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Research Article

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# Algebra, Hyperalgebra and Lie-Santilli Theory

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**Abstract**

The theory of hyperstructures can offer to the Lie-Santilli Theory a variety of models to specify the mathematical representation of the related theory. In this paper we focus on the appropriate general hyperstructures, especially on hyperstructures with hyperunits. We define a Lie hyperalgebra over a hyperfield as well as a Jordan hyperalgebra, and we obtain some results in this respect. Finally, by using the concept of fundamental relations we connect hyper algebras to Lie algebras and Lie-Santilli-addmissible algebras.

**Keywords:** Algebra; Hyperring; Hyperfield; Hypervector space; Hyper algebra; Lie hyperalgea; Lie admissible hyperalgebra; Fundamental relation

## Introduction

The structure of the laws in physics is largely based on symmetries. The objects in Lie theory are fundamental, interesting and innovating in both mathematics and physics. It has many applications to the spectroscopy of molecules, atoms, nuclei and hadrons. The central role of Lie algebra in particle physics is well known. A Lie-admissible algebra, introduced by Albert [1], is a (possibly non-associative) algebra that becomes a Lie algebra under the bracket $[a,b] = ab - ba$. Examples include associative algebras, Lie algebras and Okubo algebras. Lie admissible algebras arise in various topics, including geometry of invariant affine connections on Lie groups and classical and quantum mechanics.

For an algebra $A$ over a field $F$, the commutator algebra $A^-$ of $A$ is the anti-commutative algebra with multiplication $[a,b] = ab - ba$ defined on the vector space $A$. If $A^-$ is a Lie algebra, i.e., satisfies the Jacobi identity, then $A^-$ is called Lie-admissible. Much of the structure theory of Lie-admissible algebras has been carried out initially under additional conditions such as the flexible identity or power-associativity.

Santilli obtained Lie admissible algebras (brackets) from a modified form of Hamilton's equations with external terms which represent a general non-self-adjoint Newtonian system in classical mechanics. In 1967, Santilli introduced the product

$$(A,B) = \lambda AB - \mu BA = \alpha(AB - BA) + \beta(AB + BA), \quad (1)$$

where $\lambda = \alpha + \beta$, $\mu = \ \alpha$, which is jointly Lie admissible and Jordan admissible while admitting Lie algebras in their classification. Then, he introduced the following infinitesimal and finite generalizations of Heisenberg equations

$$i\frac{dA}{dt} = (A,H) = \lambda AH - \mu HA, \quad (2)$$

where $A(t) = U(t)A(0)V(t)^\dagger = e^{Ht\mu i}A(0)e^{-i\lambda tH}$, $U = e^{Ht\mu i}$, $V = e^{i\lambda tH}UV^\dagger \neq I$ and $H$ is the Hamiltonian. In 1978, Santilli introduced the following most general known realization of products that are jointly Lie admissible and Jordan admissible

$$\begin{aligned}(A,B) &= ARB - BSA = (ATB - BTA) + \{AWB + BWA\} \\ &= [A,B]^* + \{A,B\}^* \\ &= (ATH - HTA) + \{AWH + HWA\},\end{aligned} \quad (3)$$

where $R=T+W$, $S=W-T$ and $R,S,R\pm S$ are non-singular operators [2-25].

Algebraic hyperstructures are a natural generalization of the ordinary algebraic structures which was first initiated by Marty [11]. After the pioneered work of Marty, algebraic hyperstructuires have been developed by many researchers. A review of hyperstructures can be found in studies of Corsini [3,4,7,8,24]. This generalization offers a lot of models to express their problems in an algebraic way. Several applications appeared already as in Hadronic Mechanics, Biology, Conchology, Chemistry, and so on. Davvaz, Santilli and Vougiouklis studied multi-valued hyperstructures following the apparent existence in nature of a realization of two-valued hyperstructures with hyperunits characterized by matter-antimatter systems and their extentions where matter is represented with conventional mathematics and antimatter is represented with isodual mathematics [6,9,10]. On the other hand, the main tools connecting the class of algebraic hyperstructures with the classical algebraic structures are the fundamental relations [5,8,22,24]. In this paper, we study the notion of algebra, hyperalgebra and their connections by using the concept of fundamental relation. We introduce a special class of Lie hyperalgebra. By this class of Lie hyperalgebra, we are able to generalize the concept of Lie-Santilli theory to hyperstructure case.

## Hyperrings, Hyperfields and Hypervector Spaces

Let $H$ be a non-empty set and $\circ : H \times H \to \wp^*(H)$ be a *hyperoperation*, where $\wp^*(H)$ is the family of all non-empty subsets of $H$. The couple ($H$,o) is called a *hypergroupoid*. For any two non-empty subsets $A$ and $B$ of $H$ and $x \in H$, we define $A \circ B = \bigcup_{a\in A, b\in B} a \circ b$, $A$o$\{x\}$ = $A$o$x$ and $\{x\}$ o$A$ = $x$o$A$. A hypergroupoid ($H$,o) is called a *semihypergroup* if for all $a,b,c$ in $H$ we have ($aob$) o $c$ = $a$o ($boc$). In addition, if for every $a \in H$, $a \circ H = H = H \circ a$, then, ($H$,o) is called a *hypergroup*. A non-empty subset $K$ of a semihypergroup ($H$,o) is called a *sub-semihypergroup*

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if it is a semihypergroup. In other words, a non-empty subset $K$ of a semihypergroup $(H,o)$ is a sub-semihypergroup if $K\ K \subseteq K$. We say that a hypergroup $(H,o)$ is *canonical* if

- It is commutative;
- It has a scalar identity (also called scalar unit), which means that there exists $e \in H$, for all $x \in H$, $x$ o $e= x$;
- Every element has a unique inverse, which means that for all $x \in H$, there exists a unique $x^{-1} \in H$ such that $e \in x$ o $x^{-1}$;
- It is reversible, which means that if $x \in y$ o $z$, then $z \in y^{-1}$o $x$ and $y \in x$ o $z^{-1}$.

In liteature of Davvaz, there are several types of hyperrings and hyperfields [8]. In what follows we shall consider one of the most general types of hyperrings.

The triple $(R, +, \cdot)$ is a *hyperring* if

- $(R, +)$ is a canonical hypergroup;
- $(R, \cdot)$ is a semihypergroup such that $x\cdot 0 = 0 \cdot x = 0$ for all $x \in R$, i.e, 0 is a bilaterally absorbing element;
- the hyperoperation "." is distributive over the hyperoperation "+", which means that for all $x$, $y$, $z$ of $R$ we have:

$x\cdot(y+z) = x\cdot y + x\cdot z$ and $(x+y)\cdot z = x\cdot z + y\cdot z$.

**Example 1.** *Let R={x,y,z,t} be a set with the following hyperoperations:*

| + | x | y | z | t |
|---|---|---|---|---|
| x | x | y | z | t |
| y | y | x | t | z |
| z | z | t | {x, z} | {y, t} |
| t | t | z | {y, t} | {x, z} |

| · | x | y | z | t |
|---|---|---|---|---|
| x | x | x | x | x |
| y | x | y | x | y |
| z | x | x | {x, z} | {x, z} |
| t | x | y | {x, z} | {y, t} |

Then, $(F, +, \cdot)$ is a hyperring.

We call $(R, +, \cdot)$ a *hyperfield* if $(R, +, \cdot)$ is a hyperring and $(R-\{0\},\cdot)$ is a hypergroup.

**Example 2.** *Let F ={x,y} be a set with the following hyperoperations:*

| + | x | y |
|---|---|---|
| x | x | y |
| y | y | {x, y} |

| · | x | y |
|---|---|---|
| x | x | x |
| y | x | y |

Then, $(F, +, \cdot)$ is a hyperfield.

A *Krasner hyperring* is a hyperring such that$(R, +)$ is a canonical hypergroup with identity 0 and $\cdot$ is an operation such that 0 is a bilaterally absorbing element. An exhaustive review updated to 2007 of hyperring theory appears in [8].

**Definition 2.1.** *Let* $(R, +, \cdot)$ *be a hypering. We define the relation* $\gamma$ *as follows:*

$$x\,\gamma\, y \Leftrightarrow \exists n \in \mathbb{N}, \exists(k_1,\cdots,k_n)\in \mathbb{N}^n \quad and \quad [\exists(x_{i1},\cdots,x_{ik_i})\in R^{k_i},(i=1,\cdots,n)]$$

such that

$$x,y \in \sum_{i=1}^{n}\left(\prod_{j=1}^{k_i} x_{ij}\right).$$

**Theorem 2.2.** [24, 25] *Let* $(R,+,\cdot)$ *be a hypering* $\gamma^*$ *be the transitive closure of* $\gamma$.

- $\gamma^*$ is a strongly regular relation both on $(R, +)$ and $(R, \cdot)$.
- The quotient $R/\gamma^*$ is a ring.
- The relation $\gamma^*$ is the smallest equivalence relation such that the quotient $R/\gamma^*$ be a ring.

**Theorem 2.3.** [12] *The relation* $\gamma$ *on every hyperfield is an equivalence relation and* $\gamma=\gamma^*$.

**Remark 1.** *Let* $(F,+,\cdot)$ *be a hyperfield. Then,* $F/\gamma^*$ *is a field. If* $\varphi$ :$F\to F/\gamma^*$ *is the canonical map, then* $\omega_F = \{x \in F \mid \phi(x)=0\}$, *where 0 is the zero of the fundamental field* $F/\gamma^*$.

Let $(R,+,\cdot)$ be a hyperring, $(M, +)$ be a canonical hypergroup and there exists an external map

$$\cdot : R\times M \to \wp^*(M), (a,x)\mapsto ax$$

such that for all $a,b \in R$ and for all $x,y \in M$ we have

$$a(x+y)=ax+ay, (a+b)x = ax+bx, (ab)x = a(bx),$$

then $M$ is called a hypermodule over $R$. If we consider a hyperfield $F$ instead of a hyperring $R$, then $M$ is called a hypervector space.

**Remark 2.** Note that it is possible in a hypervector space one or more of hyperoperations be ordinary operations.

**Example 3.** Let $F$ be a field and $V$ be a vector space on $F$. If $S$ is a subspace of $V$, we consider the following external hyperoperation:*ao x=ax+S*, for all $a \in F$ and $x \in V$. Then, $V$ is a hypervector space.

## Algebra and Hyperalgebra

**Definition 3.1.** *Let* $(L,+,\cdot)$ be a hypervector space over the hyperfield $(F,+,\cdot)$. Consider the bracket (commutator) hope:

$$[\,,\,]: L\times L \to \wp^*(L) : (x,y)\to [x,y]$$

then $L$ is a Lie hyperalgebra over $F$ if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e. $[\lambda_1 x_1 + \lambda_2 x_2, y] = (\lambda_1[x_1,y]+\lambda_2[x_2,y])$, $[x,\lambda_1 y_1+\lambda_2 y_2] = (\lambda_1[x,y_1]+\lambda_2[x,y_2])$, for all $x,x_1,x_2,y,y_1,y_2 \in L$, $\lambda_1,\lambda_2 \in F$;

(L2) $0 \in [x,x]$, for all $x \in L$;

(L3) $0 \in ([x,[y,z]]+[y,[z,x]]+[z,[x,y]])$, for all $x,y \in L$.

**Definition 3.2.** Let $A$ be a hypervector space over a hyperfield $F$. Then, $A$ is called a hyperalgebra over the hyperfield $F$ if there exists a mapping $\cdot : A\times A \to \wp^*(A)$ (images to be denoted by $x\cdot y$ *for* $x,y \in A$ such that the following conditions hold:

$(x+y)\cdot z = x\cdot z + y\cdot z$ and $x\cdot(y+z) = x\cdot y + x\cdot z$;

$(cx)\cdot y = c(x\cdot y) = x\cdot(cy)$;

$0\cdot y = y\cdot 0 = 0$;

for all $x,y,z \in A$ and $c \in F$.

In the above definition, if all hyperoperations are ordinary operations, then we have an *algebra*.

**Example 4.** Let $F$ be a hyperfield and

$$A = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a,b,c,d \in F\}.$$

We define the following hyperoperations on $A$:

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$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \uplus \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \{\begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix} |$$
$$a_3 \in a_1 + a_2, b_3 \in b_1 + b_{2,3} \in c_1 + c_2, d_3 \in d_1 + d_2\},$$
$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \otimes \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \{\begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix} |$$
$$a_3 \in a_1 \cdot a_2 + b_1 \cdot d_2, b_3 \in a_1 \cdot b_2 + +b_1 \cdot d_{2,3} \in c_1 \cdot a_2 + d_1 \cdot c_2, d_3 \in c_1 \cdot b_2 + d_1 \cdot d_2\},$$
$$a \bullet \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \{\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} | \quad a_2 \in aa_1, b_2 \in ab_1, c_3 \in ac_{2,3} \in ad_2\}.$$

Then $A$ together with the above hyperoperations is a hyperalgebra over $F$.

**Example 5.** We can generalize Example 4 to $n \times n$ matrices.

A non-empty subset $A'$ of a hyperalgebra $A$ is called a *sub hyperalgebra* if it is a subhyperspace of $A$ and for all $x, y \in A'$ we have $xy \in A'$.

In connection with the explicit forms of the hyperproduct let us consider an associative hyperalgebra $A$, with hyperproduct $a{\cdot}b$, over a hyperfield $F$. It is possible to construct a new hyperalgebra, denoted by $A^-$, by means of the anti-commutative hyperproduct

$$[a,b] = a \cdot b - b \cdot a = \bigcup_{\substack{x \in a \cdot b \\ y \in b \cdot a}} x - y. \tag{4}$$

**Lemma 3.3.** *For any non-empty subset S of A, we have* $0 \in S - S$..

*Proof.* It is straightforward.

**Proposition 3.4.** *$A^-$ is a Lie hyperalgebra.*

*Proof.* For all $x, x_1, x_2, y, y_1, y_2 \in L$, $\lambda_1, \lambda_2 \in F$ ,we have

$$\begin{aligned}
[\lambda_1 x_1 + \lambda_2 x_2, y] &= (\lambda_1 x_1 + \lambda_2 x_2) \cdot y - y \cdot (\lambda_1 x_1 + \lambda_2 x_2) \\
&= (\lambda_1 x_1) \cdot y + (\lambda_2 x_2) \cdot y - y \cdot (\lambda_1 x_1) - y \cdot (\lambda_2 x_2) \\
&= ((\lambda_1 x_1) \cdot y - y \cdot (\lambda_1 x_1)) + ((\lambda_2 x_2) \cdot y - y \cdot (\lambda_2 x_2)) \\
&= [\lambda_1 x_1, y] + [\lambda_2 x_2, y], \\
&= \lambda_1 [x_1, y] + \lambda_2 [x_2, y],
\end{aligned}$$

and similarly we obtain $[x, \lambda_1 y_1 + \lambda_2 y_2] = (\lambda_1 [x, y_1] + \lambda_2 [x, y_2])$.

Now, we prove (L2). Since $x{\cdot}x$ is non-empty, there exists $a_0 \in x{\cdot}x$. hence, $-a_0 \in -xx$. Thus, $0 \in a_0 - a_0 \subseteq \bigcup_{a \in x \cdot x} a - a = x \cdot x - x \cdot x = [x, x]$.

It remains to show that (L3) is also satisfied. For,

$$\begin{aligned}
[x,[y,z]] &= x \cdot [y,z] - [y,z] \cdot x \\
&= x \cdot (y \cdot z - z \cdot y) - (y \cdot z - z \cdot y) \cdot x \\
&= x \cdot y \cdot z - x \cdot z \cdot y - y \cdot z \cdot x - z \cdot y \cdot x.
\end{aligned}$$

Hence,

$$\begin{aligned}
([x,[y,z]] + [y,[z,x]] + [z,[x,y]]) &= x \cdot y \cdot z - x \cdot z \cdot y - y \cdot z \cdot x - z \cdot y \cdot x \\
&\quad + (y \cdot z \cdot x - y \cdot x \cdot z - z \cdot x \cdot y + x \cdot z \cdot y) \\
&\quad + (z \cdot x \cdot y - z \cdot y \cdot x - x \cdot y \cdot z + y \cdot x \cdot z) \\
&= (x \cdot y \cdot z - x \cdot y \cdot z) + (x \cdot z \cdot y - x \cdot z \cdot y) \\
&\quad + (z \cdot y \cdot x - z \cdot y \cdot x) + (y \cdot z \cdot x - y \cdot z \cdot x) \\
&\quad + (y \cdot x \cdot z - y \cdot x \cdot z) + (z \cdot x \cdot y - z \cdot x \cdot y).
\end{aligned}$$

By Lemma 3.3, 0 is belong to the right hand of the abobe equality, so $0 \in ([x,[y,z]] + [y,[z,x]] + [z,[x,y]])$.

**Definition 3.5.** Corresponding to any hyperalgebra $A$ with hyperproduct a·b it is possible to define an anticommutative hyperalgebra $A^-$ which is the same hypervector space as A with the new hyperproduct

$$[a,b]_{A^-} = a \cdot b - b \cdot a. \tag{5}$$

A hyperalgebra $A$ is called *Lie-admissible* if the hyperalgebra $A^-$ is a Lie hyperalgebra.

If $A$ is an associative hyperalgebra, then the hyperproduct (5) coincide with (4) and $A^-$ is a Lie hyperalgebra in its more usual form. Thus, the associative hyperalgebras constitute a basic class of Lie-admissible hyperalgebras.

A Jordan algebra is a (non-associative) algebra over a field whose multiplication satisfies the following axioms:

(1) $xy = yx$ (commutative law);

(2) $(xy)(xx) = x(y(xx))$ (Jordan identity).

**Definition 3.6.** A Jordan hyperalgebra is a (non-associative) hyperalgebra over a hyperfield whose multiplication satisfies the following axioms:

[(J1)] $x{\cdot}\ y = y{\cdot}\ x$ (commutative law);

[(J2)] $(x{\cdot}\ y)(x{\cdot}\ x) = x{\cdot}\ (y{\cdot}\ (x{\cdot}\ x))$ (Jordan identity).

Let $A$ be an associative hyperalgebra over a hyperfield $F$. It is possible to construct a new hyperalgebra, denoted by $A^+$, by means of the commutative hyperproduct

$$\{a,b\} = a \cdot b + b \cdot a = \bigcup_{\substack{x \in a \cdot b \\ y \in b \cdot a}} x + y. \tag{6}$$

**Proposition 3.7.** *$A^+$ is a Jordan hyperalgebra.*

*Proof.* It is straightforward.

**Definition 3.8.** Corresponding to any hyperalgebra $A$ with hyperproduct $a{\cdot}\ b$ it is possible to define, as for $A^-$, a commutative hyperalgebra $A^+$ which is the same hypervector space as $A$ but with the new hyperproduct

$$\{a,b\}_{A^+} = a \cdot b + b \cdot a. \tag{7}$$

In this connection the most interesting case occure when $A^+$ is a (commutative) Jordan hyperalgebra.

A hyperalgebra $A$ is said to be *Jordan admissible* if $A^+$ is a (commutative) Jordan hyperalgebra.

If $A$ is an associative hyperalgebra, then the hyperproduct (7) reduces to (6) and $A^+$ is a special Jordan hyperalgebra. Thus, associative hyperalgebras constitute a basis class of Jordan-admissible hyperalgebras.

**Definition 3.9.** The fundamental relation $\varepsilon^*$ is defined in a hyperalgebra as the smallest equivalence relation such that the quotient is an algebra.

By using strongly regular relations, we can connect hyperalgebras to algebras. More exactly, starting with a hyperalgebra and using a strongly regular relation, we can construct an algebra structure on the quotient set. An equivalence relation $\rho$ on a hyperalgebra $A$ is called right (resp. left) strongly regular if and only if $x\rho y$ implies that $(x+z)\overline{\overline{\rho}}(y \pm z)$ and $(x\alpha z)\overline{\overline{\rho}}(y \cdot z)$ for every $z \in A$ (resp. $(z+x)\overline{\overline{\rho}}(z+y)$ and $(z \cdot x)\overline{\overline{\rho}}(z \cdot y)$ ), and $\rho$ is strongly regular if it is both left and right strongly regular.

**Theorem 3.10.** *Let A be a hyperalgebra over the hyperfield F. Denote by U the set of all finite polynomials of elements of A over F. We define the relation ε on A as follows:*

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$x\varepsilon y$ *if and only if* $\{x,y\}\subseteq u$, *where* $u\in U$.

Then, the $\varepsilon^*$ is the transitive closure of $\varepsilon$ and is called the fundamental equivalence relation on $A$.

*Proof.* The proof is similar to the proof of Theorem 3.1 [18].

**Remark 3.** Note that the relation $\varepsilon^*$ is a strongly regular relation.

**Remark 4.** In $A^-/\varepsilon^*$, the binary operations and external operation are defined in the usual manner:

$$\varepsilon^*(x)\oplus\varepsilon^*(y)=\varepsilon^*(z),\ \text{for } z\in\varepsilon^*(x)+\varepsilon^*(y),$$
$$\varepsilon^*(x)\odot\varepsilon^*(y)=\varepsilon^*(z),\ \text{for } z\in\varepsilon^*(x)\cdot\varepsilon^*(y),$$
$$\gamma^*(r)\circ\varepsilon^*(x)=\varepsilon^*(z),\ \text{for all}\in\gamma^*(r)\varepsilon^*(x).$$

**Theorem 3.11.** *Let A be an associative hyperalgebra over a hyperfield F. Then, $A^-/\varepsilon^*$ is a Lie-admissible algebra with the following product:*

$$\langle\varepsilon^*(x),\varepsilon^*(y)\rangle=\varepsilon^*(x)\odot\varepsilon^*(y)\ominus\varepsilon^*(y)\odot\varepsilon^*(x). \tag{8}$$

*Proof.* By Definition 3.9 and Theorem 3.10, $A/\varepsilon^*$ is an ordinary associative algebra. So, it is enough to show that it is a Lie algebra with the hyperproduce (8). By Proposition 3.4, $A^-$ is a Lie hyperalgebra with the hyperproduct $[a,b] = a\cdot b - b\cdot a$.

(1) By (L1), for all $x_1,x_2,y\in A$, $\lambda_1,\lambda_2\in F$, we have $[\lambda_1x_1+\lambda_2x_2,y]=\lambda_1[x_1,y]+\lambda_2[x_2,y]$. Hence,

$$(\lambda_1x_1+\lambda_2x_2)\cdot y-y\cdot(\lambda_1x_1+\lambda_2x_2)=\lambda_1(x_1\cdot y-y\cdot x_1)+\lambda_2(x_2\cdot y-y\cdot x_2),$$

and so

$$\varepsilon^*((\lambda_1x_1+\lambda_2x_2)\cdot y-y\cdot(\lambda_1x_1+\lambda_2x_2))=\varepsilon^*(\lambda_1(x_1\cdot y-y\cdot x_1)+\lambda_2(x_2\cdot y-y\cdot x_2)).$$

This implies that

$$\begin{aligned}&(\gamma^*(\lambda_1)\circ\varepsilon^*(x_1)\oplus\gamma^*(\lambda_2)\circ\varepsilon^*(x_2))\odot\varepsilon^*(y)\ominus\varepsilon^*(y)\odot(\gamma^*(\lambda_1)\circ\varepsilon^*(x_1)\oplus\gamma^*(\lambda_2)\circ\varepsilon^*(x_2))\\&=\gamma^*(\lambda_1)\circ(\varepsilon^*(x_1)\odot\varepsilon^*(y)\ominus\varepsilon^*(y)\odot\varepsilon^*(x_1))+\gamma^*(\lambda_2)\circ(\varepsilon^*(x_2)\odot\varepsilon^*(y)\ominus\varepsilon^*(y)\odot\varepsilon^*(x_2)).\end{aligned}$$

Therefore,

$$\begin{aligned}&\langle\gamma^*(\lambda_1)\circ\varepsilon^*(x_1)\oplus\gamma^*(\lambda_2)\circ\varepsilon^*(x_2),\varepsilon^*(y)\rangle\\&=\gamma^*(\lambda_1)\circ\langle\varepsilon^*(x_1),\varepsilon^*(y)\rangle\oplus\gamma^*(\lambda_2)\circ\langle\varepsilon^*(x_2),\varepsilon^*(y)\rangle.\end{aligned}$$

Similarly, for all $x,y_1,y_2\in A$, $\lambda_1,\lambda_2\in F$, we obtain

$$\begin{aligned}&\langle\varepsilon^*(x),\gamma^*(\lambda_1)\circ\varepsilon^*(y_1)\oplus\gamma^*(\lambda_2)\circ\varepsilon^*(y_2)\rangle\\&=\gamma^*(\lambda_1)\circ\langle\varepsilon^*(x),\varepsilon^*(y_1)\rangle\oplus\gamma^*(\lambda_2)\circ\langle\varepsilon^*(x),\varepsilon^*(y_2)\rangle.\end{aligned}$$

(2) By (L2), $0\in x\cdot x-x\cdot x$, so

$$\begin{aligned}\varepsilon^*(0)=\varepsilon^*(x\cdot x-x\cdot x)&=\varepsilon^*(x)\odot\varepsilon^*(x)\ominus\varepsilon^*(x)\odot\varepsilon^*(x)\\&=\langle\varepsilon^*(x),\varepsilon^*(x)\rangle.\end{aligned}$$

(3) By (L3), we have $0\in([x,[y,z]]+[y,[z,x]]+[z,[x,y]])$, for all $x$, $y\in A$. Thus,

$$\varepsilon^*(0)=\langle\varepsilon^*(x),\langle\varepsilon^*(y),\varepsilon^*(z)\rangle\rangle+\langle\varepsilon^*(y),\varepsilon^*(z),\varepsilon^*(x)\rangle\rangle+\langle\varepsilon^*(z),\langle\varepsilon^*(x),\varepsilon^*(y)\rangle\rangle.$$

**Theorem 3.12.** *Let A be an associative hyperalgebra over a hyperfield F. Then, $A^+/\varepsilon^*$ is a Jordan-admissible algebra with the following product:*

$$\lceil\varepsilon^*(x),\varepsilon^*(y)\rceil=\varepsilon^*(x)\odot\varepsilon^*(y)\oplus\varepsilon^*(y)\odot\varepsilon^*(x). \tag{9}$$

*Proof.* By Definition 3.9 and Theoremt1, $A/\varepsilon^*$ is an ordinary associative algebra. So, it is enough to show that it is a Jordan algebra with the hyperproduct (9). By Proposition 3.7, $A^+$ is a Jordan hyperalgebra with the hyperproduct $\{a,b\} = a\cdot b - b\cdot a$.

(1) By (J1), for all $x,y\in A$, $\{x,y\}=\{y,x\}$. So, $\varepsilon^*(x\cdot y+y\cdot x)=\varepsilon^*(y\cdot x+x\cdot y)$ which implies that $\varepsilon^*(x)\odot\varepsilon^*(y)\oplus\varepsilon^*(y)\odot\varepsilon^*(x)=\varepsilon^*(y)\odot\varepsilon^*(x)\oplus\varepsilon^*(x)\odot\varepsilon^*(y)$. Thus, $\lceil\varepsilon^*(x),\varepsilon^*(y)\rceil=\lceil\varepsilon^*(y),\varepsilon^*(x)\rceil$

(2) By (J2), for all $x,y\in A$, $\{\{x,y\},\{x,x\}\}=\{x,\{y,\{x,x\}\}\}$. Hence,

$$\varepsilon^*(\{\{x,y\},\{x,x\}\})=\varepsilon^*(\{x,\{y,\{x,x\}\}\}),$$

which implies that

$$\lceil\lceil\varepsilon^*(x),\varepsilon^*(y)\rceil,\lceil\varepsilon^*(x),\varepsilon^*(x)\rceil\rceil=\lceil\varepsilon^*(x),\lceil\varepsilon^*(y),\lceil\varepsilon^*(x),\varepsilon^*(x)\rceil\rceil\rceil.$$

This completes the proof.

In the same way it is possible to introduce always in terms of the associative product the following bilinear form

$$(a,b)=\lambda a\cdot b+(1-\lambda)b\cdot a=\lambda[a,b]+b\cdot a, \tag{10}$$

where $\lambda$ is a free element belonging to the hyperfield $F$, which characterizes the $\lambda$-mutations $A(\lambda)$ of $A$. Clearly, $A(1)$ is isomorphic to $A$.

In this connection, a more general bilinear form in terms of the associative hyperproduct is given by

$$(a,b)=\lambda a\cdot b+\mu b\cdot a=\alpha[a,b]+\beta\{a,b\}, \tag{11}$$

where $\lambda=\alpha+\beta$ and $\mu=\beta-\alpha$ are free elements belonging to the hyperfield $F$, which constitutes the basic hyperproduct of the $(\lambda,\mu)$-mutations $A(\lambda,\mu)$ of $A$. Clearly, $A(1,0)$ is isomorphic to $A$ and $A(1,-1)$ is isomorphic to $A^-$; $A(1,1)$ is isomorphic to $A^+$ and $A(\lambda,1-\lambda)$ is isomorphic to $A(\lambda)$.

Let $A$ be an associative hyperalgebra. We can define another hyperoperation by means an element $T$ which is denoted by $[\cdot,\cdot]_T$ and is defined by

$$\begin{aligned}&[\cdot,\cdot]_T: A\times A\to P^*(A),\\&[\cdot,\cdot]_T:(x,y)\mapsto[x,y]_T=x\cdot T\cdot y-y\cdot T\cdot x.\end{aligned} \tag{12}$$

**Proposition 3.13.** *The hyperoperation $[\cdot,\cdot]_T$ satisfies the following conditions:*

(1) $0\in[x,x]_T$, for all $x\in A$;

(2) $0\in([x,[y,z]_T]_T+[y,[z,x]_T]_T+[z,[x,y]_T]_T)$, for all $x$, $y\in T$.

*Proof.* (1) Since $x\cdot T\cdot x$ is non-empty, there exists $a_0\in x\cdot T\cdot x$. hence, $-a_0\in -x\cdot T\cdot x$.

Thus, $0\in a_0-a_0\subseteq\bigcup_{a\in x\cdot T\cdot x}a-a=x\cdot T\cdot x-x\cdot T\cdot x=[x,x]_T$.

(2) We have

$$\begin{aligned}[x,[y,z]_T]_T&=x\cdot T\cdot[y,z]_T-[y,z]_T\cdot T\cdot x\\&=x\cdot T\cdot(y\cdot T\cdot z-z\cdot T\cdot y)-(y\cdot T\cdot z-z\cdot T\cdot y)\cdot T\cdot x\\&=x\cdot T\cdot y\cdot T\cdot z-x\cdot T\cdot z\cdot T\cdot y-y\cdot T\cdot z\cdot T\cdot x-z\cdot T\cdot y\cdot T\cdot x.\end{aligned}$$

Hence,

$$\begin{aligned}&([x,[y,z]_T]_T+[y,[z,x]_T]_T+[z,[x,y]_T]_T)\\&=x\cdot T\cdot y\cdot T\cdot z-x\cdot T\cdot z\cdot T\cdot y-y\cdot T\cdot z\cdot T\cdot x-z\cdot T\cdot y\cdot T\cdot x\\&\quad+(y\cdot T\cdot z\cdot T\cdot x-y\cdot T\cdot x\cdot T\cdot z-z\cdot T\cdot x\cdot T\cdot y+x\cdot T\cdot z\cdot T\cdot y)\\&\quad+(z\cdot T\cdot x\cdot T\cdot y-z\cdot T\cdot y\cdot T\cdot x-x\cdot T\cdot y\cdot T\cdot z+y\cdot T\cdot x\cdot T\cdot z)\\&=(x\cdot T\cdot y\cdot T\cdot z-x\cdot T\cdot y\cdot T\cdot z)+(x\cdot T\cdot z\cdot T\cdot y-x\cdot T\cdot z\cdot T\cdot y)\\&\quad+(z\cdot T\cdot y\cdot T\cdot x-z\cdot T\cdot y\cdot T\cdot x)+(y\cdot T\cdot z\cdot T\cdot x-y\cdot T\cdot z\cdot T\cdot x)\\&\quad+(y\cdot T\cdot x\cdot T\cdot z-y\cdot T\cdot x\cdot T\cdot z)+(z\cdot T\cdot x\cdot T\cdot y-z\cdot T\cdot x\cdot T\cdot y).\end{aligned}$$

By Lemma 3.3, 0 is belong to the right hand of the abobe equality, so $0\in([x,[y,z]_T]_T+[y,[z,x]_T]_T+[z,[x,y]_T]_T)$.

**Remark 5.** If A is an algebra and $[\cdot,\cdot]_T: A\times A\to A$, then we have Lie-Santilli braket.

**Definition 3.14.** Corresponding to any hyperalgebra $A$ with hyperproduct $a\cdot b$ it is possible to define a $A^*$ which is the same

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hypervector space as A with the new hyperproduct

$$[a,b]_{A^\star} = a \cdot T \cdot b - b \cdot T \cdot a. \qquad (13)$$

A hyperalgebra $A$ is called *Lie-Santilli-admissible* if the hyperalgebra $A^*$ is a Lie hyperalgebra.

**Corollary 3.15.** If $A$ is an associative hyperalgebra, then the hyperproduct (13) coincide with (12) and $A^*$ is a Lie hyperalgebra.

**Theorem 3.16.** *Let A be an associative hyperalgebra over a hyperfield F. Then,* $A^\star/\varepsilon^*$ *is a Lie-Santilli-admissible algebra with the following product:*

$$\langle \varepsilon^*(x), \varepsilon^*(y)\rangle_T = \varepsilon^*(x) \odot \varepsilon^*(T) \odot \varepsilon^*(y) \ominus \varepsilon^*(y) \odot \varepsilon^*(T) \odot \varepsilon^*(x). \qquad (14)$$

*Proof.* By Definition 3.9 and Theorem 3.10, $A/\varepsilon^*$ is an ordinary associative algebra. So, it is enough to show that it is a Lie algebra with the hyperproduce (4). By corollary 3.15, $A^*$ is a Lie hyperalgebra with the hyperproduct $[a,b]_T = a \cdot T \cdot b - b \cdot T \cdot a$ .

(1) By (L1), for all $x_1, x_2, y \in$ , $\lambda_1, \lambda_2 \in F$ , we have $[\lambda_1 x_1 + \lambda_2 x_2, y]_T = \lambda_1[x_1, y]_T + \lambda_2[x_2, y]_T$ . Hence,

$$(\lambda_1 x_1 + \lambda_2 x_2) \cdot T \cdot y - y \cdot T \cdot (\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1(x_1 \cdot T \cdot y - y \cdot T \cdot x_1) + \lambda_2(x_2 \cdot T \cdot y - y \cdot T \cdot x_2),$$

and so

$$\varepsilon^*((\lambda_1 x_1 + \lambda_2 x_2) \cdot T \cdot y - y \cdot T \cdot (\lambda_1 x_1 + \lambda_2 x_2)) = \varepsilon^*(\lambda_1(x_1 \cdot T \cdot y - y \cdot T \cdot x_1) + \lambda_2(x_2 \cdot T \cdot y - y \cdot T \cdot x_2)).$$

This implies that

$$\begin{aligned}&(\gamma^*(\lambda_1) \circ \varepsilon^*(x_1) \oplus \gamma^*(\lambda_2) \circ \varepsilon^*(x_2)) \odot \varepsilon^*(T) \odot \varepsilon^*(y)\\ &\quad \ominus \varepsilon^*(y) \odot \varepsilon^*(T) \odot (\gamma^*(\lambda_1) \circ \varepsilon^*(x_1) \oplus \gamma^*(\lambda_2) \circ \varepsilon^*(x_2))\\ &= \gamma^*(\lambda_1) \circ (\varepsilon^*(x_1) \odot \varepsilon^*(T) \odot \varepsilon^*(y) \ominus \varepsilon^*(y) \odot \varepsilon^*(T) \odot \varepsilon^*(x_1))\\ &\quad \oplus \gamma^*(\lambda_2) \circ (\varepsilon^*(x_2) \odot \varepsilon^*(T) \odot \varepsilon^*(y) \ominus \varepsilon^*(y) \odot \varepsilon^*(T) \odot \varepsilon^*(x_2)).\end{aligned}$$

Therefore,

$$\begin{aligned}&\langle \gamma^*(\lambda_1) \circ \varepsilon^*(x_1) \oplus \gamma^*(\lambda_2) \circ \varepsilon^*(x_2), \varepsilon^*(y)\rangle_T\\ &= \gamma^*(\lambda_1) \circ \langle \varepsilon^*(x_1), \varepsilon^*(y)\rangle_T \oplus \gamma^*(\lambda_2) \circ \langle \varepsilon^*(x_2), \varepsilon^*(y)\rangle_T.\end{aligned}$$

Similarly, for all $x, y_1, y_2 \in A$ , $\lambda_1, \lambda_2 \in F$ , we obtain

$$\begin{aligned}&\langle \varepsilon^*(x), \gamma^*(\lambda_1) \circ \varepsilon^*(y_1) \oplus \gamma^*(\lambda_2) \circ \varepsilon^*(y_2)\rangle_T\\ &= \gamma^*(\lambda_1) \circ \langle \varepsilon^*(x), \varepsilon^*(y_1)\rangle_T \oplus \gamma^*(\lambda_2) \circ \langle \varepsilon^*(x), \varepsilon^*(y_2)\rangle_T.\end{aligned}$$

(2) By (L2), $0 \in x \cdot T \cdot x - x \cdot T \cdot x$ , so

$$\begin{aligned}\varepsilon^*(0) = \varepsilon^*(x \cdot T \cdot x - x \cdot T \cdot x) &= \varepsilon^*(x) \odot \varepsilon^*(T) \odot \varepsilon^*(x) \ominus \varepsilon^*(x) \odot \varepsilon^*(T) \odot \varepsilon^*(x)\\ &= \langle \varepsilon^*(x), \varepsilon^*(x)\rangle_T.\end{aligned}$$

(3) By (L3), we have $0 \in ([x,[y,z]] + [y,[z,x]] + [z,[x,y]])$ , for all $x, y \in A$ . Thus,

$$\varepsilon^*(0) = \langle \varepsilon^*(x), \langle \varepsilon^*(y), \varepsilon^*(z)\rangle_T\rangle_T + \langle \varepsilon^*(y), \varepsilon^*(z), \varepsilon^*(x)\rangle_T\rangle_T + \langle \varepsilon^*(z), \langle \varepsilon^*(x), \varepsilon^*(y)\rangle_T\rangle_T.$$

In some cases we can start from a hyperalgebra which is not a Lie hyperalgebra with respect to the Lie braket. However it can be Lie-Santilli hyperalgebra with respect to Lie-Santilli braket.

**Example 6.** Let $F$ be a hyperfield and

$$A = \{(a_{ij})_{m\times n} \mid a_{ij} \in F\}.$$

Similar to Examples 4 and 5, we can define the hyperoperation $\uplus$ and the external hyperproduct $\bullet$. Note that it is impossible to define the hyperoperation $\otimes$ between two elements of $A$ and so $A$ is not a Lie hyperalgebra but it is Lie-Santilli hyperalgebra with respect to

$$\begin{aligned}&[(a_{ij})_{m\times n}, (b_{ij})_{m\times n}]_{(T_{ij})_{m\times n}}\\ &= (a_{ij})_{m\times n} \otimes (T_{ij})^t_{m\times n} \otimes (a_{ij})_{m\times n} \oslash (b_{ij})_{m\times n} \otimes (T_{ij})_{m\times n} \otimes (a_{ij})_{m\times n},\end{aligned} \qquad (15)$$

where $(T_{ij})^t_{m\times n}$ denotes the transpose of the matrix $(T_{ij})_{m\times n}$ .

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