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Algebraic Structure of the Lorentz and of the Poincaré Lie Algebras

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Abstract. We start with the Lorentz algebra $\mathfrak{L} = \mathfrak{o}_{\mathbb{R}}(1, 3)$ over the reals and find a suitable basis B such that the structure constants relative to it are integers. Thus we consider the $\mathbb{Z}$-algebra $\mathfrak{L}_{\mathbb{Z}}$ which is free as a $\mathbb{Z}$-module of which B is $\mathbb{Z}$-basis. This allows us to define the Lorentz type algebra $\mathfrak{L}_K := \mathfrak{L}_{\mathbb{Z}} \otimes_{\mathbb{Z}} K$ over any field K. In a similar way, we consider Poincaré type algebras over any field K.

In this paper we study the ideal structure of Lorentz and of Poincaré type algebras over different fields. It turns out that Lorentz type algebras are simple if and only if the ground field has no square root of -1. Thus, they are simple over the reals but not over the complex. Also, if the ground field is of characteristic 2 then Lorentz and Poincaré type algebras are neither simple nor semisimple. We extend the study of simplicity of the Lorentz algebra to the case of a ring of scalars where we have to use the notion of $\mathfrak{m}$-simplicity (relative to a maximal ideal $\mathfrak{m}$ of the ground ring of scalars).

The Lorentz type algebras over a finite field $\mathbb{F}_q$ where $q = p^n$ and p is odd are simple if and only if n is odd and p of the form $p = 4k + 3$. In case $p = 2$ then the Lorentz type algebras are not simple. Once we know the ideal structure of the algebras, we get some information of their automorphism groups. For the Lorentz type algebras (except in the case of characteristic 2) we describe the affine group scheme of automorphisms and the derivation algebras. For the Poincaré algebras we restrict this program to the case of an algebraically closed field of characteristic other than 2.

1. Introduction

The algebraic structure of Lie algebras $\mathfrak{g}$ of semi-simple algebraic groups G over an algebraically closed field of prime characteristic is the main task of the paper [13]. As the author explains in that work, the focus is centered in deviations of these algebraic objects from the characteristic zero case. In the second part of [13], the structure of $\mathrm{Aut}(\mathfrak{g})$ when G is almost-simple is determined. As one can see from the latter study, the absence of simplicity in $\mathfrak{g}$ is rather moderate. For instance, the lack of simplicity in the exceptional cases is only present when $\mathfrak{g}$ is of type G_2 with $p = 3$, of type F_4 with $p = 2$, of type E_6 with $p = 3$ and of type E_7 with $p = 2$. In the cases G_2 and F_4 there is a unique proper nonzero ideal and the structure of the algebra modulo the ideal is described in [1] and [2] respectively.

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One of the properties that one would like to have in the affine group scheme of automorphisms of a finite-dimensional algebra is that of smoothness. If Φ is an arbitrary field of characteristic zero then it is well known that any affine group scheme over Φ is smooth. If Φ is of characteristic other than 2, the automorphism group scheme of split Lie algebras of type A_r ($r \geq 2$), B_r ($r \geq 2$), C_r ($r \geq 2$) and D_r ($r = 3$ or $r \geq 5$) is smooth (see [11, p. 75, ff.]). If Φ is of characteristic $p \neq 2, 3$, the automorphism group scheme of an algebra of type G_2 is smooth ([11, p.145-146]). The previous result is also true for $p = 3$ since $\mathrm{Der}(L)$ has dimension 14 for L of type G_2 and $\mathbf{Aut}(L)$ is also 14-dimensional (one can compute at once the dimension of $\mathrm{Der}(L)$ over the field of three elements, $\mathbf{F}_3$, by using *Magma*). For $p = 2$ we have $\dim(\mathrm{Der}(L)) = 21$ (again Magma), hence the group scheme $\mathbf{Aut}(L)$ is not smooth. For the Lie algebras L of type F_4 with the characteristic $p \neq 2$ of the ground field Φ, the automorphism group scheme of L is smooth ([11, p. 196]). For $p = 2$ the scheme $\mathbf{Aut}(L)$ is still smooth because $\dim(\mathrm{Der}(L)) = 52$ as one learns, for instance, from Magma by putting $\Phi = \mathbf{F}_2$ the field of two elements. For the Chevalley algebras of types E_6, E_7 and E_8, it is known that all derivations are inner except in the cases:

1. E_6 with $p = 3$,
2. E_7 with $p = 2$.

So, except in the above cases, we have smoothness of the automorphism group scheme.

In this work we depart from the semisimple scenario since we consider two algebras which are not in general semisimple. As we will see, the Poincaré algebra has a 4-dimensional radical and the Lorentz algebra in characteristic two is not semisimple. We will consider the Lie algebras of Lorentz and Poincaré groups defined over general fields, and even on general rings. We call these algebras, Lorentz type algebras or Poincaré type algebras. The main motivation of this seeming obstinacy to work over general rings (instead of over fields) comes from one of our objectives: that of describing the algebraic group of automorphism of the Lorentz and Poincaré type algebras. Since we adhere to the idea of studying algebraic groups from the viewpoint of affine groups schemes, this implies the necessity of considering the groups $\mathrm{Aut}_R(\mathfrak{L}_R)$ of automorphisms of the scalar extensions $\mathfrak{L}_R$ where R in an associative commutative algebra over a fixed field Φ.

It is known that the Lorentz algebra $\mathfrak{L}_{\mathbb{R}}$ is simple while its complexification $\mathfrak{L}_{\mathbb{C}}$ is not. In the case in which a given algebra A is not simple, the group of automorphisms G acts on the set S of ideals of A. So G acts as a permutation group on S and this provides a certain information on G (roughly speaking, the group would be a semidirect product of a permutation group times automorphisms groups of the ideals). Thus, the ideal structure of the algebra directly influences the automorphism group of it.

In this way we arrive at the two main topics of the work: ideal structure and automorphism group. It should be mentioned that the study of the automorphism group is almost mandatory for some tasks: for instance, to classify all the possible gradings on the algebra under scope. It was in this way that we were motivated to the study of the automorphism group scheme. However the task is interesting in itself and moves a series of algebraic results, some of them linear algebra of course, but some others are of very different nature.

The study of orthogonal groups in prime characteristic is not recent. As far as we know there are papers on this matter at the end of the XIX century. One of the relevant related works that we would like to mention is that of Dickson ([9]) who, as early as in 1899, makes a detailed study of groups defined by quadratic forms in prime characteristic. As a by-product of his research, he identifies the Icosahedral group as a subgroup of a Lorentz group in characteristic 2.

The interest on physical applications of discrete Lorentz groups comes from the middle of the last century. Some scientists considered the idea of a finite Minkowski space, that is a four-dimensional space over a finite field $\mathbb{F}_p$ endowed with a suitable quadratic form. The idea is to have a Relativity Theory over a finite field such that the Theory of Special Relativity (as we know it today) arises as a limit when $p \to \infty$. In this finite theory, the Lorentz group plays an essential role. The papers [4], [5], [8] go in this direction. This poses the question on how we can relate a finite Lorentz group with the real Lorentz group. Some people have studied this question and two key words emerge: approximation and local isomorphism. Perhaps the more recent work in this line is the one by [12], dealing with approximation results of the Lorentz groups with its finite counterpart (defined over finite fields).

Some recent developments on applications of finite Lorentz groups to Signal and Image Processing, seem to be under research.

To end this introduction we give an idea of the organization of this work.

Section 1. We give some motivations for this work. We summarize some known results about ideal structure and smoothness of the affine group scheme of automorphisms of Lie algebras. In our study we want to initiate an exploration of non-semisimple algebras (as the Poincaré algebra turns out to be) but still with an eye in its group of automorphisms, seen as an affine group scheme.

Section 2. We introduce a categorical language that we will use all through the paper. When one speaks, for instance, about the algebra of $n \times n$ zero-trace matrices over rings of scalars it might be convenient to have a functor $\mathbf{sl}_n$ from the category of rings (or a suitable subcategory of this) to the category of Lie algebras. So, for any ring R we may apply the functor and consider the Lie algebra $\mathbf{sl}_n(R)$. These algebra functors allow us to use notions of category theory such as natural transformations, isomorphisms and others. We put, also in this section, some results that are known in case the ground ring of scalars is a field. However we need them in the more general setting of algebras over a ring.

Section 3. In order to study ideal structure of algebras over rings one must rule out those ideals of the algebra coming from ideals of the ring of scalars. These ideals are not present (except for the trivial ideal and the whole algebra) when the ground ring is a field. We introduce some notions to handle this situation. Roughly speaking, in Proposition 2 we prove that the unique algebra functors of type $\mathfrak{o}_{pq}$ which are not simple are $\mathfrak{o}_{13}$ and $\mathfrak{o}_{22}$. This suggests focusing our attention on the Lorentz type algebra $\mathfrak{o}_{13}$. Theorem 2 says in particular that a 6-dimensional perfect Lie algebra over a field has not ideals of dimension 4 or 5. Of course this is the case of the Lorentz type algebras and in fact, we detect 3-dimensional ideals in Proposition 4 and Theorem 3.

Section 4. In this section we consider the possibility of decomposing the Lie algebra $\mathfrak{sl}_2(R) \times \mathfrak{sl}_2(R)$ as a direct sum of ideals. We provide a more general result: If we have an algebra functor $\mathfrak{g}$ from a category of rings to a category of algebras, any decomposition of the ring of scalars as a direct sum of ideals, induces a decomposition of $\mathfrak{g}(R)$. In Theorem 5 we study the converse.

Section 5. We prove that if R is a commutative associative unital algebra with a square root of -1 and such that $1/2 \in R$, the Lorentz type algebra $\mathfrak{L}_R$ is isomorphic to $\mathfrak{sl}_2(R) \oplus \mathfrak{sl}_2(R)$. We investigate also the case $\sqrt{-1} \notin R$.

Section 6. This is a brief incursion in the case of a ground field of scalars of characteristic two. Here the Lorentz type algebra behaves in a very different way as in other characteristics: it has an ideal which as an algebra in itself, is abelian and the quotient of the algebra modulo the ideal is simple and isomorphic to the orthogonal algebra $\mathfrak{o}_3$.

Section 7. We deal with the case in which the ground field of scalars $\mathbb{F}_q$ is finite and give a full account of the simpleness of the Lorentz type algebra in terms of q.

Section 8. We study the automorphism group of Lorentz type algebra applying results of G. Benkart and E. Neher ([6, Corollary 2.28 (b)]), to obtain group scheme-theoretic versions of them. Thus we describe the group scheme of automorphisms of Lorentz type algebras.

Section 9 and 10 deal with automorphisms and derivations in the case of characteristic 2. We prove non-smoothness of the automorphism group scheme in this case.

Section 11. We focus on the Poincaré algebra and study its ideal structure.

Section 12. We prove that the derivation algebra of Poincaré algebras fits in the middle term of a certain short exact sequence. From this, we are able to describe the algebra of derivations of Poincaré type algebras.

Section 13. We consider the description of automorphisms and of the affine group scheme of automorphisms of Poincaré type algebras.

Section 14. Paraphrasing N. Jacobson [15, p. 185], in many connections in which Lie algebras arise naturally, one encounters in the prime characteristic case, structures that are somewhat richer than that of ordinary Lie algebras. This is the case of restricted Lie algebras. We discuss briefly, restricted Lorentz and Poincaré type algebras in this section.

2. Preliminary definitions

2.1. Category language. All through this paper Φ will denote an associative, commutative ring with unit and $\mathbf{alg}_\Phi$ the category whose objects are the associative, commutative and unital Φ-algebras. On the other hand, $\mathbf{Lie}_\Phi$ will denote the category of Lie Φ-algebras. We will have the occasion to deal with (covariant) functors $\mathcal{F}\colon \mathbf{alg}_\Phi \to \mathbf{Lie}_\Phi$. These functors will be called Lie algebra functors since they take values in $\mathbf{Lie}_\Phi$. Given two Lie algebra functors $\mathcal{F}, \mathcal{G}\colon \mathbf{alg}_\Phi \to \mathbf{Lie}_\Phi$ a homomorphism $\eta\colon \mathcal{F} \to \mathcal{G}$ is a natural transformation from $\mathcal{F}$ to $\mathcal{G}$, that is, a family $\{\eta_R\}$ where:

1. R ranges in the class of objects of $\mathbf{alg}_\Phi$,
2. $\eta_R\colon \mathcal{F}(R) \to \mathcal{G}(R)$ is a homomorphism of Lie Φ-algebras.
3. For any two objects R and S in $\mathbf{alg}_\Phi$ and any homomorphism of Φ-algebras $\alpha\colon R \to S$, the following squares commute:

$$\begin{tikzcd} \mathcal{F}(R) \arrow[r, "\eta_R"] \arrow[d, "\mathcal{F}(\alpha)"'] & \mathcal{G}(R) \arrow[d, "\mathcal{G}(\alpha)"] \\ \mathcal{F}(S) \arrow[r, "\eta_S"'] & \mathcal{G}(S)\,. \end{tikzcd}$$

We will say that $\mathcal{F}$ is isomorphic to $\mathcal{G}$ if all the η_R are isomorphisms (in this case we will use any of the notations $\eta\colon \mathcal{F} \cong \mathcal{G}$, $\mathcal{F} \overset{\eta}{\cong} \mathcal{G}$ or $\mathcal{F} \cong \mathcal{G}$).

DEFINITION 1. Consider next the full subcategory $\sqrt{-1}_\Phi$ of $\mathbf{alg}_\Phi$ whose objects are the Φ-algebras R such that $\sqrt{-1} \in R$. Denote by $\mathcal{I}$ the inclusion functor $\mathcal{I}\colon \sqrt{-1}_\Phi \to \mathbf{alg}_\Phi$.

2.2. The Lorentz functor. The Lorentz algebra over the reals, denoted by $\mathfrak{o}(1,3)$, is the Lie algebra of the orthogonal Lie group O(1, 3):

$$\mathfrak{o}(1,3) = \mathrm{Lie}(O(1,3)) = \{M \in \mathfrak{gl}_4(\mathbb{R})\colon MI_{13} + I_{13}M^t = 0\}\,,$$

where M^t denotes matrix transposition of M and $I_{13} = \mathrm{diag}(-1,1,1,1)$ (some authors take $I_{13} = \mathrm{diag}(1,1,1,-1)$ which is equivalent). A straightforward computation reveals that a generic element of $\mathfrak{o}(1,3)$ is of the form

$$\begin{pmatrix} 0 & x_1 & x_2 & x_3 \\ x_1 & 0 & x_4 & x_5 \\ x_2 & -x_4 & 0 & x_6 \\ x_3 & -x_5 & -x_6 & 0 \end{pmatrix}$$

and then denoting by e_{ij} the elementary matrix with 1 in the entry (i,j) and 0 elsewhere we have a basis of $\mathfrak{o}(1,3)$ given by $B = \{s_{12}, s_{13}, s_{14}, a_{23}, a_{24}, a_{34}\}$ where $s_{ij} := e_{ij} + e_{ji}$ and

[,]	s_{12}	s_{13}	s_{14}	a_{23}	a_{24}	a_{34}
s_{12}	0	a_{23}	a_{24}	s_{13}	s_{14}	0
s_{13}	$-a_{23}$	0	a_{34}	$-s_{12}$	0	s_{14}
s_{14}	$-a_{24}$	$-a_{34}$	0	0	$-s_{12}$	$-s_{13}$
a_{23}	$-s_{13}$	s_{12}	0	0	$-a_{34}$	a_{24}
a_{24}	$-s_{14}$	0	s_{12}	a_{34}	0	$-a_{23}$
a_{34}	0	$-s_{14}$	s_{13}	$-a_{24}$	a_{23}	0

FIGURE 1. Multiplication table of $\mathfrak{o}(1,3)$.

$a_{ij} = e_{ij} - e_{ji}$. Relative to this basis the structure constants are 0, 1 or -1. Thus we can construct the $\mathbb{Z}$-algebra $\mathfrak{L}_\mathbb{Z} := \mathbb{Z}s_{12} \oplus \mathbb{Z}s_{13} \oplus \mathbb{Z}s_{14} \oplus \mathbb{Z}a_{23} \oplus \mathbb{Z}a_{24} \oplus \mathbb{Z}a_{34}$ whose multiplication table is given in Figure 1. Fix now an associative, commutative and unital ring Φ and consider the category $\mathbf{alg}_\Phi$ defined above.

Then for any object R in $\mathbf{alg}_\Phi$ we may define the Lorentz type algebra $\mathfrak{L}_R := \mathfrak{L}_\mathbb{Z} \otimes_\mathbb{Z} R$. This is nothing but the free R-module with basis s_{12}, s_{13}, s_{14}, a_{23}, a_{24} and a_{34}, enriched with an R-algebra structure by the multiplication table as in Figure 1. As a free R-module we have

$$\dim \mathfrak{L}_R = 6\,.$$

Of course if we take $R = \mathbb{R}$ then $\mathfrak{L}_R \cong \mathfrak{o}(1, 3)$, the Lorentz algebra. If $R = \mathbb{C}$ then $\mathfrak{L}_R$ is the complexified Lorentz algebra. If R and S are objects in $\mathbf{alg}_\Phi$ and $f : R \to S$ a Φ-algebras homomorphism, then we may define a Lie Φ-algebras homomorphism $\mathfrak{L}_f : \mathfrak{L}_R \to \mathfrak{L}_S$ in an obvious way. Thus we have defined a covariant functor $\mathfrak{L}\colon\ \mathbf{alg}_\Phi \to \mathbf{Lie}_\Phi$ (where $\mathbf{Lie}_\Phi$ is the category of Lie Φ-algebras).

Let $\mathrm{O}(n)$ be the orthogonal Lie group over the reals: the group of all matrices M in $\mathrm{GL}_n(\mathbb{R})$ such that $MM^t = 1_n$. Then, its Lie algebra $\mathfrak{o}(n)$ consists of all matrices M in $\mathfrak{gl}_n(\mathbb{R})$ such that $M + M^t = 0$. This is generated (as a vector space) by the matrices $e_{ij} - e_{ji}$ where $i < j$ with $i, j \in \{1, \dots, n\}$ and the structure constants relative to the basis of these elements are again 0 or ± 1. Thus, we can consider as before the $\mathbb{Z}$-algebra $\mathfrak{o}(n; \mathbb{Z}) := \oplus_{i<j} \mathbb{Z}(e_{ij} - e_{ji})$. Fix as before a ring Φ and then, for any algebra R in $\mathbf{alg}_\Phi$ we may define the scalar extension $\mathfrak{o}(n; R) := \mathfrak{o}(n; \mathbb{Z}) \otimes_\mathbb{Z} R$. So, this is the Lie R-algebra with basis $e_{ij} - e_{ji}$ as before and multiplication table as the one for $\mathfrak{o}(n)$ in the corresponding basis.

Thus we have $\dim_R(\mathfrak{o}(n; R)) = n(n-1)/2$ and we have again a functor

$$\mathfrak{o}(n)\colon\ \mathbf{alg}_\Phi \to \mathbf{Lie}_\Phi$$

such that $R \mapsto \mathfrak{o}(n; R)$. If $f : R \to S$ is a homomorphism of algebras in $\mathbf{alg}_\Phi$ then we will denote by $\mathfrak{o}(n; f)\colon\ \mathfrak{o}(n; R) \to \mathfrak{o}(n; S)$ the homomorphism of Lie algebras $\mathfrak{o}(n; f) := 1 \otimes f$. Along this work, the alternative notation $\mathfrak{o}_n(R)$ (meaning $\mathfrak{o}(n; R)$) will be used eventually.

REMARK 1. If Φ is a ring agreeing with its 2-torsion, that is, $1 + 1 = 0$, then for any Φ-algebra R in $\mathbf{alg}_\Phi$, the Lie algebra $\mathfrak{o}(4; R)$ agrees with the Lorentz type Lie algebra $\mathfrak{L}_R$. In particular this is the case for a field $\mathbb{K}$ of characteristic two: $\mathfrak{L}_\mathbb{K} = \mathfrak{o}(4; \mathbb{K})$. A more general result is the following.

LEMMA 1. *For any Φ, the functors $\mathfrak{L} \circ \mathcal{I}$ and $\mathfrak{o}(4) \circ \mathcal{I} : \sqrt{-1}_\Phi \to \mathbf{Lie}_\Phi$ are isomorphic. More precisely* (i) *for any algebra R in $\mathbf{alg}_\Phi$ such that the equation $x^2 + 1 = 0$ has a solution in R, there is an isomorphism $\eta_R\colon \mathfrak{L}_R \cong \mathfrak{o}(4; R)$;* (ii) *If $f : R \to S$ is a* *homomorphism of Φ-algebras and $\sqrt{-1} \in R$, the following diagram commutes*:

$$\begin{tikzcd} \mathfrak{L}_R \arrow[r, "\eta_R"] \arrow[d, "\mathfrak{L}_f"'] & \mathfrak{o}(4;R) \arrow[d, "\mathfrak{o}(4;f)"] \\ \mathfrak{L}_S \arrow[r, "\eta_S"'] & \mathfrak{o}(4;S)\,. \end{tikzcd}$$

PROOF. Take $\mathbb{i} \in R$ such that $\mathbb{i}^2 = -1$. Starting from the standard basis B of $\mathfrak{L}_R$, we define a new basis $C = \{a'_{ij} : i, j \in \{1,2,3,4\}, i < j\}$ where $a'_{12} := \mathbb{i}s_{12}$, $a'_{13} := \mathbb{i}s_{13}$, $a'_{14} := \mathbb{i}s_{14}$ and $a'_{ij} := a_{ij}$ for the remaining elements. Then the isomorphism $\mathfrak{L}_R \to \mathfrak{o}(4; R)$ is the induced by $a'_{ij} \mapsto e_{ij} - e_{ji}$ for $i < j$. On the other hand, the commutativity of the square above is straightforward. □

For any object R of $\mathbf{alg}_\Phi$, the Lie algebra of the linear special group, $\mathfrak{sl}_2(R)$, is defined by $\mathfrak{sl}_2(R) = \{A \in \mathfrak{gl}_2(R) : \operatorname{Tr}(A) = 0\}$, where Tr denotes the matrix trace. The system $\{h := e_{11} - e_{22}, e := e_{12}, f := e_{21}\}$, where e_{ij} is the elementary matrix with 1 in the position (i, j) and 0 in the others, is a basis of $\mathfrak{sl}_2(R)$ and their elements satisfy the following identities:

$$[h, f] = -2f,\ [h, e] = 2e,\ [e, f] = h\,. \tag{1}$$

Consider now algebras R and S in the category $\mathbf{alg}_\Phi$ such that R is a subalgebra of S. Denote by $\mathfrak{sl}_2(S)$ the Lie algebra of 2×2 matrices with entries in S of zero trace. Any $\alpha \in \operatorname{Aut}_R(S)$ induces an automorphism $\hat{\alpha} \in \operatorname{Aut}_R(\mathfrak{sl}_2(S))$ by applying α componentwise. Also for any $P \in \mathrm{GL}_2(R)$ the map $M \to PMP^{-1}$ gives an automorphism of $\mathfrak{sl}_2(S)$ which is denoted by $\operatorname{Ad}(P)$. More generally, for any P in a linear algebraic group G, the adjoint action of G on its Lie algebra $\mathfrak{g}$ will be denoted $\operatorname{Ad}\colon G \to \operatorname{Aut}(\mathfrak{g})$, so that for any $M \in \mathfrak{g}$ we have $\operatorname{Ad}(P)M := PMP^{-1}$.

LEMMA 2. *Under the condition in the above paragraph if $\operatorname{Ad}(P) = \hat{\alpha}$, then $\alpha = 1$ and consequently $\operatorname{Ad}(P) = 1$.*

PROOF. We know that $PMP^{-1} = \hat{\alpha}(M)$ for any $M \in \mathfrak{sl}_2(S)$, in particular since $R \subset S$ we may take $M \in \mathfrak{sl}_2(R)$ and so $PMP^{-1} = M$ hence $PM = MP$ for any $M \in \mathfrak{sl}_2(R)$. This implies $P = k\,\mathrm{id}$ for some invertible $k \in R$. Thus $\operatorname{Ad}(P) = 1$ which implies $\hat{\alpha} = 1$. □

LEMMA 3. *For any algebra R in $\mathbf{alg}_\Phi$, if $\beta\colon \mathfrak{o}(4; R) \to R^4$ is an R-linear map such that $\beta([M, M']) = \beta(M)M' - \beta(M')M$ for any $M, M' \in \mathfrak{o}(4; R)$, then there is a unique $v \in R^4$ such that $\beta(M) = vM$ for any $M \in \mathfrak{o}(4; R)$.*

PROOF. When R is a field, this is a cohomological result (a version of Whitehead's lemma). It is well-known in characteristic zero and in prime characteristic may be seen as a consequence of [10, Theorem 1]. We include here a proof for a general algebra in $\mathbf{alg}_\Phi$.

Fix the basis of $\mathfrak{o}(4; R)$ given above (see Figure 1) and consider the coordinate map $\chi\colon \mathfrak{o}(4; R) \to R^6$ such that the components of $\chi(M)$ are the coordinates of M relative to the fixed basis. Since β is R-linear there is a 6×4 matrix L with entries in R such that $\beta(M) = \chi(M)L$. Now, one can see that the conditions $\beta([M, M']) = \beta(M)M' - \beta(M')M$ for any $M, M' \in \mathfrak{o}(4; \mathbb{K})$ imply that there are only 4 free (independent) parameters in L. Indeed, L is of the form:

$$L = \begin{pmatrix} a & b & 0 & 0 \\ c & 0 & b & 0 \\ g & 0 & 0 & b \\ 0 & c & -a & 0 \\ 0 & g & 0 & -a \\ 0 & 0 & g & -c \end{pmatrix}.$$

Now, it is immediate to check that defining $v = (b, -a, -c, g)$, one has $\chi(M)L = vM$ for any $M \in \mathfrak{o}(4; R)$. Thus $\beta(M) = vM$ for any $M \in \mathfrak{o}(4; R)$. □

3. Simplicity results

We would like to study under what conditions the Lorentz functor $\mathfrak{L}$: $\mathbf{alg}_\Phi \to \mathbf{Lie}_\Phi$ produces simple Lie algebras. The hidden motivation for this study is that when $\mathfrak{L}_R$ is not simple, under suitable conditions, we can decompose $\mathfrak{L}_R$ as a certain direct sum of two ideals. These ideals are very special and our expectation on them is that the automorphisms of $\mathfrak{L}_R$ either fix the ideals or swap them. Thus, a knowledgement of the ideal structure of $\mathfrak{L}_R$ immediately produces information about the affine group scheme $R \mapsto \mathbf{aut}(\mathfrak{L}_R)$.

To shorten the notations, we write $b_1 := a_{12}$, $b_2 := a_{13}$, $b_3 := a_{14}$, $b_4 := s_{23}$, $b_5 := s_{24}$, $b_6 := s_{34}$ so that the basis B of $\mathfrak{L}_R$ is now $B = \{b_i\}_1^6$ and has the multiplication table given in Figure 2. Also for any R in $\mathbf{alg}_\Phi$ we will denote by $\mathrm{Max}(R)$ the maximal spectrum of R (the set of maximal ideals of R).

In this section we study the simplicity of Lorentz type algebras $\mathfrak{L}_R$ where R is an algebra in $\mathbf{alg}_\Phi$. Since any ideal I of R induces trivially an ideal $I\mathfrak{L}_R$ of $\mathfrak{L}_R$ we will pay no attention to this class of ideals. One way to rule out such ideals is to rule out the ideals contained in those of the kind $\mathfrak{m}\mathfrak{L}_R$, where $\mathfrak{m} \in \mathrm{Max}(R)$ is a maximal ideal of R (since any proper ideal of R is contained in some maximal one). The ideals contained in some $\mathfrak{m}\mathfrak{L}_R$ will be termed $\mathfrak{m}$-null ideals (we will define them formally later). Other class of ideals, that we shall exclude of our study, are the $\mathfrak{m}$-total ideals (the ideals which agree with $\mathfrak{L}_R/\mathfrak{m}\mathfrak{L}_R$ when passing to the quotient).

We start by considering an algebra R in $\mathbf{alg}_\Phi$ and an R-module M. For any maximal ideal $\mathfrak{m} \in \mathrm{Max}(R)$ we may consider the epimorphism $\phi\colon M \to M \otimes_R \mathbb{K} =: M_\mathbb{K}$ where $\mathbb{K}$ is the field $\mathbb{K} := R/\mathfrak{m}$. Then $M_\mathbb{K}$ is a vector space over $\mathbb{K}$ and $\ker\phi = \mathfrak{m}M$ (see [18, Lemma 5, p.215]), so that $M/\mathfrak{m}M \cong M_\mathbb{K}$.

DEFINITION 2. A collection of elements $m_1, \dots, m_n \in M$ is said to be $\mathfrak{m}$-free if for any $r_1, \dots, r_n \in R$ the equality $\sum_i r_i m_i \in \mathfrak{m}M$ implies $r_i \in \mathfrak{m}$ for all i. An R-module M is said to have an $\mathfrak{m}$-free part of cardinal n is there is an $\mathfrak{m}$-free subset $\{m_1, \dots, m_n\} \subset M$.

If R happens to be a field, then $\mathfrak{m} = 0$ and a set is $\mathfrak{m}$-free if and only if it is linearly independent. In general, a set $\{m_1, \ldots, m_n\} \subset M$ is $\mathfrak{m}$-free if and only if the set of equivalence classes $\{\overline{m_1}, \ldots, \overline{m_n}\} \subset M/\mathfrak{m}M$ is a linearly independent subset of the $\mathbb{K}$-vector space $M/\mathfrak{m}M$.

Fix a ring Φ and a Lie Φ-algebra V. For any associative commutative and unital Φ-algebra R denote by V_R the scalar extension $V_R := V \otimes_\Phi R$. If S is another algebra in $\mathbf{alg}_\Phi$ and S is an R-algebra we may consider also the scalar extension $V_S := V \otimes_\Phi S$. The reader can easily check the existence of an isomorphism

$$V_R \otimes_R S \cong V_S$$

such that $(v \otimes_\Phi r) \otimes_R s \mapsto v \otimes_\Phi rs$.

Furthermore, if $\mathfrak{m} \in \mathrm{Max}(R)$ is a maximal ideal and I a submodule of V_R with an $\mathfrak{m}$-free part of cardinal n, then its image $\phi(I)$ under the canonical epimorphism $\phi\colon V_R \to V_R \otimes_R \mathbb{K}$ (where $\mathbb{K} = R/\mathfrak{m}$) contains a linearly independent set of cardinal n hence $\dim_{\mathbb{K}} \phi(I) \geq n$.

Definition 3. An ideal $I \lhd V_R$ such that its image under the epimorphism $\phi\colon V_R \to V_{\mathbb{K}}$ (as above) is the whole $V_{\mathbb{K}}$ is said to be $\mathfrak{m}$-total. An ideal $I \lhd V_R$ such that $\phi(I) = 0$ (equivalently $I \subset \mathfrak{m}V_R$) is said to be $\mathfrak{m}$-null. The algebra V_R is said to be $\mathfrak{m}$-simple if $V_R^2 \not\subset \mathfrak{m}V_R$ and its unique ideals are the $\mathfrak{m}$-null and the $\mathfrak{m}$-total ones.

Again, when R is a field $\mathfrak{m} = 0$ and so $\mathfrak{L}_R$ is $\mathfrak{m}$-simple if and only if it is simple in the usual sense. Thus, in our study on the simplicity of Lorentz type algebras, we will replace simplicity with $\mathfrak{m}$-simplicity.

As an example of $\mathfrak{m}$-total ideal consider the Lorentz algebra $\mathfrak{L}_{\mathbb{Z}}$, the ideal $\mathfrak{m} := 3\mathbb{Z}$ of $\mathbb{Z}$ and define $I := \mathbb{Z}(b_1+b_6)+\mathbb{Z}(b_1-b_6)+\mathbb{Z}(b_2+b_5)+\mathbb{Z}(b_2-b_5)+\mathbb{Z}(b_3+b_4)+\mathbb{Z}(b_3-b_4)$. This is an ideal of $\mathfrak{L}_{\mathbb{Z}}$. It is proper since $b_1 \notin I$ but its image in $\mathfrak{L}_{\mathbb{K}}$ (where $\mathbb{K} = \mathbb{Z}/3\mathbb{Z}$) is the whole algebra. The reader can check that $2\mathfrak{L}_{\mathbb{Z}}$ is also an $\mathfrak{m}$-total ideal of $\mathfrak{L}_{\mathbb{Z}}$ and that $3\mathfrak{L}_{\mathbb{Z}}$ is an $\mathfrak{m}$-null ideal of $\mathfrak{L}_{\mathbb{Z}}$.

Consider an algebra U over a ring Φ (commutative and unital) and assume that U is a free Φ-module with basis $\{u_i\}$. Let γ_{ij}^k be the structure constants relative to the previous basis. So $u_i u_j = \gamma_{ij}^k u_k$ (using Einstein sum convention). If we are lucky, the ideal of Φ generated by the structure constants might be the whole Φ but of course this is not the general case. However there are many interesting circumstances in which this is true.

Lemma 4. *If U is a Φ-algebra which is free as Φ-module and if $U^2 = U$, then the ideal generated by the structure constant relative to any basis of U, is Φ.*

Proof. Take a basis $\{u_i\}$ so that $u_i u_j = \gamma_{ij}^k u_k$. For any u_i in the basis we have $u_i \in U^2$ so $u_i = \lambda^{pq} u_p u_q$ for some $\lambda^{pq} \in \Phi$. Thus $u_i = \lambda^{pq} \gamma_{pq}^k u_k$ hence $1 = \lambda^{pq} \gamma_{pq}^i$ and so 1 is in the ideal generated by the structure constants. □

The condition $V_R^2 \not\subset \mathfrak{m}V_R$ in the definition of $\mathfrak{m}$-simplicity is automatically satisfied under certain mild circumstances:

LEMMA 5. *Consider an algebra U over a ring Φ (commutative and unital) and assume that U is a free Φ-module with basis $\{u_i\}$. Assume also that the ideal generated by the structure constants is Φ. Take any Φ-algebra R in $\mathbf{alg}_\Phi$ and any maximal ideal $\mathfrak{m} \in \mathrm{Max}(R)$. Then $U_R^2 \not\subset \mathfrak{m}U_R$. In particular if Φ is a field and $U^2 \neq 0$, for any R in alg_Φ and $m \in \mathrm{Max}(R)$ we have again $U_R^2 \not\subset \mathfrak{m}U_R$.*

PROOF. We have a basis $\{u_i \otimes 1\}$ of $U_R = U \otimes R$ as R-algebra, and the structure constants are $\gamma_{ij}^k \otimes 1$ which we identity with γ_{ij}^k. The ideal of R generated by the structure constants is R. If we had $U_R^2 \subset \mathfrak{m}U_R$, then $\gamma_{ij}^k \in \mathfrak{m}$ for any i, j, k. Thus $\mathfrak{m} = (1)$ which is a contradiction. □

3.1. On the $\mathfrak{m}$-simplicity of certain algebra functors. Consider an algebra U over a commutative unitary ring Φ. Take on the one hand the category $\mathbf{alg}_\Phi$, and on the other $\mathbf{Alg}_\Phi$, the category of Φ-algebras where no special identity (or unit element) is required to exist. For any R in $\mathbf{alg}_\Phi$ we may define the Φ-algebra $U_R := U_\Phi \otimes R$. This enables us to define a Φ-functor $\mathbf{U}_\Phi\colon \mathbf{alg}_\Phi \to \mathbf{Alg}_\Phi$ given by $\mathbf{U}_\Phi(R) := U_R$ where $U_R = U_\Phi \otimes R$. In some cases we will drop the index Φ and so we will speak of the algebra functor $\mathbf{U}$ associated to U.

PROPOSITION 1. *Let U be an algebra over a field Φ with $U^2 \neq 0$. Assume that for any field extension K of Φ, the K-algebra $\mathbf{U}_\Phi(K) = U_K$ is simple. Then for any algebra R in $\mathbf{alg}_\Phi$ the R-algebra $\mathbf{U}_\Phi(R) = U_R$ is $\mathfrak{m}$-simple for any $\mathfrak{m} \in \mathrm{Max}(R)$.*

PROOF. Take any algebra R in $\mathbf{alg}_\Phi$ and any maximal ideal $\mathfrak{m} \in \mathrm{Max}(R)$. To prove that U_R is $\mathfrak{m}$-simple, the first thing we need to prove is that $U_R^2 \not\subset \mathfrak{m}U_R$, but this is given by Lemma 5. Denote by $\mathbb{K}$ the residue field $\mathbb{K} := R/\mathfrak{m}$. As in previous case we use the canonical isomorphism $\phi\colon U_\mathbb{K} \cong U_R/\mathfrak{m}U_R$ to identify both algebras. By the hypothesis, $U_\mathbb{K}$ is simple so for any ideal $I \lhd U_R$ the ideal $p(I) = 0$ or $p(I) = U_R/\mathfrak{m}U_R$ being $p\colon U_R \to U_R/\mathfrak{m}U_R$ the canonical projection. In the first case I is $\mathfrak{m}$-null and in the second is $\mathfrak{m}$-total. □

Now we apply the results above to a specific kind of algebras. Consider the real orthogonal group $\mathrm{O}(p, q)$ of all matrices M in $\mathrm{GL}_n(\mathbb{R})$ (being $n = p + q$) such that $MI_{pq}M^t = I_{pq}$. Here

$$I_{pq} = \mathrm{diag}(\overbrace{-1, \dots, -1}^{p}, \overbrace{1, \dots, 1}^{q}) .$$

Its Lie algebra is $\mathfrak{o}(p, q)$ the orthogonal Lie algebra, given by all matrices M such that $MI_{pq} + I_{pq}M^t = 0$. As in previous section we can consider this algebra over any commutative unitary ring Φ and denote it by $\mathfrak{o}_{p,q}(\Phi)$. Of course the ideal generated by the structure constants in Φ is the whole Φ. Also we have a functor $\mathfrak{o}_{p,q}\colon \mathbf{alg}_\Phi \to \mathbf{Lie}_\Phi$ such that $\mathfrak{o}_{p,q}(R)$ is the scalar extension of $\mathfrak{o}_{p,q}(\Phi)$ to R. We will use the notation $\mathfrak{o}_n(\ )$ to denote $\mathfrak{o}_{n,0}(\ )$.

PROPOSITION 2. *Let Φ be a field and R any algebra in $\mathbf{alg}_\Phi$. Take any $\mathfrak{m} \in \mathrm{Max}(R)$. Then if $p + q \neq 2, 4$, the Lie algebra $\mathfrak{o}_{p,q}(R)$ is $\mathfrak{m}$-simple.*

PROOF. We may take the algebraic closure K of Φ and apply the results in [13] or in [14]. Alternatively it is an easy exercise that if $n > 4$ or $n = 3$, the algebra $\mathfrak{o}_n(K)$ is simple. For $n = 2$ the algebra $\mathfrak{o}_2(K)$ is one-dimensional. Since the scalar extension $\mathfrak{o}_{p,q}(\Phi) \otimes K \cong \mathfrak{o}_{p,q}(K) \cong \mathfrak{o}_n(K)$, and this is simple for $n \neq 2, 4$, we conclude that $\mathfrak{o}_{p,q}(\Phi)$ is simple if $p + q \neq 2, 4$. The isomorphism $\mathfrak{o}_{p,q}(K) \cong \mathfrak{o}_n(K)$ is given by $\alpha\colon \mathfrak{o}_{p,q}(K) \to \mathfrak{o}_n(K)$ such that $\alpha(M) := PMP^{-1}$ where $P = \mathrm{diag}(i, \overset{p)}{\ldots}, i, 1, \overset{q)}{\ldots}, 1)$ being $i^2 = -1$. □

This results justifies that, from a structural viewpoint, the unique algebras which may present some nontrivial ideal structure are the given by the functors $\mathfrak{o}_{1,3}$ and $\mathfrak{o}_{2,2}$.

3.2. On $\mathfrak{m}$-null and $\mathfrak{m}$-total ideals. Consider any Φ-algebra V, and an algebra R in $\mathbf{alg}_\Phi$. In special cases, it is easy to describe those ideals of V_R which are $\mathfrak{m}$-total for any $\mathfrak{m} \in \mathrm{Max}(R)$:

PROPOSITION 3. *Assume that V is a perfect algebra over Φ, that is, $V^2 = V$. Assume that R in $\mathbf{alg}_\Phi$ is artinian. Then $I \lhd V_R$ is $\mathfrak{m}$-total for any $\mathfrak{m} \in \mathrm{Max}(R)$ if and only if $I = V_R$.*

PROOF. For any $\mathfrak{m} \in \mathrm{Max}(R)$ we have $V_R = I + \mathfrak{m}V_R$. Since $V = V^2$, the same holds for V_R. If we take two ideals $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathrm{Max}(R)$, we have $V_R = I + \mathfrak{m}_1 V_R = I + \mathfrak{m}_2 V_R$ by the uniform $\mathfrak{m}$-totality of I when $\mathfrak{m} \in \mathrm{Max}(R)$. Thus $V_R = V_R^2 = I + \mathfrak{m}_1\mathfrak{m}_2 V_R$. Consequently, for any finite subset $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\} \subset \mathrm{Max}(R)$ we have $V_R = I + \mathfrak{m}_1 \cdots \mathfrak{m}_n V_R$. Since R is artinian it has only a finite number of maximal ideals (see [3, Proposition 8.3, p. 89]). Thus $V_R = I + \mathrm{rad}(R)V_R$ (where $\mathrm{rad}(\cdot)$ denotes Jacobson's radical) and since V is perfect we get $V_R = I + \mathrm{rad}(R)^k V_R$ for any positive integer k. Also the artinian character of R implies that $\mathrm{rad}(R)$ is nilpotent (take into account also that every prime ideal is maximal and so the Jacobson radical agrees with the nilradical, [3, p.89]). So, for some k one has $\mathrm{rad}(R)^k = 0$ implying $V_R = I$. □

REMARK 2. It is standard result that if A is a free R-algebra and $\{I_\alpha\}$ a collection of ideals of the ring of scalars R, then $(\cap_\alpha I_\alpha)A = \cap_\alpha(I_\alpha A)$. It is easy to see that if an ideal $I \lhd V_R$ is $\mathfrak{m}$-null for any $\mathfrak{m} \in \mathrm{Max}(R)$, then $I \subset \bigcap_{\mathfrak{m}}(\mathfrak{m}V_R) = (\bigcap_{\mathfrak{m}}\mathfrak{m})V_R = \mathrm{rad}(R)V_R$. Thus in the case in which the Jacobson radical of R is null, the unique ideal which is $\mathfrak{m}$-null for every $\mathfrak{m} \in \mathrm{Max}(R)$ is the 0 ideal.

As a consequence of Proposition 3 and of the previous paragraph we can state:

THEOREM 1. *Assume as before that V is a perfect algebra over Φ, R is an artinian algebra in $\mathbf{alg}_\Phi$, and V_R is free as an R-module. Then the following are equivalent*:

1. *V_R is $\mathfrak{m}$-simple for any $\mathfrak{m} \in \mathrm{Max}(R)$.*
2. *Any proper nonzero ideal I of V_R satisfies some of the following*:
   (a) *$I \subset rad(R)V_R$.*
   (b) *The maximal spectrum* (*which agrees with the prime spectrum and is finite*) *is not Zariski connected*: *There is a partition of* $\mathrm{Max}(R)$ *into two closed nonempty*

subsets S_1 and S_2 where $S_1 = \{\mathfrak{m}\colon I \subset \mathfrak{m}V_R\}$ and $S_2 = \{\mathfrak{m}\colon V_R = I + \mathfrak{m}V_R\}$ such that defining $\mathfrak{i} := \cap_{\mathfrak{m}\in S_1}\mathfrak{m}$, $\mathfrak{j} := \cap_{\mathfrak{m}\in S_2}\mathfrak{m}$, one has $V_R = \mathfrak{i}V_R + \mathfrak{j}V_R$, with $\mathfrak{i}\cap\mathfrak{j} = rad(R)$ and $I \subset \mathfrak{i}V_R$.

In particular if $rad(R) = 0$ the second item reduces to the assertion that any proper nonzero ideal I of V_R is of the form $I = \mathfrak{i}V_R$ where $\mathfrak{i}$ is a product of maximal ideals $\mathfrak{i} = \mathfrak{m}_{i_1}\cdots\mathfrak{m}_{i_k}$, $\mathfrak{m}_{i_q} \in \mathrm{Max}(R)$. Furthermore, V_R splits in the form $V_R = I \oplus J$ where $J = \mathfrak{j}V_R$ and $\mathfrak{j} = \mathfrak{m}_{j_1}\cdots\mathfrak{m}_{j_q}$ where each factor $\mathfrak{m}_{j_r} \in \mathrm{Max}(R)$, and $\mathrm{Max}(R)$ is the disjoint union of $\{\mathfrak{m}_{i_1},\dots,\mathfrak{m}_{i_k}\}$ and $\{\mathfrak{m}_{j_1},\dots,\mathfrak{m}_{j_q}\}$.

PROOF. Take $I \lhd V_R$ which is nonzero and proper. Since V_R is $\mathfrak{m}$-simple for any $\mathfrak{m}$, then it may not happen that I is $\mathfrak{m}$-total for any $\mathfrak{m}$ (see Proposition 3). If I happens to be $\mathfrak{m}$-null for every $\mathfrak{m} \in \mathrm{Max}(R)$, then $I \subset \cap_{\mathfrak{m}}\mathfrak{m}V_R = (\cap\mathfrak{m})V_R = \mathrm{rad}(R)V_R$. So assume that I is not $\mathfrak{m}$-null (for every $\mathfrak{m}$). Then we may decompose $\mathrm{Max}(R)$ as a disjoint union of nonempty subsets $\mathrm{Max}(R) = S_1\dot{\cup}S_2$ where $S_1 := \{\mathfrak{m} \in \mathrm{Max}(R)\colon I \text{ is } \mathfrak{m}\text{-null}\}$ and $S_2 := \{\mathfrak{m} \in \mathrm{Max}(R)\colon I \text{ is } \mathfrak{m}\text{-total}\}$. Of course $\mathfrak{m} \in S_1$ if and only if $I \subset \mathfrak{m}V_R$ and $\mathfrak{m} \in S_2$ if and only if $V_R = I + \mathfrak{m}V_R$. Also recall that $\mathrm{Max}(R)$ is finite. Now, for any $\mathfrak{m}, \mathfrak{m}' \in S_2$ with $\mathfrak{m} \neq \mathfrak{m}'$ we have $V_R = I + \mathfrak{m}V_R = I + \mathfrak{m}'V_R$ and by the perfection of V we have $V_R = I + \mathfrak{m}\mathfrak{m}'V_R$. So we conclude that

$$V_R = I + \prod_{\mathfrak{m}\in S_2}\mathfrak{m}V_R\,.$$

If $\mathfrak{j} = \prod_{\mathfrak{m}\in S_2}\mathfrak{m} = \cap_{\mathfrak{m}\in S_2}\mathfrak{m}$, then $V_R = I + \mathfrak{j}V_R$. On the other hand, $I \subset \cap_{\mathfrak{m}\in S_1}(\mathfrak{m}V_R) = (\cap_{\mathfrak{m}\in S_1}\mathfrak{m})V_R = \mathfrak{i}V_R$ where $\mathfrak{i} := \cap_{\mathfrak{m}\in S_1}\mathfrak{m} = \prod_{\mathfrak{m}\in S_1}\mathfrak{m}$. Then $\mathfrak{i}\cap\mathfrak{j} = (\cap_{\mathfrak{m}\in S_1}\mathfrak{m}) \cap (\cap_{\mathfrak{m}\in S_2}\mathfrak{m}) = \mathrm{rad}(R)$. Thus $V_R = \mathfrak{i}V_R + \mathfrak{j}V_R$.

In case $\mathrm{rad}(R) = 0$ everything follows from what we have already proved. But in this case we can show that $I = \mathfrak{i}V_R$ (we already had $I \subset \mathfrak{i}V_R$). Take $z \in \mathfrak{i}V_R$, since $V_R = I \oplus \mathfrak{j}V_R$ we have $z = i + w$ where $i \in I$ and $w \in \mathfrak{j}V_R$. Then $w = z - i \in \mathfrak{j}V_R \cap \mathfrak{i}V_R = 0$. Thus $z = i \in I$ which completes the proof. □

LEMMA 6. *For R in $\mathbf{alg}_\Phi$ such that $\frac{1}{2} \in R$, we consider the Lie R-algebra $\mathfrak{sl}_2(R)$. Let I be a proper nonzero ideal of the Lie ring $\mathfrak{sl}_2(R)$ (so we do not assume $RI \subset I$). Then I is of the form $\mathfrak{i}\,\mathfrak{sl}_2(R)$ for a nonzero proper ideal $\mathfrak{i}$ of R. Consequently:* (1) *any ideal of the Lie ring $\mathfrak{sl}_2(R)$ is an R-algebra ideal, and* (2) *any nonzero proper ideal of $\mathfrak{sl}_2(R)$ is $\mathfrak{m}$-null for some maximal ideal $\mathfrak{m} \in \mathrm{Max}(R)$.*

PROOF. This is a consequence of [13] or of [14] (if 3 is not a zero divisor in R). However a simple selfcontained proof can be given. Take $0 \neq I \lhd \mathfrak{sl}_2(R)$. We consider the usual basis $\{h, e, f\}$ of the free R-module $\mathfrak{sl}_2(R)$ such that $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$. It is easy to check that for any scalar $x \in R$ we have $xh \in I \Leftrightarrow xe \in I \Leftrightarrow xf \in I$. Then we may define $\mathfrak{i} := \{x \in R\colon xh \in I\}$ which is an ideal of R and $I = \mathfrak{i}\,\mathfrak{sl}_2(R) = \mathfrak{sl}_2(\mathfrak{i})$. Now if $\mathfrak{i} = R$, then $I = \mathfrak{sl}_2(R)$ contradicting the fact that I is proper. Thus $\mathfrak{i} \neq R$ and there is a maximal ideal $\mathfrak{m} \in \mathrm{Max}(R)$ such that $\mathfrak{i} \subset \mathfrak{m}$. Consequently $I \subset \mathfrak{m}\,\mathfrak{sl}_2(R)$ hence I is $\mathfrak{m}$-null. □

REMARK 3. Take as before $\mathfrak{m} \in \mathrm{Max}(R)$ where R is in $\mathbf{alg}_\Phi$ and let $I \lhd \mathfrak{L}_R$ be an ideal of the Lorentz type algebra $\mathfrak{L}_R$. Then I is not $\mathfrak{m}$-null if and only if it has an $\mathfrak{m}$-free subset of cardinal ≥ 1.

THEOREM 2. *Let V be a Lie Φ-algebra which is a free Φ-module of dimension* 6 *and satisfies $[V, V] = V$. Take R to be an algebra in $\mathbf{alg}_\Phi$ and $\mathfrak{m} \in \mathrm{Max}(R)$. Denote by $\mathbb{K} := R/\mathfrak{m}$ the residue field. Let $\phi\colon V_R \to V_{\mathbb{K}} := V_R \otimes_R \mathbb{K}$ be the canonical epimorphism $x \mapsto x \otimes \bar{1}$ whose kernel is $\mathfrak{m}V_R$. Assume that I is an R-submodule with an $\mathfrak{m}$-free part of cardinal* 4 *or* 5. *Then either $\phi(I) = V_{\mathbb{K}}$ or I is not an ideal of V_R. In particular, if V is a Lie algebra over a field $\mathbb{K}$ and $[V, V] = V$ with* $\dim V = 6$, *then V does not have ideals of dimension* 4 *or* 5.

PROOF. First we prove the particular case. So we take V to be a perfect Lie algebra over a field $\mathbb{K}$ and prove that V has no ideal of dimension 5 or 4 . Assume that I is a 5-dimensional ideal, then V/I has dimension 1 hence it is abelian. Thus $[V, V] \subset I$ but since $[V, V] = V$, we conclude that $V = I$ a contradiction. Next we prove that V has no ideal of dimension 4. If J is such an ideal, there is a maximal ideal M of V such that $J \subset M$. We know that M is not 5-dimensional hence $J = M$. Thus taking into account the maximality of M, we conclude that V/M is a simple Lie algebra of dimension 2. This is a contradiction because no simple Lie algebra can be 2-dimensional.

Now we prove the result for a general R in $\mathbf{alg}_\Phi$. Assume that I is an ideal of V_R with an $\mathfrak{m}$-free part of cardinal 4 or 5. Consider the R-algebra $\mathbb{K} := R/\mathfrak{m}$ which is a field. Thus $\phi(I)$ is a $\mathbb{K}$-vector subspace of $V_{\mathbb{K}}$ and if $\phi(I) \neq V_{\mathbb{K}}$, then $\dim_{\mathbb{K}} \phi(I) \in \{4, 5\}$. So $\phi(I)$ is not an ideal (hence I is not an ideal given the epimorphic character of ϕ). □

COROLLARY 1. *For any field $\mathbb{K}$, the Lorentz type algebra $\mathfrak{L}_{\mathbb{K}}$ has no ideals of dimension* 4 *or* 5.

Next we recall an elementary result on field theory: assume $\mathbb{K}$ to be a field such that $\sqrt{-1} \in \mathbb{K}$. Then putting $x := \sqrt{-1}$ and $y = 1$ one has $x^2 + y^2 = 0$ where $x, y \neq 0$. Reciprocally if $\mathbb{K}$ is a field such that there are nonzero elements $x, y \in \mathbb{K}$ such that $x^2 + y^2 = 0$, then $(y/x)^2 = -1$ and so $\sqrt{-1} \in \mathbb{K}$. Thus for a field $\mathbb{K}$ the following assertions are equivalent:

1. $\mathbb{K}$ has a square root of -1.
2. There are nonzero elements $x, y \in \mathbb{K}$ such that $x^2 + y^2 = 0$.

DEFINITION 4. We say that a field $\mathbb{K}$ is 2-formally real if for any $x, y \in \mathbb{K}$, the equality $x^2 + y^2 = 0$ implies $x = y = 0$. More generally for an ideal $\mathfrak{i}$ of a ring R we say that R is $\mathfrak{i}$-2-formally real if for any $x, y \in R$, the fact $x^2 + y^2 \in \mathfrak{i}$ implies $x, y \in \mathfrak{i}$.

For instance $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$ (with p a prime of the form $p = 4k + 3$) are 2-formally real while $\mathbb{C}$ and $\mathbb{Z}_p$ (with p a prime of the form $p = 4k+1$) are not. We will devote a section to finite fields and there, we will test the 2-formally real character of these fields. On the other hand, if D is a product of 2-formally real fields, then the ring D is $\mathfrak{m}$-2-formally real for any maximal ideal $\mathfrak{m} \triangleleft D$. We will see that in this case the Lorentz type algebra $\mathfrak{L}_D$ is $\mathfrak{m}$-simple for every maximal ideal.

Consider now the basis $\{b_i : i = 1, \dots, 6\}$ of $\mathfrak{L}_R$ such that $b_1 = s_{12}$, $b_2 = s_{13}$, $b_3 = s_{14}$, $b_4 = a_{23}$, $b_5 = a_{24}$ and $b_6 = a_{35}$. The multiplication table of $\mathfrak{L}_R$ relative to this basis is the transcription of the one in Figure 1:

[,]	b_1	b_2	b_3	b_4	b_5	b_6
b_1	0	b_4	b_5	b_2	b_3	0
b_2	$-b_4$	0	b_6	$-b_1$	0	b_3
b_3	$-b_5$	$-b_6$	0	0	$-b_1$	$-b_2$
b_4	$-b_2$	b_1	0	0	$-b_6$	b_5
b_5	$-b_3$	0	b_1	b_6	0	$-b_4$
b_6	0	$-b_3$	b_2	$-b_5$	b_4	0

FIGURE 2. Second version of table in Figure 1.

PROPOSITION 4. *Let R be a commutative unitary Φ-algebra and $\mathfrak{m} \in \max(R)$ such that R is not $\mathfrak{m}$-2-formally real, that is, there are $x, y \in R$ satisfying $x^2 + y^2 \in \mathfrak{m}$ but $x, y \notin \mathfrak{m}$. Define the elements*

$$\begin{cases} a_1 := xb_1 + yb_6, \\ a_2 := xb_2 - yb_5, \\ a_3 := xb_3 + yb_4 \end{cases}$$

of $\mathfrak{L}_R$, then $I := Ra_1 + Ra_2 + Ra_3 + \mathfrak{m}\mathfrak{L}_R$ is a nontrivial nontotal ideal of $\mathfrak{L}_R$ with an $\mathfrak{m}$-free part of cardinal 3. Consequently $\mathfrak{L}_R$ is not $\mathfrak{m}$-simple in this case.

PROOF. Since $x \notin \mathfrak{m}$ and $R/\mathfrak{m}$ is a field, there is an $x' \in R$ such that $xx' \in 1 + \mathfrak{m}$. Next we compute the products $[a_1, b_i]$ using the symbol $\equiv$ for the relation of congruence module the ideal $\mathfrak{m}\mathfrak{L}_R$:

- $[a_1, b_1] = 0$.
- $[a_1, b_2] = xb_4 - yb_3 \equiv x(b_4 - x'yb_3)$ and since $y^2 \equiv -x^2$, then $x'y^2 \equiv -x'x^2 \equiv -x$. Thus $[a_1, b_2] \equiv x(b_4 - x'yb_3) \equiv xy'(yb_4 - x'y^2b_3) \equiv xy'(yb_4 + x'x^2b_3) \equiv xy'(yb_4 + xb_3)$. So $[a_1, b_2] \in Ra_3 + \mathfrak{m}\mathfrak{L}_R$. Similarly:
- $[a_1, b_3] \in Ra_2 + \mathfrak{m}\mathfrak{L}_R$, $[a_1, b_4] = a_2$, $[a_1, b_5] = a_3$ and $[a_1, b_6] = 0$.

So far we have proved $[a_1, \mathfrak{L}_R] \subset I$ and using the same type of computations we can also prove that $[a_i, \mathfrak{L}_R] \subset I$ for $i = 2, 3$. Thus, I is a nonzero ideal of $\mathfrak{L}_R$. To prove that I is nontotal consider the epimorphism $\phi\colon \mathfrak{L}_R \to \mathfrak{L}_R \otimes_R R/\mathfrak{m}$. Let $\mathbb{K} := R/\mathfrak{m}$ be the corresponding field, then $\mathfrak{L}_R \otimes_R R/\mathfrak{m} = \mathfrak{L}_\mathbb{K}$ and $\phi(I) \subset \mathbb{K}(a_1 \otimes 1) + \mathbb{K}(a_2 \otimes 1) + \mathbb{K}(a_3 \otimes 1)$ so that $\dim_\mathbb{K} \phi(I) \leq 3$ hence I is nontotal. Finally we prove that $\{a_1, a_2, a_3\}$ is a $\mathfrak{m}$-free part: assume that $\sum r_i a_i \in \mathfrak{m}$ (we must prove that each $r_i \in \mathfrak{m}$). Then $r_1(xb_1 + yb_6) + r_2(xb_2 - yb_5) + r_3(xb_3 + yb_4) \in \mathfrak{m}\mathfrak{L}_R$ and so there are $m_i \in \mathfrak{m}$ such that $r_1(xb_1 + yb_6) + r_2(xb_2 - yb_5) + r_3(xb_3 + yb_4) = \sum m_i b_i$. Thus $r_1 x \in \mathfrak{m}$ and $\mathfrak{m}$ being a maximal ideal it is prime. Since $x \notin \mathfrak{m}$ then $r_1 \in \mathfrak{m}$. Similarly the remaining r_j's are in $\mathfrak{m}$. □

To finish this section we establish the dichotomy theorem

THEOREM 3. *Let $\mathfrak{m} \in \mathrm{Max}(R)$ be a maximal ideal. Then the Lorentz type algebra $\mathfrak{L}_R$ is $\mathfrak{m}$-simple if and only if R is $\mathfrak{m}$-2-formally real. In particular, the Lorentz type algebra $\mathfrak{L}_\mathbb{K}$ over a field $\mathbb{K}$ is simple if and only if $\mathbb{K}$ is 2-formally real (equivalently, if and only if $\sqrt{-1} \notin \mathbb{K}$).*

PROOF. If R is not $\mathfrak{m}$-2-formally real, we have seen in Proposition 4 that $\mathfrak{L}_R$ is not $\mathfrak{m}$-simple. Next we prove that in case that R is $\mathfrak{m}$-2-formally real, then the Lorentz type algebra $\mathfrak{L}_R$ is $\mathfrak{m}$-simple. For that, we start with the simpler case when R is a field $\mathbb{K}$. Under the hypothesis in the theorem we know that $\sqrt{-1} \notin \mathbb{K}$. Take the basis $B = \{b_i\}$ of $\mathfrak{L}_\mathbb{K}$ defined in the introduction, and an arbitrary nonzero element $g = \sum \lambda_i b_i$ where $\lambda_i \in \mathbb{K}$. Let $I = (g)$ be the ideal generated by g. It is easy to see that $I \ni z := g - [[g, b_1], b_1] = xb_1 + yb_6$ where $x = \lambda_1$ and $y = \lambda_6$.

- Assume that x or y is nonzero. Then, if we compute the matrix whose rows are the coordinates of $[z, b_i]$ for $i = 1, \dots, 6$, we get (removing the null rows) the matrix

$$\begin{pmatrix} 0 & 0 & -y & x & 0 & 0 \\ 0 & y & 0 & 0 & x & 0 \\ 0 & x & 0 & 0 & -y & 0 \\ 0 & 0 & x & y & 0 & 0 \end{pmatrix}, \text{ and since } \begin{vmatrix} 0 & -y & x & 0 \\ y & 0 & 0 & x \\ x & 0 & 0 & -y \\ 0 & x & y & 0 \end{vmatrix} = (x^2 + y^2)^2 \neq 0$$

  we get that the dimension of the image of $\mathrm{ad}(z)$ is 4 and taking into account Theorem 2, we have that I is the whole algebra.
- If $x = y = 0$, then $g = \sum_{i=2}^{5} \lambda_i b_i$. One can see that defining now $z := [g, b_2] = xb_1 + yb_6$ where $x = \lambda_4$ and $y = -\lambda_3$, and repeating the argument above, we get that the ideal I is the whole algebra or $x = y = 0$. In this last case $g = \lambda_2 b_2 + \lambda_5 b_5$. But defining now $z := [g, b_3] = xb_1 + yb_6$ for $x = \lambda_5$ and $y = \lambda_2$ we get that again the ideal I is the whole algebra or $x = y = 0$ which is a contradiction since $g \neq 0$.

This proves the simplicity of the Lorentz algebra if R is a field. In the general case, take an ideal $I \triangleleft \mathfrak{L}_R$. Consider as usual the homomorphism $\phi\colon \mathfrak{L}_R \to \mathfrak{L}_\mathbb{K}$ for $\mathbb{K} := R/\mathfrak{m}$ and $\phi(I)$, which is an ideal of $\mathfrak{L}_\mathbb{K}$. Since $\sqrt{-1} \notin \mathbb{K}$ the $\mathbb{K}$-algebra $\mathfrak{L}_\mathbb{K}$ is simple and so $\phi(I) = 0$ or $\phi(I) = \mathfrak{L}_\mathbb{K}$. Thus I is either $\mathfrak{m}$-null or $\mathfrak{m}$-total. This finishes the proof. □

REMARK 4. Any field $\mathbb{K}$ of characteristic two has a square root of $-1 = 1$. Therefore, the Lorentz type algebras over fields of characteristic two are not simple.

To finish this section we would like to describe those semisimple algebras R in $\mathbf{alg}_\Phi$ such that the Lorentz type algebra $\mathfrak{L}_R$ is $\mathfrak{m}$-simple for any $\mathfrak{m} \in \mathrm{Max}(R)$. Here "semisimple" means $\mathrm{rad}(R) = 0$ where $\mathrm{rad}(\cdot)$ denotes the Jacobson radical. Applying Theorem 3 the necessary and sufficient condition for this is that R must be $\mathfrak{m}$-2-formally real for any $\mathfrak{m} \in \mathrm{Max}(R)$. Since there is a monomorphism $j : R \to \prod_{\mathfrak{m}\in\mathrm{Max}(R)} R/\mathfrak{m}$ mapping any x to $(x + \mathfrak{m})_{\mathfrak{m}\in\mathrm{Max}(R)}$, we have that R is a subalgebra of a product of fields $\mathbb{K}_\mathfrak{m} := R/\mathfrak{m}$ which are 2-formally real. This is of course the usual description of R as a subdirect product of fields $\prod_\mathfrak{m} \mathbb{K}_\mathfrak{m}$, but each factor $\mathbb{K}_\mathfrak{m}$ is 2-formally real. However not every subdirect product R of 2-formally real fields gives a Lorentz type algebra $\mathfrak{L}_R$ with the property that this algebra is $\mathfrak{m}$-simple for every $\mathfrak{m}$. We need a further property that we explain in the following result:

THEOREM 4. *Let R be an algebra in $\mathbf{alg}_\Phi$ with Jacobson radical $rad(R) = 0$. Then $\mathfrak{L}_R$ is $\mathfrak{m}$-simple for any $\mathfrak{m} \in \mathrm{Max}(R)$ if and only if $R \subset \prod_{i\in I} \mathbb{K}_i$ is a subdirect product of 2-formally real field $\{\mathbb{K}_i\}_{i\in I}$ and for any $\mathfrak{m} \in \mathrm{Max}(R)$ there is some $j \in I$ such that $\pi_j(\mathfrak{m}) = 0$, being $\pi_j \colon \prod_{i\in I} \mathbb{K}_i \to \mathbb{K}_j$ the canonical projection onto the field $\mathbb{K}_j$.*

PROOF. If $\mathfrak{L}_R$ is $\mathfrak{m}$-simple for every $\mathfrak{m}$, we take $\{\mathbb{K}_i\}_{i\in I} = \{R/\mathfrak{m}\}_{\mathfrak{m}\in\mathrm{Max}(R)}$ and the property is satisfied. Reciprocally assume $R \subset \prod_i \mathbb{K}_i$ is a subdirect product of $\{\mathbb{K}_i\}$ with the additional property on the maximal ideals $\mathfrak{m}$. Let us prove that R is $\mathfrak{m}$-2-formally real for each $\mathfrak{m} \in \mathrm{Max}(R)$. If we assume $x^2 + y^2 \in \mathfrak{m}$ then, since there is some j such that $\pi_j(\mathfrak{m}) = 0$, we have $\pi_j(x)^2 + \pi_j(y)^2 = 0$ in the field $\mathbb{K}_j$ which is 2-formally real. So $\pi_j(x), \pi_j(y) = 0$ and $x, y \in \ker \pi_j = \mathfrak{m}$. □

REMARK 5. When R turns out to be a field the above necessary and sufficient condition is that R must be 2-formally real.

4. Decomposability of $\mathfrak{sl}_2(R) \times \mathfrak{sl}_2(R)$

In this section we consider the possibility of decomposing the Lie algebra $\mathfrak{sl}_2(R) \times \mathfrak{sl}_2(R)$ as a direct sum of ideals which are $\mathfrak{m}$-simple algebras for any $\mathfrak{m} \in \mathrm{Max}(R)$. Define $R^2 := R \times R$ with componentwise operations. We will identify $\mathfrak{sl}_2(R^2)$ with $\mathfrak{sl}_2(R) \times \mathfrak{sl}_2(R)$ in the standard way.

LEMMA 7. *Let I be an ideal of the Lie ring $\mathfrak{sl}_2(R^2)$, that is, $I + I \subset I$ and $[I, \mathfrak{sl}_2(R^2)] \subset I$. Assume that $\frac{1}{2} \in R$. Then there are ideals $\mathfrak{i}, \mathfrak{j} \lhd R$ such that $I = \mathfrak{sl}_2(\mathfrak{i}) \times \mathfrak{sl}_2(\mathfrak{j})$.*

PROOF. Identifying $\mathfrak{sl}_2(R^2)$ with $\mathfrak{sl}_2(R) \times \mathfrak{sl}_2(R)$, we may take $I_1 := \{x \in \mathfrak{sl}_2(R) \colon (x, 0) \in I\}$ and $I_2 := \{x \in \mathfrak{sl}_2(R) \colon (0, x) \in I\}$. Then these are ideals of $\mathfrak{sl}_2(R)$ hence by Lemma 6 there are ideals $\mathfrak{i}, \mathfrak{j} \lhd R$ such that $I_1 = \mathfrak{i}\,\mathfrak{sl}_2(R)$ and $I_2 = \mathfrak{j}\,\mathfrak{sl}_2(R)$. It is easy to see that

$\mathfrak{sl}_2(\mathfrak{i})\times\mathfrak{sl}_2(\mathfrak{j}) \subset I$. Assume now that $(a, b) \in I$, then $[(a, b), \mathfrak{sl}_2(R)\times 0] = [a, \mathfrak{sl}_2(R)]\times 0 \subset I$ hence $[a, \mathfrak{sl}_2(R)] \subset \mathfrak{i}$. From this, we can prove that $a \in \mathfrak{i}$ in the following manner: write $a = r_1 h + r_2 e + r_3 f$ where $\{h, e, f\}$ is the standard basis of $\mathfrak{sl}_2(R)$ such that $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$. Then $I \ni [[e, [a, h]], f] = [[e, -2r_2 e + 2r_3 f], f] = 2r_3[h, f] = -4r_3 f$. Thus $r_3 f \in I$. With similar arguments we prove $r_2 e, r_1 h \in I$. Thus $a \in \mathfrak{i}$. Dually, $b \in \mathfrak{j}$ and so $I \subset \mathfrak{sl}_2(\mathfrak{i}) \times \mathfrak{sl}_2(\mathfrak{j})$. □

LEMMA 8. *Let Φ be a ring and $\mathfrak{g}$ a Φ-algebra (not necessarily a Lie algebra) which is free as Φ-module. Let R be an algebra in $\mathbf{alg}_\Phi$ that decomposes as a direct sum of two ideals $R = \mathfrak{a} \oplus \mathfrak{b}$. We write $\mathfrak{g}(R) = \mathfrak{g} \otimes_\Phi R$ and $\mathfrak{g} \otimes_\Phi \mathfrak{a} = \mathfrak{g}(\mathfrak{a})$ for any ideal $\mathfrak{a} \triangleleft R$. Define $I = \mathfrak{g}(\mathfrak{a}) \times \mathfrak{g}(\mathfrak{b})$ and $J = \mathfrak{g}(\mathfrak{b}) \times \mathfrak{g}(\mathfrak{a})$. Then I and J are ideals of $\mathfrak{g}(R)^2$ such that $\mathfrak{g}(R^2) = I \oplus J$ and:*

1. *$Ann_R(I) = \{0\} = Ann_R(J)$, and*
2. *if $\mathfrak{m} \in \mathrm{Max}(R)$ is a maximal ideal of R for which $\mathfrak{g}(R)$ is a simple $R/\mathfrak{m}$-algebra, then I and J are $\mathfrak{m}$-simple.*

PROOF. Everything is straightforward but we will explicit the last assertion for the ideal I:

$$I/\mathfrak{m}I \cong \frac{\mathfrak{g}(\mathfrak{a}) \times \mathfrak{g}(\mathfrak{b})}{\mathfrak{m}\mathfrak{g}(\mathfrak{a}) \times \mathfrak{m}\mathfrak{g}(\mathfrak{b})} \cong \frac{\mathfrak{g}(\mathfrak{a})}{\mathfrak{m}\mathfrak{g}(\mathfrak{a})} \times \frac{\mathfrak{g}(\mathfrak{b})}{\mathfrak{m}\mathfrak{g}(\mathfrak{b})} \cong$$

$$(\frac{\mathfrak{g}}{\mathfrak{m}\mathfrak{g}} \otimes \mathfrak{a}) \times (\frac{\mathfrak{g}}{\mathfrak{m}\mathfrak{g}} \otimes \mathfrak{b}) \cong \frac{\mathfrak{g}}{\mathfrak{m}\mathfrak{g}} \otimes R \cong \frac{\mathfrak{g}(R)}{\mathfrak{m}\mathfrak{g}(R)}$$

and since $\mathfrak{g}(R)$ is $\mathfrak{m}$-simple, the $R/\mathfrak{m}$-algebra $\frac{\mathfrak{g}(R)}{\mathfrak{m}\mathfrak{g}(R)}$ is simple. □

THEOREM 5. *Let Φ be a ring, $\mathfrak{g}$ a perfect Φ-algebra, which is free as Φ-module, and R in $\mathbf{alg}_\Phi$ such that every perfect ideal I of $\mathfrak{g}(R)$ has the form $\mathfrak{g}(\mathfrak{a})$ for some ideal $\mathfrak{a} \triangleleft R$. Assume $\mathfrak{g}(R^2) = I \oplus J$ where $I, J \triangleleft \mathfrak{g}(R^2)$ such that $Ann_R(I) = Ann_R(J) = 0$. Then there exist ideals $\mathfrak{a}, \mathfrak{b} \triangleleft R$ such that $R = \mathfrak{a} \oplus \mathfrak{b}$, $I = \mathfrak{g}(\mathfrak{a}) \times \mathfrak{g}(\mathfrak{b})$ and $J = \mathfrak{g}(\mathfrak{b}) \times \mathfrak{g}(\mathfrak{a})$.*

PROOF. Since $\mathfrak{g}^2 = \mathfrak{g}$ then the algebras $\mathfrak{g}(R)$ and $\mathfrak{g}(R^2)$ are also perfect. Then from the decomposition $\mathfrak{g}(R^2) = I \oplus J$ we get that I and J are perfect ideals and from here $I = I \cap (\mathfrak{g}(R) \times 0) + I \cap (0 \times \mathfrak{g}(R))$ and similarly for J. From the assumption on the perfect ideals of $\mathfrak{g}(R)$ we get the existence of ideals $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \triangleleft R$ such that $I = \mathfrak{g}(\mathfrak{a}) \times \mathfrak{g}(\mathfrak{b})$ and $J = \mathfrak{g}(\mathfrak{c}) \times \mathfrak{g}(\mathfrak{d})$. But since $IJ = 0$, then $\mathfrak{g}(\mathfrak{a})\mathfrak{g}(\mathfrak{c}) = 0$ and $\mathfrak{g}(\mathfrak{b})\mathfrak{g}(\mathfrak{d}) = 0$, which implies $\mathfrak{a}\mathfrak{c} = 0$ and $\mathfrak{b}\mathfrak{d} = 0$. From here it is easy to see that $\mathrm{ann}(\mathfrak{a}) = \mathfrak{c}$, $\mathrm{ann}(\mathfrak{c}) = \mathfrak{a}$, $\mathrm{ann}(\mathfrak{b}) = \mathfrak{d}$ and $\mathrm{ann}(\mathfrak{d}) = \mathfrak{b}$. On the other hand, $R = \mathfrak{a} \oplus \mathfrak{c} = \mathfrak{b} \oplus \mathfrak{d}$. Also $0 = \mathrm{ann}(I) = \mathrm{ann}(\mathfrak{a}) \cap \mathrm{ann}(\mathfrak{b}) = \mathfrak{c} \cap \mathfrak{d}$ and similarly $0 = \mathrm{ann}(J) = \mathrm{ann}(\mathfrak{c}) \cap \mathrm{ann}(\mathfrak{d}) = \mathfrak{a} \cap \mathfrak{b}$. Thus $\mathfrak{c}\mathfrak{d} = 0$ and $\mathfrak{a}\mathfrak{b} = 0$ implying $\mathfrak{c} \subset \mathrm{ann}(\mathfrak{d}) = \mathfrak{b}$ and $\mathfrak{b} \subset \mathrm{ann}(\mathfrak{a}) = \mathfrak{c}$ hence $\mathfrak{c} = \mathfrak{b}$ and symmetrically $\mathfrak{a} = \mathfrak{d}$. So $I = \mathfrak{g}(\mathfrak{a}) \times \mathfrak{g}(\mathfrak{b})$ and $J = \mathfrak{g}(\mathfrak{b}) \times \mathfrak{g}(\mathfrak{a})$ being $R = \mathfrak{a} \oplus \mathfrak{b}$. □

REMARK 6. Along the ideas in the proof of Theorem 5, it is easy to prove the following result: If R^2 splits in form $R^2 = I \oplus J$ where $I, J \lhd R^2$ such that $\mathrm{ann}_R(I) = \mathrm{ann}_R(J) = 0$, then there are ideals $\mathfrak{a}, \mathfrak{b} \lhd R$ such that $R = \mathfrak{a} \oplus \mathfrak{b}$, $I = \mathfrak{a} \times \mathfrak{b}$ and $J = \mathfrak{b} \times \mathfrak{a}$. The ideals $\mathfrak{a}$ and $\mathfrak{b}$ are necessarily unique. It is also clear that we can apply Lemma 8 to $\mathfrak{sl}_2(\Phi)$ and then, any decomposition of $R = \mathfrak{a} \oplus \mathfrak{b}$ as s direct sum of ideals produces a decomposition $\mathfrak{sl}_2(R^2) = I \oplus J$ also as a direct sum of ideals $I = \mathfrak{sl}_2(\mathfrak{a}) \times \mathfrak{sl}_2(\mathfrak{b})$, $J = \mathfrak{sl}_2(\mathfrak{b}) \times \mathfrak{sl}_2(\mathfrak{a})$ with $\mathrm{Ann}_R(I) = \mathrm{Ann}_R(J) = 0$ and I and J being $\mathfrak{m}$-simple for any $\mathfrak{m} \in \mathrm{Max}(R)$ (taking into account Lemma 6). Reciprocally, applying Theorem 5, if $\mathfrak{sl}_2(R^2) = I \oplus J$ for some ideals with $\mathrm{Ann}_R(I) = \mathrm{Ann}_R(J) = 0$, then there is a decomposition of R as a direct sum of ideals $R = \mathfrak{a} \oplus \mathfrak{b}$ such that $I = \mathfrak{sl}_2(\mathfrak{a}) \times \mathfrak{sl}_2(\mathfrak{b})$ and $J = \mathfrak{sl}_2(\mathfrak{b}) \times \mathfrak{sl}_2(\mathfrak{a})$.

5. Lorentz type algebra over rings of characteristic other than 2

In the first part of this section we consider an algebra R in $\mathbf{alg}_\Phi$ such that $1/2 \in R$ and also $\sqrt{-1} \in R$. We have seen in Lemma 1 that under these hypothesis, the Lorentz type algebra over R is isomorphic to $\mathfrak{o}(4; R)$. We have proved also in Lemma 6 that the R-algebra $\mathfrak{sl}_2(R)$ is basically simple. We have also the following:

LEMMA 9. *Let $\frac{1}{2} \in R$ and $J \lhd \mathfrak{sl}_2(R)$ an ideal. Assume $x \in \mathfrak{sl}_2(R)$ satisfies $[x, \mathfrak{sl}_2(R)] \subset J$. Then $x \in J$.*

PROOF. Let $x = r_1h + r_2e + r_3f$ where $\{h, e, f\}$ is the standard basis of the free R-algebra $\mathfrak{sl}_2(R)$. Then $[x, h] = -2r_2e + 2r_3f \in J$ and $J \ni [e, [x, h]] = 2r_3h$ hence $r_3h \in J$ implying $r_3f \in J$. With similar arguments $r_2e \in J$ and $r_1h \in J$. Consequently $x \in J$. □

The following result is standard over algebraically closed fields of characteristic other than 2.

PROPOSITION 5. *Let R be an algebra with a square root of -1 and such that $1/2 \in R$. Then for the Lorentz type algebra $\mathfrak{L}_R$ we have*:

$$\mathfrak{L}_R \cong \mathfrak{sl}_2(R) \oplus \mathfrak{sl}_2(R)\,.$$

PROOF. Consider the Lie algebra $\mathfrak{sl}_2(R)$ with its usual basis $\{h := e_{11} - e_{22}, e := e_{12}, f := e_{21}\}$, which satisfies $[h, f] = -2f$, $[h, e] = 2e$ and $[e, f] = h$. Now, for $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$, define in $\mathfrak{o}(4)$ the elements

$$h_\alpha = -\mathbb{i}\left(a_{1,2} + a_{3,4}\right), \qquad h_\beta = \mathbb{i}\left(a_{1,2} - a_{3,4}\right),$$

$$v_\alpha = \frac{1}{2}(a_{1,3} - a_{2,4}) + \frac{\mathbb{i}}{2}(a_{1,4} + a_{2,3}), \quad v_\beta = \frac{1}{2}\left(-a_{1,3} - a_{2,4}\right) + \frac{\mathbb{i}}{2}\left(-a_{1,4} + a_{2,3}\right),$$

$$v_{-\alpha} = \frac{1}{2}\left(-a_{1,3} + a_{2,4}\right) + \frac{\mathbb{i}}{2}\left(a_{1,4} + a_{2,3}\right), \quad v_{-\beta} = \frac{1}{2}\left(a_{1,3} + a_{2,4}\right) + \frac{\mathbb{i}}{2}\left(-a_{1,4} + a_{2,3}\right),$$

where $a_{i,j} := e_{ij} - e_{ji}$. These elements verify

$$[h_\alpha, v_\alpha] = 2v_\alpha,\ [h_\alpha, v_{-\alpha}] = -2v_{-\alpha},\ [v_\alpha, v_{-\alpha}] = h_\alpha\,, \tag{2}$$

that is to say, the identities (1). The isomorphism is now clear:

$$\begin{array}{rcl} \alpha : \mathfrak{sl}_2(R) & \longrightarrow & I = \langle h_\alpha, v_\alpha, v_{-\alpha} \rangle \\ h & \longmapsto & h_\alpha \\ e & \longmapsto & v_\alpha \\ f & \longmapsto & v_{-\alpha} \end{array}$$

We do the same to prove that $J = \langle h_\beta, v_\beta, v_{-\beta} \rangle$ is also isomorphic to $\mathfrak{sl}_2(R)$. Thus, any of the 3-dimensional R-submodules $\langle h_\alpha, v_\alpha, v_{-\alpha} \rangle$ and $\langle h_\beta, v_\beta, v_{-\beta} \rangle$ is a subalgebra isomorphic to $\mathfrak{sl}_2(R)$ and they satisfy $[x, y] = 0$ for any $x \in \langle h_\alpha, v_\alpha, v_{-\alpha} \rangle$ and any $y \in \langle h_\beta, v_\beta, v_{-\beta} \rangle$. Thus, we have $\mathfrak{o}(4) = I \oplus J$ and $I, J \lhd \mathfrak{o}(4)$ (ideals of $\mathfrak{o}(4)$). Moreover, $I \cong \mathfrak{sl}_2(R) \cong J$ and so we have proved the proposition. □

REMARK 7. For a complex vector space V we are denoting by $V^{\mathbb{R}}$ its "reallification", that is the underlying real vector space of V. If A is a complex algebra, by $A^{\mathbb{R}}$ we denote the underlying real algebra of A. An easy observation is that simplicity of the complex algebra A implies simplicity of the real algebra $A^{\mathbb{R}}$. In a more general fashion assume that R is an algebra in $\mathbf{alg}_\Phi$. If S is an algebra in $\mathbf{alg}_\Phi$ and R a subalgebra of S, then for any S-algebra V we will denote by V^R the restriction of scalars algebra of V.

For any algebra R in $\mathbf{alg}_\Phi$, we may construct $\bar{R} := R \times R$ where the product in $\bar{R}$ is given by

$$(x, y)(u, v) := (xu - yv, xv + yu)\,. \tag{3}$$

We have a canonical monomorphism $R \to \bar{R}$ such that $r \mapsto (r, 0)$. Regardless of the fact that $\sqrt{-1} \in R$ or not, in the new algebra $\bar{R}$ we always have $\sqrt{-1} \in \bar{R}$. Note that if $\frac{1}{2} \in R$ and the starting algebra R has a square root of -1 then $\bar{R} \cong R \times R$ where $R \times R$ is the R-algebra with componentwise product (in fact the isomorphism $\bar{R} \to R \times R$ is $(a, b) \mapsto (a + \sqrt{-1}\, b, a - \sqrt{-1}\, b)$).

At this point we will need to use the (algebraic) groups μ_n. For any algebra R in the category $\mathbf{alg}_\Phi$, the group $\mu_n(R)$ is nothing but the group of elements $x \in R$ such that $x^n = 1$ (so $x \in R^\times$). However for future reference we must introduce this as an algebraic group in the sense of affine group schemes. Thus, denoting by **Grp** the category of groups, the affine group scheme $\boldsymbol{\mu_n} \colon \mathbf{alg}_\Phi \to \mathbf{Grp}$ is defined as the one such that for any algebra R in $\mathbf{alg}_\Phi$, we have $R \mapsto \mu_n(R) := \{x \in R^\times : x^n = 1\}$. It is an algebraic group whose representing Hopf algebra is $\Phi[x]/(x^n - 1)$.

LEMMA 10. *If $\frac{1}{2} \in R$ the automorphism group $\mathrm{Aut}_R(\bar{R})$ is isomorphic to $\mu_2(R)$ (the group of order-two elements in R). The isomorphism $\theta_R \colon \mathrm{Aut}_R(\bar{R}) \cong \mu_2(R)$ may be chosen to be natural in R: for any homomorphism of Φ-algebras $f \colon R \to S$ there is a commutative* *diagram*

$$\begin{array}{ccc} \mathrm{Aut}_S(\bar{S}) & \xrightarrow{\theta_S} & \mu_2(S) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f_*} \\ \mathrm{Aut}_R(\bar{R}) & \xrightarrow[\theta_R]{} & \mu_2(R) \end{array}$$

where f_* is the restriction of f from $\mu_2(R)$ to $\mu_2(S)$ and f^* maps any R-automorphism α of $\bar{R}$ to the S-automorphism β of $\bar{S}$ such that $\beta(0,1) = (f(x), f(y))$ where $\alpha(0,1) = (x, y)$.

PROOF. Let $\alpha \in \mathrm{Aut}_R(\bar{R})$, then $\alpha(0,1) = (a,b)$ such that $a^2 - b^2 = -1$ and $ab = 0$. Then $\alpha^{-1}(0,1) = (c,d)$ where $c^2 - d^2 = -1$ and $cd = 0$. Since $\alpha^{-1}\alpha = 1$ we have $(0,1) = \alpha^{-1}(a,b) = (a,0) + b(c,d) = (a+bc, bd)$. Thus $b, d \in R^\times$ and since $ab = 0$, we have $a = 0$ and $b^2 = 1$. Thus any automorphism α acts in the form $\alpha(0,1) = (0,b)$ where $b^2 = 1$. So the map $\mathrm{Aut}_R(\bar{R}) \to \mu_2(R)$ such that $\alpha \mapsto b$ is a group isomorphism that we can denote by θ_R. The assertion on the naturality of θ_R is straightforward. □

REMARK 8. In the language of group schemes, the above lemma says that there is an isomorphism of affine group schemes $\theta\colon \mathbf{Aut}(\bar{\Phi}) \cong \mu_2$.

LEMMA 11. *If $\frac{1}{2} \in R$ the ideals of the R-algebra $\mathfrak{sl}_2(\bar{R})^R$ are exactly the ideals of the $\bar{R}$-algebra $\mathfrak{sl}_2(\bar{R})$.*

PROOF. We have to prove that if $I \lhd \mathfrak{sl}_2(\bar{R})^R$ then $\mathbb{i}I \subset I$ where $\mathbb{i} = (0,1)$. Take $x \in I$, then $x = \alpha h + \beta e + \gamma f$ for some $\alpha, \beta, \gamma \in \bar{R}$. So $I \ni [x, \mathbb{i}h] = -2\beta\mathbb{i}e + 2\gamma\mathbb{i}f$ and also $I \ni [e, [x, \mathbb{i}h]] = 2\gamma\mathbb{i}h$ hence $I \ni [f, [e, [x, \mathbb{i}h]]] = 4\gamma\mathbb{i}f$ and so $\mathbb{i}\gamma f \in I$. □

THEOREM 6. *Let R be an algebra in $\mathbf{alg}_\Phi$ with $\frac{1}{2} \in R$. Take as before $\bar{R}$ and consider $\mathfrak{sl}_2(\bar{R})$. Then*

$$\mathfrak{L}_R \cong \mathfrak{sl}_2(\bar{R})^R\,.$$

In particular, the (real) Lorentz algebra $\mathfrak{L}_{\mathbb{R}}$ is isomorphic to the reallification of $\mathfrak{sl}_2(\mathbb{C})$.

PROOF. One of the square roots of -1 in $\bar{R}$ is $\mathbb{i} = (0,1)$ and indeed $\bar{R} = R \oplus \mathbb{i}R$ (as R-modules). Then also $\mathfrak{sl}_2(\bar{R}) = \mathfrak{sl}_2(R) \oplus \mathbb{i}\,\mathfrak{sl}_2(R)$ as R-modules. Thus, the system $\{h, e, f, \mathbb{i}h, \mathbb{i}e, \mathbb{i}f\}$ is a basis of $\mathfrak{sl}_2(\bar{R})^R$.

We can define now a new basis in the Lie algebra $\mathfrak{L}_R$, denoted by $\mathcal{B} = \{x_i\}_{i=1}^6$, as follows:

$$\begin{aligned} &x_1 := -2(\alpha + \beta + w),\ x_2 := 2(\alpha + v),\ x_3 := \alpha - u\,, \\ &x_4 := 2(\alpha - u + v),\ x_5 := 2(\beta + w),\ x_6 := \gamma + w\,, \end{aligned} \tag{4}$$

where $\{\alpha, \beta, \gamma, u, v, w\}$ is the obvious basis obtained from the generic expression of an element in $\mathfrak{o}(1,3)$:

$$\begin{aligned} \alpha = e_{12} + e_{21},\ \beta = e_{13} + e_{31},\ \gamma = e_{14} + e_{41}\,, \\ u = e_{23} - e_{32},\ v = e_{24} - e_{42},\ w = e_{34} - e_{43}\,. \end{aligned} \tag{5}$$

We can verify now that denoting $h' := \mathbb{i}h$, $e' := \mathbb{i}e$, $f' := \mathbb{i}f$, the multiplication table in $\mathfrak{sl}_2(\bar{R})^R$ is

	h	e	f	h'	e'	f'
h	0	$2e$	$-2f$	0	$2e'$	$-2f'$
e	$-2e$	0	h	$-2e'$	0	h'
f	$2f$	$-h$	0	$2f'$	$-h'$	0
h'	0	$2e'$	$-2f'$	0	$-2e$	$2f$
e'	$-2e'$	0	h'	$2e$	0	$-h$
f'	$2f'$	$-h'$	0	$-2f$	h	0

and the new one in $\mathfrak{L}_R$, with the new basis, is

	x_1	x_2	x_3	x_4	x_5	x_6
x_1	0	$2x_2$	$-2x_3$	0	$2x_5$	$-2x_6$
x_2	$-2x_2$	0	x_1	$-2x_5$	0	x_4
x_3	$2x_3$	$-x_1$	0	$2x_6$	$-x_4$	0
x_4	0	$2x_5$	$-2x_6$	0	$-2x_2$	$2x_3$
x_5	$-2x_5$	0	x_4	$2x_2$	0	$-x_1$
x_6	$2x_6$	$-x_4$	0	$-2x_3$	x_1	0

It is clear then that $\mathfrak{sl}_2(\bar{R})^R \cong \mathfrak{L}_R$. □

6. Structure of Lorentz type algebras in characteristic two

In order to study Lorentz type algebras in characteristic two we recall the functor $\mathfrak{o}(3)\colon \mathbf{alg}_\Phi \to \mathbf{Lie}_\Phi$ such that, for any algebra R in $\mathbf{alg}_\Phi$, we define $\mathfrak{o}(3; R)$ as the free R-module $\mathfrak{o}(3; R) = R(e_{12} - e_{21}) \oplus R(e_{13} - e_{31}) \oplus R(e_{23} - e_{32})$ with the Lie algebra structure induced by $[e_{12} - e_{21}, e_{13} - e_{31}] = -(e_{23} - e_{32})$, $[e_{12} - e_{21}, e_{23} - e_{32}] = e_{13} - e_{31}$ and $[e_{13} - e_{31}, e_{23} - e_{32}] = -(e_{12} - e_{21})$. We will call this, the $\mathfrak{o}(3)$-type functor.

LEMMA 12. *For a field $\mathbb{K}$ of characteristic 2, the derivation algebra of $\mathfrak{o}(3,\mathbb{K})$ is isomorphic to the Lie algebra of symmetric 3×3 matrices of zero trace with entries in $\mathbb{K}$. Hence* $\dim \operatorname{Der} \mathfrak{o}(3,\mathbb{K}) = 5$.

PROOF. The basis $b_1 := e_{12} + e_{21}$, $b_2 = e_{13} + e_{31}$ and $b_3 = e_{23} + e_{32}$ of $\mathfrak{o}(3,\mathbb{K})$ multiplies according to the rule $[b_i, b_j] = b_k$ (where $i, j, k \in \{1, 2, 3\}$ cyclically). Taking $d \in \operatorname{Der} \mathfrak{o}(3,\mathbb{K})$ and writing $d(b_i) = \sum x_{ij} b_j$, the equations $d([b_i, b_j]) = [d(b_i), b_j] +$

$[b_i, d(b_j)]$ give:

$$x_{ij} = x_{ji}, (i \neq j), \sum x_{ii} = 0\,.$$

□

PROPOSITION 6. *We consider now the Lorentz algebra* $\mathfrak{L} := \mathfrak{L}_{\mathbb{K}}$ *over fields* $\mathbb{K}$ *of characteristic two. Then* $\mathfrak{L}$ *is not simple*: *it has a three dimensional ideal* I *which is minimal and maximal*, *satisfies* $[I, I] = 0$ *and* $\mathfrak{L}/I \cong \mathfrak{o}(3; \mathbb{K})$ *which is a simple Lie algebra. This ideal is unique.*

PROOF. We consider the system $\{e_{ij} + e_{ji}\}$ such that $1 \leq i \neq j \leq 4$, basis of $\mathfrak{L}$ and denote by $b_1 := s_{12} := e_{12} + e_{21}$, $b_2 := s_{13} := e_{13} + e_{31}, \ldots, b_6 := s_{34} := e_{34} + e_{43}$. We take $I = \langle x_1, x_2, x_3 \rangle$, where $x_1 = b_1 + b_6$, $x_2 = b_2 + b_5$ and $x_3 = b_3 + b_4$. It is easy to check that I satisfies the following conditions:

- $[I, I] = 0$.
- $[I, \mathfrak{L}] \subset I$

Then I is an ideal of $\mathfrak{L}$. To prove that I is minimal we will prove that the ideal generated by any element $g \in I$ is I itself. This is trivial if $g = x_1, x_2$ or x_3. So we take a generic element $0 \neq g = \sum_i \lambda_i x_i \in I$. Denote the ideal generated by g as (g). We have $[[g, b_1], b_3] = \lambda_3 x_1$. So if $\lambda_3 \neq 0$ we have $x_1 \in (g)$ and therefore $I = (x_1) \subset (g)$ hence $(g) = I$. If on the contrary $\lambda_3 = 0$, then we consider the relation $[[g, b_1], b_2] = \lambda_2 x_1$. Thus, if $\lambda_2 \neq 0$ we have $x_1 \in (g)$ and again $I = (x_1) \subset (g)$ implying $(g) = I$. If $\lambda_2 = 0$, then $g = \lambda_1 x_1 \neq 0$, and so $(x_1) = (g) = I$. Summarizing: the ideal generated by any nonzero element in I is I itself. So I is minimal. Let us prove now that I is also maximal. For this, we will prove that the quotient algebra $\mathfrak{L}/I$ is simple. Let us denote by $\bar{x}$ the class of $x \in \mathfrak{L}$ module I. Then a basis of the quotient algebra is $\{\bar{b}_1, \bar{b}_2, \bar{b}_3\}$ and the multiplication table of this algebra is

	$\bar{b}_1$	$\bar{b}_2$	$\bar{b}_3$
$\bar{b}_1$	0	$\bar{b}_3$	$\bar{b}_2$
$\bar{b}_2$		0	$\bar{b}_1$
$\bar{b}_3$			0

On the other hand, the Lie algebra $\mathfrak{o}(3; \mathbb{K})$, as a vector space is the three-dimensional span $\mathfrak{o}(3; \mathbb{K}) = \langle e_{12} + e_{21}, e_{13} + e_{31}, e_{23} + e_{32} \rangle$ and if we make the multiplication table of $\mathfrak{o}(3; \mathbb{K})$ relative to the specified basis we will see immediately the isomorphism $\mathfrak{L}/I \cong \mathfrak{o}(3; \mathbb{K})$. Since $\mathfrak{o}(3; \mathbb{K})$ is simple we have also the maximality of the ideal I. To prove that I is unique, assume that J is a different ideal satisfying the same properties as I. Then J is also minimal and maximal, so that $\mathfrak{L} = I \oplus J$. Consequently

$$[\mathfrak{L}, \mathfrak{L}] = [I + J, I + J] = [I, I] + [I, J] + [J, J] = 0$$

which is a contradiction. □

7. Lorentz algebra over finite fields

In this section we investigate the simplicity of $\mathfrak{o}(1,3)$ over a finite field $\mathbb{F}_q$. Specifically, we consider the Lie algebra

$$L := \mathfrak{L}_{\mathbb{F}_q} = \left\{ \begin{pmatrix} 0 & x & y & z \\ x & 0 & s & t \\ y & -s & 0 & u \\ z & -t & -u & 0 \end{pmatrix} : x, y, z, s, t, u \in \mathbb{F}_q \right\}.$$

We know that q must be a power $q = p^n$ of some prime number p (and $n > 0$). Since we have already studied the Lorentz type algebra over fields of characteristic 2 we assume that p is odd. By Theorem 3 the simplicity of L is equivalent to the fact that the field $\mathbb{F}_q$ has no square root of -1.

We start this subsection investigating under what conditions a finite field $\mathbb{F}_q$ (of characteristic other than 2) has a square root of -1. This is included here only for selfcontainedness reasons.

LEMMA 13. *If q is a positive integer, the factorization $x^q - x = (x^2+1)(x^{q-2} - x^{q-4} + \cdots + x^3 - x)$ is only possible when q is of the form $q = 4n+1$.*

PROOF. If $q = 4n+1$ we can define the polynomial

$$\sum_{k=1}^{2n} (-1)^{k+1} x^{q-2k} = x^{q-2} - x^{q-4} + \cdots + x^3 - x$$

and the factorization holds (observe that we have used the fact that q is of the form $4n+1$). On the other hand, if $x^q - x$ is divisible by x^2+1, then the quotient is $x^{q-2} - x^{q-4} + \cdots + x^3 - x$, so that the different summands are $(-1)^{k+1}x^{q-2k}$. Equating $(-1)^{k+1}x^{q-2k} = -x$ we get that k must be an even number $k = 2n$ and $q - 2k = 1$. Hence $q = 4n+1$. □

PROPOSITION 7. *Let $q = p^n$ where p is an odd prime number, then $\mathbb{F}_q$ contains a square root of -1 if and only if q is of the form $q = 4n+1$.*

PROOF. Recall that $\mathbb{F}_q$ is the splitting field of $x^q - x$ over $\mathbb{Z}_p$. Thus all the elements in $\mathbb{F}_q$ satisfy $x^q - x = 0$. If $q = 4n+1$ the factorization in Lemma 13 holds and so there is a square root of -1 in the field. Reciprocally, if $\sqrt{-1} \in \mathbb{F}_q$ then there is an element $q \in \mathbb{F}_q^\times$ of order 4 (of course $q = \sqrt{-1}$). Therefore the order of the group $\mathbb{F}_q^\times$ is a multiple of 4. But this order is $q-1$ whence q is of the form $4n+1$. □

COROLLARY 2. *The Lorentz type algebra $\mathfrak{L}_{\mathbb{Z}_p}$ over the field $\mathbb{Z}_p$ is simple if and only if p is odd and of the form $p = 4k+3$.*

Finally we investigate when is an odd prime power p^n of the form $4n+1$.

PROPOSITION 8. *Let p be an odd prime number. Then*:

- *If $p = 4k + 1$, then p^n is also of the form $4m + 1$.*
- *If $p = 4k + 3$, then p^n is of the form $4m + 1$ if and only if n is even.*

PROOF. Define A to be the positive integers of the form $4n+1$ and B those of the form $4n + 3$. Since A is closed under multiplication the first assertion is trivial. On the other hand $BB \subset A$ and $AB \subset B$ hence multiplying an even number of elements of B we get an element of A:

$$\overbrace{B \cdots B}^{2k} \subset A, \quad \text{and} \quad \overbrace{B \cdots B}^{2k+1} \subset B\,.$$

□

COROLLARY 3. *Consider the Lorentz type algebra $\mathfrak{L}_{\mathbb{F}_q}$ over a finite field $\mathbb{F}_q$ where $q = p^n$ and p is odd. Then $\mathfrak{L}_{\mathbb{F}_q}$ is simple if and only if n is odd and p of the form $p = 4k+3$.*

8. Automorphisms and derivations of the Lorentz type algebra $\mathfrak{L}_R$ if $\frac{1}{2} \in R$

In this section we pursue to describe the algebraic group $\mathbf{Aut}(\mathfrak{L}_\Phi)$ when $\frac{1}{2} \in \Phi$. So consider the category of groups **Grp** and the group functor $\mathbf{Aut}(\mathfrak{L}_\Phi)\colon \mathbf{alg}_\Phi \to \mathbf{Grp}$ such that $R \mapsto \mathrm{Aut}_R(\mathfrak{L}_R)$. By section 5, Theorem 6, we know that $\mathfrak{L}_R$ can be identified with $\mathfrak{sl}_2(\bar{R})^R$. Now there are two possibilities for $\bar{R}$: if $\sqrt{-1} \in R$ then $\bar{R} \cong R^2 = R \times R$ with componentwise product. In this case $\mathfrak{L}_R$ can be identified with $\mathfrak{sl}_2(R^2)^R \cong \mathfrak{sl}_2(R)^2$ (as R-algebra). If $\sqrt{-1} \notin R$ then $\bar{R}$ is an R-algebra with $\sqrt{-1} \in \bar{R}$. In any case

$$\mathfrak{L}_R = \mathfrak{sl}_2(\bar{R})^R \cong \mathfrak{sl}_2(R) \otimes_R \bar{R} \tag{6}$$

and we would like to describe $\mathrm{Aut}_R(\mathfrak{L}_R)$ in terms of the automorphism group $\mathrm{Aut}_{\bar{R}}(\mathfrak{sl}_2(\bar{R}))$ (automorphisms of $\bar{R}$-algebras) and of $\mathrm{Aut}_R(\bar{R})$. In order to do that we may get inspired by the result :

THEOREM 7 ([6, Corollary 2.28 (b)]). *Let A be a perfect, central algebra over a field Φ and B be a unital commutative associative Φ-algebra. Then, after identifying* $\mathrm{Aut}_\Phi(B)$ *with a subgroup of* $\mathrm{Aut}_\Phi(A \otimes_\Phi B)$ *via $g \mapsto id \otimes g$, we have* $\mathrm{Aut}_\Phi(A \otimes_\Phi B) = \mathrm{Aut}_B(A \otimes_\Phi B) \rtimes \mathrm{Aut}_\Phi(B)$ (*semidirect product*).

But since our aim is the description of the algebraic group $\mathbf{Aut}(\mathfrak{L}_\Phi)$, we need to translate the mentioned result of [6] to the setting of algebraic groups (in affine group schemes ambient).

8.1. On a result of G. Benkart and E. Neher for algebraic groups. In this subsection we give a version of the result of G. Benkart and E. Neher ([6, Corollary 2.28 (b)]) for algebraic groups. Let Φ be a field, A a Φ-algebra and B an algebra in $\mathbf{alg}_\Phi$. We consider the following group functors:

1. $\mathbf{Aut}_\Phi(A \otimes B)\colon \mathbf{alg}_\Phi \to \mathbf{Grp}$ such that $R \mapsto \mathrm{Aut}_R(A \otimes B \otimes R) = \mathrm{Aut}_R(A_{R_B})$, where $\otimes$ denotes $\otimes_\Phi$.
2. $\mathbf{Aut}_\Phi(B)\colon \mathbf{alg}_\Phi \to \mathbf{Grp}$ such that $R \mapsto \mathrm{Aut}_R(B_R) = \mathrm{Aut}_R(R_B)$.
3. $\mathbf{Aut}_B(A \otimes B)\colon \mathbf{alg}_B \to \mathbf{Grp}$ such that $S \mapsto \mathrm{Aut}_S((A \otimes B) \otimes_B S)$.
4. Composing the functor $\mathbf{alg}_\Phi \to \mathbf{alg}_B$ such that $R \mapsto R \otimes B$ with $\mathbf{Aut}_B(A \otimes B)$, we get a group functor $\mathcal{R}\colon \mathbf{alg}_\Phi \to \mathbf{Grp}$ such that $\mathcal{R}(R) = \mathrm{Aut}_{R_B}((A \otimes B) \otimes_B R_B)$. This is a corestriction functor $\mathcal{R} = \mathcal{R}_{B/\Phi}(\mathbf{Aut}_B(A \otimes B))$ (see [16, p. 329]). Taking into account that there is a canonical isomorphism $A_B \otimes_B R_B \cong A_{R_B}$ of R_B-algebras, we may assume that under the suitable identification, $\mathcal{R}(R) = \mathrm{Aut}_{R_B}(A_{R_B})$.

Of course we have a monomorphism of group functors $j\colon \mathbf{Aut}_\Phi(B) \to \mathbf{Aut}_\Phi(A \otimes B)$ such that for any R in $\mathbf{alg}_\Phi$, the map $j_R\colon \mathrm{Aut}_R(B_R) \to \mathrm{Aut}_R(A \otimes B_R)$ is given by $z \mapsto 1 \otimes z$. There is also a homomorphism of group functors $i\colon \mathcal{R} \to \mathbf{Aut}_\Phi(A \otimes B)$ such that for any R in $\mathbf{alg}_\Phi$, we have $i_R\colon \mathrm{Aut}_{R_B}(A_{R_B}) \to \mathrm{Aut}_R(A_{R_B})$ where for any R_B-automorphism f of A_{R_B}, we have $i_R(f) = f$ considered as an R-automorphism of A_{R_B}. Since i is also a monomorphism of group functors we have two subgroups $\mathbf{Aut}_\Phi(B)$ (identified with the image of j) and $i(\mathcal{R})$ of $\mathbf{Aut}_\Phi(A \otimes B)$.

LEMMA 14. *For any R in $\mathbf{alg}_\Phi$ we have $im(i_R) \cap im(j_R) = 1$.*

PROOF. Assume $1 \otimes f \in \mathrm{Aut}_R(A_{B_R}) \cap \mathrm{Aut}_{R_B}(A_{R_B})$ where $f \in \mathrm{Aut}_R(B_R)$. Then, since $1 \otimes f \in \mathrm{Aut}_{R_B}(A_{R_B})$ we have

$$(1 \otimes f)(a \otimes r \otimes b) = (1 \otimes f)[(r \otimes b)(a \otimes 1 \otimes 1)] = (r \otimes b)(1 \otimes f)(a \otimes 1 \otimes 1)$$

$$(r \otimes b)(a \otimes 1 \otimes 1) = a \otimes r \otimes b\,.$$

□

For any algebra R in $\mathbf{alg}_\Phi$, there is an action of $\mathrm{im}(j_R)$ on $\mathrm{im}(i_R)$ by automorphisms. To be more precise we may define $\varphi\colon \mathrm{im}(j_R) \to \mathrm{Aut}(\mathrm{im}(i_R))$ such that for any $f \in \mathrm{Aut}_R(B_R)$, we have $\varphi(1 \otimes f)\colon \mathrm{im}(i_R) \to \mathrm{im}(i_R)$ given by $\varphi(1 \otimes f)(g) = (1 \otimes f)g(1 \otimes f^{-1})$ for any $g \in \mathrm{Aut}_{R_B}(A_{R_B})$. It is not difficult to realize that this map is well defined. So we may consider the semidirect product

$$\mathrm{im}(j_R) \rtimes \mathrm{im}(i_R)$$

with multiplication

$$(g_1, 1 \otimes f_1)(g_2, 1 \otimes f_2) = (g_1(1 \otimes f_1)g_2(1 \otimes f_1^{-1}), 1 \otimes f_1 f_2)\,,$$

and the map $\Omega_R\colon \mathrm{im}(j_R) \rtimes \mathrm{im}(i_R) \to \mathrm{Aut}_R(A_{R_B})$ such that $(g, 1 \otimes f) \mapsto g(1 \otimes f)$. This map is a group homomorphism and a monomorphism by Lemma 14. So we have a group monomorphism Ω. If we want to make of Ω_R an epimorphism, we need extra hypothesis. So assume that A is perfect and central (see [6, Subsection 2.1] for definitions). We must also take into account item (b) of [6, Corollary 2.23] according to which, the centralizer of the tensor product $A \otimes R_B$ is $R_B \cong B_R$. Thus, assume $h \in \mathrm{Aut}_R(A_{R_B})$, then this h induces an automorphism of the centralizer $C(A_{R_B})$ of A_{R_B} as Φ-algebra. Indeed, the induced automorphism is $\bar{h}\colon C(A_{R_B}) \to C(A_{R_B})$ where $\bar{h}(T) = hTh^{-1}$ for any $T \in C(A_{R_B})$. Since $C(A_{R_B}) \cong B_R$ we may consider $\bar{h} \in \mathrm{Aut}_R(B_R)$. On the other hand h is $\bar{h}$-semilinear in the sense that $h(T \cdot a) = \bar{h}(T) \cdot h(a)$ for any $a \in A$ and $T \in C(A_{R_B})$. Since $1 \otimes \bar{h}^{-1}$ is $\bar{h}^{-1}$ semilinear, the composition $g := h(1 \otimes \bar{h}^{-1})$ is an R_B-linear automorphism of A_{R_B}. Moreover, $\Omega_R(g, 1 \otimes \bar{h}) = h$. Summarizing:

PROPOSITION 9. *Let A be a central perfect Lie algebra over a field Φ. If R and B are algebras in $\mathbf{alg}_\Phi$ then there is a group isomorphism $\Omega_R\colon \mathrm{Aut}_R(B_R) \rtimes \mathrm{Aut}_{R_B}(A_{R_B}) \to \mathrm{Aut}_R(A_{R_B})$ which is natural in R.*

Now we interpret this result in terms of an isomorphism of group functors. More precisely we consider the group functors $\mathbf{Aut}_\Phi(A \otimes B)$, $\mathbf{Aut}_\Phi(B)$ and the corestriction group functor $\mathcal{R}$ previously defined. Define

$$\Omega\colon \mathbf{Aut}_\Phi(B) \rtimes \mathcal{R} \to \mathbf{Aut}_\Phi(A \otimes B)$$

where for each algebra R in $\mathbf{alg}_\Phi$, the map Ω_R is the previously defined. Then, this is an isomorphism of group functors (and if B is a finite-dimensional Φ-algebra, Ω is an isomorphism of algebraic groups in the sense of affine group schemes with finitely generated representing Hopf algebra).

Now we are willing to apply the isomorphism

$$\Omega\colon \mathbf{Aut}_\Phi(B) \rtimes \mathcal{R} \cong \mathbf{Aut}_\Phi(A \otimes B) \tag{7}$$

taking $B = \bar{\Phi}$ (recall 3)) and $A = \mathfrak{sl}_2(\Phi)$. Thus we get

THEOREM 8. *For the algebraic group of the Lorentz type algebra $\mathfrak{L}_\Phi$ we have*

$$\mathbf{Aut}_\Phi(\mathfrak{L}_\Phi) \cong \mathbf{Aut}_\Phi(\mathfrak{sl}_2(\bar{\Phi}^\Phi) \cong \mathbf{Aut}_\Phi(\mathfrak{sl}_2(\Phi) \otimes \bar{\Phi}) \cong \mathbf{Aut}_\Phi(\bar{\Phi}) \rtimes \mathcal{R}\,. \tag{8}$$

Thus the description of the algebraic group $\mathbf{Aut}(\mathfrak{L}_\Phi)$ will be fully achieved once we study:

1. The algebraic group $\mathbf{Aut}_\Phi(\bar{\Phi})\colon \mathbf{alg}_\Phi \to \mathbf{Grp}$, such that $R \mapsto \mathrm{Aut}_R(\bar{R})$. This has been described in Lemma 10 and turns out to be isomorphic to μ_2.
2. The corestriction group functor $\mathcal{R}\colon \mathbf{alg}_\Phi \to \mathbf{Grp}$ such that for any R in $\mathbf{alg}_\Phi$, we have $R \mapsto \mathrm{Aut}_{R_B}(A_{R_B})$. Since $B = \bar{\Phi}$ we have $R_B \cong \bar{R}$ and so $A_{R_B} \cong \mathrm{Aut}_{\bar{R}}(sl_2(\bar{R}))$. So we may take the corestriction functor to be $\mathcal{R}\colon R \mapsto \mathrm{Aut}_{\bar{R}}(\mathfrak{sl}_2(\bar{R}))$.

Consequently we must focus on $\mathrm{Aut}_R(\mathfrak{sl}_2(R))$ for any R in $\mathbf{alg}_\Phi$ (so the conclusions of this description will be applied to $\mathrm{Aut}_{\bar{R}}(\mathfrak{sl}_2(\bar{R}))$). Our philosophy is that the natural ambient to study $\mathrm{Aut}_R(\mathfrak{sl}_2(R))$ is that of the algebraic group $\mathbf{Aut}(\mathfrak{sl}_2(\Phi))\colon \mathbf{alg}_\Phi \to \mathbf{Grp}$ which is the affine group scheme $R \mapsto \mathrm{Aut}_R(\mathfrak{sl}_2(R))$. For this we will need the paraphernalia in the next paragraph.

Consider as before, the category of groups **Grp** and the affine group scheme

$$\mathbf{PGL}_n\colon \mathbf{alg}_\Phi \to \mathbf{Grp}$$

such that $\mathbf{PGL}_n := \mathbf{GL}_n \,/\, \mathbf{G_m}$ (see [11, Appendix A.2, p. 307]), where $\mathbf{G_m}(R) := R^\times$ is the affine group scheme whose associated Hopf algebra is $\Phi[x, x^{-1}]$ (Laurent polynomials in x) and $\mathbf{GL}_n$ the general linear affine group scheme. Following [11], we have a morphism $\mathbf{Ad}\colon \mathbf{GL}_n \to \mathbf{GL}(M_n(\Phi))$ whose kernel is $\mathbf{G_m}$ and its image $\mathbf{PGL}_n$. Composing the injection $\mathbf{PGL}_n \to \mathbf{GL}(M_n(\Phi))$ with the canonical morphism $\mathbf{GL}(M_2(\Phi)) \to \mathbf{Aut}(\mathfrak{sl}_2(\Phi))$ we have a morphism $\mathbf{PGL}_2 \to \mathbf{Aut}(\mathfrak{sl}_2(\Phi))$ which by abuse of notation we also denote by $\mathbf{Ad}\colon \mathbf{PGL}_2 \to \mathbf{Aut}(\mathfrak{sl}_2(\Phi))$. The following result is a consequence of [11, Theorem 3.9, p.77] and, as we have been informed, can be traced to a 1961 paper by Steinberg ([19]).

LEMMA 15. *If Φ is a field of characteristic other than 2, the adjoint map* **Ad** *is an isomorphism of algebraic group schemes* $\mathbf{PGL}_2 \to \mathbf{Aut}(\mathfrak{sl}_2(\Phi))$.

PROOF. We know that $\dim \mathbf{Aut}(\mathfrak{sl}_2(\Phi)) = 3$ hence this is a smooth affine group scheme. We know that the differential $\mathrm{ad}\colon \mathrm{pgl}_2(\Phi) \to \mathrm{Der}(\mathfrak{sl}_2(\Phi))$ of **Ad** is an isomorphism (under the hypothesis in the Lemma $\mathrm{pgl}_2(\Phi) \cong \mathfrak{sl}_2(\Phi)$). If K is the algebraic closure of Φ, applying [19, 4.6, 4.7, p. 1123] and [16, Proposition (22.5), p. 340] we get the required isomorphism. □

As a corollary $\mathrm{Aut}(\mathfrak{sl}_2(R)) \cong \mathrm{PGL}_2(R)$ for any algebra R in $\mathbf{alg}_\Phi$ and in particular: $\mathrm{Aut}(\mathfrak{sl}_2(\Phi)) \cong \mathrm{PGL}_2(\Phi)$ for a field of characteristic other than 2, and $\mathrm{Der}(\mathfrak{sl}_2(\Phi)) \cong \mathfrak{sl}_2(\Phi)$ applying the Lie functor. Of course this allows us to explicit the group of automorphisms of the Lorentz algebra over fields of characteristic not two.

Recalling the isomorphism

$$\mathbf{Aut}_\Phi(\mathfrak{L}_\Phi) \cong \mathbf{Aut}_\Phi(\bar{\Phi}) \rtimes \mathcal{R} \cong \mu_2 \rtimes \mathcal{R}$$

given in (8) of Theorem 8, we get $\mathrm{Aut}_R(\mathfrak{L}_R) \cong \mathrm{Aut}_R(\bar{R}) \rtimes \mathrm{Aut}_{\bar{R}}(\mathfrak{sl}_2(\bar{R}))$, for any R in $\mathbf{alg}_\Phi$. Taking into account Lemma 15 and Lemma 10, we get $\mathrm{Aut}_R(\mathfrak{L}_R) \cong \mathrm{PGL}_2(\bar{R}) \rtimes \mu_2(R)$. Now, if $\sqrt{-1} \in \Phi$ we know that $\bar{R} \cong R^2$ and so $\mathrm{PGL}(\bar{R}) \cong \mathrm{PGL}(R)^2$ which allows us to write

$$\mathrm{Aut}_R(\mathfrak{L}_R) \cong \mathrm{PGL}_2(R)^2 \rtimes \mu_2(R)\,.$$

Summarizing these comments we have

THEOREM 9. *For a field Φ with $\frac{1}{2}, \sqrt{-1} \in \Phi$ we have*

$$\mathbf{Aut}_\Phi(\mathfrak{L}_\Phi) \cong \mathbf{PGL}_2^2 \rtimes \mu_2\,.$$

In particular the group of Φ-points of $\mathrm{Aut}(\mathfrak{L}_\Phi)$ *is* $\mathrm{PGL}_2(\Phi)^2 \rtimes \mu_2(\Phi)$. *The derivation algebra of the Lorentz type algebra $\mathfrak{L}_\Phi$ is isomorphic to $\mathfrak{sl}_2(\Phi)^2$. Thus the derivations of $\mathfrak{L}_\Phi$*

are inner. The automorphisms of $\mathfrak{L}_\Phi$ are inner, more precisely if we identify $\mathfrak{L}_\Phi$ with $I \oplus J$ where $I = J = \mathfrak{sl}_2(\Phi)$, any $\phi \in \mathrm{Aut}(\mathfrak{L}_\Phi)$ fixing I and J is of the form

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mapsto \begin{pmatrix} pxp^{-1} & 0 \\ 0 & qyq^{-1} \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} p^{-1} & 0 \\ 0 & q^{-1} \end{pmatrix},$$

and any $\phi \in \mathrm{Aut}(\mathfrak{L}_\Phi)$ swapping I and J is of the form

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mapsto \begin{pmatrix} pyp^{-1} & 0 \\ 0 & qxq^{-1} \end{pmatrix} = \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} 0 & q^{-1} \\ p^{-1} & 0 \end{pmatrix},$$

for some $p, q \in \mathrm{GL}_2(\Phi)$.

In this paragraph we use the finite constant group (see [20, 2.3, p. 16]) Z_2. This is the affine group scheme whose representing Hopf algebra is $\Phi^2 := \Phi \times \Phi$ with componentwise product and Hopf algebra structure define in the reference above. Thus, for R in $\mathbf{alg}_\Phi$ we have $Z_2(R) := \hom_\Phi(\Phi^2, R)$ and so any element $f \in Z_2(R)$ is completely determined by $e_1 := f(1,0)$ and $e_2 := f(0,1)$ which are orthogonal idempotents in R. This produces a decomposition $R = Re_1 \oplus Re_2$ as a direct sum of two ideals and reciprocally any set $\{e_1, e_2\}$ of orthogonal idempotents of R, whose sum is 1, gives a decomposition $R = Re_1 \oplus Re_2$, hence produces a homomorphism $f\colon \Phi^2 \to R$ such that $f(1,0) = e_1$ and $f(0,1) = e_2$. Of course, if R has no idempotents others than 0 and 1, the abstract group of $Z_2(R)$ is isomorphic to the group $\mathbb{Z}_2$ of integers module 2. The set $Z_2(R)$ is in one-to-one correspondence with the set of decompositions of R as a direct sum of ideals.

REMARK 9. It is a standard result that if $\frac{1}{2} \in \Phi$, there is an isomorphism of affine group schemes $\mathbf{Z_2} \cong \boldsymbol{\mu_2}$.

To finish this section we apply Theorem 8 to the case in which $\frac{1}{2} \in \Phi$ but $\sqrt{-1} \notin \Phi$. Observe that the correstriction functor $\mathcal{R}$ in this case is $\mathcal{R}(R) = \mathrm{Aut}_{\bar{R}}(\mathfrak{sl}_2(\bar{R}))$. Since $\mathbf{Aut}(\bar{\Phi}) \cong \mu_2$ (Remark 8), we have:

THEOREM 10. *Let Φ be a field with $\frac{1}{2} \in \Phi$ and with no square root of -1. Then the affine group scheme $\mathbf{Aut}(\mathfrak{L}_\phi)$ is $\mathcal{R} \rtimes Z_2$ and so for any algebra R in $\mathbf{alg}_\Phi$ we have $\mathrm{Aut}_R(\mathfrak{L}_R) \cong \mathbf{PGL}_2(\bar{R}) \rtimes Z_2(R)$. In particular, the automorphism group of the (real) Lorentz algebra is $\mathrm{PGL}_2(\mathbb{C}) \rtimes \mu_2(\mathbb{R})$.*

COROLLARY 4. *If Φ is a field of characteristic other than 2, $\dim(\mathbf{Aut}(\mathfrak{L}_\Phi)) = 6$. The affine group scheme $\mathbf{Aut}(\mathfrak{L}_\Phi)$ is smooth.*

PROOF. If $\sqrt{-1} \in \Phi$ we apply Theorem 9 and then $\mathbf{Aut}(\mathfrak{L}_\Phi) \cong \mathbf{PGL}_2^2 \rtimes Z_2$ hence $\dim \mathbf{Aut}(\mathfrak{L}_\Phi) = \dim \mathbf{PGL}_2^2 = 6$ since $\dim Z_2 = 0$. If $\sqrt{-1} \notin \Phi$ we apply Theorem 10 and so $\mathbf{Aut}(\mathfrak{L}_\Phi) \cong \mathcal{R} \rtimes Z_2$. Thus $\dim \mathbf{Aut}(\mathfrak{L}_\Phi) = \dim \mathcal{R}$ where $\mathcal{R}$ is the corestriction functor $\mathcal{R} = \mathcal{R}_{\bar{\Phi}/\Phi}(\mathbf{Aut}_{\bar{\Phi}}(\mathfrak{sl}_2(\bar{\Phi})))$. Applying now [16, (21.7) Proposition, p.337] we have

$\dim \mathcal{R} = [\bar{\Phi} : \Phi] \cdot \dim(\mathbf{Aut}_{\bar{\Phi}}(\mathfrak{sl}_2(\bar{\Phi}))) = 2 \cdot 3 = 6$. Since the derivation algebra $\mathrm{Der}(\mathfrak{L}_\Phi)$ is also 6-dimensional we get the required smoothness. □

9. Automorphisms of $\mathfrak{L}_{\mathbb{K}}$ in the case of characteristic 2

In this section the ring of scalars Φ is taken to be a field $\mathbb{K}$, algebraically closed of characteristic 2 and the Lie $\mathbb{K}$-algebra $\mathfrak{o}(3; \mathbb{K})$ (as introduced at the beginning of section 6), will be denoted simply by $\mathfrak{o}(3)$ if the reference to the field is not crucial . Also the notation O(3) will stand for the algebraic group of 3×3 matrices $M \in \mathrm{GL}_3(\mathbb{K})$ such that $MM^t = 1$.

PROPOSITION 10. *The algebraic group* O(3) *is connected and of dimension* 3.

PROOF. First of all, note that O(3) is a connected algebraic group. In fact, $M \in \mathrm{O}(3)$ if and only if

$$M = \begin{pmatrix} a & b & 1+a+b \\ s & t & 1+s+t \\ 1+a+s & 1+b+t & 1+a+b+s+t \end{pmatrix},$$

where $1 + a + b + s + t + at + bs = 0$. Thus, as an affine variety the representing Hopf algebra of the affine group scheme whose group of $\mathbb{K}$-points is O(3) is $H := \mathbb{K}[a, b, s, t]/I$ where I is the ideal generated by $1 + a + b + s + t + at + bs$. But since the ideal I is prime H has no idempotents other that 0 and 1. Thus, taking into account [17, Proposition 3.2 and Definition 3.3, p. 208], O(3) is connected. Also by using this algebraic geometry ideas we recognize the fact that

$$\dim \mathrm{O}(3) = 3\,. \tag{9}$$

Indeed, the set $S = \{a, b, s\}$ satisfies that $\mathbb{K}[S]$ is a polynomial algebra and H is finitely generated as a $\mathbb{K}[S]$-module. (see [17, 16.1 (a)]). □

Consider the three-dimensional ideal $I = \langle x_1, x_2, x_3\rangle$ of $\mathfrak{L}_{\mathbb{K}}$ (see Proposition 6) and $f : \mathfrak{L}_{\mathbb{K}} \to \mathfrak{L}_{\mathbb{K}}$ an automorphism. Then $f(I) = I$ and we can consider the induced map $\bar{f} : \mathfrak{L}_{\mathbb{K}}/I \to \mathfrak{L}_{\mathbb{K}}/I$. Since $\mathfrak{L}_{\mathbb{K}}/I \cong \mathfrak{o}(3)$ we may identify these algebras and consider $\bar{f} : \mathfrak{o}(3) \to \mathfrak{o}(3)$. It is routinary to prove that $\bar{f} \in \mathrm{Aut}(\mathfrak{o}(3)) \cong \mathrm{O}(3)$. Thus we have a map $\phi : \mathrm{Aut}(\mathfrak{L}_{\mathbb{K}}) \to \mathrm{O}(3)$ such that $\phi(f)$ is the image of $\bar{f}$ in O(3) and it is straightforward to see that ϕ is a group homomorphism. We want to prove that ϕ is an epimorphism.

PROPOSITION 11. *ϕ is an epimorphism.*

PROOF. Consider the 3-dimensional $\mathbb{K}$-vector space $V := \mathbb{K}^3$ endowed with the cross-product, that is $x \wedge y = (s_2t_3 + s_3t_2, s_1t_3 + s_3t_1, s_1t_2 + s_2t_1)$ where $x = (s_1, s_2, s_3)$, $y = (t_1, t_2, t_3)$. We will have the occasion to use also the inner product $\langle x, y\rangle := s_1t_1 + s_2t_2 + s_3t_3$. We know that V is a Lie algebra relative to the product $\wedge$. Furthermore, $V \times V$ is a Lie algebra relative to

$$[(x, y), (z, t)] := (x \wedge z, x \wedge z + x \wedge t + y \wedge z), \forall x, y, z, t \in V\,.$$

It is easy to see that $V \times V \cong \mathfrak{L}_{\mathbb{K}}$ where $0 \times V$ corresponds to the 3-dimensional ideal I of $\mathfrak{L}_{\mathbb{K}}$. More precisely, if we denote by i, j, k the vectors of the canonical basis of V, then the isomorphism acts in the form

$$\begin{aligned}(i, 0) &\mapsto b_1,\\ (j, 0) &\mapsto b_2,\\ (k, 0) &\mapsto b_3,\\ (0, i) &\mapsto x_1,\\ (0, j) &\mapsto x_2,\\ (0, k) &\mapsto x_3\,.\end{aligned} \tag{10}$$

So take now an automorphism $g\colon \mathfrak{o}(3) \to \mathfrak{o}(3)$. We know that relative to the basis $\{b_1, b_2, b_3\}$ of $\mathfrak{o}(3)$ such that $[b_i, b_j] = b_k$ (cyclically), the matrix of g is orthogonal. So the rows of the matrix of g relative to the mentioned basis are three vectors $a_i \in V$, $i = 1, 2, 3$ such that $a_i \wedge a_j = a_k$ (cyclically) and $\langle a_i, a_i\rangle = 1$ (also $\langle a_i, a_j\rangle = 0$ if $i \neq j$). Now define $f\colon V \times V \to V \times V$ such that

$$\begin{aligned}f(i, 0) &= (a_1, \alpha_1 a_1), \text{ where } \alpha_1 \in \mathbb{K}^\times,\\ f(j, 0) &= (a_2, \alpha_2 a_2), \text{ where } \alpha_2 \in \mathbb{K}^\times,\\ f(k, 0) &= (a_3, \alpha_3 a_3), \text{ where } \alpha_3 \in \mathbb{K}^\times\,,\end{aligned} \tag{11}$$

and the scalars satisfy $\alpha_1 + \alpha_2 + \alpha_3 \neq 1$ (this choice is possible since $\mathbb{K}$ is an infinite field). Next take $\alpha_0 = \alpha_1 + \alpha_2 + \alpha_3 + 1 \neq 0$ and define

$$\begin{aligned}f(0, i) &= (0, \alpha_0 a_1),\\ f(0, j) &= (0, \alpha_0 a_2),\\ f(0, k) &= (0, \alpha_0 a_3)\,.\end{aligned} \tag{12}$$

Now it can be proved that f induces an automorphism of $\mathfrak{L}_{\mathbb{K}} \cong V \times V$ and that $\phi(f) = \bar{f} = g$. Thus ϕ is an epimorphism. □

Thus ϕ is an epimorphism and we define $G := \ker(\phi)$. So, we have a short exact sequence

$$1 \to G \to \operatorname{Aut}(\mathfrak{L}_{\mathbb{K}}) \to \mathrm{O}(3) \to 1 \tag{13}$$

and we would like to find out more about the group G. The elements $f \in G$ verify that $f(b_i) = b_i + x^{(i)}$ where each $x^{(i)} \in I$ ($i = 1, 2, 3$). If we assume that $f(x_i) = \sum_j a_{ij} x_j$ and take into account the conditions $[x_i, b_i] = 0$ and $[x_i, b_j] = x_k$ (this last assertion meaning that $i \neq j$ and $k \in \{1, 2, 3\} \setminus \{i, j\}$), then we find that $a_{ij} = 0$ for $i \neq j$ and $a_{11} = a_{22} = a_{33}$.

Thus, the matrix of f in the basis $\{x_1, x_2, x_3, b_1, b_2, b_3\}$ is of the form

$$\left(\begin{array}{ccc|ccc} a_{11} & 0 & 0 & & & \\ 0 & a_{11} & 0 & & 0 & \\ 0 & 0 & a_{11} & & & \\ \hline & & & 1 & 0 & 0 \\ & * & & 0 & 1 & 0 \\ & & & 0 & 0 & 1 \end{array}\right).$$

We assume now that $x^{(i)} = \sum_j \lambda_{ij} x_j$. Then, since $[b_1, b_2] = b_3$, applying f we have $[b_1 + x^{(1)}, b_2 + x^{(2)}] = b_3 + x^{(3)}$. Thus, $[b_1, x^{(2)}] + [b_2, x^{(1)}] = x^{(3)}$ and from here we get

$$\begin{cases} \lambda_{31} = \lambda_{13}, \\ \lambda_{32} = \lambda_{23}, \\ \lambda_{33} = \lambda_{11} + \lambda_{22}\,. \end{cases}$$

Similarly applying f to $[b_2, b_3] = b_1$ we get

$$\begin{cases} \lambda_{12} = \lambda_{21}, \\ \lambda_{13} = \lambda_{31}, \\ \lambda_{11} = \lambda_{22} + \lambda_{33}\,, \end{cases}$$

and so cyclically we conclude that the matrix of f is

$$\left(\begin{array}{ccc|ccc} a_{11} & 0 & 0 & & & \\ 0 & a_{11} & 0 & & 0 & \\ 0 & 0 & a_{11} & & & \\ \hline \lambda_{11} & \lambda_{12} & \lambda_{13} & 1 & 0 & 0 \\ \lambda_{12} & \lambda_{22} & \lambda_{23} & 0 & 1 & 0 \\ \lambda_{13} & \lambda_{23} & \lambda_{11} + \lambda_{22} & 0 & 0 & 1 \end{array}\right).$$

So using 3×3 blocks we see that $G = \left\{\begin{pmatrix} a \cdot 1_3 & 0 \\ M & 1_3 \end{pmatrix} : a \in \mathbb{K}^\times, M^t = M, \operatorname{tr}(M) = 0\right\}$. Thus, taking into account equation (9), since $\dim(G) = 6$, from the existence of the short exact sequence (13), we conclude that

$$\dim(\operatorname{Aut}(\mathfrak{L}_{\mathbb{K}})) = 9\,.$$

This fact is in contrast with the results for characteristic $\neq 2$, where as a consequence of Corollary 4, the dimension of the automorphism group of Lorentz type algebras is 6.

THEOREM 11. *The algebraic group* $\operatorname{Aut}(\mathfrak{L}_{\mathbb{K}})$ *is connected and* 9*-dimensional.*

PROOF. We already know that $\dim \operatorname{Aut}(\mathfrak{L}_{\mathbb{K}}) = 9$. For the connectedness property we will apply [17, Proposition 3.11, p. 210] to the above short exact sequence (13). We need to show that G is connected. But, $G \cong \mathbb{K}^{\times} \times \mathcal{E}$ where $\mathcal{E}$ is a vector space (regarded as an affine variety). Concretely, $\mathcal{E}$ is the space of traceless symmetric matrices. Thus $\mathcal{E}$ is connected and so is G. Consequently [17, Proposition 3.11, p. 210] implies the connectedness of $\mathrm{Aut}(\mathfrak{L}_{\mathbb{K}})$. □

10. Derivation algebra of $\mathfrak{L}_{\mathbb{K}}$ in characteristic two

In this section, we will consider the identification of $\mathfrak{L}_{\mathbb{K}}$ with $V \times V$ as in the previous section.

THEOREM 12. *For a field $\mathbb{K}$ of characteristic 2 we have*:

$$\dim(\mathrm{Der}(\mathfrak{L}_{\mathbb{K}})) = 12\,.$$

PROOF. Let $d \in \mathrm{Der}(\mathfrak{L}_{\mathbb{K}})$ where $\mathbb{K}$ is a field of characteristic 2, so $d\colon V \times V \to V \times V$. There are four linear maps $\alpha, \beta, \gamma, \delta$ such that $d(x, 0) = (\alpha(x), \beta(x))$ and $d(0, x) = (\gamma(x), \delta(x))$. Taking in to account that d is a derivation, we get the following set of identities

1. $x \wedge \alpha(y) + \alpha(x) \wedge y + \alpha(x \wedge y) + \gamma(x \wedge y) = 0$
2. $x \wedge \alpha(y) + \alpha(x) \wedge y + x \wedge \beta(y) + \beta(x) \wedge y + \beta(x \wedge y) + \delta(x \wedge y) = 0$,
3. $x \wedge \gamma(y) + \gamma(x \wedge y) = 0$,
4. $\alpha(x) \wedge y + x \wedge \gamma(y) + x \wedge \delta(y) + \delta(x \wedge y) = 0$,
5. $x \wedge \gamma(y) + \gamma(x) \wedge y = 0$.

A straightforward computation reveals that the identity (5) implies the existence of a scalar $\lambda \in \mathbb{K}$ such that $\gamma(x) = \lambda x$ for all $x \in V$. As a consequence the identity (3) is automatically satisfied. Now, since we now γ, we may determine α using equation (1). From this, we get that the matrix of α, in canonical basis, is

$$\begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{1,2} & b_{2,2} & b_{2,3} \\ b_{1,3} & b_{2,3} & \lambda + b_{1,1} + b_{2,2} \end{pmatrix}.$$

From the equation (4) we obtain that the matrix of δ is of the form:

$$\begin{pmatrix} c_{1,1} & b_{1,2} & b_{1,3} \\ b_{1,2} & b_{1,1} + b_{2,2} + c_{1,1} & b_{2,3} \\ b_{1,3} & b_{2,3} & \lambda + c_{1,1} + b_{2,2} \end{pmatrix}.$$

Finally, equation (2) gives that the matrix of β is of the form:

$$\begin{pmatrix} f_{1,1} & f_{1,2} & f_{1,3} \\ f_{1,2} & f_{2,2} & f_{2,3} \\ f_{1,3} & f_{2,3} & \lambda + b_{1,1} + c_{1,1} + f_{1,1} + f_{2,2} \end{pmatrix}.$$

So the free parameters appearing in d are:

$$\lambda, b_{11}, b_{22}, b_{12}, b_{13}, b_{23}, c_{11}, f_{11}, f_{22}, f_{12}, f_{13}, f_{23}\,,$$

hence

$$\dim(\mathrm{Der}(\mathfrak{L}_{\mathbb{K}})) = 12\,.$$

□

10.1. Non-smoothness of $\mathrm{Aut}(\mathfrak{L}_\Phi)$ **in characteristic** 2. Consider now the affine group scheme $\mathbf{Aut}(\mathfrak{L}_\Phi)\colon\ \mathbf{alg}_\Phi \to \mathbf{Grp}$, where Φ is a field of characteristic 2. The dimension of this algebraic group is $\dim \mathrm{Aut}(\mathfrak{L}_{\mathbb{K}})$ where $\mathbb{K}$ is the algebraic closure of Φ. As we have proved in previous sections $\dim \mathrm{Aut}(\mathfrak{L}_{\mathbb{K}}) = 9$ hence $\dim(\mathbf{Aut}(\mathfrak{L}_\Phi)) = 9$. Of course, we have $\mathbf{Lie}(\mathbf{Aut}(\mathfrak{L}_\Phi)) = \mathrm{Der}(\mathfrak{L}_\Phi)$ and $\dim(\mathbf{Lie}(\mathbf{Aut}(\mathfrak{L}_\Phi))) = \dim(\mathrm{Der}(\mathfrak{L}_\Phi)) = \dim(\mathrm{Der}(\mathfrak{L}_{\mathbb{K}})) = 12$ by Theorem 12. Thus we have

$$\dim(\mathbf{Lie}(\mathbf{Aut}(\mathfrak{L}_\Phi))) = 12$$

and so the affine group scheme $\mathbf{Aut}(\mathfrak{L}_\Phi)$ is not smooth.

11. Structure of the Poincaré algebra

The Poincaré group is the inhomogeneous Lorentz group, that is, the group generated by the Lorentz group plus translations. Similarly the Poincaré algebra $\mathfrak{p}$ over the field $\mathbb{K}$ is the inhomogeneous Lorentz algebra. We can define it as the direct sum $\mathfrak{p} = \begin{pmatrix} 0 & \mathbb{K}^4 \\ 0 & \mathfrak{L}_{\mathbb{K}} \end{pmatrix}$ but since in this section we assume the ground field $\mathbb{K}$ to be algebraically closed and of characteristic other than 2, we identify $\mathfrak{L}_{\mathbb{K}}$ with $\mathfrak{o}_4(\mathbb{K})$ all through the section (see Lemma 1). Thus, the Poincaré algebra $\mathfrak{p}$ is the direct sum

$$\mathfrak{p} = \begin{pmatrix} 0 & \mathbb{K}^4 \\ 0 & \mathfrak{o}_4(\mathbb{K}) \end{pmatrix},$$

with the product in this Lie algebra $\left[\begin{pmatrix} 0 & v \\ 0 & M \end{pmatrix}, \begin{pmatrix} 0 & v' \\ 0 & M' \end{pmatrix}\right] = \begin{pmatrix} 0 & vM' - v'M \\ 0 & [M, M'] \end{pmatrix}$ for any $v, v' \in \mathbb{K}^4$, $M, M' \in \mathfrak{o}_4(\mathbb{K})$. Moreover, $\mathfrak{r} = \begin{pmatrix} 0 & \mathbb{K}^4 \\ 0 & 0 \end{pmatrix}$ is an abelian ideal (in fact the radical of $\mathfrak{p}$). The first result we need to prove is:

Lemma 16. *The ideal* $\mathfrak{r}$ *is minimal.*

Proof. The group $\mathrm{O}_4(\mathbb{K})$ of orthogonal matrices acts by conjugation on its Lie algebra $\mathfrak{o}_4(\mathbb{K})$ of skew-symmetric matrices. Also $\mathrm{O}_4(\mathbb{K})$ acts naturally on $\mathbb{K}^4$ and if two vectors $v_1, v_2 \in \mathbb{K}^4$ are in the same orbit under the action of $\mathrm{O}_4(\mathbb{K})$ (so $v_2 = v_1 P$ for some $P \in \mathrm{O}_4(\mathbb{K})$), then the ideals of $\mathfrak{p}$ generated by $\begin{pmatrix} 0 & v_i \\ 0 & 0 \end{pmatrix}$, $(i = 1.2)$ are conjugated by the automorphism $\Omega \in \mathrm{Aut}(\mathfrak{p})$ given by

$$\Omega\left[\begin{pmatrix} 0 & v \\ 0 & M \end{pmatrix}\right] := \begin{pmatrix} 0 & vP \\ 0 & P^{-1}MP \end{pmatrix}.$$

Consequently, when studying the ideal generated by an element $x = \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}$ up to isomorphism, we may replace v with any vector in its orbit under the action of $\mathrm{O}_4(\mathbb{K})$. To prove that $\mathfrak{r}$ is minimal, we show that the ideal generated by any nonzero element in $\mathfrak{r}$ is $\mathfrak{r}$. Thus, take $x = \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \in \mathfrak{r} \setminus \{0\}$. There are two possibilities to analyze:

1. Assume first that $v \in \mathbb{K}^4$ is nonisotropic relative to the quadratic form $q(x, y, z, t) := x^2 + y^2 + z^2 + t^2$ of $\mathbb{K}^4$. We may take $q(v) = 1$ since the ideal generated by x is the same that the ideal generated by any nonzero scalar multiple of x. In this case, applying Witt's Theorem, v is in the same orbit as $(1, 0, 0, 0)$ under the action of the orthogonal group. Hence without loss in generality we may take v to be $(1, 0, 0, 0)$. But then, it is easy to prove that the ideal generated by x is the radical $\mathfrak{r}$.
2. Let us assume now that v is isotropic. Again by Witt's theorem, any two isotropic vectors are in the same orbit under the action of $\mathrm{O}_4(\mathbb{K})$. So we may take v to be $v = (1, i, 0, 0)$. In this case, the ideal generated by x contains all the elements $\begin{pmatrix} 0 & vM \\ 0 & 0 \end{pmatrix}$ with $M \in \mathfrak{o}_4(\mathbb{K})$. But the subspace vM is 3-dimensional hence it contains a nonisotropic vector (the Witt index of q is 2). Thus the ideal generated by x contains an element $\begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix}$ with w nonisotropic. Hence this ideal is $\mathfrak{r}$.

$\square$

We know that $\mathfrak{p}/\mathfrak{r} \cong \mathfrak{o}_4(\mathbb{K})$ and by Lemma 1 and Proposition 5, this algebra is a direct sum $\mathfrak{o}_4(\mathbb{K}) = I \oplus J$ of two isomorphic ideals $I \cong J \cong \mathfrak{sl}_2(\mathbb{K})$. Thus we have seven-dimensional ideals

$$\mathfrak{i} = \begin{pmatrix} 0 & \mathbb{K}^4 \\ 0 & I \end{pmatrix}, \quad \mathfrak{j} = \begin{pmatrix} 0 & \mathbb{K}^4 \\ 0 & J \end{pmatrix},$$

such that $\mathfrak{i} + \mathfrak{j} = \mathfrak{p}$ and $\mathfrak{i} \cap \mathfrak{j} = \mathfrak{r}$. Furthermore, $\mathfrak{p}/\mathfrak{i} \cong \mathfrak{sl}_2(\mathbb{K}) \cong \mathfrak{p}/\mathfrak{j}$ so $\mathfrak{i}$ and $\mathfrak{j}$ are maximal ideals of $\mathfrak{p}$. We may represent this by

$$\begin{tikzcd}[column sep=small, row sep=small] & \mathfrak{p} \arrow[dl, dash] \arrow[dr, dash] & \\ \mathfrak{i} \arrow[dr, dash] & & \mathfrak{j} \arrow[dl, dash] \\ & \mathfrak{r} & \end{tikzcd}$$

LEMMA 17. *If K is an ideal of $\mathfrak{p}$ such that $[K, \mathfrak{p}] \subset \mathfrak{r}$, then $K = 0$ or $K = \mathfrak{r}$.*

PROOF. If $K \neq 0$ take a nonzero element $x_0 = \begin{pmatrix} 0 & v_0 \\ 0 & M_0 \end{pmatrix} \in K$. Then for any element $x = \begin{pmatrix} 0 & v \\ 0 & M \end{pmatrix} \in \mathfrak{p}$ we have $[x_0, x] = \begin{pmatrix} 0 & v_0 M - v M_0 \\ 0 & [M_0, M] \end{pmatrix} \in \mathfrak{r}$ hence M_0 is in the center of $\mathfrak{o}_4(\mathbb{K})$ which is null. Thus $x_0 \in \mathfrak{r}$ and we have proved $K \subset \mathfrak{r}$. Finally the minimality of $\mathfrak{r}$ implies $K = \mathfrak{r}$. □

LEMMA 18. *The only ideals of* $\mathfrak{p}$ *are*: $0, \mathfrak{r}, \mathfrak{i}, \mathfrak{j}$ *and* $\mathfrak{p}$.

PROOF. Let $K \lhd \mathfrak{p}$ such that $K \notin \{0, \mathfrak{r}, \mathfrak{i}, \mathfrak{j}, \mathfrak{p}\}$. Observe that by maximality of $\mathfrak{i}$ we must have either $K \subsetneq \mathfrak{i}$ or $K + \mathfrak{i} = \mathfrak{p}$ (and similarly $K \subsetneq \mathfrak{j}$ or $K + \mathfrak{j} = \mathfrak{p}$). Thus we have four possibilities:

1. $K \subsetneq \mathfrak{i}$, $K \subsetneq \mathfrak{j}$.
2. $K \subsetneq \mathfrak{i}$, $K + \mathfrak{j} = \mathfrak{p}$.
3. $K + \mathfrak{i} = \mathfrak{p}$, $K \subsetneq \mathfrak{j}$.
4. $K + \mathfrak{i} = \mathfrak{p}$, $K + \mathfrak{j} = \mathfrak{p}$.

Since $\mathfrak{r}$ is minimal $K \cap \mathfrak{r} = \mathfrak{r}$ or $K \cap \mathfrak{r} = 0$. In the first case $\mathfrak{r} \subset K$, and $\mathfrak{r} \subset K \cap \mathfrak{i} \subset \mathfrak{i}$ implying $K \cap \mathfrak{i} = \mathfrak{r}$ or $K \cap \mathfrak{i} = \mathfrak{i}$. Similarly $K \cap \mathfrak{j} = \mathfrak{r}$ or $K \cap \mathfrak{j} = \mathfrak{j}$.

- If $K \cap \mathfrak{i} = \mathfrak{i}$, then $\mathfrak{i} \subset K$ and by maximality of $\mathfrak{i}$ we have $K = \mathfrak{i}$ or $K = \mathfrak{p}$, a contradiction.
- If $K \cap \mathfrak{j} = \mathfrak{j}$, then we get as above a contradiction.
- If $K \cap \mathfrak{i} = \mathfrak{r} = K \cap \mathfrak{j}$, we analyze the compatibility of this conditions with (1)-(4). In case that (1) or (2) holds, we have $\mathfrak{r} \subset K \subsetneq \mathfrak{i}$ which implies $K = \mathfrak{r}$ a contradiction. If (3) holds, then $\mathfrak{r} \subset K \subsetneq \mathfrak{j}$ implying $K = \mathfrak{r}$ a contradiction again. So the situation now is: $K + \mathfrak{i} = K + \mathfrak{j} = \mathfrak{p}$ and $K \cap \mathfrak{i} = K \cap \mathfrak{j} = \mathfrak{r}$. But then $[K, \mathfrak{p}] \subset [K, \mathfrak{i}] + [K, \mathfrak{j}] \subset (K \cap \mathfrak{i}) + (K \cap \mathfrak{j}) \subset \mathfrak{r}$ and by Lemma 17 we get $K = 0$ or $K = \mathfrak{r}$ a contradiction again.

Now we must analyze the case $K \cap \mathfrak{r} = 0$. If $K \neq 0$, take a nonzero element $x_0 = \begin{pmatrix} 0 & v_0 \\ 0 & M_0 \end{pmatrix} \in K$. Consider now any $v \in K^4$ and $x = \begin{pmatrix} 0 & v \\ 0 & M_0 \end{pmatrix}$. Then $[x_0, x] = \begin{pmatrix} 0 & (v_0 - v)M_0 \\ 0 & 0 \end{pmatrix} \in K \cap \mathfrak{r} = 0$. So $(v_0 - v)M_0 = 0$ for any v which implies $M_0 = 0$, a contradiction. □

COROLLARY 5. *The Poincaré algebra* $\mathfrak{p}$ *is centerless.*

11.1. Some properties of the Poincaré algebra over rings. Fix an algebraically closed field Φ of characteristic other than 2 and consider, as in previous sections, the functor $\mathfrak{p}\colon \mathbf{alg}_\Phi \to \mathbf{Lie}_\Phi$ such that $\mathfrak{p}(R) := \mathfrak{p}_R$. Thus we have

$$\mathfrak{p}_R = \begin{pmatrix} 0 & \mathfrak{r}_R \\ 0 & \mathfrak{o}_4(R) \end{pmatrix}.$$

Some properties of these algebras $\mathfrak{p}_R$ can be given. For instance any abelian ideal of $\mathfrak{p}_R$ is contained in $\mathfrak{r}_R$. Indeed if $I \lhd \mathfrak{p}_R$ is abelian, we may define the ideal J of $\mathfrak{o}_4(R)$ whose elements are all the $m \in \mathfrak{o}_4(R)$ such that there is some $v \in \mathfrak{r}_R$ with $\left(\begin{smallmatrix} 0 & v \\ 0 & m \end{smallmatrix}\right) \in I$. But being I abelian, J is an abelian ideal in $\mathfrak{o}_4(R)$ and we know (as a consequence of Proposition 5) that then $J = 0$ hence $I \subset \mathfrak{r}_R$. Since $\mathfrak{r}_R$ is abelian we conclude that $\mathfrak{r}_R$ is the maximum abelian ideal of $\mathfrak{p}_R$.

REMARK 10. Take R to be an algebra in $\mathbf{alg}_\Phi$ and $f \in \mathrm{Aut}(\mathfrak{p}_R)$ any automorphism. Since $\mathfrak{r}_R$ is the maximum abelian ideal of $\mathfrak{p}_R$ we have $f(\mathfrak{r}_R) = \mathfrak{r}_R$, that is, $\mathfrak{r}_R$ is fixed by any automorphism of $\mathfrak{p}_R$.

12. Derivation algebra of $\mathfrak{p}$

We analyze in this section the Lie algebra $\mathrm{Der}(\mathfrak{p})$ of derivations of the Poincaré algebra. The ground field $\mathbb{K}$ is supposed to be algebraically closed and of characteristic other than 2. First of all since $\mathrm{Z}(\mathfrak{p}) = 0$ the map $\mathrm{ad}\colon \mathfrak{p} \to \mathrm{Der}(\mathfrak{p})$ is a monomorphism and the ideal of inner derivations is 10-dimensional. We have a linear map $q\colon \mathrm{Der}(\mathfrak{p}) \to \mathrm{Der}(\mathfrak{o}_4(\mathbb{K}))$ defined by $q(d) = \pi d i$ where $i\colon \mathfrak{o}_4(\mathbb{K}) \to \mathfrak{p}$ is the natural injection $M \mapsto \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix}$, and $\pi : \mathfrak{p} \to \mathfrak{o}_4(\mathbb{K})$ the map $\begin{pmatrix} 0 & v \\ 0 & M \end{pmatrix} \mapsto M$ which is an epimorphism of Lie algebras.

LEMMA 19. *The linear map q is an epimorphism of Lie algebras whose kernel is isomorphic to the Lie algebra $\mathbb{K} \times \mathbb{K}^4$ with product given by* $[(\lambda, v_0), (\lambda', v_0')] := (0, \lambda v_0' - \lambda' v_0)$. *Thus* $\mathrm{Der}(\mathfrak{p})$ *fits in a short exact sequence*

$$0 \to \mathbb{K} \times \mathbb{K}^4 \to \mathrm{Der}(\mathfrak{p}) \xrightarrow{q} \mathrm{Der}(\mathfrak{o}_4(\mathbb{K})) \to 0$$

and therefore $\dim \mathrm{Der}(\mathfrak{p}) = 11$.

PROOF. The proof that q is a Lie algebra homomorphism is easy if we take into account that the radical $\mathfrak{r}$ of $\mathfrak{p}$ is d-invariant for any derivation d of $\mathfrak{p}$ (see [7]). Next we prove that q is an epimorphism. Let $\alpha \in \mathrm{Der}(\mathfrak{o}_4(\mathbb{K}))$, since the derivations of $\mathfrak{o}_4(\mathbb{K})$ are inner (see Theorem 9), there is some $x_0 \in \mathfrak{o}_4(\mathbb{K})$ such that $\alpha = \mathrm{ad}(x_0)$. Then define $d\colon \mathfrak{p} \to \mathfrak{p}$ such that

$$d\left[\begin{pmatrix} 0 & v \\ 0 & x \end{pmatrix}\right] := \begin{pmatrix} 0 & -vx_0 \\ 0 & \alpha(x) \end{pmatrix}.$$

It is easy to check that $d \in \mathrm{Der}(\mathfrak{p})$ and $q(d) = \alpha$. In order to determine the kernel of q, assume that $q(d) = 0$ for a derivation d. Then $d\left[\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}\right] = \begin{pmatrix} 0 & \beta(x) \\ 0 & 0 \end{pmatrix}$ for some linear map $\beta\colon \mathfrak{o}_4(\mathbb{K}) \to \mathbb{K}^4$ which must satisfy the hypothesis on Lemma 3. Therefore there is an element $v_0 \in \mathbb{K}^4$ such that $\beta(x) = v_0 x$ for any $x \in \mathfrak{o}_4(\mathbb{K})$. Since any derivation preserves the radical $\mathfrak{r}$ of $\mathfrak{p}$ (again [7]), we also must have $d\left[\begin{pmatrix}0 & v\\ 0 & 0\end{pmatrix}\right] = \begin{pmatrix}0 & \alpha(v)\\ 0 & 0\end{pmatrix}$ for some linear map $\alpha\colon \mathbb{K}^4 \to \mathbb{K}^4$. So there is a matrix $P \in \mathcal{M}_4(\mathbb{K})$ such that $\alpha(v) = vP$ for any v. Then $d\left[\begin{pmatrix}0 & v\\ 0 & x\end{pmatrix}\right] = \begin{pmatrix}0 & vP + v_0x\\ 0 & 0\end{pmatrix}$ and imposing the condition that d to be a derivation we get that P must commute with any element in $\mathfrak{o}_4(\mathbb{K})$ but this implies that $P = \lambda 1_4$ where $\lambda \in \mathbb{K}$ and 1_4 denotes de identity matrix 4×4. Summarizing: $d\left[\begin{pmatrix}0 & v\\ 0 & x\end{pmatrix}\right] = \begin{pmatrix}0 & \lambda v + v_0x\\ 0 & 0\end{pmatrix}$ and the scalar λ as well as the vector v_0 are uniquely determined by d. Thus the map $\ker(q) \to \mathbb{K} \times \mathbb{K}^4$ such that $d \mapsto (\lambda, v_0)$ is a Lie algebras homomorphism where $\mathbb{K} \times \mathbb{K}^4$ is provided with a Lie algebra structure whose product is $[(\lambda, v_0), (\lambda', v_0')] := (0, \lambda v_0' - \lambda' v_0)$. Thus, the exact sequence of Lie algebras

$$0 \to \mathbb{K} \times \mathbb{K}^4 \to \mathrm{Der}(\mathfrak{p}) \xrightarrow{q} \mathrm{Der}(\mathfrak{o}_4(\mathbb{K})) \to 0 \tag{14}$$

exists and $\dim \mathrm{Der}(\mathfrak{p})) = 11$. □

DEFINITION 5. For $(\lambda, v_0) \in \mathbb{K} \times \mathbb{K}^4$ define the derivation $d_{\lambda,v_0} \in \ker(q)$ by

$$d_{\lambda,v_0}\left[\begin{pmatrix}0 & v\\ 0 & x\end{pmatrix}\right] = \begin{pmatrix}0 & \lambda v + v_0x\\ 0 & 0\end{pmatrix}.$$

PROPOSITION 12. *The short exact sequence* (14) *is split*: *there is a monomorphism*

$$j\colon\ \mathrm{Der}(\mathfrak{o}_4(\mathbb{K})) \to \mathrm{Der}(\mathfrak{p})$$

such that for $\alpha \in \mathrm{Der}(\mathfrak{o}_4(\mathbb{K}))$ *one has* $j(\alpha)\colon \begin{pmatrix}0 & v\\ 0 & x\end{pmatrix} \mapsto \begin{pmatrix}0 & -vx_0\\ 0 & \alpha(x)\end{pmatrix}$ *being* $\alpha = ad(x_0)$. *This map satisfies* $qj = 1$. *Consequently any derivation* $d \in \mathrm{Der}(\mathfrak{p})$ *can be uniquely written as a sum* $j(\alpha) + d_{\lambda,v}$. *Furthermore*:

1. $\mathbb{K} \times \mathbb{K}^4 \cong \{d_{\lambda,v}\colon (\lambda, v) \in \mathbb{K} \times \mathbb{K}^4\}$ *is the radical* $\mathfrak{r} := \mathfrak{r}(\mathrm{Der}(\mathfrak{p}))$.
2. $\mathrm{Der}(\mathfrak{p}) = \mathfrak{r} \oplus j(\mathrm{Der}(\mathfrak{o}_4(\mathbb{K}))) \cong \mathfrak{r} \oplus \mathfrak{sl}_2(\mathbb{K})^2$.

PROOF. It is straightforward to check that j is a monomorphism and that $qj = 1$. So $\mathrm{Der}(\mathfrak{p}) = j(\mathrm{Der}(\mathfrak{o}_4(\mathbb{K}))) \oplus \ker(q)$. To finish the proof, it only remains to check that $\ker(q) \cong \mathbb{K} \times \mathbb{K}^4$ is the radical of $\mathrm{Der}(\mathfrak{p})$. By the definition of the product of the Lie algebra $\mathbb{K} \times \mathbb{K}^4$ we see that it is solvable and so $\ker(q)$ is a solvable ideal and $\mathrm{Der}(\mathfrak{p})/\ker(q) \cong \mathfrak{sl}_2(\mathbb{K})^2$ is semisimple (take into account Lemma 1 and Proposition 5). Thus, $\ker(q)$ is the radical of $\mathrm{Der}(\mathfrak{p})$. □

13. Automorphism group of $\mathfrak{p}$

The ground field $\mathbb{K}$ is supposed to be algebraically closed and of characteristic other than 2. Consider the group homomorphism $Q\colon\ \mathrm{Aut}(\mathfrak{p}) \to \mathrm{Aut}(\mathfrak{o}_4(\mathbb{K}))$ such that $Q(f) =$

$\pi f i$ where π and i have been defined at the beginning of the previous section. Given $\theta \in \mathrm{Aut}(\mathfrak{o}_4(\mathbb{K}))$ there is an $x_0 \in \mathrm{GL}_4(\mathbb{K})$ such that $\theta = \mathrm{Ad}(x_0)$. The map $f : \mathfrak{p} \to \mathfrak{p}$ such that $f : \begin{pmatrix} 0 & v \\ 0 & x \end{pmatrix} \mapsto \begin{pmatrix} 0 & vx_0^{-1} \\ 0 & \theta(x) \end{pmatrix}$ is an automorphism of $\mathfrak{p}$ and $Q(f) = \theta$. Therefore Q is an epimorphism whose kernel is described in the following

LEMMA 20. *The kernel* $\ker(Q)$ *is the subgroup of automorphisms* f_{λ, v_0} *where* $\lambda \in \mathbb{K}^\times$ *and* $v_0 \in \mathbb{K}^4$; *such that*

$$f_{\lambda, v_0} : \begin{pmatrix} 0 & v \\ 0 & x \end{pmatrix} \mapsto \begin{pmatrix} 0 & \lambda v + v_0 x \\ 0 & x \end{pmatrix}.$$

PROOF. Let $f \in \mathrm{Aut}(\mathfrak{p})$, since $\mathfrak{r}$ is f-invariant, we have

$$f\left[\begin{pmatrix} 0 & v \\ 0 & x \end{pmatrix}\right] = \begin{pmatrix} 0 & \mu(v) + \alpha(x) \\ 0 & \beta(x) \end{pmatrix}$$

for some linear maps $\mu : \mathbb{K}^4 \to \mathbb{K}^4$, $\alpha : \mathfrak{o}_4(\mathbb{K}) \to \mathbb{K}^4$, $\beta : \mathfrak{o}_4(\mathbb{K}) \to \mathfrak{o}_4(\mathbb{K})$. If we assume that $f \in \ker(Q)$ then $\pi f i = 1$ and this implies $\beta = 1$. Now imposing the condition that f is a Lie algebra homomorphism, we get:

1. $\alpha([x, y]) = \alpha(x)y - \alpha(y)x$ for any $x, y \in \mathfrak{o}_4(\mathbb{K})$.
2. $\mu(vx) = \mu(v)x$ for any $v \in \mathbb{K}^4$ and $x \in \mathfrak{o}_4(\mathbb{K})$.

Applying Lemma 3 to α, there is some $v_0 \in \mathbb{K}^4$ such that $\alpha(x) = vx$ for any x. Also, since μ is linear there must be a matrix P such that $\mu(v) = vP$ for any v. But imposing the condition in the second item above we find that P must be a scalar multiple of the identity. So $\mu(v) = \lambda v$ for any $v \in \mathbb{K}^4$. Finally the fact that f is injective implies $\lambda \neq 0$. □

Thus $\ker(Q) \cong \mathbb{K}^\times \times \mathbb{K}^4$ with multiplication $(\lambda, v_0)(\mu, w_0) := (\lambda\mu, \lambda w_0 + v_0)$. Consequently $\mathrm{Aut}(\mathfrak{p})$ fits in a exact sequence

$$1 \to \mathbb{K}^\times \times \mathbb{K}^4 \to \mathrm{Aut}(\mathfrak{p}) \xrightarrow{Q} \mathrm{Aut}(\mathfrak{o}_4(\mathbb{K})) \to 1 \tag{15}$$

which is also split: the map $J : \mathrm{Aut}(\mathfrak{o}_4(\mathbb{K})) \to \mathrm{Aut}(\mathfrak{p})$ such that $J(\theta) : \begin{pmatrix} 0 & v \\ 0 & x \end{pmatrix} \mapsto \begin{pmatrix} 0 & vx_0^{-1} \\ 0 & \theta(x) \end{pmatrix}$ for any automorphism $\theta = \mathrm{Ad}(x_0)$ of $\mathfrak{o}_4(\mathbb{K})$, is a monomorphism and $QJ = 1$. As a corollary of this, we have

$$\dim \mathrm{Aut}(\mathfrak{p}) = 11$$

since $\dim \mathrm{Aut}(\mathfrak{o}_4(\mathbb{K})) = 6$ (see Lemma 1 and Corollary (4)).

PROPOSITION 13. *The automorphism group* $\mathrm{Aut}(\mathfrak{p})$ *agrees with the group*

$$\mathrm{Aut}(\mathfrak{p}) = \{f_{\lambda, v_0, x_0} : \lambda \in \mathbb{K}^\times, v_0 \in \mathbb{K}^4, Ad(x_0) \in \mathrm{Aut}(\mathfrak{o}_4(\mathbb{K}))\}$$

where

$$f_{\lambda,v_0,x_0}\left[\begin{pmatrix}0 & v\\ 0 & x\end{pmatrix}\right] := \begin{pmatrix}0 & \lambda v x_0^{-1} + v_0 x_0 x x_0^{-1}\\ 0 & x_0 x x_0^{-1}\end{pmatrix}.$$

Some multiplicative relations of these elements are:

$$f_{\lambda,v_0,x_0} f_{\mu,v_1,x_1} = f_{\lambda\mu,\lambda v_1 x_0^{-1}+v_0,x_0x_1},$$

$$f_{\lambda,v_0,x_0}^{-1} = f_{\lambda^{-1},-\lambda^{-1}v_0x_0,x_0^{-1}}.$$

PROOF. Since the exact sequence (15) is split, any automorphism of $\mathfrak{p}$ is of the form $f_{\lambda,v_0}J(\theta)$ and this gives the description of $\mathrm{Aut}(\mathfrak{p})$ claimed in the statement of the Proposition. The multiplicative relations are straightforward to check. □

13.1. Translation to affine groups. In this section we extend some of the previous results on automorphisms of $\mathfrak{p}$ to the algebraic group $\mathbf{alg}_\Phi(\mathfrak{p})$ where Φ is an algebraically closed field of characteristic not two. Thus $\mathbf{Aut}_\Phi(\mathfrak{p})\colon \mathbf{alg}_\Phi \to \mathbf{Lie}_\Phi$ is the affine group scheme such that $R \mapsto \mathrm{Aut}_R(\mathfrak{p}_R)$ (we will drop the index R in $\mathrm{Aut}_R(\,)$ in the cases in which no ambiguity is possible). If we denote by $i\colon \mathfrak{o}_4(R) \to \mathfrak{p}_R$ the natural injection $M \mapsto \begin{pmatrix}0 & 0\\ 0 & M\end{pmatrix}$, and by $\pi\colon \mathfrak{p}_R \to \mathfrak{o}_4(R)$ the map $\begin{pmatrix}0 & v\\ 0 & M\end{pmatrix} \mapsto M$ (which is an epimorphism of Lie algebras), we can as before define a group homomorphism $Q_R\colon \mathrm{Aut}(\mathfrak{p}_R) \to \mathrm{Aut}(\mathfrak{o}_4(R))$ given by $f \mapsto \pi f i$. This enables us to define a homomorphism of algebraic groups $\mathbf{Q}\colon \mathbf{Aut}_\Phi(\mathfrak{p}) \to \mathbf{Aut}_\Phi(o_4(\Phi))$ such that $\mathbf{Q}(R) := Q_R$.

LEMMA 21. *The homomorphism* $\mathbf{Q}$ *is surjective.*

PROOF. By Lemma 1, $\mathfrak{o}_4(\Phi) \cong \mathfrak{L}_\Phi$ and by Corollary 4 the algebraic group $\mathbf{Aut}(\mathfrak{L}_\Phi)$ is smooth. Taking into account that $\mathbf{Q}_\Phi$ is the epimorphism Q introduced in the first paragraph of Section 13, applying [16, Proposition (22.3), p. 339] we get that $\mathbf{Q}$ is surjective. □

Next we compute the kernel $\ker(\mathbf{Q})$. In order to do this, we describe $\ker(Q_R)$. This is formed by all the automorphisms f of $\mathfrak{p}_R$ of the form

$$\begin{pmatrix}0 & v\\ 0 & x\end{pmatrix} \mapsto \begin{pmatrix}0 & \mu(v)+\alpha(x)\\ 0 & x\end{pmatrix}$$

for some R-linear maps $\mu\colon R^4 \to R^4$, $\alpha\colon \mathfrak{o}_4(R) \to R^4$, $\beta\colon \mathfrak{o}_4(R) \to \mathfrak{o}_4(R)$, which must satisfy :

1. $\alpha([x,y]) = \alpha(x)y - \alpha(y)x$ for any $x, y \in \mathfrak{o}_4(R)$.
2. $\mu(vx) = \mu(v)x$ for any $v \in R^4$ and $x \in \mathfrak{o}_4(R)$.

But μ must be invertible since it is the restriction of f to $\mathfrak{r}_R$. So there is an invertible matrix $P \in \mathrm{GL}_4(R)$ such that $\mu(v) = vP$ for any v. Now, condition (2) implies that $P = \lambda 1$ for some $\lambda \in R^{\times}$. On the other hand, Lemma 3 applied to α gives that there is a $v \in R^4$ such that $\alpha(x) = vx$ for any x. Thus we can describe $\ker(Q_R)$ as before, $\ker(Q_R) = R^{\times} \times R^4$ with multiplication $(\lambda, v_0)(\mu, w_0) := (\lambda\mu, \lambda w_0 + v_0)$. So we define the algebraic group $\ker(\mathbf{Q})$ such that $R \mapsto R^{\times} \times R^4$, and there is a sequence

$$1 \to \ker(\mathbf{Q}) \to \mathbf{Aut}_{\Phi}(\mathfrak{p}) \overset{\mathbf{Q}}{\to} \mathbf{Aut}(\mathfrak{o}_4(\Phi)) \to 1 \tag{16}$$

which is not a priori exact. But applying [16, Proposition 22.10, p. 341], we have the exactness of the sequence (16).

REMARK 11. As a corollary of the exact sequence (16) and the previous results on derivations, under the assumptions of this section on the ground field, the group $\mathbf{Aut}_{\Phi}(\mathfrak{p})$ is smooth.

14. Restricted Lorentz and Poincaré algebras

In this section we consider fields Φ of prime characteristic and add some information about the restricted Lie algebras $\mathfrak{L}_{\Phi}$ and $\mathfrak{p}_{\Phi}$. So assume that Φ is a field of prime characteristic $p \neq 2$. Then $\mathfrak{L}_{\Phi}$ coincides with the Lie algebra of matrices $M \in \mathfrak{gl}_4(\Phi)$ such that $MI_{13}+I_{13}M^t = 0$. So, if $M \in \mathfrak{L}_{\Phi}$ we have $M^2I_{13}+MI_{13}M^t = 0$ or $M^2I_{13}-I_{13}(M^2)^t = 0$. More generally it is easy to check that

$$M^kI_{13} + (-1)^{k+1}I_{13}(M^k)^t = 0$$

for any natural number k. In particular $M^pI_{13} + I_{13}(M^p)^t = 0$ hence $M^p \in \mathfrak{L}_{\Phi}$. Consequently, $\mathfrak{L}_{\Phi}$ is a restricted Lie algebra relative to the p-operation $\mathfrak{L}_{\Phi} \to \mathfrak{L}_{\Phi}$ such that $M \mapsto M^p$.

By Theorem 3, we know that $\mathbb{i} \in \Phi$ if and only if the Lorentz type algebra $\mathfrak{L}$ is not simple. Therefore we must consider the nonsimple case: $\mathbb{i} \in \Phi$. Under this hypothesis we identify $\mathfrak{L}_{\Phi}$ with $\mathfrak{o}_4(\Phi)$. To study the p-structure of $\mathfrak{L}_{\Phi}$ we consider the basis $\{h_\alpha, v_\alpha, v_{-\alpha}, h_\beta, v_\beta, v_{-\beta}\}$ defined in the proof of Proposition 5. We know that $\mathfrak{L}_{\Phi} = I \oplus J$ where I is the subspace generated by $\{h_\alpha, v_\alpha, v_{-\alpha}\}$ and J the one generated by $\{h_\beta, v_\beta, v_{-\beta}\}$. Moreover, these are ideals of $\mathfrak{L}_{\Phi}$ and $I \cong J \cong \mathfrak{sl}_2(\Phi)$. Now, we must check if these two ideals are p-ideals, that is, $I^p \subset I$ and $J^p \subset J$. But this is easy because if we take, for instance, a generic element $g = xh_\alpha + yv_\alpha + zv_{-\alpha}$ in I, then for any natural n we have

$$\begin{cases} g^{2n} = (x^2 + yz)^n I \\ g^{2n+1} = (x^2 + yz)^n g\,, \end{cases}$$

where I denotes the identity 4×4 matrix. In particular $g^p \in I$ hence $I^p \subset I$ and similarly $J^p \subset J$. Thus, the restricted algebra structure of $\mathfrak{L}_{\Phi}$ agrees with its algebra structure.

We would like also to comment that $\mathfrak{sl}_2(\Phi)$ is simple and a restricted Lie algebra whose p-operation is again $x \mapsto x^p$. So, $\mathfrak{sl}_2(\Phi) \oplus \mathfrak{sl}_2(\Phi)$ is a restricted algebra relative to $(x, y) \mapsto$

(x^p, y^p). The unique nonzero proper ideals of $\mathfrak{sl}_2(\Phi)\oplus\mathfrak{sl}_2(\Phi)$ are $\mathfrak{sl}_2(\Phi)\oplus 0$ and $0\oplus\mathfrak{sl}_2(\Phi)$ and of course these are p-ideals. Moreover, the isomorphism $\mathfrak{L}_\Phi \cong \mathfrak{sl}_2(\Phi)\oplus\mathfrak{sl}_2(\Phi)$ is an isomorphism of restricted Lie algebras (which amounts to the same work as proving that the ideals I and J above are p-ideals). Thus, we identify in the sequel the ideal I with the one generated by $\{h_\alpha, v_\alpha, v_{-\alpha}\}$ and J with the one generated by $\{h_\beta, v_\beta, v_{-\beta}\}$.

Consider now the Poincaré algebra $\mathfrak{p}_\Phi = \begin{pmatrix} 0 & \Phi^4 \\ 0 & \mathfrak{L}_\Phi \end{pmatrix}$. For any element in this algebra and any natural n, we have

$$\begin{pmatrix} 0 & v \\ 0 & x \end{pmatrix}^n = \begin{pmatrix} 0 & vx^{n-1} \\ 0 & x^n \end{pmatrix}. \tag{17}$$

The p-operation is again given by $\mathfrak{p}_\Phi \to \mathfrak{p}_\Phi$ such that $m \mapsto m^p$. We know by Lemma 18 that the unique proper and nonzero ideals of $\mathfrak{p}_\Phi$ are its radical $\mathfrak{r} = \begin{pmatrix} 0 & \Phi^4 \\ 0 & 0 \end{pmatrix}$, $\mathfrak{i} = \begin{pmatrix} 0 & \Phi^4 \\ 0 & I \end{pmatrix}$ and $\mathfrak{j} = \begin{pmatrix} 0 & \Phi^4 \\ 0 & J \end{pmatrix}$. So the formula (17) proves that all of them are p-ideals. Again the restricted algebra structure of $\mathfrak{p}_\Phi$ coincides with its algebra structure.

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