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Complete Left-Invariant Affine Structures on Solvable Non-Unimodular Three-Dimensional Lie Groups

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Abstract

In this paper, we shall use a method based on the theory of extensions of left-symmetric algebras to classify complete left-invariant affine real structures on solvable non-unimodular three-dimensional Lie groups.

Keywords: Extensions of left-symmetric algebras; Left-invariant affine connections; Novikov algebras

Introduction

The notion of a left-symmetric algebra appeared for the first time in the work of Koszul [1] and Vinberg [2] concerning bounded homogeneous domains and convex homogeneous cones, respectively. Over the field of real numbers, left-symmetric algebras are of special interest because of their role in the differential geometry of affine manifolds (i.e. smooth manifolds with flat torsion-free affine connections), and in the representation theory of Lie groups [3,4]. In fact, for a given simply connected Lie group G with Lie algebra $\mathcal{G}$, the left-invariant affine structures on $\mathcal{G}$ are in one-to-one correspondence with the left-symmetric structures on G compatible with the Lie structure [5].

On the other hand, it is well known that there is a one-to-one correspondence between left-invariant affine structures on a Lie group G and locally simply transitive affine actions of G on an n-dimensional real vector space V [5]. The classification of left-invariant affine structures on a given Lie group G is then reduced to the classification of compatible left-symmetric products on the Lie algebra $\mathcal{G}$ of G. It has been proved [6] that a simply connected Lie group G which acts simply transitively on $\mathbb{R}^n$ by affine transformations is necessarily solvable. Since a few years, there has been a growing interest in the study of simply transitive affine actions of Lie groups on $\mathbb{R}^n$. This interest is mostly due to the example of Benoist [7], who constructed a simply connected nilpotent Lie group not admitting any locally simply transitive affine action on $\mathbb{R}^n$. This example provided a negative answer to the following question of Milnor [3]. Does any simply connected solvable Lie group admit a simply transitive affine action on $\mathbb{R}^n$?

From another point of view, there is also the question of classifying all simply transitive affine actions of a given solvable Lie group G admitting such an action. This question, even in the abelian case $G=\mathbb{R}^k$, seems to be very hard. When G is nilpotent, the classification has been completely achieved up to dimension four [8,9].

Recently, a method based on the theory of extensions of left-symmetric algebras has been proposed [10] to classify complete left-invariant affine real structures on a given solvable Lie group of low dimension. Since the classification in the case of solvable unimodular Lie groups of dimension three was obtained [8], we will use that method to carry out in this paper the classification of complete left-invariant affine structures on three-dimensional solvable non-unimodular Lie groups.

The paper is organized as follows. In section 2, we will briefly recall some necessary definitions and basic results on left-symmetric algebras and their extensions. In section 3, using the classification of the three-dimensional complex simple left-symmetric algebras given [11] and a result [12], we shall first show that any complete real left-symmetric algebra A_3 of dimension 3 whose Lie algebra is solvable and non-unimodular is not simple. Therefore, we can get A_3 as an extension of complete left-symmetric algebras. By using the Lie group exponential maps, we shall deduce the classification of all complete left-invariant affine structures on solvable non-unimodular Lie groups of dimension 3 in terms of simply transitive actions of subgroups of the affine group $Aff(\mathbb{R}^3)=GL(\mathbb{R}^3)\times\mathbb{R}^3$ (see Theorem 13).

Throughout this paper, all considered vector spaces, Lie algebras, and left-symmetric algebras are supposed to be over the field $\mathbb{R}$. We shall also suppose that all considered Lie groups are simply connected.

Left-symmetric Algebras and their Extensions

Let A be a finite-dimensional vector space over $\mathbb{R}$. A left-symmetric product on A is a bilinear product that we denote by $x\cdot y$ satisfying

$$(x\cdot y)\cdot z-(y\cdot x)\cdot z=x\cdot(y\cdot z)-y\cdot(x\cdot z), \tag{1}$$

for all $[x,y]=x\cdot y-y\cdot x$. In this case, A together with a left-symmetric product is called left-symmetric algebra.

Now if A is a left-symmetric algebra, then the commutator

$$[x,y]=x\cdot y-y\cdot x \tag{2}$$

defines a structure of Lie algebra on A, called the associated Lie algebra. On the other hand, if $\mathcal{G}$ is a Lie algebra with a left-symmetric product satisfying (2), then we say that this left-symmetric structure is compatible with the Lie structure on $\mathcal{G}$.

Let G be a simply connected Lie group with a left-invariant affine connection ∇. Define a product • on the Lie algebra $\mathcal{G}$ of G by

$$x\cdot y=\nabla_x y,$$

for all $x,\ y\in\mathcal{G}$. Then, the flat and torsion-free conditions on ∇

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correspond to conditions (1) and (2), respectively.

Conversely, If G is a simply connected Lie group with Lie algebra $\mathcal{G}$ and $x \bullet y$ denotes a left-symmetric product on $\mathcal{G}$ compatible with the Lie bracket, then the left-invariant connection given by $\nabla_x y = x \cdot y$ defines a left-invariant affine structure ∇ on $\mathcal{G}$. We deduce that if G is a simply connected Lie group with Lie algebra $\mathcal{G}$, then the study of left-invariant affine structures on G is equivalent to the study of left-symmetric structures on G compatible with the Lie structure.

Let A be a left-symmetric algebra whose associated Lie algebra is $\mathcal{G}$, and let L_x and R_x denote the left and right multiplications, respectively i.e. $L_x y = x \cdot y$ and $R_x y = x \cdot y$. The identity in (1) is now equivalent to the formula

$$\left[L_x, L_y\right] = L_{[x,y]}, \quad \text{for all } x,y \in A,$$

or, in other words, the linear map $L: \mathcal{G} \to End(A)$ is a representation of Lie algebras.

If a left-symmetric algebra A has no proper two-sided ideal and it is not the zero algebra of dimension 1, then A is called simple. A is called semi simple, if it is a direct sum of simple left-symmetric algebras.

We say that A is complete if R_x is a nilpotent operator for all $x \in A$. It turns out that, for a given simply connected Lie group G with Lie algebra $\mathcal{G}$, the complete left-invariant affine structures on G are in one-to-one correspondence with the complete left-symmetric structures on $\mathcal{G}$ compatible with the Lie structure. It is also known that an n-dimensional simply connected Lie group admits a complete left-invariant affine structure if and only if it acts simply transitively on $\mathbb{R}^n$ by affine transformations [9]. A simply connected Lie group which is acting simply transitively on $\mathbb{R}^n$ by affine transformations must be solvable according to [6]. It is well known that not every solvable (even nilpotent) Lie group can admit an affine structure [7].

We say that A is Novikov algebra if it satisfies the identity

$$(x \cdot y) \cdot z = (x \cdot z) \cdot y, \qquad \text{for all } x,y \in A. \tag{3}$$

In terms of left and right multiplications, (3) is equivalent to the formula

$$\left[R_x, R_y\right] = 0, \quad \text{for all } x,y \in A.$$

The left-symmetric algebra A is called derivation algebra if it satisfies the identity

$$(x \cdot y) \cdot z = (z \cdot y), \quad \text{for all } x,y,z \in A$$

or, equivalently, all left and right multiplications L_x and R_x are derivations of g.

Recall that a Lie algebra $\tilde{\mathcal{G}}$ is an extension of the Lie algebra $\mathcal{G}$ by the Lie algebra A if there exists a short exact sequence of Lie algebras

$$0 \to \mathcal{A} \xrightarrow{i} \tilde{\mathcal{G}} \xrightarrow{\pi} \mathcal{G} \to 0.$$

In other words, A is an ideal of $\tilde{\mathcal{G}}$ such that $\tilde{\mathcal{G}} / A \cong \mathcal{G}$.

For (x, a) and (y, b) in $\tilde{\mathcal{G}} \cong \mathcal{G} \oplus A$, the extended Lie bracket is given by

$$\left[(x,a),(y,b)\right] = \left([x,y],[a,b] + \phi(x)b - \phi(y)a + \omega(x,y)\right), \tag{4}$$

where $\phi : \mathcal{G} \to Der(A)$ is a linear map and $w : \mathcal{G} \times \mathcal{G} \to A$ is an alternating bilinear map such that

$$\left[\phi(x), \phi(y)\right] = \phi([x,y]) + ad_{\omega(x,y)},$$

and

$$\omega([x,y],z) - \omega(x,[y,z]) + \omega(y,[x,z]) = \phi(x)\omega(y,z) + \phi(y)\omega(z,x) + \phi(z)\omega(x,y).$$

Note here that if A is abelian, then ω is a 2-cocycle [13,14].

Now we shall briefly discuss the problem of extension of a left-symmetric algebra by another left-symmetric algebra. To our knowledge, the notion of extensions of left-symmetric algebras has been considered for the first time in [9], to which we refer the reader for more details [15].

Suppose that a vector space extension of a left-symmetric algebra A by another left-symmetric algebra E is given. We want to define a left-symmetric structure on $\tilde{A}$ in terms of the left-symmetric structures given on A and E. In other words, we want to define a left-symmetric product on $\tilde{A}$ for which E becomes a two-sided ideal in $\tilde{A}$ such that $\tilde{A} / \mathrm{E} \cong A$; or equivalently,

$$0 \to E \to \tilde{A} \to A \to 0$$

Becomes a short exact sequence of left-symmetric algebras.

Theorem 1: *There exists a left-symmetric structure on* $\tilde{A}$ *extending a left-symmetric algebra A by a left-symmetric algebra E if and only if there exist two linear maps* $\lambda, \rho : A \to \mathrm{End(E)}$ *and a bilinear map* $g : A \times A \to E$ *suct that for all* $x, y, z \in A$ *A and* $a, b \in E$, *the following conditions are satisfied* [9].

1 $\lambda_x(a \cdot b) = \lambda_x(a) \cdot b + a \cdot \lambda_x(b) - \rho_x(a) \cdot b,$

2 $\rho_x([a,b]) = a \cdot \rho_x(b) - b \cdot \rho_x(a),$

3 $[\lambda_x, \lambda_y] - \lambda_{[x,y]} = L_{g(x,y) - g(y,x)},$

4 $[\lambda_x, \rho_y] + \rho_y \circ \rho_x - \rho_{x \cdot y} = R_{g(x,y)}$

5 $g(x, y \cdot z) - g(y, x \cdot z) + \lambda_x(g(y,z)) - \lambda_y(g(x,z)) - g([x,y],z)$

$-\rho_z(g(x,y) - g(y,x)) = 0.$

If the conditions of the above theorem are fulfilled, then the extended left-symmetric product on $A \cong A \times E$ is given by

$$(x,a) \cdot (y,b) = \left(x \cdot y, a \cdot b + \lambda_x(b) + \rho_y(a) + g(x,y)\right). \tag{5}$$

It is remarkable that if the left-symmetric product of E is trivial, then the conditions of the above theorem simplify to the following three conditions:

(i) $\left[\lambda_x, \lambda_y\right] = \lambda_{[x,y\%]}$, i.e. λ is a representation of Lie algebras,

(ii) $\left[\lambda_x, \rho_y\right] = \rho_{x \cdot y} - \rho_y \circ \rho_x.$

(iii) $g(x, y \cdot z) - g(y, x \cdot z) + \lambda_x(g(y,z)) - \lambda_y(g(x,z)) - g([x,y],z)$

$-\rho_z(g(x,y) - g(y,x)) = 0.$

In this case, E becomes an A-bimodule and the extended product given in (5) simplifies too. Recall that if K is a left-symmetric algebra and V is a vector space, then we say that V is a K-bimodule if there exist two linear maps $\lambda, \rho : K \to End(V)$ which satisfy the conditions (i) and (ii) stated above.

Let K be a left-symmetric algebra, and suppose that a K-bimodule V is known. We denote by $L^p(K, V)$ the space of all p-linear maps from K to V, and we define two co-boundary operators $\delta_1 : L^1(K,V) \to L^2(K,V)$ and $\delta_2 : L^2(K,V) \to L^3(K,V)$ as follows:

For a linear map $h \in L^1(K,V)$ we set

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$$\delta_1 h(x,y) = \rho_y\left(h(x)\right) + \lambda_x\left(h(y)\right) - h(x \cdot y), \tag{6}$$

and for a bilinear map $g \in L^2(K,V)$ we set

$$\delta_2 g(x,y,z) = g(x, y\cdot z) - g(y, x\cdot z) + \lambda_x\left(g(y,z)\right) - \lambda_y\left(g(x,z)\right) - g([x,y],z) - \rho_z\left(g(x,y) - g(y,x)\right) \tag{7}$$

where λ and ρ are linear maps $\lambda, \rho : K \to End(V)$.

It is straightforward to check that $\delta_2 o \delta_1 = 0$. Therefore, if we set $Z^2_{\lambda,\rho}(K,V) = \ker \delta_2$ and $B_{\lambda,\rho}(K,V)$ Im $_1$, we can define a notion of second co-homology for the actions λ and ρ by simply setting $H^2_{\lambda,\rho}(K,V) = Z^2_{\lambda,\rho}(K,V) / B^2_{\lambda,\rho}(K,V)$. As in the case of Lie algebras, we can prove the following [9].

Proposition 2: *For given linear maps* $\lambda, \rho : K \to End(V)$, *the equivalent classes of extensions*

$$0 \to V \to A \to K \to 0$$

of K by V are in one-to - one correspondence with the elements of the second co-homology group $H^2_{\lambda,\rho}(K,V)$.

A left-symmetric algebras extension

$$0 \to E \xrightarrow{i} \tilde{A} \xrightarrow{\pi} A \to 0$$

is called central if and only if $i(E) \subseteq C(\tilde{A})$ where

$$C(\tilde{A}) = \left\{x \in \tilde{A} : x \cdot y = y \cdot x = 0\right\}$$

is the center of $\tilde{A}$. In particular, the extension is central whenever E is a trivial A-bimodule (i.e. $\lambda = \rho = 0$). We say that the extension is exact if and only if $i(E) = C(\tilde{A})$. It is easy to verify [9] that the extension is exact if and only if $I_{[g]=}0$, where

$$I_{[g]} = \left\{x \in A : x \cdot y = y \cdot x = 0 \text{ and } g(x,y) = g(y,x) = 0 \,\text{for all } \ \mathrm{y} \in \mathrm{A}\right\}$$

We observe that $I_{[g]}$ is depends only on the co-homology class of g, that is $I_{[g]}$ is well defined. In case E is a trivial A - bimodule, we denote the central extension corresponding to the class $[g] \in H^2(A,E) by \left(\tilde{A}, [g]\right)$.

Let $\left(\tilde{A}, [g]\right)$ and $\left(\tilde{A}, [g']\right)$ be two central extensions of A by E, $\mu \in Aut(E) = GL(E)$ and $\eta \in Aut(A)$, where *Aut (E)* and *Aut (A)* are the groups of left-symmetric automorphisms of E and K, respectively. It is clear that if, $h \in L^1(A,E)$, then the linear mapping $\psi : \tilde{A} \to \tilde{A}'$ defined by

$$\psi(x,a) = \left(\eta(x), \mu(a) + h(x)\right)$$

is an isomorphism provided

$$g'\left(\eta(x), \eta(y)\right) = \mu\left(g(x,y)\right) + \delta_1 h(x,y) \, \textit{forall} \, (x,y) \in A \times A, \text{ i.e., } \eta^*[g'] = \mu_*[g].$$

This allows us to define an action of the group *G=Aut (E) x Aut (A)* on *H* ^{2} *(A, E)* by setting

$$(\mu, \eta) \cdot [g] = \mu_* \eta^*[g]$$

or equivalently, $(\mu,\eta) \cdot g(x,y) = \mu\left(g\left(\eta(x), \eta(y)\right)\right)$ for all $x, y \in A$.

Denoting the set of all exact central extensions of A by E by

$$H^2_{ex}(A,E) = \left\{[g] \in H^2(A,E) : I_{[g]} = 0\right\}$$

and the orbit of [g] by $G_{[g]}$, it turns out that the following result is valid [9].

Proposition 3: *Let [g] and* $[g']$ *be two classes in* $H^2_{ex}(A,E)$. *Then, the central extensions* $\left(\tilde{A}, [g]\right)$ *and* $(\tilde{A}', [g'])$ *are isomorphic if and only if* $G_{[g]} = G[g']$. *In other words, the classification of the exact central extensions of A by E is, up to left-symmetric isomorphism the orbit space of* $H^2_{ex}(A,E)$ *under the natural action of G=Aut (E) x Aut(A).*

We close this section by the following important result [15].

Proposition 4: *Let* $0 \to I \to A\% \to J \to 0$ *be an exact sequence of left-symmetric algebras such that A is complete then I and J are complete*

Proof: Let A be a complete left-symmetric algebra. Then R_x is nilpotent for all $x \in A$, . Since J is an ideal of A, then R_x is nilpotent for all $x \in I$, that is I is complete. On the other hand, Since $J \cong A / I$, we can define for $x \in A$, $R_x|_J : J \to J$, by $R_x|_J(\overline{y}) = R_x y + I$ for all $y \in A$, $\overline{y} = y + I$. Since for all $y_1, y_2 \in A$ such that $y_1 + I = y_2 + I$ there exists $z \in I$ so that $y_2 = y_1 + z$, and

$$
\begin{aligned}
R_x(y_2 + I) &= R_x y_2 + I \\
&= R_x(y_1 + z) + I \\
&= R_x y_1 + R_x z + I \\
&= R_x y_1 + I \\
&= R_x(y_1 + I)
\end{aligned}
$$

then, $R_x|_J$ is well defined. We also have, for all $x, y \in A$, , that

$$
\begin{aligned}
R_{\overline{x}}\overline{y} &= (y + I) \cdot (x + I) \\
&= y \cdot x + I \\
&= R_x y + I \\
&= R_x \overline{y}
\end{aligned}
$$

Thus, to prove that J is complete, it is enough to prove that $R_x|_J$ is nilpotent for all $x \in A$. Since R_x is nilpotent, then $R_x^k = 0$ for some $k \in \mathbb{N}$. This implies that

$$R_x^k(y) + I = I = \overline{0}$$

for all $y \in A$ Hence, $R_x^k(\overline{y}) = 0$ for all $\overline{y} \in J$, that is $R_x|_J$ is nilpotent for all $x \in A$, and hence J is complete.

Complete Left-Symmetric Structures on Solvable Non-Unimodular Lie Algebras of Dimension 3

Recall that a lie algebra $\mathcal{G}$ is unimodular if and only if tr(ad_x)=0 for all $x \in \mathcal{G}$. The classification of solvable non unimodular Lie algebras of dimension 3 can be found [16].

Lemma 5: *Let g be solvable non-unimodular Lie algebra of dimension 3. Then there is a basis* $\{e_1, e_2, e_3\}$ *of* $\mathcal{G}$ *so that*

$$[e_1, e_2] = \alpha e_2 + \beta e_3$$

$$[e_1, e_3] = \gamma e_2 + (2 - \alpha) e_3$$

If we exclude the case where D is the identity matrix then the determinant $\det D = \alpha(2-\alpha) - \beta\gamma$ *provides a complete isomorphism invariant for this Lie algebra.*

According to this result, we can, by simple computations, find that there are five possibilities for D:

$$D \cong \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad D \cong \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D \cong \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

$$D \cong \begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix}, \ \textit{where } 0 < |\mu| < 1 \text{ or } D \cong \begin{pmatrix} 0 & -\varsigma \\ \varsigma & 1 \end{pmatrix} \text{ where } \varsigma > 0$$

This implies that any solvable non-unimodular Lie algebra of dimension 3 is isomorphic to one and only one of the following Lie algebras

$$\mathcal{G}_{3,1}:[e_1,e_2]=e_2$$
$$\mathcal{G}_{3,2}:[e_1,e_2]=e_2,[e_1,e_3]=e_3$$
$$\mathcal{G}_{3,3}:[e_1,e_2]=e_2+e_3,[e_1,e_3]=e_3$$
$$\mathcal{G}_{3,4}^{\mu}:[e_1,e_2]=e_2,[e_1,e_3]=\mu e_3,0<|\mu|<1$$
$$\mathcal{G}_{3,5}^{\zeta}:[e_1,e_2]=e_2+\zeta e_3,[e_1,e_3]=-\zeta e_2+e_3,\zeta>0$$

Now let $\mathcal{G}$ be real solvable non-unimodular Lie algebra of dimension 3. Let A_3 be a complete left- symmetric algebra whose associated Lie algebra is $\mathcal{G}$.

We shall first recall the following result from [12].

Lemma 6: *Only the complex sim le left-symmetric algebras and even-dimensional complex semisim le left- symmetric algebras may have simple real forms, where a real form of a complex left-symmetric algebra A is sub algebra A_0 of $A^{\mathbb{R}}$ such that $A_0^{\mathbb{C}}=A$. Here $A^{\mathbb{R}}$ is A regarded as a real left-symmetric algebra.*

Now, we can prove the following

Proposition 7: *A_3 is not simple. In other words, any complete left-symmetric structure on a solvable non- unimodular Lie algebra of dimension 3 is not simple.*

Proof: Assume to the contrary that A_3 is simple. Then, Lemma 6 shows that complexification $A_3^{\mathbb{C}}$ of A_3 is simple as the dimension of $A_3^{\mathbb{C}}$ is odd. We can now apply Corollary 4.2 in [11] to deduce that $A_3^{\mathbb{C}}$ is isomorphic to the complex left-symmetric algebra A_1^{-1} having a basis {e_p e_2, e_3} such that the only non-trivial products are

$$e_1\cdot e_2=e_2,$$
$$e_1\cdot e_3=-e_3,$$
$$e_2\cdot e_3=e_3\cdot e_2=e_1.$$

Thus, the complex lie algebra $\mathcal{G}_3$ associated to $A_3^{\mathbb{C}}\cong A_1^{-1}$ is unimodular and hence $\mathcal{G}$ must be unimodular. This contradiction shows that A_3 is not simple

Before returning to the left-symmetric algebra A_3, we need to state the following facts without proofs.

Lemma 8: *Let A be a left-symmetric algebra with associated Lie algebra $\mathcal{G}$ and R a two -sided ideal in A. Then the lie algebra R associated to R is an ideal in $\mathcal{G}$*

Lemma 9: *Let $\mathcal{G}$ be solvable non-unimodular Lie algebra of dimension 3 and let $\mathcal{I}$ be a proper ideal of $\mathcal{G}$. Then $\mathcal{I}$ is isomorphic to* $\mathbb{R}$ $\mathbb{R}^2$, $aff(\mathbb{R})=\langle e_1,e_2:[e_1,e_2]=e_2\rangle$.

By Proposition 7, A_3 is not simple and hence it has a proper two-sided ideal *I*, so we get a short exact sequence of left-symmetric algebras

$$0\to I\xrightarrow{i}A_3\xrightarrow{\pi}J\to 0 \tag{8}$$

If $\mathcal{I}$ is the Lie sub algebra associated to *I* then, by Lemma 8, $\mathcal{I}$ is an ideal in $\mathcal{G}$. From Lemma 9 it follows that there are three cases to be considered according to weather $\mathcal{I}$ is isomorphic to $\mathbb{R}$, $\mathbb{R}^2$, or *aff* ($\mathbb{R}$).

Case 1: $\mathcal{I}\cong\mathbb{R}$.

In this case, the short exact sequence (8) becomes

$$0\to\mathbb{R}_0\to A_3\to I_2\to 0$$

where I_2 is a complete left-symmetric algebra of dimension 2 and $\mathbb{R}_0$ is $\mathbb{R}$ with the trivial product. At the Lie algebra level, we have a short exact sequence of Lie algebras of the form

$$0\to\mathbb{R}\to\tilde{\mathcal{G}}\to\mathcal{H}_2\to 0 \tag{9}$$

where $\mathcal{H}_2$ denotes the associated Lie algebra of I_2 and $\tilde{\mathcal{G}}$ is an extension of $\mathcal{H}_2$ by $\mathbb{R}$.

Since $\mathcal{H}_2$ is of dimension 2, then $\mathcal{H}_2$ is either isomorphic to $\mathbb{R}^2$ or *off* ($\mathbb{R}$).

Assume first that $\mathcal{H}_2\cong\mathbb{R}^2$. Then, the short exact sequence (9) becomes

$$0\to\mathbb{R}\to\tilde{\mathcal{G}}\to\mathbb{R}^2\to 0$$

Let {e_1, e_2} be a basis for $\mathbb{R}^2$. On $\mathbb{R}^2\times\mathbb{R}$, the extended Lie bracket given by (4) takes the simplified form

$$[(x,a),(y,b)]=(0,\phi(x)b-\phi(y)a+\omega(x,y)), \tag{10}$$

for all $a,b\in\mathbb{R}$, $x,y\in\mathbb{R}^2$.

Setting $\tilde{e}_i=(e_i,0)$, i=1, 2 and $\tilde{e}_3=(0,1)$ we get

$$[\tilde{e}_1,\tilde{e}_2]=\omega(e_1,e_2)\tilde{e}_3$$
$$e_3[\tilde{e}_1,\tilde{e}_3]=\phi(e_1)\tilde{e}_3$$
$$[\tilde{e}_2,\tilde{e}_3]=\phi(e_2)\tilde{e}_3$$

Since $\mathcal{G}$ is solvable and non-unimodular, we can, without loss of generality, assume that $\phi(e_2)=0$. That is

$$D=\begin{pmatrix}0 & \omega(e_1,e_2)\\ 0 & \phi(e_1)\end{pmatrix}$$

Notice that $\phi(e_1)$ should be non- zero, since otherwise $\mathcal{G}$ becomes unimodular. In other words,

$$D\cong\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$$

Now, we shall determine all the complete left-symmetric structures on $\mathbb{R}^2$. These are described by the following lemma that we state without proof.

Lemma 10: *Up to left-symmetric isomorphism, there are two complete left-symmetric structures on $\mathbb{R}^2$ given, in a basis {e_p, e_2} of $\mathbb{R}^2$, by either*

(i) $e_i\cdot e_j=0$ i,j=1,2

(ii) $e_2\cdot e_2=e_1$.

From now on, A_2 will denote the vector space $\mathbb{R}^2$ endowed with one of the complete left-symmetric structures described in Lemma 10.

The extended left-symmetric product on $A_2\times\mathbb{R}_0$ given by (5) turns out to take the simplified form

$$(x,a)\cdot(y,b)=(x\cdot y,b\lambda_x+a\rho_y+g(x,y)), \tag{11}$$

for all $x,y\in A_2$ and $a,b\in\mathbb{R}$. Indeed, $\rho_x,\lambda_x\in End(\mathbb{R})\cong\mathbb{R}$ for all $x\in A_2$. So, we can identify ρ_x and λ_x with real numbers that we denote by ρ_x and λ_x, respectively.

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Note here that $\lambda_x = \phi(x) + \rho_x$, for all $x \in \mathbb{R}^2$ whereas $\phi : \mathbb{R}^2 \to End(\mathbb{R}) \cong \mathbb{R}$ in (10).

The conditions in Theorem 1 can be simplified to the following conditions

$$\rho_{(x\cdot y)} = \rho_y {}^\circ \rho_x \tag{12}$$

$$\begin{aligned} & g(x, y.z) - g(y, x.z) + \lambda_x(g(y,z)) - \lambda_y(g(y,z)) \\ & -\rho_z(g(x,y) - g(y,x)) = 0 \end{aligned} \tag{13}$$

By using (10) and (11), we deduce from

$$\left[(x,a),(y,b)\right] = (x,a)\cdot(y,b) - (y,b)\cdot(x,a), \tag{14}$$

that

$$\omega(x,y) = g(x,y) - g(y,x) \ .$$

Since $\omega(e_1,e_2) = 0$, then $g(e_1, e_2)=g(e_2, e_1)$. Since $\phi(e_2) = 0$, then $\lambda_{e_2} = \rho_{e_2}$. Also, since $\phi(e_1) \neq 0$, then $\lambda_{e_1} - \rho_{e_1} \neq 0$. By applying identity (12) to $e_i \cdot e_i$, i=1,2, we deduce that $\rho = 0$. Hence $\lambda_{e_2} = 0$ and $\lambda_{e_1} \neq 0$, say $\lambda_{e_1} = \alpha$, $\alpha \in \mathbb{R}^*$.

In this case, the formula (6) and (7) become

$$\delta_1 h(x,y) = \lambda_x\left(h(y)\right) - h(x\cdot y)$$

And

$$\delta\, g(x,y,z) = g(x,y\cdot z) - g(y,x\cdot z) + \lambda_x\left(g(y,z)\right) - \lambda_y\left(g(x,z)\right)$$

where $h \in \mathcal{L}^1(A_2,\mathbb{R})$ and $g \in \mathcal{L}^2(A_2,\mathbb{R})$.

According to Lemma 10, there are two cases to be considered.

10.1. $A_2 = \left\langle e_1, e_2 : e_i \cdot e_j = 0, i,j = 1,2 \right\rangle$.

In this case, using the first formula above for δ_1 , we get

$$\delta_1 h = \begin{pmatrix} h_{11} & h_{12} \\ 0 & 0 \end{pmatrix},$$

Where $h_{11} = \alpha h(e_1)$ and $h_{12} = \alpha h(e_2)$. Similarly, using the second formula above for δ_2 , we verify easily that if g is a cocycle (i.e. $\delta_2 g = 0$) and $g_{ij} = g(e_i, e_j)$, then

$$g = \begin{pmatrix} g_{11} & 0 \\ 0 & 0 \end{pmatrix}$$

that is $g_{12} = g_{21} = g_{22} = 0$. In this case, the class $[g] \in H^2_{\lambda,\rho}(A_2,\mathbb{R})$ of a cocycle g may be represented, in the basis above, by a matrix of the simplified form

$$g = \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix}$$

We can now determine the extended complete left-symmetric structures on A_3. By setting $\tilde{e}_i = (e_i, 0)$, i=1, 2 and $\tilde{e}_3 = (0,1)$ and using formula (11) we obtain that the non- zero relations in A_3 are

$$\tilde{e}_1 \cdot \tilde{e}_2 = s\tilde{e}_3,$$
$$\tilde{e}_1 \cdot \tilde{e}_3 = \alpha\tilde{e}_3,$$

with $\alpha = \lambda_{e_1} \neq 0$

By setting $e_1 = \frac{1}{\alpha}\tilde{e}_1$, $e_2 = \tilde{e}_3$ and $\tilde{e}_3 = e_2$, and $t = \frac{s}{\alpha}$ we see that the new basis $\{e_1, e_2, e_3\}$ of A_3 satisfies

$$e_1 . e_2 = e_2$$
$$e_1 . e_3 = te_2$$

and all other products are zero. We can easily see that this product is isomorphic to

$$e_1 . e_2 = e_2$$

We set $N_{3,0} = \left\langle e_1, e_2, e_3 : e_1 \cdot e_2 = e_2 \right\rangle$.

10.2. $A_2 = \left\langle e_1, e_2 : e_2 \cdot e_2 = e_1 \right\rangle$.

We obtain, as above, that A_3 is isomorphic to one of the following complete left-symmetric algebras

(i) $N_{3,2} = \left\langle e_1, e_2, e_3 : e_1 \cdot e_2 = e_2, e_3 \cdot e_3 = e_1 \right\rangle$,

(ii) $N_{3,3} = \left\langle e_1, e_2, e_3 : e_1 \cdot e_2 = e_2, e_3 \cdot e_3 = -e_1 \right\rangle$.

Assume now that $\mathcal{H}_2 \cong aff(\mathbb{R})$. Then the extended Lie bracket on *aff* (ℝ) X ℝ given by (4) takes the form

$$\left[(x,a),(y,b)\right] = \left([x,y], \phi(x)b - \phi(y)a + \omega(x,y)\right),$$

for all $a\ b \in \mathbb{R}$, $x, y \in \mathrm{aff}(\mathbb{R})$.

Let $\{e_1 , e_2\}$ be a basis of *aff* (ℝ) satisfying $\left[e_1, e_2\right] = e_2$. By setting $\tilde{e}_i = (e_i, 0)$, i=1, 2 and $\tilde{e}_3 = (0,1)$

we get

$$\left[\tilde{e}_1, \tilde{e}_2\right] = e + \omega(e_1,e_2)\tilde{e}_3$$
$$e_3\left[\tilde{e}_1, \tilde{e}_3\right] = \phi(e_1)\tilde{e}_3$$
$$\left[\tilde{e}_2, \tilde{e}_3\right] = \phi(e_2)\tilde{e}_3$$

Since $\mathcal{G}$ is solvable and non-unimodular, then as above, we can assume that $\phi(e_2) = 0$. That is,

$$D = \begin{pmatrix} 0 & \omega(e_1,e_2) \\ 0 & \phi(e_1) \end{pmatrix}$$

Notice that $\phi(e_1) + 1 \neq 0$, since otherwise g becomes unimodular. Now, we have the following cases.

1. If $\det D = 0$, then $D \cong \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ that is, $\phi(e_1) = 0$ and $\omega(e_1,e_2) = 0$. This means that ϕ is identically zero, i.e. $\tilde{\mathcal{G}}$ is a central extension of *aff* (ℝ) by ℝ.

2. If $\det D \neq 0, D \cong \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\ or \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}$, with $0 < |\mu| < 1$.

It is not hard to prove the following

Lemma 11: *Up to left-symmetric isomorphisms, there is a unique complete left-symmetric structure on aff (ℝ) which is given, relative to a basis e_1, e_2 of aff (ℝ)* $\left[e_1, e_2\right] = e_2$, *by* $e_1 \cdot e_2 = e_2$.

We will denote by N_2 the vector space *aff* (ℝ) endowed with the complete left-symmetric product given in Lemma 11.

On the other hand, the extended left-symmetric product on $N_2 \times \mathbb{R}_0$ is given by

$$(x,a)\cdot(y,b) = \left(x\cdot y, b\lambda(x) + a\rho(y) + g(x,y)\right), \tag{15}$$

for all $a, b \in \mathbb{R}$, $x, y \in (\mathbb{R})$.

The conditions in Theorem 1 can be simplified to the following conditions

$$\lambda_{[x,y]}=0 \tag{16}$$

$$\rho_{(x\cdot y)}=\rho_y\circ\rho_x \tag{17}$$

$$g(x,y\cdot z)-g(y,x\cdot z)+\lambda_x\big(g(y,z)\big)-\lambda_y\big(g(x,z)\big)-g\big([x,y],z\big)$$

$$-\rho_z\big(g(x,y)-g(y,x)\big)=0$$

By using (10) and (11), we deduce from

$$\big[(x,a),(y,b)\big]=(x,a)\cdot(y,b)-(y,b)\cdot(x,a),$$

that

$$\omega(x,y)=g(x,y)-g(y,x)$$

From condition (16), we get $\lambda_{e_2}=0$. Applying the identity (17) above to $e_i\cdot e_i$, i=1, 2, we deduce that $\rho=0$ and hence $\lambda_{e_1}=\phi(e_1)$.

Assume first that $D\cong\begin{pmatrix}1&0\\0&0\end{pmatrix}$, that is, $\omega(e_1,e_2)=0$ and $\phi(e_1)=0$, then $\lambda=\rho=0$. Thus, the extension is central.

We know that the classification of the exact central extension of N_2 by $\mathbb{R}_0$ is, up to left-symmetric isomorphism, the orbit space of $H^2_{ex}(N_2,\mathbb{R}_0)$ under the natural action of $G=Aut(\mathbb{R}_0)\times Aut(N_2)$ (Proposition3). So, we must compute $H^2_{ex}(N_2,\mathbb{R}_0)$. Since $\mathbb{R}_0$ is a trivial N_2-bimodule, then

$$\delta_1 h(x,y)=-h(x\cdot y),$$

$$\delta_2 g(x,y,z)=g(x,y\cdot z)-g(y,x\cdot z)-g\big([x,y],z\big),$$

where $h\in\mathcal{L}^1(N_2,\mathbb{R})$ and $g\in\mathcal{L}^2(N_2,\mathbb{R})$. This implies that, with respect to the basis e_1,e_2 of N_2, $\delta_1 h$ is of the form

$$\delta_1 h=\begin{pmatrix}0&h_{12}\\0&0\end{pmatrix}$$

where $h_{12}=-h(e_2)$.

Observe that if g is a 2-cocycle (i.e. $\delta_2 g=0$), then

$$g=\begin{pmatrix}g_{11}&0\\0&0\end{pmatrix}$$

where $g_{ij}=g(e_i,e_j)$. Hence, $[g]\in H^2(N_2,\mathbb{R})$ can be represented as a matrix with respect to $\{e_1,e_2\}$ by

$$g=\begin{pmatrix}t&0\\0&0\end{pmatrix},t\in\mathbb{R}$$

We determine, in this case, the extended left-symmetric structure on A_3. By setting $\tilde{e}_i=(e_i,0)$, i=1, 2

and $\tilde{e}_3=(0,1)$, and using formula (15), we find

$$\tilde{e}_1\cdot\tilde{e}_1=t\tilde{e}_3,\qquad \tilde{e}_1\cdot\tilde{e}_2=\tilde{e}_2.$$

and all other products are zero, $t\in\mathbb{R}$. We denote $\mathcal{G}$ endowed with this structure by $N_{3,t}$.

Recall that the extension

$$0\to\mathbb{R}_0\to A_3\to N_2\to 0$$

is exact (i.e. $i(\mathbb{R}_0)=C(A_2)$) if and only if $I_{[g]}=\{0\}$.

Let $x=ae_1+be_2\in I_{[g]}$. Then computing all the products $x\cdot e_i=e_i\cdot x=0$, we deduce that x=0, that is the extension is exact.

Let $N_{3,t}$, $N_{3,t'}$ be two left-symmetric algebras as above. We know that $N_{3,t}$ is isomorphic to $N_{3,t}$ if and only if there exists $(\alpha,\eta)\in Aut(\mathbb{R}_0)\times Aut(N_2)=\mathbb{R}^*\times Aut(N_2)$ such that for all $x,y\in N_2$, we have

$$g'(x,y)=\alpha g\big(\eta(x),\eta(y)\big). \tag{18}$$

Now, we have to calculate $Aut(N_2)$. Let $\eta\in Aut(N_2)$ so that, with respect to the basis e_1,e_2 of N$_2$with $e_1\cdot e_2=e_2$,

$$\eta=\begin{pmatrix}a&b\\c&d\end{pmatrix}$$

Since $\eta(e_2)=\eta(e_1\cdot e_2)=\eta(e_1)\cdot\eta(e_2)$, then b=0 and d=ad. Also $0=\eta(e_1\cdot e_1)=\eta(e_1)\cdot\eta(e_1)$ which implies that a= 0 or c= 0. Since $\det\eta\neq 0$, then $d\neq 0$ and hence a=1 and c=0. This means that

$$\eta=\begin{pmatrix}1&0\\0&d\end{pmatrix}$$

with $d\neq 0$. We shall now apply formula (18). For this we recall first that in the basis e_1,e_2, the classes g and g' corresponding to $N_{3,t}$ and $N_{3,t'}$ have, respectively, the forms

$$g=\begin{pmatrix}t&0\\0&0\end{pmatrix}\text{and } g'=\begin{pmatrix}t'&0\\0&0\end{pmatrix}$$

From $g'(e_1,e_1)=\alpha g\big(\eta(e_1),\eta(e_1)\big)$, we get

$$t'=\alpha t$$

Hence $N_{3,t}$ and $N_{3,t'}$ are isomorphic if and only if $t'=\alpha t$, for some $\alpha\in\mathbb{R}^*$.

Notice that if t= 0, we obtain the complete left-symmetric algebra $N_{3,0}$ described above. If $t\neq 0$, we obtain, by setting $e_i=\tilde{e}_i$, i=1, 2, and $e_3=t\tilde{e}_3$, the complete left-symmetric algebra

$$N_{3,1}=\langle e_1,e_2,e_3:e_1\cdot e_1=e_3,e_1\cdot e_2=e_2\rangle$$

Assume now that $D\cong\begin{pmatrix}1&0\\0&1\end{pmatrix}$, that is, $\omega(e_1,e_2)=0$ and $\phi(e_1)=1$. Then $\lambda(e_1)=\phi(e_1)=1$. We deduce, in this case, that, in the basis e_1, e_2 of N_2, the $[g]\in H^2_{\lambda,\rho}(N_2,\mathbb{R})$ of a cocycle g may berepresented by a matrix of the simplified form

$$g=\begin{pmatrix}0&t\\t&0\end{pmatrix}$$

We determine, in this case, the extended complete left-symmetric structure on A_3. By setting $\tilde{e}_i=(\tilde{e}_i,0)$,$i$=1, 2 and $\tilde{e}_3=(0,1)$and using formula (15), we obtain

$$\tilde{e}_1\cdot\tilde{e}_2=\tilde{e}_2+t\tilde{e}_3$$

$$\tilde{e}_2\cdot\tilde{e}_1=t\tilde{e}_3$$

$$\tilde{e}_1\cdot\tilde{e}_3=\tilde{e}_3$$

We denote this left-symmetric algebra by $B_{3,t}$. Notice that if t=0, we obtain the complete left-symmetric algebra $B_{3,0}$ with the non- zero relations

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$$e_1 \cdot e_2 = e_2,$$
$$e_1 \cdot e_3 = e_3.$$

If $t \neq 0$; we obtain, by setting $e_i = \tilde{e}_i$, i=1,2 and $e_3 = t\tilde{e}_3$; the complete left-symmetric algebra $B_{3,1}$ with the non-zero relations

$$e_1 \cdot e_2 = e_2 + e_3$$
$$e_2 \cdot e_1 = e_3$$
$$e_1 \cdot e_3 = e_3$$

Assume now that $D \cong \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ that is, $\omega(e_1, e_2) = 1$ and $\phi(e_1) = 1$. Hence $\lambda(e_1) = \phi(e_1) = 1$. Usingthe same method as above, it follows that the class $[g] \in H^2_{\lambda,\rho}(N_2, \mathbb{R})$ of a co-cycle g takes the reduced form

$$g = \begin{pmatrix} 0 & t \\ t-1 & 0 \end{pmatrix}$$

We determine, in this case, the extended complete left-symmetric structures on A_3. By setting $\tilde{e}_i = (e_i, 0)$,i=1, 2 and $\tilde{e}_3 = (0,1)$ and using formula (15), we obtain

$$\tilde{e}_1 \cdot \tilde{e}_2 = \tilde{e}_2 + te_3$$
$$\tilde{e}_2 \cdot \tilde{e}_1 = (t-1)\tilde{e}_3$$
$$\tilde{e}_1 \cdot \tilde{e}_3 = \tilde{e}_3$$

We denote such a left-symmetric algebra by $C_{3,t}$. Notice that if t=1, we obtain the complete left- symmetric algebra $C_{3,1}$ with the non-zero relations

$$e_1 \cdot e_2 = e_2 + e_3,$$
$$e_1 \cdot e_3 = e_3,$$

and if $t \neq 1$, we obtain the complete left-symmetric algebra $C_{3,t}$ with the non-zero relations

$$e_1 \cdot e_2 = e_2 + te_3$$
$$e_2 \cdot e_1 = (t-1)e_3$$
$$e_1 \cdot e_3 = e_3$$

where different values of t give non-isomorphic complete left-symmetric algebras.

Assume finally that $D \cong \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}$, with $0 < |\mu| < 1$, that is $\omega(e_1, e_2) = 0$ and $\phi(e_1) = \mu$. Hence $\lambda(e_1) = \phi(e_1) = \mu$. It follows that the class $[g] \in H^2_{\lambda,\rho}(N_2, \mathbb{R})$ of a co-cycle g is identically zero.

We determine, in this case, the extended complete left-symmetric structures on A_3. By setting $\tilde{e}_i = (e_i, 0)$, i=1, 2 and $\tilde{e}_3 = (0,1)$ and using formula (15), we obtain

$$\tilde{e}_1 \cdot \tilde{e}_2 = \tilde{e}_2,$$
$$\tilde{e}_1 \cdot \tilde{e}_3 = \mu\tilde{e}_3.$$

where $0 < |\mu| < 1$. We set

$$D_{3,1}(\mu) = \langle e_1, e_2, e_3 . e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = \mu e_3 \rangle$$

where $0 < |\mu| < 1$.

Case 2: $\mathcal{I} \cong \mathit{aff}(\mathbb{R})$.

In this case, the short exact sequence (8) becomes

$$0 \to N_2 \to A_3 \to \mathbb{R}_0 \to 0 \tag{19}$$

where N_2 is the complete left-symmetric algebra whose associated Lie algebra is *aff* ($\mathbb{R}$) and $\mathbb{R}_0$ is the trivial left-symmetric algebra over $\mathbb{R}$.

Let $\sigma : \mathbb{R}_0 \to A_3$ be a section and set $\sigma(1) = x_\circ \in A_3$ and define two linear maps $\lambda, \rho \in End(N_2)$ by putting $\lambda(y) = x_\circ \cdot y$ and $\rho(y) = y \cdot x_\circ$. By setting $e = x_\circ \cdot x_\circ$, we see that $e \in N_2$. Let $g : \mathbb{R}_0 \times \mathbb{R}_0 \to N_2$ be the bilinear map defined by $g(a,b) = \sigma(a) \cdot \sigma(b) - \sigma(a \cdot b)$. Since the complete left-symmetric structure on $\mathbb{R}$ is trivial, then $g(a,b) = abe$, or equivalently $g(1,1) = e$. Also we can show that $\delta_2 g = 0$, i.e. $g \in Z^2_{\lambda,\rho}(\mathbb{R}_0, N_2)$.

In this case, the extended left-symmetric product on $\mathbb{R}_0 \oplus N_2$ given by (5) takes the simplified form

$$(a,x) \cdot (b,y) = (0, x \cdot y + a\lambda(y) + b\rho(x) + abe),$$

for all $a,b \in \mathbb{R}$ and $x,y \in N_2$.

The conditions in Theorem 1 can be simplified to the following conditions

$$\lambda(x \cdot y) = \lambda(x) \cdot y + x \cdot \lambda(y) - \rho(x) \cdot y \tag{20}$$

$$\rho([x,y]) = x \cdot \rho(y) - y \cdot \rho(x) \tag{21}$$

$$[\lambda, \rho] + \rho^2 = R_e \tag{22}$$

Let $\phi : \mathbb{R} \to Der(\mathrm{aff}(\mathbb{R}))$, be a derivation of *aff* ($\mathbb{R}$). Set

$$\phi(1) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

relative to a basis e_1, e_2 of *aff* ($\mathbb{R}$) satisfying $[e_1, e_2] = e_2$. From the identity $\phi(1)e_2 = [\phi(1)e_1, e_2] + [e_1, \phi(1)e_2]$, we deduce that a=c=0, hence

$$\phi(1) = \begin{pmatrix} 0 & 0 \\ b & d \end{pmatrix}$$

Let

$$\rho = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}$$

relative to a basis e_1, e_2 of *aff* ($\mathbb{R}$) satisfying $[e_1, e_2] = e_2$. Applying formula (21) to e_2, we get $\beta_1 = 0$. Since $\phi(1) = \lambda - \rho$, we deduce that, relative to the basis e_1, e_2, we have

$$\lambda = \begin{pmatrix} \alpha_1 & 0 \\ \alpha_2 + b & \beta_2 + d \end{pmatrix}$$

Applying formula (20) to all products of the form e_i, e_j, i=1, 2, we get $\alpha_2 + b = 0$. Moreover, by applying formula (22) to e_1 and e_2, we get $\alpha_1 = \beta_2 = 0$. Thus

$$\rho = \begin{pmatrix} 0 & 0 \\ -b & 0 \end{pmatrix} \text{ and } \lambda = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$$

Now, since $e \in N_2$, then $e = te_1 + se_2$ for some $t,s \in \mathbb{R}$. Formula (22) when applied to e_1 gives

$$-bde_2 = se_2$$

for which we get that $e = x_\circ \cdot x_\circ = te_1 - bde_2$, $t \in \mathbb{R}$. Hence we get a left-symmetric product on A_3. Now, let us write down the structure of A_3 using a basis. From above we have

$$e_1 \cdot e_2 = e_2, \qquad e_1 \cdot x_\circ = -be_2 \cdot$$

$$x_\circ e_2 = de_2, \qquad x_\circ \cdot x_\circ = te_1 - bde_2,\ t \in \mathbb{R}$$

Since $x_0 \in A_3$ and $\pi(x_0) = 1$, then $x_0 \in A_3 \setminus N_2$. Indeed if $x_0 \in N_2$, then the exactness of the short sequence (19) implies that $x_0 \in i(N_2) = \ker \pi$, a contradiction. This implies that, relative to a basis $\{e_1, e_2, e_3\}$ of A_3, x_0 is of the form $x_0 = \alpha e_1 + \beta e_2 + \gamma e_3$, where $\alpha, \beta, \gamma \in \mathbb{R}$ with $\gamma \neq 0$. In this case, we can, without loss of generality, assume that $\gamma = 1$. Thus, $e_3 = x_0 - \alpha e_1 - \beta e_2$. Since $e_1 \cdot x_\circ = -be_2$ we get that

$$e_1 \cdot e_3 = -(b + \beta) e_2,$$

also since $x_\circ \cdot e_2 = de_2$ we get that

$$e_3 \cdot e_2 = (d - \alpha) e_2.$$

since $x_\circ \cdot x_\circ = te_1 - bde_{2,}$, we deduce that

$$e_3 \cdot e_3 = te_1 + (\alpha b + \alpha\beta - bd - \beta d) e_2.$$

Since α, β are arbitrary, we can choose α, β so that $e_3 = x_\circ - de_1 - be_2$. Hence the left-symmetric product on A_3 is given, relative the basis $\{e_1, e_2, e_3\}$, by the non- zero relations

$$e_1 \cdot e_2 = e_2$$
$$e_3 \cdot e_3 = te_1,$$

Notice that if t=0, we obtain the complete left-symmetric algebra $N_{3,0}$. If $t \neq 0$, we obtain, by setting $e_i = \tilde{e}_i$; i=1, 2 and $\tilde{e}_3 = \frac{1}{\sqrt{|t|}} e_3$; that A_3 is isomorphic to one of the left-symmetric algebras $N_{3,2}$ or $N_{3,3}$ given above

Case 3: $\mathcal{I} \cong \mathbb{R}^2$.

In this case, the short exact sequence (8) becomes

$$0 \to A_2 \to A_3 \to \mathbb{R}_0 \to 0$$

where A_2 is a complete left-symmetric algebra whose lie algebra is ^{2} and $\mathbb{R}_0$ is the trivial left-symmetric algebra over ℝ.

At the lie algebra level, we have a short exact sequence of lie algebras of the form

$$0 \to \mathbb{R}^2 \to \tilde{\mathcal{G}} \to \mathbb{R} \to 0$$

Let $\phi : \mathbb{R} \to Der(\mathbb{R}^2) \cong End(\mathbb{R}^2)$, be a derivation of $\mathbb{R}^2$. Relative to a basis e_1, e_2 of $\mathbb{R}^2$ set

$$\phi(1) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

In this case, the extended Lie bracket on $\mathbb{R} \times \mathbb{R}^2$, given by (4), takes the simplified form

$$[(a,x),(b,y)] = (0, \phi(a)y - \phi(b)x + \omega(a,b))$$

for all $x, y \in \mathbb{R}^2$ and $a, b \in \mathbb{R}$. By setting $\tilde{e}_1 = (1,0)$ and $\tilde{e}_{i+1} = (0, e_i)$, i=1, 2 we obtain

$$[\tilde{e}_1, \tilde{e}_2] = a\tilde{e}_1 + b\tilde{e}_2$$
$$[\tilde{e}_1, \tilde{e}_3] = c\tilde{e}_1 + d\tilde{e}_2$$
$$[\tilde{e}_2, \tilde{e}_3] = 0$$

By Lemma 5, we obtain that, relative to the basis e_1, e_2,

$$D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a + d \neq 0$. Note that, in this case, ω may not be zero, that is, the extensions of ℝ by $\mathbb{R}^2$are not necessarily semi direct products of ℝ by $\mathbb{R}^2$.

According to Lemma 5, there are five cases to be considered

$$D \cong \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix} \text{ or } \begin{pmatrix} 1 & -\zeta \\ \zeta & 1 \end{pmatrix},$$

Where $\zeta > 0$ and $0 < |\mu| < 1$.

Let $\sigma : \mathbb{R}_0 \to A_3$ be a section and set $\sigma(1) = x_\circ \in A_3$ and define two linear maps $\lambda, \rho \in End(A_2)$ by putting $\lambda(y) = x_\circ \cdot y$ and $\rho(y) = y \cdot x_\circ$. By setting $e = x_\circ \cdot x_\circ$, we see that $e \in A_2$. Let $g : \mathbb{R}_0 \times \mathbb{R}_0 \to A_2$ be the bilinear map defined by $g(a,b) = \sigma(a) \cdot \sigma(b) - \sigma(a \cdot b)$. Since the complete left-symmetric structure on ℝ is trivial, then $g(a,b) = abe$, or equivalently $g(1,1) = e$. Also we can show that $\delta_2 g = 0$, i.e. $g \in Z^2_{\lambda,\rho}(\mathbb{R}_0, A_2)$.

The extended left-symmetric product on $\mathbb{R}_0 \oplus A_2$ given by (5) is then takes the simplified form

$$(a,x) \cdot (b,y) = (0, x \cdot y + a\lambda(y) + b\rho(x) + abe) \tag{23}$$

for all $x, y \in A_2$ and $a, b \in \mathbb{R}$.

The conditions in Theorem 1 can be simplified to the following conditions

$$\lambda(x \cdot y) = \lambda(x) \cdot y + x \cdot \lambda(y) - \rho(x) \cdot y \tag{24}$$

$$x \cdot \rho(y) - y \cdot \rho(x) = 0 \tag{25}$$

$$[\lambda, \rho] + \rho^2 = R_e \tag{26}$$

According to Lemma 10, we have the following cases of A_2

1. $A_2 = \langle e_1, e_2 : e_i \cdot e_j = 0, i, j = 1, 2 \rangle$.

Assume first that $D \cong \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

and let

$$\rho = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}$$

relative to the basis e_1, e_2 of A_2. Since $\phi(1) = \lambda - \rho$, we deduce that, relative to the basis e_1, e_2, we have

$$\lambda = \begin{pmatrix} \alpha_1 + 1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}$$

Applying formula (26) to e_2, we obtain $\beta_1 = \beta_2 = 0$. The same formula when applied to e_1 yields $\alpha_1 = \alpha_2 = 0$. It follows that ρ is identically zero and

$$\lambda = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

We can easily show that the condition (26) above is satisfied for all $e = x_\circ \cdot x_\circ = se_1 + te_2$, $s\ t \in \mathbb{R}$. Hence we get a left-symmetric product on A_3.

Now, let us write down the structure of A_3 using a basis. From

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above we have

$$x_\circ \cdot e_1 = e_1, \qquad x_\circ \cdot x_\circ = se_1 + te_2.$$

We can easily prove that $x_0 \in A_3 \setminus A_2$. This implies that, relative to a basis $\{e_1, e_2, e_3\}$ of A_3, x_0 is of the form $x_0 = \alpha e_1 + \beta e_2 + \gamma e_3$, where $\alpha, \beta, \gamma \in \mathbb{R}$ with $\gamma \neq 0$. In this case, we can, without loss of generality, assume that $\gamma = 1$. Thus, $e_3 = x_0 - \alpha e_1 - \beta e_2$. Since $x_\circ \cdot e_1 = e_1$ we get that

$$e_3 \cdot e_1 = e_1$$

also since $x_\circ \cdot x_\circ = se_1 + te_2$, we deduce that

$$e_3 \cdot e_3 = (s - \alpha)e_1 + te_2 .$$

Since α, β are arbitrary, we can choose α, β so that $e_3 = x_\circ - se_1$. Hence the left-symmetric product on A_3 is given, relative to the basis $\{e_1, e_2, e_3\}$ of A_3, by the non- zero relations

$$e_3 \cdot e_1 = e_1$$
$$e_3 \cdot e_3 = te_2$$

Notice that if $t = 0$, we find the complete left-symmetric algebra $N_{3,0}$. If $t \neq 0$, we get, by setting $\tilde{e}_1 = e_3, \tilde{e}_2 = e_1$, and $\tilde{e}_3 = te_2$ that A_3 is isomorphic to the complete left-symmetric algebra $N_{3,1}$.

Assume then that $D \cong \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and let

$$\rho = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}$$

relative to the basis e_1, e_2 of A_2. Since $\phi(1) = \lambda - \rho$, we deduce that, relative to the basis e_1, e_2, we have

$$\lambda = \begin{pmatrix} \alpha_1 + 1 & \beta_1 \\ \alpha_2 & \beta_2 + 1 \end{pmatrix}$$

By applying formula (26) to e_1 and e_2, we get

$$\rho = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix},$$
$$\lambda = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \ \alpha \in \mathbb{R}$$

and $e = x_\circ \cdot x_\circ = \alpha^2 e_1 + \alpha e_2$.

Similarly, we find that, relative to the basis $\{e_1, e_2, e_3\}$ of A_3 with $e_3 = x_\circ + \alpha^2 e_1 - \alpha e_2$, the left- symmetric product on A_3 is given by the non- zero relations

$$e_3 \cdot e_1 = e_1$$
$$e_3 \cdot e_2 = \alpha e_1 + e_2$$
$$e_2 \cdot e_3 = \alpha e_1.$$

Notice that if $\alpha = 0$, we get, by setting $\tilde{e}_1 = e_3, \tilde{e}_2 = e_1$ and $\tilde{e}_3 = e_2$, the complete left-symmetric algebra $B_{3,0}$. If $t \neq 0$ we get, by setting $\tilde{e}_1 = e_3$; $\tilde{e}_2 = e_2$ $\tilde{e}_3 = \alpha e_1$; that A_3 is isomorphic to the complete left-symmetric algebras $B_{3,1}$.

Assume now that $D \cong \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and let

$$\rho = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}$$

relative to the basis e_1, e_2 of A_2. Since $D = \lambda - \rho$, we deduce that, relative to the basis e_1, e_2, we have

$$\lambda = \begin{pmatrix} \alpha_1 + 1 & \beta_1 + 1 \\ \alpha_2 & \beta_2 + 1 \end{pmatrix}$$

By applying formula (26) to e_1 and e_2, we get

$$\rho = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}, \ \lambda = \begin{pmatrix} 1 & \alpha + 1 \\ 0 & 1 \end{pmatrix}, \ \alpha \in \mathbb{R}$$

and $e = x_\circ \cdot x_\circ = \alpha e_1 + \alpha e_2$.

Similarly, we find that, relative to a basis $\{e_1, e_2, e_3\}$ of A_3 with $e_3 = x_\circ + 2\alpha^2 e_1 - \alpha e_2$, the left-symmetric product on A_3 is given by the non-zero relations

$$e_3 \cdot e_1 = e_1$$
$$e_3 \cdot e_2 = (\alpha + 1)e_1 + e_2$$
$$e_2 \cdot e_3 = \alpha e_1.$$

Notice that if $\alpha = 0$, we get, by setting $\tilde{e}_1 = e_3, \tilde{e}_2 = e_1$ and $\tilde{e}_3 = e_2$ the complete left-symmetric algebra $C_{3,1}$. If $\alpha \neq 0$, we get, by setting $\alpha = t - 1$ with $t \neq 1$, the complete left-symmetric algebra $C_{3,1}$ where different values of t give non-isomorphic complete left-symmetric algebras.

Assume then that $D \cong \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, where $0 < |\mu| < 1$, and let

$$\rho = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}$$

relative to the basis e_1, e_2 of A_2. Since $\phi(1) = \lambda - \rho$, we deduce that, relative to the basis e_1, e_2, we have

$$\lambda = \begin{pmatrix} \alpha_1 + 1 & \beta_1 \\ \alpha_2 & \beta_2 + \mu \end{pmatrix}$$

By applying formula (26) to e_1 and e_2, we obtain that ρ is identically zero,

$$\lambda = \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}$$

and $e = x_\circ \cdot x_\circ = e_1 + \mu e_2$.

Similarly, we find that, relative to a basis $\{e_1, e_2, e_3\}$ of A_3 with $e_3 = x_\circ - e_1 - e_2$, the left-symmetric product on A_3 is given by the non- zero relations

$$e_3 \cdot e_1 = e_1$$
$$e_3 \cdot e_2 = \mu e_2.$$

By setting $\tilde{e}_1 = e_3$, $\tilde{e}_2 = e_1$ and $\tilde{e}_3 = e_2$, we get the complete left-symmetric algebra $D_{3,l}(\mu)$ where $0 < |\mu| < 1$

Assume finally that $D \cong \begin{pmatrix} 1 & -\zeta \\ \zeta & 1 \end{pmatrix}$, where $\zeta > 0$, and let

$$\rho = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}$$

relative to the basis e_1, e_2 of A_2. Since $\phi(1) = \lambda - \rho$, we deduce that, relative to the basis e_1, e_2 above, we have

$$\lambda = \begin{pmatrix} \alpha_1 + 1 & \beta_1 - \zeta \\ \alpha_2 + \zeta & \beta_2 + 1 \end{pmatrix}$$

By applying formula (26) to e_1 and e_2, we obtain that ρ is identically zero,

$$\lambda = \begin{pmatrix} 1 & -\zeta \\ \zeta & 1 \end{pmatrix}$$

and $e = x_\circ \cdot x_\circ = 2\zeta e_1 + \left(\zeta^2 - 1\right) e_2$.

Similarly, we find that, relative to a basis $\{e_1, e_2, e_3\}$ of A_3 with $e_3 = x_\circ - \zeta e_1 + e_2$, the left-symmetric product on A_3 is given by the non- zero relations

$$e_3 \cdot e_1 = e_1 + \zeta e_2$$
$$e_3 \cdot e_2 = -\zeta e_1 + e_2$$

Set $\tilde{e}_1 = e_3$, $\tilde{e}_2 = e_1$ and $\tilde{e}_3 = e_2$. Then, the non- zero relations above become

$$\tilde{e}_1 \cdot \tilde{e}_2 = \tilde{e}_2 + \zeta \tilde{e}_3$$
$$\tilde{e}_1 \cdot \tilde{e}_3 = -\zeta \tilde{e}_2 + \tilde{e}_3$$

We set

$$E_{3,\zeta} = \langle e_1, e_2, e_3 : e_1 \cdot e_2 = e_2 + \zeta e_3, e_1 \cdot e_3 = -\zeta e_2 + e_3, \zeta > 0 \rangle.$$

2. $A_2 = \langle e_1, e_2 : e_2 \cdot e_2 = e_1 \rangle$..

Let

$$\rho = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}$$

relative to the basis e_1, e_2 of A_2 . By applying formula (25) to e_1 and e_2, we get that $\alpha_2 = 0$

Assume first that $D \cong \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then, as $\phi(1) = \lambda - \rho$, we deduce that, relative to the basis e_1 , e_2, we have

$$\lambda = \begin{pmatrix} \alpha_1 + 1 & \beta_1 \\ 0 & \beta_2 \end{pmatrix}$$

By applying formula (26) to e_1 and e_2, we get that $\alpha_1 = \beta_2 = 0$. Moreover, by applying formula (24) to all products of the form $e_i \cdot e_j, i, j = 1,2$, we get that 1=0, a contradiction. Thus D cannot be of this form. Similarly, we can prove that D cannot be of the forms $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, or \begin{pmatrix} 1 & -\zeta \\ \zeta & 1 \end{pmatrix}, \zeta > 0.$

where $\zeta > 0$.

Assume that $D \cong \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}$, where $0 < |\mu| < 1$, Then, as $\phi(1) = \lambda - \rho$, we deduce that

$$\lambda = \begin{pmatrix} \alpha_1 + 1 & \beta_1 \\ 0 & \beta_2 + \mu \end{pmatrix}$$

By applying formula (26) to e_1 and e_2, we get that $\alpha_1 = \beta_2 = 0$. Moreover, by applying formula (24) to all products of the form $e_i \cdot e_j, i, j = 1,2,$ we get that $\mu = \frac{1}{2}$. Thus

$$\rho = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}, \lambda = \begin{pmatrix} 1 & \alpha \\ 0 & \frac{1}{2} \end{pmatrix}, \alpha \in \mathbb{R}$$

and $e = x_\circ \cdot x_\circ = t e_1 + \frac{1}{2} \alpha e_2, t \in \mathbb{R}$.

Similarly, we find that, relative to a basis $\{e_1, e_2, e_3\}$ of A_3 with $e_3 = x_\circ + \left(\alpha^2 - t\right) e_1 - \alpha e_2$, the left-symmetric product on A_3 is given by the non-zero relations

$$e_2 \cdot e_2 = e_1,$$
$$e_3 \cdot e_1 = e_1,$$
$$e_3 \cdot e_2 = \frac{1}{2} e_2,$$

Set $\tilde{e}_1 = e_3$, $\tilde{e}_2 = e_1$ and $\tilde{e}_3 = e_2$. Then the non- zero relations above become

$$\tilde{e}_2 \cdot \tilde{e}_2 = \tilde{e}_1,$$
$$\tilde{e}_1 \cdot \tilde{e}_2 = \tilde{e}_2,$$
$$\tilde{e}_1 \cdot \tilde{e}_3 = \frac{1}{2} \tilde{e}_3$$

We set

$$D_{3,2} = \left\langle e_1, e_2, e_3 : e_2 \cdot e_2 = e_1, e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = \frac{1}{2} e_3 \right\rangle.$$

Conclusion

We can now state the main result of this paper

Theorem 12: *Let A_3 be a three dimensional complete left-symmetric algebra whose associated Lie algebra $\mathcal{G}$ is solvable and non-unimodular. Then A_3 is isomorphic to one of the following left-symmetric algebras* (Table 1).

Name	Non-zero product	Lie algebra	Remarks
$N_{3,0}$	$e_1 \cdot e_2 = e_2$	$\mathcal{G}_{3,1}$	N,D,S
$N_{3,1}$	$e_1 \cdot e_1 = e_3, e_1 \cdot e_2 = e_2$	$\mathcal{G}_{3,1}$	N,D,S
$N_{3,2}$	$e_1 \cdot e_2 = e_2, e_3 \cdot e_3 = e_1$	$\mathcal{G}_{3,1}$	S
$N_{3,3}$	$e_1 \cdot e_2 = e_2, e_3 \cdot e_3 = -e_1$	$\mathcal{G}_{3,1}$	S
$B_{3,0}$	$e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = e_3$	$\mathcal{G}_{3,2}$	N,D,S
$B_{3,1}$	$e_1 \cdot e_2 = e_2 + e_3,$ $e_2 \cdot e_1 = e_3, e_1 \cdot e_3 = e_3$	$\mathcal{G}_{3,2}$	D
$C_{3,1}$	$e_1 \cdot e_2 = e_2 + e_3,$ $e_1 \cdot e_3 = e_3$	$\mathcal{G}_{3,3}$	N,D,S
$C_{3,t}$	$e_1 \cdot e_2 = e_2 + t e_3,\ e_1 \cdot e_3 = e_3,$ $e_2 \cdot e_1 = (t-1) e_3,, t \neq 1$	$\mathcal{G}_{3,3}$	D
$D_{3,1}(\mu)$	$e_1 \cdot e_2 = e_2,$ $e_1 \cdot e_3 = \mu e_3,\ 0 < \lvert\mu\rvert < 1$	$\mathcal{G}_{3,4}^{\mu}$	N,D,S
$D_{3,2}$	$e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = \frac{1}{2} e_3,$ $e_2 \cdot e_2 = e_1$	$\mathcal{G}_{3,4}^{\frac{1}{2}}$	N
$E_{3,1}(\zeta)$	$e_1 \cdot e_2 = e_2 + \zeta e_3,$ $e_1 \cdot e_3 = -\zeta e_2 + e_3,\ \zeta > 0$	$\mathcal{G}_{3,5}^{\zeta}$	N,D,S

Table 1: Left-symmetric algebras.

Here, the letter N that the left-symmetric algebra A_3 is Novikov, the letter D means that A_3 is derivation and the letter S means that A_3satisfying $[x, y] \cdot z = 0$ for all $x, y, z \in A_3$.

Remark 1: *We note that left-symmetric algebras satisfying the identity $(x \cdot y) \cdot z = (y \cdot x) \cdot z$ for all $x, y, z \in A$ (or equivalently, the* *identity* $[x,y]\cdot z=0$ *for all* $x,y,z\in A$ *are of special interest because they correspond to locally simply transitive affine actions of Lie groups G on a vector space E such that the commutator subgroup [G,G] is acting by translations. These left-symmetric algebras have been considered and studied in [7].*

We note that the mapping $X\to(L_X,X)$ is a Lie algebra representation of $\mathcal{G}$ in $\mathit{aff}(\mathbb{R}^3)=\mathit{End}(\mathbb{R}^3)\oplus\mathbb{R}^3$.

By using the exponential maps, Theorem 12 can now be stated, in terms of simply transitive actions of subgroups of the affine group $\mathit{Aff}(\mathbb{R}^3)=GL(\mathbb{R}^3)\mathbb{R}^3$, as follows

To state it, define the continuous functions *f, g, h, k* and ϕ by

$$f(x)=\begin{cases}\dfrac{e^x-1}{x}, & x\neq 0\\ 1 & x=0\end{cases},\quad g(x)=\begin{cases}\dfrac{e^x-x-1}{x^2}, & x\neq 0\\ \dfrac{1}{2} & x=0\end{cases},$$

$$h(x)=\begin{cases}\dfrac{\cos x-1}{x}+\dfrac{x}{2}, & x\neq 0\\ 0 & x=0\end{cases},\quad k(x)=\begin{cases}\dfrac{\sin x-x}{x}, & x\neq 0\\ 0 & x=0\end{cases},$$

$$\phi(x)=\sum_{n=1}^{\infty}\frac{nx^n}{(n+1)!}$$

Theorem 13: *Suppose that the Lie group G of the non-unimodular Lie algebra* $\mathcal{G}$ *of dimension 3 acts simply transitively by affine transformations on* $\mathbb{R}^3$*. Then, as a subgroup of* $\mathit{Aff}(\mathbb{R}^3)$*, G is conjugate to one of the following sub groups:*

$$G_{A_{3,0}}=\left\{\begin{pmatrix}1&0&0\\0&e^a&0\\0&0&1\end{pmatrix}\begin{bmatrix}a\\bf(a)\\c\end{bmatrix},\ a,b,c\in\mathbb{R}\right\}$$

$$G_{A_{3,1}}=\left\{\begin{pmatrix}1&0&0\\0&e^a&0\\a&0&1\end{pmatrix}\begin{bmatrix}a\\bf(a)\\c+\frac{1}{2}a^2\end{bmatrix},\ a,b,c\in\mathbb{R}\right\}$$

$$G_{A_{3,2}}=\left\{\begin{pmatrix}1&0&c\\0&e^a&0\\0&0&1\end{pmatrix}\begin{bmatrix}a+\frac{1}{2}c^2\\bf(a)\\c\end{bmatrix},\ a,b,c\in\mathbb{R}\right\}$$

$$G_{A_{3,3}}=\left\{\begin{pmatrix}1&0&-c\\0&e^a&0\\0&0&1\end{pmatrix}\begin{bmatrix}a-\frac{1}{2}c^2\\bf(a)\\c\end{bmatrix},\ a,b,c\in\mathbb{R}\right\}$$

$$G_{B_{3,0}}=\left\{\begin{pmatrix}1&0&0\\0&e^a&0\\0&0&e^a\end{pmatrix}\begin{bmatrix}a\\bf(a)\\cf(a)\end{bmatrix},\ a,b,c\in\mathbb{R}\right\}$$

$$G_{B_{3,1}}=\left\{\begin{pmatrix}1&0&0\\0&e^a&0\\bf(a)&ae^a&e^a\end{pmatrix}\begin{bmatrix}a\\bf(a)\\(ab+c)f(a)\end{bmatrix},\ a,b,c\in\mathbb{R}\right\}$$

$$G_{C_{3,1}}=\left\{\begin{pmatrix}1&0&0\\0&e^a&0\\0&ae^a&e^a\end{pmatrix}\begin{bmatrix}a\\bf(a)\\cf(a)+b\phi(a)\end{bmatrix},\ a,b,c\in\mathbb{R}\right\}$$

$$G_{C_{3,t}}=\left\{\begin{pmatrix}1&0&0\\0&e^a&0\\(t-1)bf(a)&tae^a&e^a\end{pmatrix}\begin{bmatrix}a\\bf(a)\\(tab+c-b)f(a)+b\end{bmatrix},\ a,b,c\in\mathbb{R},\ t\neq 1\right\}$$

$$G_{D_{3,1}(\mu)}=\left\{\begin{pmatrix}1&0&0\\0&e^a&0\\0&0&e^{\mu a}\end{pmatrix}\begin{bmatrix}a\\bf(a)\\cf(\mu a)\end{bmatrix},\ a,b,c\in\mathbb{R}\ ,0<|\mu|<1\right\}$$

$$G_{D_{3,2}}=\left\{\begin{pmatrix}1&bf(a)&0\\0&e^a&0\\0&0&e^{\frac{1}{2}a}\end{pmatrix}\begin{bmatrix}a+b^2g(a)\\bf(a)\\cf\left(\frac{a}{2}\right)\end{bmatrix},\ a,b,c\in\mathbb{R}\right\}$$

$$G_{E_3(\zeta)}=\left\{\begin{matrix}\begin{pmatrix}1&0&0\\0&e^a\cos\zeta a&-e^a\sin\zeta a\\0&e^a\sin\zeta a&e^a\cos\zeta a\end{pmatrix}\\ \begin{bmatrix}a\\b(f(a)+k(\zeta a))+c(h(\zeta a)-\zeta\phi(a))\\b(\zeta\phi(a)-h(\zeta a))+c(f(a)+k(\zeta a))\end{bmatrix},\ a,b,c\in\mathbb{R},\zeta>0\end{matrix}\right\}$$

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