isomorphism:

$$\mathrm{HH}^2\big(U(L), \mathrm{End}_k(A)\big) \cong \mathrm{Ext}^2_{U(L)}(A, A)$$

of abelian groups. The exact sequence 2.15 gives a sequence:

$$\begin{aligned} 0 &\longrightarrow \mathrm{Exan}_k^{inn}\big(U(L), \mathrm{End}_k(A)\big) \longrightarrow \mathrm{Exan}_k\big(U(L), \mathrm{End}_k(A)\big) \\ &\longrightarrow \mathrm{Ext}^2_{U(L)}(A, A) \longrightarrow 0 \end{aligned}$$

with $A = U(L)$ and $I = \mathrm{End}_k(A)$. If we can construct a lifting:

$$\tilde{c}_1(W) \in \mathrm{Exan}_k\big(U(L), \mathrm{End}_k(A)\big)$$

of the class:

$$c_1(W) \in \mathrm{Ext}^2_{U(L)}(A, A) = \mathrm{HH}^2\big(U(L), \mathrm{End}_k(A)\big),$$

we get a generalization of the characteristic class from [6] to arbitrary Lie-Rinehart algebras L. This problem will be studied in a future paper on the subject (see [7]).

Example 2.17. Non-commutative Kodaira-Spencer maps.

Let A be an associative k-algebra, and let M be a left A-module. Let $D^1(A) \subseteq \mathrm{End}_k(A)$ be the *module of first-order differential operators* on A. It is defined as follows: an element $\partial \in \mathrm{End}_k(A)$ is in $D^1(A)$ if and only if $[\partial, a] \in D^0(A) = A \subseteq \mathrm{End}_k(A)$ for all $a \in A$. Define the following map:

$$f : D^1(A) \longrightarrow \mathrm{Hom}_k\big(A, \mathrm{End}_k(M)\big)$$

by

$$f(\partial)(a, m) = [\partial, a]m = \big(\partial(a) - a\partial(1)\big)m.$$

Here, $\partial \in D^1(A)$, $a \in A$, and $m \in M$. Since $[\partial, a] \in A$, we get a well-defined map. Let for any $a \in A$ and $m \in M$ $\phi_a(m) = am$. It follows $\phi_a \in \mathrm{End}_k(M)$ is an endomorphism of M. We get

$$\begin{aligned} f(\partial)(ab, m) &= \big(\partial(ab) - ab\partial(1)\big)m = \big(\partial\phi_{ab} - \phi_{ab}\partial\big)(1)m \\ &= \big(\partial\phi_{ab} - \phi_a\partial\phi_b + \phi_a\partial\phi_b - \phi_{ab}\partial\big)(1)m \\ &= \big(\partial\phi_a - \phi_a\partial\big)\phi_b(1)m + \phi_a\big(\partial\phi_b - \phi_b\partial\big)(1)m \\ &= f(\partial)(a, bm) + af(\partial)(b, m). \end{aligned}$$

Hence,

$$f(\partial)(ab) = af(\partial)(b) + f(\partial)(a)b$$

for all $\partial \in D^1(A)$ and $a, b \in A$. The Hochschild complex gives a map:

$$d^1 : \mathrm{Hom}_k\big(A, \mathrm{End}_k(M)\big) \longrightarrow \mathrm{Hom}_k\big(A \otimes A, \mathrm{End}_k(M)\big),$$

and

$$\ker\big(d^1\big) = \mathrm{Der}_k\big(A, \mathrm{End}_k(M)\big).$$

It follows that we get a map:

$$f : D^1(A) \longrightarrow \mathrm{Der}_k\big(A, \mathrm{End}_k(M)\big).$$

We get an induced map:

$$f : D^1(A) \longrightarrow \mathrm{HH}^1\big(A, \mathrm{End}_k(M)\big) = \mathrm{Ext}^1_A(M, M).$$

Lemma 2.18. *The following holds* $f(D^0(A)) = f(A) = 0$

Proof. The proof is left to the reader as an exercise. □

One checks that $D^1(A)/D^0(A) = D^1(A)/A \cong \mathrm{Der}_k(A)$. It follows that we get an induced map:

$$g : \mathrm{Der}_k(A) = D^1(A)/D^0(A) \longrightarrow \mathrm{Ext}^1_A(M, M),$$

the *non-commutative Kodaira-Spencer map.*

Lemma 2.19. *Assume A is commutative. The following hold:*

$$\mathbb{V}_M = \ker(g) \subseteq \mathrm{Der}_k(A) \text{ is a Lie-Rinehart algebra,} \tag{2.3}$$

$$g(\delta) = 0 \iff \exists \phi \in \mathrm{End}_k(M),\ \phi(am) = a\phi(m) + \delta(a)m, \tag{2.4}$$

$$\exists \nabla \in \mathrm{Hom}_k\big(\mathbb{V}_M, \mathrm{End}_k(M)\big) \text{ with } \nabla(\delta)(am) = a\nabla(\delta)(m) + \delta(a)m, \tag{2.5}$$

$$\mathbb{V}_M \text{ is the maximal Lie-Rinehart algebra satisfying (2.5).} \tag{2.6}$$

Proof. We first prove (2.3): assume $g(\delta) = g(\eta) = 0$. By definition, this is if and only if there are maps $\phi, \psi \in \mathrm{End}_k(M)$ such that the following hold:

$$d^0\phi = g(\delta), \tag{2.7}$$

$$d^0\psi = g(\eta). \tag{2.8}$$

One checks that conditions (2.7) and (2.8) hold if and only if the following hold:

$$\phi(am) = a\phi(m) + \delta(a)m,$$

and

$$\psi(am) = a\psi(m) + \eta(a)m.$$

We claim $d^0[\delta,\eta]=g([\delta,\eta])$. We get

$$\begin{aligned}[\phi,\psi](am) &= \phi\psi(am)-\psi\phi(am)=\phi\big(a\psi(m)+\eta(a)m\big)-\psi\big(a\phi(m)+\delta(a)m\big)\\ &= a\phi\psi(m)+\delta(a)\psi(m)+\eta(a)\phi(m)+\delta\eta(a)m-a\psi\phi(m)-\eta(a)\phi(m)\\ &\quad -\delta(a)\psi(m)-\eta\delta(a)m=a[\phi,\psi](m)+[\delta,\eta](a)m.\end{aligned}$$

Hence, $g([\delta,\eta])=0$ and $\mathbb{V}_M\subseteq \mathrm{Der}_k(A)$ is a k-Lie algebra. It is an A-module since g is A-linear, hence it is a Lie-Rinehart algebra. Claim (2.3) is proved. Claim (2.4) and (2.5) follows from the proof of (2.3). Claim (2.6) is obvious and the lemma is proved. □

The Lie-Rinehart algebra $\mathbb{V}_M$ is the *linear Lie-Rinehart algebra* of M.

Let in the following E be a left and right A-module.

Definition 2.20. Let

$$\mathcal{J}^1_I(E)=I\otimes_A E\oplus E$$

be the *first-order I-jet bundle of E*.

Pick a derivation $d\in \mathrm{Der}_k(A,I)$ of left and right modules. This means that

$$d(xy)=xd(y)+d(x)y$$

for all $x,y\in A$. Let $B^C=I\oplus^C A$ and define the following left B^C-action on $\mathcal{J}^1_I(E)$:

$$(u,x)(w\otimes e,f)=\big(u\otimes f+xw\otimes e+d(x)\otimes f,xf\big)$$

for any elements $(u,x)\in B^C$ and $(w\otimes e,f)\in\mathcal{J}^1_I(E)$.

Proposition 2.21. *The abelian group $\mathcal{J}^1_I(E)$ is a left B^C-module if and only if $C(y,x)\otimes f=0$ for all $y,x\in A$ and $f\in E$.*

Proof. One easily checks that for any $a,b\in B^C$ and $l,j\in\mathcal{J}^1_I(E)$ the following hold:

$$(a+b)i=ai+bi,$$
$$a(i+j)=ai+aj.$$

Moreover,

$$\mathbf{1}i=i.$$

It remains to check that $a(bi)=(ab)i$. Let $a=(v,y)\in B^C$ and $b=(u,x)\in B^C$. Let also $i=(w\otimes e,f)\in\mathcal{J}^1_I(E)$. We get

$$a(bi)=(v,y)\big((u,x)(w\otimes e,f)\big)=\big(vx\otimes f+yu\otimes f+yxw\otimes e+d(yx)\otimes f,yxf\big).$$

We also get

$$(ab)i=\big(vx\otimes f+yu\otimes f+yxw\otimes e+d(yx)\otimes f+C(y,x)\otimes f,yxf\big).$$

It follows that

$$(ab)i-a(bi)=0,$$

if and only if

$$C(y,x)\otimes f=0,$$

and the claim of the proposition follows. □

Note the abelian group $\mathcal{J}^1_I(E)$ is always a left A-module and there is an exact sequence of left A-modules:

$$0 \longrightarrow I \otimes E \longrightarrow \mathcal{J}^1_I(E) \longrightarrow E \longrightarrow 0,$$

defining a characteristic class:

$$c_I(E) \in \mathrm{Ext}^1_A(E, E \otimes I).$$

The class $c_I(E)$ has the property that $c_I(E) = 0$ if and only if E has an I-connection:

$$\nabla : E \longrightarrow I \otimes E$$

with

$$\nabla(xe) = x\nabla(e) + d(x) \otimes e.$$

Let $J \subseteq I \subseteq B^C$ be the smallest two-sided ideal containing $Im(C)$, where $C : A \otimes_k A \to I$ is the cocycle defining B^C. Let $D^C = B^C/J$ and $I^C = I/J$. We get a square zero extension:

$$0 \longrightarrow I^C \longrightarrow D^C \longrightarrow A \longrightarrow 0$$

of A by the square zero ideal I^C. It follows that $D^C = I^C \oplus A$ as abelian group. Since $C(x, y) = 0$ in I^C, it follows that D^C has a well-defined associative multiplication defined by

$$(u, x)(v, y) = (uy + xv, xy).$$

Also D^C is the largest quotient of B^C such that the ring homomorphism $B^C \to D^C$ fits into a commutative diagram of square zero extensions:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & B^C & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{=} & & \\
0 & \longrightarrow & I^C & \longrightarrow & D^C & \longrightarrow & A & \longrightarrow & 0.
\end{array}$$

Definition 2.22. Let

$$\mathcal{J}^1_{I^C}(E) = I^C \otimes E \oplus E$$

be the *first-order I^C-jet bundle of E*.

Example 2.23. First-order commutative jets.

Let $k \to A$ be a commutative k-algebra, and let $I \subseteq A \otimes_k A$ be the ideal of the diagonal. Let $\mathcal{J}^1_A = A \otimes A/I^2$ and $\Omega^1_A = I/I^2$. We get an exact sequence of left A-modules:

$$0 \longrightarrow \Omega^1_A \longrightarrow \mathcal{J}^1_A \longrightarrow A \longrightarrow 0. \tag{2.9}$$

It follows that $\mathcal{J}^1_A \cong \Omega^1_A \oplus A$ with the following product:

$$(\omega, a)(\eta, b) = (\omega a + b\eta, ab),$$

hence the sequence (2.9) splits. Let $\mathcal{J}^1_A(E) = \Omega^1_A \otimes E \oplus E$ be the first-order Ω^1_A-jet of E. We get an exact sequence of left A-modules:

$$0 \longrightarrow \Omega^1_A \otimes E \longrightarrow \mathcal{J}^1_A(E) \longrightarrow E \longrightarrow 0.$$

Since the sequence (2.9) splits, it follows that $\mathcal{J}^1_A(E)$ is a lifting of E to the first-order jet $\mathcal{J}^1_A$.

3 Atiyah classes and Kodaira-Spencer classes

In this section, we define and prove some properties of Atiyah classes and Kodaira-Spencer classes.

Let X be any scheme defined over an arbitrary basefield F and let $\operatorname{Pic}(X)$ be the *Picard group* of X. Let $\mathcal{O}^* \subseteq \mathcal{O}_X$ be the following subsheaf of abelian groups: for any open set $U \subseteq X$, the group $\mathcal{O}(U)^*$ is the multiplicative group of units in $\mathcal{O}_X(U)$. Define for any open set $U \subseteq X$ the following morphism:

$$\operatorname{dlog} : \mathcal{O}(U)^* \longrightarrow \Omega^1_X(U),$$

defined by

$$\operatorname{dlog}(x) = d(x)/x,$$

where d is the universal derivation and $x \in \mathcal{O}(U)^*$.

Lemma 3.1. *The following holds:*

$$\operatorname{dlog}(xy) = \operatorname{dlog}(x) + \operatorname{dlog}(y)$$

for $x, y \in \mathcal{O}(U)^*$

Proof. The proof is left to the reader as an exercise. □

Hence, $\operatorname{dlog} : \mathcal{O}^* \to \Omega^1_X$ defines a map of sheaves of abelian groups. The map dlog induces a map on cohomology

$$\operatorname{dlog} : \operatorname{Pic}(X) = \mathrm{H}^1\left(X, \mathcal{O}^*\right) \longrightarrow \mathrm{H}^1\left(X, \Omega^1_X\right),$$

and by definition

$$\operatorname{dlog}(\mathcal{L}) = c_1(\mathcal{L}) \in \mathrm{H}^1\left(X, \Omega^1_X\right).$$

Let $\mathcal{I} \subseteq \Omega^1_X$ be any sub $\mathcal{O}_X$-module, and let $\mathcal{F} = \Omega^1_X/\mathcal{I}$ be the quotient sheaf. We get a derivation:

$$d : \mathcal{O}_X \longrightarrow \mathcal{F}$$

by composing with the universal derivation. We get a canonical map:

$$\mathrm{H}^1\left(X, \Omega^1_X\right) \longrightarrow \mathrm{H}^1(X, \mathcal{F}),$$

and we let

$$\overline{c}_1(\mathcal{L}) \in \mathrm{H}^1(X, \mathcal{F})$$

be the image of $c_1(\mathcal{L})$ under this map.

Definition 3.2. The class $c_1(\mathcal{L}) \in \mathrm{H}^1(X, \Omega^1_X)$ is the *first Chern class* of the line bundle $\mathcal{L} \in \operatorname{Pic}(X)$. The class $\overline{c}_1(\mathcal{L}) \in \mathrm{H}^1(X, \mathcal{F})$ is the *generalized first Chern class* of $\mathcal{L}$.

Let $\mathcal{E}$ be any $\mathcal{O}_X$-module and consider the following sequence of sheaves of abelian groups:

$$0 \longrightarrow \mathcal{F} \otimes \mathcal{E} \longrightarrow \mathcal{J}^1_{\mathcal{F}}(\mathcal{E}) \longrightarrow \mathcal{E} \longrightarrow 0,$$

where

$$\mathcal{J}^1_{\mathcal{F}}(\mathcal{E}) = \mathcal{F} \otimes \mathcal{E} \oplus \mathcal{E}$$

as sheaf of abelian groups. Let s be a local section of $\mathcal{O}_X$, and let $(x \otimes e, f)$ be a local section of $\mathcal{J}^1_{\mathcal{F}}(\mathcal{E})$ over some open set U. Make the following definition:

$$s(x \otimes e, f) = (sx \otimes e + ds \otimes f, sf).$$

It follows that the sequence

$$0 \longrightarrow \mathcal{F} \otimes \mathcal{E} \longrightarrow \mathcal{J}^1_{\mathcal{F}}(\mathcal{E}) \longrightarrow \mathcal{E} \longrightarrow 0$$

is a short exact sequence of sheaves of abelian groups. It is called the *Atiyah-Karoubi sequence.*

Definition 3.3. An $\mathcal{F}$-connection ∇ is a map:

$$\nabla : \mathcal{E} \longrightarrow \mathcal{F} \otimes \mathcal{E}$$

of sheaves of abelian groups with

$$\nabla(se) = s\nabla(e) + d(s) \otimes e.$$

Proposition 3.4. *The Atiyah-Karoubi sequence is an exact sequence of left $\mathcal{O}_X$-modules. It is left split by an $\mathcal{F}$-connection.*

Proof. We first show that it is an exact sequence of left $\mathcal{O}_X$-modules. The $\mathcal{O}_X$-module structure is twisted by the derivation d, hence we must verify that this gives a well-defined left $\mathcal{O}_X$-structure on $\mathcal{J}^1_{\mathcal{F}}(\mathcal{E})$. Let $\omega = (x \otimes e, f)$ be a local section of $\mathcal{J}^1_{\mathcal{F}}(\mathcal{E})$, and let s, t be local sections of $\mathcal{O}_X$. We get the following calculation:

$$\begin{aligned}(st)\omega &= (st)(x \otimes e, f) = \big((st)x \otimes e + d(st) \otimes f, (st)f\big)\\ &= \big(stx \otimes e + sdt \otimes f + (ds)t \otimes f, stf\big) = \big(s(tx \otimes e + dt \otimes f) + ds \otimes tf, s(tf)\big)\\ &= s(tx \otimes e + dt \otimes f, tf) = s\big(t(x \otimes e, f)\big) = s(t\omega).\end{aligned}$$

It follows that $\mathcal{J}^1_{\mathcal{F}}(\mathcal{E})$ is a left $\mathcal{O}_X$-module and the sequence is left exact. Assume that

$$s : \mathcal{E} \longrightarrow \mathcal{J}_{\mathcal{F}}(\mathcal{E}) = \mathcal{F} \otimes \mathcal{E} \oplus \mathcal{E}$$

is a left splitting. It follows that $s(e) = (\nabla(e), e)$ for e a local section of $\mathcal{E}$. It follows that ∇ is a generalized connection and the theorem is proved. □

Note: If $\mathcal{I} = 0$, we get that $\mathcal{J}^1_{\mathcal{F}}(\mathcal{E}) = \mathcal{J}^1_X(\mathcal{E})$ is the first-order jet bundle of $\mathcal{E}$ and the exact sequence above specializes to the well-known *Atiyah sequence*:

$$0 \longrightarrow \Omega^1_X \otimes \mathcal{E} \longrightarrow \mathcal{J}^1_X(\mathcal{E}) \longrightarrow \mathcal{E} \longrightarrow 0.$$

The Atiyah sequence is left split by a connection:

$$\nabla : \mathcal{E} \longrightarrow \Omega^1_X \otimes \mathcal{E}.$$

The $\mathcal{O}_X$-module $\mathcal{J}^1_{\mathcal{F}}(\mathcal{E})$ is the *generalized first-order jet bundle of $\mathcal{E}$.*

Definition 3.5. The characteristic class:

$$\mathrm{AT}(\mathcal{E}) \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F} \otimes \mathcal{E})$$

is called the *Atiyah class* of $\mathcal{E}$.

The class $\mathrm{AT}(\mathcal{E})$ is defined for an arbitrary $\mathcal{O}_X$-module $\mathcal{E}$ and an arbitrary sub module $\mathcal{I} \subseteq \Omega^1_X$.

Assume that $\mathcal{E} = \mathcal{L} \in \mathrm{Pic}(X)$ is a line bundle on X. We get isomorphisms:

$$\begin{aligned}\mathrm{Ext}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L} \otimes \mathcal{F}) &\cong \mathrm{Ext}^1_{\mathcal{O}_X}\big(\mathcal{O}_X, \mathcal{L}^* \otimes \mathcal{L} \otimes \mathcal{F}\big) \\ &\cong \mathrm{Ext}^1_{\mathcal{O}_X}\big(\mathcal{O}_X, \mathcal{F}\big) \longrightarrow \mathrm{H}^1(X, \mathcal{F}).\end{aligned}$$

We get a morphism:

$$\phi : \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L} \otimes \mathcal{F}) \longrightarrow \mathrm{H}^1(X, \mathcal{F}).$$

Proposition 3.6. *The following holds:*

$$\phi\big(\,\mathrm{AT}(\mathcal{L})\big) = \overline{c}_1(\mathcal{L}).$$

Hence, the Atiyah class calculates the generalized first Chern class of a line bundle.

Proof. Let $\mathcal{I} = 0$. It is well known that $\mathrm{AT}(\mathcal{L})$ calculates the first Chern class $c_1(\mathcal{L})$. From this, the claim of the proposition follows. □

Let T_X be the tangent sheaf of X. It has the property that for any open affine set $U = \mathrm{Spec}(A) \subseteq X$ the local sections $T_X(U)$ equal the module $\mathrm{Der}_F(A)$ of derivations of A. Let $\mathbb{V}_{\mathcal{E}} \subseteq T_X$ be the subsheaf of local sections ∂ of T_X with the following property: the section $\partial \in T_X(U)$ lifts to a local section $\nabla(\partial)$ of $\mathrm{End}_F(\mathcal{E}|_U)$ with the following property:

$$\nabla(\partial) : \mathcal{E}|_U \longrightarrow \mathcal{E}|_U$$

which satisfies

$$\nabla(\partial)(se) = s\nabla(\partial)(e) + \partial(s)e.$$

It follows that $\mathbb{V}_{\mathcal{E}} \subseteq T_X$ is a subsheaf of Lie algebras – the *Kodaira-Spencer sheaf* of $\mathcal{E}$.

Define for any local sections a, b of $\mathcal{O}_X$, ∂ of $\mathbb{V}_{\mathcal{E}}$ and e of $\mathcal{E}$ the following:

$$L(a, \partial)(e) = a\nabla(\partial)(e) - \nabla(a\partial)(e).$$

Lemma 3.7. *It follows that* $L(a, \partial) \in \mathrm{End}_{\mathcal{O}_U}(\mathcal{E}|_U)$.

Proof. The following holds:

$$\begin{aligned}L(a, \partial)(be) &= a\nabla(\partial)(be) - \nabla(a\partial)(be) \\ &= a\big(b\nabla(\partial)(e) + \partial(b)e\big) - b\nabla(a\partial)(e) - a\partial(b)e \\ &= ab\nabla(\partial)(e) + a\partial(b)e - b\nabla(a\partial)(e) - a\partial(b)e \\ &= b\big(a\nabla(\partial)(e) - \nabla(a\partial)(e)\big) = b\big(a\nabla(\partial) - \nabla(a\partial)\big)(e) \\ &= bL(a, \partial)(e),\end{aligned}$$

and the lemma is proved. □

Lemma 3.8. *The following formula holds:*

$$L(ab,\partial) = aL(b,\partial) + L(a,b\partial)$$

for all local sections a, b, and ∂.

Proof. We get

$$\begin{aligned} L(ab,\partial) &= ab\nabla(\partial) - \nabla(ab\partial) \\ &= ab\nabla(\partial) - a\nabla(b\partial) + a\nabla(b\partial) - \nabla(ab\partial) \\ &= a\big(b\nabla(\partial) - \nabla(b\partial)\big) + (a\nabla - \nabla a)(b\partial) \\ &= aL(b,\partial) + L(a,b\partial), \end{aligned}$$

and the lemma is proved. □

Let $\mathrm{LR}(\mathbb{V}_{\mathcal{E}}) = \mathrm{End}_{\mathcal{O}_X}(\mathcal{E}) \oplus \mathbb{V}_{\mathcal{E}}$ be the *linear Lie-Rinehart algebra* of $\mathcal{E}$. Let $\mathrm{LR}(\mathbb{V}_{\mathcal{E}})$ have the following left $\mathcal{O}_X$-module structure:

$$a(\phi,\partial) = \big(a\phi + L(a,\partial), a\partial\big).$$

Here, a, ϕ, and ∂ are local sections of $\mathcal{O}_X$, $\mathrm{End}_{\mathcal{O}_X}(\mathcal{E})$, and $\mathbb{V}_{\mathcal{E}}$. We twist the trivial $\mathcal{O}_X$ structure on $\mathrm{End}_{\mathcal{O}_X}(\mathcal{E}) \oplus \mathbb{V}_{\mathcal{E}}$ with the element L. We get a sequence of sheaves of abelian groups:

$$0 \longrightarrow \mathrm{End}_{\mathcal{O}_X}(\mathcal{E}) \xrightarrow{i} \mathrm{LR}\left(\mathbb{V}_{\mathcal{E}}\right) \xrightarrow{p} \mathbb{V}_{\mathcal{E}} \longrightarrow 0,$$

where i and p are the canonical maps. An $\mathcal{O}_X$-linear map:

$$\nabla : \mathbb{V}_{\mathcal{E}} \longrightarrow \mathrm{End}_F(\mathcal{E}),$$

satisfying

$$\nabla(\partial)(ae) = a\nabla(\partial)(e) + \partial(a)e$$

is a $\mathbb{V}_{\mathcal{E}}$*-connection* on $\mathcal{E}$.

Proposition 3.9. *The sequence defined above is an exact sequence of left $\mathcal{O}_X$-modules. It is left split by a $\mathbb{V}_{\mathcal{E}}$-connection ∇.*

Proof. We need to check that $\mathrm{LR}(\mathbb{V}_{\mathcal{E}})$ has a well-defined left $\mathcal{O}_X$-module structure. By definition,

$$a(\phi,\partial) = \big(a\phi + L(a,\partial), a\partial\big).$$

We get

$$\begin{aligned} (ab)x = (ab)(\phi,\partial) &= \big((ab)\phi + L(ab,\partial), (ab)\partial\big) \\ &= \big(ab\phi + aL(b,\partial) + L(a,b\partial), ab\partial\big) \\ &= a\big(b\phi + L(b,\partial), b\partial\big) = a\big(b(\phi,\partial)\big) = a(bx), \end{aligned}$$

and it follows that the sequence is a left exact sequence of $\mathcal{O}_X$-modules. If

$$s : \mathbb{V}_{\mathcal{E}} \longrightarrow \mathrm{End}_{\mathcal{O}_X}(\mathcal{E}) \oplus \mathbb{V}_{\mathcal{E}} = \mathrm{LR}\left(\mathbb{V}_{\mathcal{E}}\right)$$

is a section, it follows that $s(e) = (\nabla(e), e)$. One checks that ∇ is a $\mathbb{V}_{\mathcal{E}}$-connection, and the theorem is proved. □

Definition 3.10. We get a characteristic class:

$$\mathrm{KS}(\mathcal{E}) \in \mathrm{Ext}^1_{\mathcal{O}_X}\big(\mathbb{V}_{\mathcal{E}}, \mathrm{End}_{\mathcal{O}_X}(\mathcal{E})\big),$$

the *Kodaira-Spencer class* of $\mathcal{E}$.

Assume that $\mathbb{V}_{\mathcal{E}}$ is locally free and $\mathcal{E} = \mathcal{L} \in \mathrm{Pic}(X)$ is a line bundle on X. Assume also that $\mathbb{V}^*_{\mathcal{E}} = \mathcal{F} = \Omega^1_X/\mathcal{I}$ for some submodule $\mathcal{I}$. We get the following calculation:

$$\begin{aligned}\mathrm{Ext}^1_{\mathcal{O}_X}\big(\mathbb{V}_{\mathcal{E}}, \mathrm{End}_{\mathcal{O}_X}(\mathcal{L})\big) &\cong \mathrm{Ext}^1_{\mathcal{O}_X}\big(\mathcal{O}_X, \mathrm{End}_{\mathcal{O}_X}(\mathcal{L}) \otimes \mathbb{V}^*_{\mathcal{E}}\big)\\ &\cong \mathrm{Ext}^1_{\mathcal{O}_X}\big(\mathcal{O}_X, \mathrm{End}_{\mathcal{O}_X}(\mathcal{L}) \otimes \mathcal{F}\big) \longrightarrow \mathrm{H}^1(X, \mathcal{F}).\end{aligned}$$

We get a map:

$$\psi : \mathrm{Ext}^1_{\mathcal{O}_X}\big(\mathbb{V}_{\mathcal{E}}, \mathrm{End}_{\mathcal{O}_X}(\mathcal{L})\big) \longrightarrow \mathrm{H}^1(X, \mathcal{F})$$

of sheaves.

Proposition 3.11. *The following holds: there is an equality:*

$$\psi\big(\,\mathrm{KS}(\mathcal{L})\big) = \overline{c}_1(\mathcal{L})$$

in $\mathrm{H}^1(X, \mathcal{F})$*. Hence the Kodaira-Spencer class calculates the class* $\overline{c}_1(\mathcal{L})$*.*

Proof. The proof is left to the reader as an exercise. □

We get the following diagram expressing the relationship between the characteristic classes defined above:

$$\begin{array}{ccc} \mathrm{Ext}^1_{\mathcal{O}_X}\big(\mathbb{V}_{\mathcal{L}}, \mathrm{End}_{\mathcal{O}_X}(\mathcal{L})\big) & & \\ & \searrow^{\psi} & \\ & & \mathrm{H}^1(X, \mathcal{F}) \xleftarrow{\overline{c}_1(-)} \mathrm{Pic}(X)\,. \\ & \nearrow^{\phi} & \\ \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{L}, \mathcal{F} \otimes \mathcal{L}) & & \end{array}$$

The following equation holds in $\mathrm{H}^1(X, \mathcal{F})$:

$$\phi\big(\,\mathrm{AT}(\mathcal{L})\big) = \psi\big(\,\mathrm{KS}(\mathcal{L})\big) = \overline{c}_1(\mathcal{L}).$$

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