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On the Functional Relations for the Euler-Zagier Multiple Zeta-functions

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Abstract. The purpose of this paper is to formulate problem about existence or non-existence of functional relations for the Euler-Zagier multiple zeta-functions and solve this problem. This problem is a functional analogue of the problem about existence or non-existence of relations among the multiple zeta values. By our results, we can solve a functional analogue of the problem about the dimension of the $\mathbb{Q}$-vector space spanned by the multiple zeta values.

1. Introduction

Let $d \in \mathbb{N}$ and $s_1, \dots, s_d \in \mathbb{C}$. The Euler-Zagier multiple zeta-functions are defined by

$$\zeta_d(s_1, \dots, s_d) = \sum_{n_1=1}^{\infty} \frac{1}{n_1^{s_1}} \sum_{n_2=n_1+1}^{\infty} \frac{1}{n_2^{s_2}} \cdots \sum_{n_d=n_{d-1}+1}^{\infty} \frac{1}{n_d^{s_d}},$$

where the sum is absolutely convergent if $\Re s_d > 1, \Re s_{d-1} + \Re s_d > 2, \dots, \Re s_1 + \cdots + \Re s_d > d$ and ζ_1 is equal to the Riemann zeta-function ζ. The values of $\zeta_d(s_1, \dots, s_d)$ at positive integer points are called the multiple zeta values. These values have been investigated a great deal in recent years. There are plenty of relations among them, for example $\zeta_2(1, 2) = \zeta(3)$ and $\zeta(5) = \zeta_3(1, 1, 3) + \zeta_3(2, 1, 2) + \zeta_3(1, 2, 2)$. These relations have appeared in various fields of mathematics and physics (see, for example, [2], [4], [5], [19] and [22]).

Let $\zeta_{MT,r}$ be the Mordell-Tornheim multiple zeta-functions defined by

$$\zeta_{MT,r}(s_1, \dots, s_r, s_{r+1}) = \sum_{m_1, \dots, m_r=1}^{\infty} \frac{1}{m_1^{s_1} \cdots m_r^{s_r} (m_1 + \cdots + m_r)^{s_{r+1}}}.$$

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Subbarao-Sitaramachandrarao [20] showed the explicit formula, for $k \in \mathbb{N}$

$$\zeta_{MT,2}(2k, 2k, 2k) = \frac{4}{3}\sum_{j=0}^{k}\binom{4k-2j-1}{2k-1}\zeta(2j)\zeta(6k-2j)\,.$$

We note that Huard, Williams and Zhang [7] showed some relations between ζ and $\zeta_{MT,2}$.

On the other hand, these functions are continued meromorphically (see [1] or [23], [12]). Recently analytic properties of $\zeta_d(s_1, \ldots, s_d)$ were investigated by various authors (see, for example, [8], [10], [11], [13] [16] and [18]).

From an analytic point of view, Matsumoto suggested the following problem [14, p. 161].

MATSUMOTO'S PROBLEM. Reveal whether the various relations among the multiple zeta values are valid only at integer points, or valid also at other values as functional relations.

Matsumoto's problem may be regarded as a problem about existence or non-existence of functional relations for the various multiple zeta-functions.

It is well known that the Euler-Zagier multiple zeta-functions satisfy harmonic product formulas. By using these formulas, we can express a product of multiple zeta-functions as a linear combination of some multiple zeta-functions. For example

$$\zeta(s_1)\zeta(s_2) = \zeta_2(s_1, s_2) + \zeta_2(s_2, s_1) + \zeta(s_1 + s_2) \tag{1}$$

and

$$\begin{aligned}\zeta(s_1)\zeta_2(s_2, s_3) = \zeta_3(s_1, s_2, s_3) &+ \zeta_2(s_1 + s_2, s_3) + \zeta_3(s_2, s_1, s_3) \\ &+ \zeta_2(s_2, s_1 + s_3) + \zeta_3(s_2, s_3, s_1)\,.\end{aligned} \tag{2}$$

These formulas are valid for any complex numbers. Tsumura [21] discovered some relations between $\zeta_{MT,2}$ and ζ which are satisfied for any complex variable. Tsumura proved that the results of Huard-Williams-Zhang [7] and Subbarao-Sitaramachandrarao [20] follow from his formula. Tsumura's formula may be regarded as functional relations among the Mordell-Tornheim multiple zeta-functions and the Riemann zeta-function. Relations among various multiple zeta-functions were further studied by some mathematicians. For example, see Matsumoto and Tsumura [15], Nakamura [17], Bradley, Cai and Zhou [3] and Ikeda and Matsuoka [9].

The purpose of this paper is to formulate the problem about existence or non-existence of functional relations for the Euler-Zagier multiple zeta-functions and solve this problem.

2. Statement of the theorem

In this section we state our main theorem and some corollaries. In order to state our main result, we give some definitions.

DEFINITION 1. Let $n \in \mathbb{N}$. We define T_n as the additive semigroup freely generated by coordinate functions $s_1, \dots, s_n$ on $\mathbb{C}^n$:

$$T_n = \{t = a_1 s_1 + \cdots + a_n s_n | a_1, \dots, a_n \in \mathbb{N} \cup \{0\}, a_1 + \cdots + a_n > 0\}.$$

We define

$$T_n^d = \{\mathbf{t} = (t_1, \dots, t_d) | t_1, \dots, t_d \in T_n\}, \qquad (d \in \mathbb{N}).$$

DEFINITION 2. Let $n \in \mathbb{N}$. We define the set U_n of functions of several complex variables by

$$U_n = \{\zeta_d(\mathbf{t}) = \zeta_d(t_1, \dots, t_d) | d \in \mathbb{N}, \mathbf{t} = (t_1, \dots, t_d) \in T_n^d\}$$

and

$$\mathscr{U} = \bigcup_{n=1}^{\infty} U_n,$$

where we regard U_m as the subset of U_n naturally for $m < n$.

Note that we may write

$$U_1 = \{\zeta_d(a_1 s, \dots, a_d s) | d \in \mathbb{N}, a_1, \dots, a_d \in \mathbb{N}\}.$$

DEFINITION 3. We define

$$H(t) = \max\{a_i | t = a_1 s_1 + \cdots + a_n s_n\} \qquad (\text{for } t \in T_n).$$

We define

$$\text{height}(\mathbf{t}) = \max\{H(t_i) | \mathbf{t} = (t_1, \dots, t_d)\} \qquad (\text{for } \mathbf{t} \in T_n^d).$$

DEFINITION 4. For $\mathbf{t} = (t_1, \dots, t_d) \in T_n^d$, we define the depth of $\mathbf{t}$ by $\text{depth}(\mathbf{t}) = d$. We define the weight of $\mathbf{t} = (t_1, \dots, t_d) \in T_1^d$ by

$$\text{weight}(\mathbf{t}) = \frac{1}{s_1}(t_1 + \cdots + t_d)(\in \mathbb{N}).$$

Let $\overline{\{1\} \cup U_n}$ be the $\mathbb{C}$-algebra generated by $\{1\} \cup U_n$ in the space of meromorphic functions on $\mathbb{C}^n$. We regard Matsumoto's problem as a problem about existence or non-existence of relations among the elements of $\overline{\{1\} \cup U_n}$, and we formulate that problem in the following form.

PROBLEM. *Prove whether the following statement* (P) *is true or false.*
(P): *For all $n \in \mathbb{N}$, the set of functions of several complex variables $\{1\} \cup U_n$ is a basis of $\overline{\{1\} \cup U_n}$ as a $\mathbb{C}$-vector space.*

By the harmonic product formulas, we can express every element of $\overline{\{1\} \cup U_n}$ as a linear combination of some elements of $\{1\} \cup U_n$ over $\mathbb{C}$. If the statement (P) is true, then that expression is unique. Therefore that the statement (P) is true implies that we can obtain every relation among the elements of $\overline{\{1\} \cup U_n}$ from the harmonic product formulas.

Now we state our main theorem.

THEOREM 1. *The set of functions of one complex variable* $\{1\} \cup U_1$ *is linearly independent over* $\mathbb{C}$.

From this theorem and the following proposition, we can prove the following corollaries.

PROPOSITION 1. *Let* $t_1, t_2 \in T_m$ *with* $t_1 = a_1 s_1 + \cdots + a_m s_m$, $t_2 = b_1 s_1 + \cdots + b_m s_m$ *and* $g = \max\{H(t_1), H(t_2)\} + 1$. *If we define* $s_i = g^i s$ $(1 \le i \le m)$, *then we have*

$$t_1 = (a_1 g + a_2 g^2 + \cdots + a_m g^m)s$$

$$t_2 = (b_1 g + b_2 g^2 + \cdots + b_m g^m)s\,.$$

Then $a_1 g + a_2 g^2 + \cdots + a_m g^m = b_1 g + b_2 g^2 + \cdots + b_m g^m$ *if and only if* $a_i = b_i$ *for all* $i \in \{1, \ldots, m\}$.

PROOF. By the uniqueness of base g expansion for positive integers, we can obtain the proposition. □

COROLLARY 1. *The set of functions of several complex variables* $\{1\} \cup \mathscr{U}$ *is linearly independent over* $\mathbb{C}$.

PROOF. Suppose that we have

$$c_0 + \sum_{j \le d} \sum_{1 \le l \le L_j} c_{l,j} \zeta_j(\mathbf{t}_{l,j}) = 0\,,$$

where $m \in \mathbb{N}$, $L_j \in \mathbb{N} \cup \{0\}$, $c_0 \in \mathbb{C}$, $c_{l,j} \in \mathbb{C}$ $(1 \le l \le L_j, 1 \le j \le d)$, $\mathbf{t}_{l,j} \in T_m^j$ $(1 \le j \le d, 1 \le l \le L_j)$ and $\mathbf{t}_{k,j} \ne \mathbf{t}_{l,j}$ for $k \ne l$. Let $g = \max\{\mathrm{height}(\mathbf{t}_{l,j}) \mid 1 \le l \le L_j, 1 \le j \le d\} + 1$. By setting $s_i = g^i s$ $(1 \le i \le m)$, we have

$$c_0 + \sum_{j \le d} \sum_{1 \le l \le L_j} c_{l,j} \zeta_j(\mathbf{t}'_{l,j}) = 0\,,$$

where $\mathbf{t}'_{l,j} \in T_1^j$ $(1 \le j \le d, 1 \le l \le L_j)$. Since $\mathbf{t}'_{k,j} \ne \mathbf{t}'_{l,j}$ holds for $k \ne l$ by Proposition 1, we have $c_0 = 0$ and $c_{l,j} = 0$ for all l, j by Theorem 1. □

COROLLARY 2. *The statement* (P) *is true.*

By Corollary 1, we can define the depth of the element of $\mathscr{U}$ and the weight of the element of U_1 as follows :

$$\mathrm{depth}(\zeta_d(\mathbf{t})) = \mathrm{depth}(\mathbf{t}) = d \qquad (\mathbf{t} \in T_m^d)$$

$$\mathrm{weight}(\zeta_d(\mathbf{t})) = \mathrm{weight}(\mathbf{t}) \qquad (\mathbf{t} \in T_1^d)\,.$$

The depth and weight of the element of $\mathscr{U}$ are natural analogue of those of the multiple zeta value. We show some examples of $z \in U_1$ as follows.

	depth 1	depth 2	depth 3	depth 4
weight 1	$\zeta(s)$			
weight 2	$\zeta(2s)$	$\zeta_2(s, s)$		
weight 3	$\zeta(3s)$	$\zeta_2(s, 2s), \zeta_2(2s, s)$	$\zeta_3(s, s, s)$	
weight 4	$\zeta(4s)$	$\zeta_2(s, 3s)$, $\zeta_2(2s, 2s), \zeta_2(3s, s)$	$\zeta_3(s, s, 2s)$, $\zeta_3(s, 2s, s), \zeta_3(2s, s, s)$	$\zeta_4(s, s, s, s)$

COROLLARY 3. *Let* $Z_w = \{z \in U_1 \mid \mathrm{weight}(z) = w\}$, $V'_w = \mathrm{span}\, Z_w$ *and*

$$V_w = \mathrm{span}\left(\bigcup_{j=1}^{w} Z_j\right).$$

Then we have $\dim_{\mathbb{C}} V'_w = 2^{w-1}$, $\dim_{\mathbb{C}} V_w = 2^w - 1$, $\dim_{\mathbb{C}}(\mathrm{span}(V_w \cup \{1\})) = 2^w$ *and*

$$\mathrm{span}\, U_1 = \bigoplus_{j=1}^{\infty} V'_j .$$

In the theory of multiple zeta values, the dimension of the $\mathbb{Q}$-vector space spanned by the multiple zeta values are quite important. There are some open problems (see, for example, [22], [5]). On the other hand, we can easily calculate the dimension of the $\mathbb{C}$-vector space spanned by the elements of U_1 by Theorem 1. We can regard this result as the solution to a functional analogue of the problem about the dimension of the $\mathbb{Q}$-vector space spanned by the multiple zeta values.

3. Some definitions and lemmas

We collect some definitions and lemmas for the proof of the theorem.

DEFINITION 5. Let $a_1, a_2, \ldots, a_j \in \mathbb{N}$. We define

$$A_n(a_1, a_2, \ldots, a_j) = \#\{(q_1, q_2, \ldots, q_j) \in \mathbb{N}^j \mid q_1^{a_1} q_2^{a_2} \cdots q_j^{a_j} = n, q_1 < q_2 < \cdots < q_j\}.$$

For $\Re s > 1$, $\zeta_j(a_1 s, a_2 s, \ldots, a_j s) \in U_1$ can be expressed by the following Dirichlet series:

$$\zeta_j(a_1 s, a_2 s, \ldots, a_j s) = \sum_{n=1}^{\infty} \frac{A_n(a_1, a_2, \ldots, a_j)}{n^s}.$$

DEFINITION 6. Let $a_1, a_2, \ldots, a_j, a \in \mathbb{N}$. We define

$$B_n(a_1, a_2, \ldots, a_j; a) = \#\{(q_1, \ldots, q_j, q) \in \mathbb{N}^{j+1} \mid q_1^{a_1} \cdots q_j^{a_j} q^a = n, q_1 < q_2 < \cdots < q_j\}.$$

For $\Re s > 1$, $\zeta_j(a_1 s, a_2 s, \ldots, a_j s)\zeta(as)$ can be expressed by the following Dirichlet series:

$$\zeta_j(a_1 s, a_2 s, \ldots, a_j s)\zeta(as) = \sum_{n=1}^{\infty} \frac{B_n(a_1, a_2, \ldots, a_j; a)}{n^s}.$$

LEMMA 1. *Let $j \geq 2$, $m, N \in \mathbb{N}$ and let p be a prime. If $N < p$, then we have*

$$A_{Np^m}(a_1, a_2, \ldots, a_j) = \begin{cases} B_N(a_1, \ldots, a_{j-1}; a_j) & (m = a_j) \\ 0 & (m < a_j). \end{cases}$$

PROOF. We consider the first case. Write $Np^{a_j} = q_1^{a_1} \cdots q_j^{a_j}$, where $q_1, \ldots, q_j \in \mathbb{N}$ such that $q_1 < \cdots < q_j$. We first prove $p | q_j$. Suppose $p \nmid q_j$. Then $q_j \leq N$ and there exists $q_i (i < j)$ which satisfies $p | q_i$. Therefore $q_j \leq N < p \leq q_i$. This contradicts the supposition. Hence we have

$$\begin{aligned}
&A_{Np^{a_j}}(a_1, a_2, \ldots, a_j) \\
&= \#\{(q_1, \ldots, q_{j-1}, q_j) \in \mathbb{N}^j | q_1^{a_1} \cdots q_j^{a_j} = Np^{a_j}, q_1 < q_2 < \cdots < q_j\} \\
&= \#\{(n_1, \ldots, n_{j-1}, pn_j) \in \mathbb{N}^j | n_1^{a_1} \cdots n_j^{a_j} = N, n_1 < n_2 < \cdots < n_{j-1} < pn_j\} \\
&\qquad (\text{We set } n_1 = q_1, \ldots, n_{j-1} = q_{j-1}, q_j = pn_j.) \\
&= \#\{(n_1, \ldots, n_{j-1}, n_j) \in \mathbb{N}^j | n_1^{a_1} \cdots n_j^{a_j} = N, n_1 < n_2 < \cdots < n_{j-1} < pn_j\} \\
&= \#\{(n_1, \ldots, n_{j-1}, n_j) \in \mathbb{N}^j | n_1^{a_1} \cdots n_j^{a_j} = N, n_1 < n_2 < \cdots < n_{j-1}\}.
\end{aligned}$$

The last equation follows because we have $n_{j-1} \leq N < p \leq pn_j$ for any $n_{j-1}, n_j \leq N$. This completes the proof of the first case.

We deal with the second case. If there exists $(q_1, \ldots, q_j) \in \mathbb{N}^j$ satisfying $Np^m = q_1^{a_1} \cdots q_j^{a_j}$, then we have $q_j \leq N$ and there exists $q_i (i < j)$ which satisfies $p | q_i$. Therefore $q_j \leq N < p \leq q_i$. This contradicts the supposition. This completes the proof of the second case. □

LEMMA 2. *Let $m, N \in \mathbb{N}$ and let p be a prime. If $N < p$, then we have*

$$A_{Np^m}(a_1) = \begin{cases} A_N(a_1) & (m = a_1) \\ 0 & (m < a_1). \end{cases}$$

PROOF. The first equation follows because

$$A_{Np^{a_1}}(a_1) = \#\{q \in \mathbb{N} | q^{a_1} = Np^{a_1}\} = \#\{n \in \mathbb{N} | n^{a_1} = N\}.$$

The second equation follows because the equation $q^{a_1} = Np^m$ has no solutions in positive integers. □

LEMMA 3. *The set of functions of one complex variable* $\{1\} \cup \{\zeta(a_l s) | a_l \in \mathbb{N}\}$ *is linearly independent over* $\mathbb{C}$.

PROOF. We take the negation of the given statement. Suppose that there exists an equation

$$c_0 + \sum_{1 \le l \le L} c_l \zeta(a_l s) = 0\,,$$

where $L \in \mathbb{N}$, $c_0, \dots, c_L \in \mathbb{C}$, $c_l \neq 0 (1 \le l \le L)$ and $a_i \neq a_j$ for $i \neq j$. Let $M = \min_l a_l$ and let k be the integer which satisfies $a_k = M$. We define $n = n(N) = N p_N^M$, where p_N is the smallest prime number greater than N. By Lemma 2

$$\sum_{1 \le l \le L} c_l A_n(a_l) = c_k A_n(a_k) = c_k A_N(a_k) = 0$$

holds for any $N \in \mathbb{N}$. Hence we have

$$c_k \zeta(a_k s) = 0\,.$$

This is impossible. This completes the proof. □

REMARK 1. Alternatively we can easily prove Lemma 3 by checking a pole of $\zeta(a_l s)$.

4. Proof of the theorem

PROOF. We prove that a set of vectors

$$\bigcup_{n=1}^{m} \{\zeta_n(\mathbf{t}) \in U_1 \mid \mathbf{t} \in T_1^n\} \cup \{1\} \tag{3}$$

is linearly independent over $\mathbb{C}$ for all m. We prove this assertion by mathematical induction. For $m = 1$ we can easily check that (3) is linearly independent by Lemma 3. Suppose that (3) is linearly independent for $m \le d - 1$. If (3) is not linearly independent for $m = d$, then there exists an equation

$$c_0 + \sum_{j \le d} \sum_{1 \le l \le L_j} c_{l,j} \zeta_j(a_{l,j,1}s, a_{l,j,2}s, \dots, a_{l,j,j}s) = 0\,, \tag{4}$$

where $L_j \in \mathbb{N} \cup \{0\}$, $a_{l,j,i} \in \mathbb{N}$ $(1 \le j \le d, 1 \le l \le L_j, 1 \le i \le j)$, $c_0 \in \mathbb{C}$, $c_{l,j} \neq 0$ $(1 \le l \le L_j, 1 \le j \le d)$ and $(a_{k,j,1}s, a_{k,j,2}s, \dots, a_{k,j,j}s) \neq (a_{l,j,1}s, a_{l,j,2}s, \dots, a_{l,j,j}s)$ for $k \neq l$. From the supposition, it is justified that $L_d \neq 0$ holds. Let $M = \min_{l,j} a_{l,j,j}$. We define $n = n(N) = N p_N^M$, where p_N is the smallest prime number greater than N. By (4)

$$\sum_{j \le d} \sum_{1 \le l \le L_j} c_{l,j} A_n(a_{l,j,1}, a_{l,j,2}, \dots, a_{l,j,j}) = 0$$

holds for $N \in \mathbb{N}$. By Lemma 1 and Lemma 2

$$c_{k,1}\chi(M; a_{k,1,1})A_N(M) + \sum_{l,j} c_{l,j} B_N(a_{l,j,1}, a_{l,j,2}, \ldots, a_{l,j,j-1}; M) = 0$$

holds for $N \in \mathbb{N}$, where the summation is taken over l and j which satisfy $j \neq 1$ and $a_{l,j,j} = M$, and $a_{k,1,1} = \min_{1 \le l \le L_1}\{a_{l,1,1}\}$ and

$$\chi(M; a_{k,1,1}) = \begin{cases} 1 & (M = a_{k,1,1}) \\ 0 & (M < a_{k,1,1})\,. \end{cases}$$

Hence we have

$$c_{k,1}\chi(M; a_{k,1,1})\zeta(Ms) + \sum_{l,j} c_{l,j}\zeta_{j-1}(a_{l,j,1}s, a_{l,j,2}s, \ldots, a_{l,j,j-1}s)\zeta(Ms) = 0\,.$$

Therefore we have

$$c_{k,1}\chi(M; a_{k,1,1}) + \sum_{l,j} c_{l,j}\zeta_{j-1}(a_{l,j,1}s, a_{l,j,2}s, \ldots, a_{l,j,j-1}s) = 0\,.$$

This contradicts the supposition that (3) is linearly independent for $m \le d-1$. Hence (3) is linearly independent for $m \le d$. This completes the proof. □

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