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Quasi Contact Metric Manifolds with Constant Sectional Curvature

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Abstract. A quasi contact metric manifold is a natural generalization of a contact metric manifold based on the geometry of the corresponding quasi Kahler cones. In this paper we prove that if a quasi contact metric manifold has constant sectional curvature c, then $c = 1$; additionally, if the characteristic vector field is Killing, then the manifold is Sasakian. These facts are some generalizations of Olszak's theorem to quasi contact metric manifolds.

1. Introduction

A $(2n + 1)$-dimensional smooth manifold M is said to have an almost contact structure (J. W. Gray [3]) if the structural group of its tangent bundle reduces to $U(n) \times 1$; equivalently (S. Sasaki and S. Hatakeyama [7], [8]) an almost contact structure is given by a triple (ϕ, ξ, η) satisfying certain conditions (see Section 2). The almost contact structure is comparable to the almost complex structure of even dimensional manifolds. Many different types of almost contact structures are defined by D. Chinea and C. Gonzalez ([2]), and Y. Tashiro ([10]). Tashiro in [9] showed that an orientable hypersurface in an almost complex manifold has an almost contact structure and that a hypersurface in an almost Hermitian manifold has an almost contact metric structure. For almost contact metric manifold (M, ϕ, ξ, η, g) Tashiro in [10] defined on $\overline{M} = M \times \mathbb{R}$ the almost complex structure J by the natural extension of ϕ into $\overline{M}$ and the metric tensor G by

$$G := e^{-2t} \begin{pmatrix} g_{ij} & 0 \\ 0 & 1 \end{pmatrix}, \qquad \text{with } t \in \mathbb{R}.$$

He proved that if M is almost contact metric with $N_\phi = 0$, the contact metric or Sasakian, then the manifold $\overline{M}$ constructed above is Hermitian, almost Kahlerian or Kahlerian respectively, and vice versa. He called M an O^*-manifold if $\overline{M}$ is quasi Kahlerian. Later J. H. Kim, J. H. Park and K. Sekigawa in [4] called this kind of manifold *a quasi contact metric manifold*.

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They studied this kind of manifolds from the view point of a generalization of contact metric manifolds and gave an equivalent condition for this class of almost contact metric manifolds based on the property of quasi Kahler manifold $\overline{M}$ that we explain in Section 2. Kim, Park and W. Shin in [5] proved that quasi contact metric manifolds are contact.

For a contact metric manifold (M, ϕ, ξ, η, g) with $\dim M = 2n+1 \geq 5$, Olszak has proved in [6] the following important theorem:

THEOREM 1. *If a contact metric manifold M^{2n+1} is of constant curvature c and* $\dim M \geq 5$, *then $c = 1$ and the structure is Sasakian.*

In this paper we describe a quasi contact metric manifold with constant sectional curvature as follows:

MAIN THEOREM. *If a quasi contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is of constant sectional curvature c and $n \geqslant 2$, then $c = 1$ and $\|\nabla\phi\|^2 - 4n = 3\|d\Phi\|^2$. Furthermore, if characteristic vector field, ξ, is a unit Killing field, then M is Sasakian.*

This paper is organized as follows: In Section 2, we give some primary concepts needed to start. In Section 3, a relationship is established among scalar curvature, $*$-scalar curvature, $\|\nabla\phi\|^2$ and $\|d\Phi\|^2$ on quasi contact metric manifolds. In fact, this relation generalizes one made by Z. Olszak in [6] for contact metric manifolds. Finally, in Section 4, we prove the Main Theorem.

2. Preliminaries

A $(2n+1)$-dimensional smooth manifold equipped with a triple (ϕ, ξ, η) of a (1,1)-tensor field ϕ, a vector field ξ and a 1-form η on M satisfying

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1$$

for $X \in \chi(M)$, is called an *almost contact manifold* with the *almost contact structure* (ϕ, ξ, η). Further, an almost contact manifold $M = (M, \phi, \xi, \eta)$ equipped with a Riemannian metric g satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(\xi, X)$$

for $X, Y \in \chi(M)$, is called an *almost contact metric manifold* with *almost contact metric structure* (ϕ, ξ, η, g). An almost contact metric manifold (M, ϕ, ξ, η, g) is called a *contact metric manifold* if $d\eta(X, Y) = g(X, \phi Y)$ for every $X, Y \in \chi(M)$.

If (M, ϕ, ξ, η, g) is an almost contact metric manifold, we can define on $\overline{M} = M \times \mathbb{R}$ an almost Hermitian structure (J, G) as

$$JX = \phi X - \eta(X)\frac{\partial}{\partial t}, \qquad J\frac{\partial}{\partial t} = \xi,$$

$$G(X, Y) = e^{-2t} g(X, Y), \qquad G(X, \frac{\partial}{\partial t}) = 0, \qquad G(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = e^{-2t},$$

for $X, Y \in \chi(M)$. M is called a *quasi contact metric manifold* if $\overline{M}$ is quasi Kahlerian. It is well known that M is a quasi contact metric manifold if and only if:

$$(\nabla_X\phi)Y + (\nabla_{\phi X}\phi)\phi Y = 2g(X, Y)\xi - \eta(Y)(X + \eta(X)\xi + hX) \tag{1}$$

for $X, Y \in \chi(M)$ ([4]), in which $h = \frac{1}{2}L_\xi\phi$ (L_ξ is the Lie differentiation along ξ). Therefore

$$hX = \frac{1}{2}\big((\nabla_\xi\phi)X - \nabla_{\phi X}\xi + \phi\nabla_X\xi\big)\,. \tag{2}$$

We can rewrite equation (1) as

$$(\nabla_{\phi X}\Phi)(\phi Y, Z) + (\nabla_X\Phi)(Y, Z) = -2\eta(Z)g(\phi X, \phi Y) + \eta(Y)g(\phi X + hX, \phi Z)\,,$$

where $\Phi(X, Y) := g(X, \phi Y)$. It is well known ([5]) that (φ, ξ, η, g)-quasi contact metric manifolds are contact with contact form η, i.e., $\eta \wedge (d\eta)^n \neq 0$ everywhere on M, and contact metric manifolds are quasi contact metric manifolds ([1], [2]), thus quasi contact metric manifolds can be thought as a generalization of contact metric manifolds. Now we recall some basic properties satisfied by quasi contact metric structures which will be used in this paper ([1], [4]):

$$h\xi = 0\,, \tag{3}$$

$$\eta \circ h = 0\,, \tag{4}$$

$$\nabla_\xi\xi = 0\,, \tag{5}$$

$$\nabla_X\xi = -\phi X - \phi hX\,, \tag{6}$$

$$\nabla_\xi\phi = 0\,, \tag{7}$$

$$\phi h + h\phi = 0\,, \tag{8}$$

$$(\nabla_X\eta)Y + (\nabla_{\phi X}\eta)\phi Y + 2g(\phi X, Y) = 0\,, \tag{9}$$

$$d\eta(X, Y) = \Phi(X, Y) + \frac{1}{2}[\Phi(hX, Y) + \Phi(X, hY)]\,, \tag{10}$$

$$(\nabla_{\phi X}\phi)Y - (\nabla_X\phi)\phi Y = 2g(\phi X, Y)\xi - \eta(Y)[\phi X + h\phi X]\,. \tag{11}$$

By (3), (4) and (8) and taking a ϕ-basis $\{e_1, ..., e_n, \phi e_1, ..., \phi e_n, \xi\}$ we can show that

$$\mathrm{tr}h = 0\,. \tag{12}$$

Olszak in [6] proved that for an orthonormal ϕ-basis $\{e_i\}$ on a contact metric manifold, we have

$$\sum_{i=1}^{2n+1}(\nabla_{e_i}\phi)e_i = 2n\xi\,, \tag{13}$$

$$\sum_{i=1}^{2n+1}(\nabla_{e_i}\phi)\phi e_i = 0\,. \tag{14}$$

Using (1) and (11) one can easily show that these relations hold on a quasi contact metric manifold, too. Blair in [1] proved that a contact metric manifold satisfies the following relations:

$$(\nabla_\xi h)X = \phi X - h^2\phi X - \phi R_{X\xi}\xi\,, \tag{15}$$

$$\frac{1}{2}(R_{\xi X}\xi - \phi R_{\xi\phi X}\xi) = h^2 X + \phi^2 X\,, \tag{16}$$

$$\mathrm{Ric}(\xi) = 2n - \mathrm{tr}h^2\,. \tag{17}$$

For proving these relations he used only relations (6), (7) and (8), thus a quasi contact metric manifold also satisfies the same formulas.

LEMMA 1. *On a quasi contact metric manifold* (M, ϕ, ξ, η, g) *we have the following relations*:

$$(\nabla_X\Phi)(\phi Y, Z) - (\nabla_X\Phi)(Y, \phi Z) = -g(X + hX, \phi(\eta(Y)Z + \eta(Z)Y))\,, \tag{18}$$

$$(\nabla_X\Phi)(\phi Y, \phi Z) + (\nabla_X\Phi)(Y, Z) = g(X + hX, \eta(Y)Z - \eta(Z)Y)\,, \tag{19}$$

$$(\nabla_Y\phi h)Z - \phi(\nabla_Y\phi h)\phi Z = (\nabla_Y\phi)hZ - h(\nabla_Y\phi)Z\,. \tag{20}$$

PROOF. It is easy to show that on an almost contact metric manifold $M = (M, \phi, \xi, \eta, g)$ we have

$$(\nabla_X\phi)\phi Y = -\phi(\nabla_X\phi)Y + \eta(Y)\nabla_X\xi + ((\nabla_X\eta)Y)\xi\,.$$

This relation and (6), imply (18), and by substituting ϕZ for Z in (18) we get (19). Finally straightforward computation, by using (8) proves the last equation. □

3. Curvature of quasi contact metric manifolds

In this section we obtain some properties on curvature of a quasi contact metric manifold that we need in the sequel. We also give a relationship among scalar curvature, $*$-scalar curvature, $\|d\Phi\|^2$ and $\|\nabla\phi\|^2$ in a (ϕ, ξ, η, g) quasi contact metric manifold.

LEMMA 2. *The curvature tensor of a quasi contact metric manifold satisfies*:

$$\begin{aligned} g(R_{\xi X}Y, Z) &= (\nabla_Y\Phi)(Z, X) + (\nabla_Z\Phi)(X, Y) \\ &\quad - g(X, (\nabla_Y\phi h)Z - (\nabla_Z\phi h)Y)\,, \end{aligned} \tag{21}$$

and

$$\begin{aligned} &g(R_{\xi X}Y, Z) - g(R_{\xi X}\phi Y, \phi Z) + g(R_{\xi\phi X}Y, \phi Z) + g(R_{\xi\phi X}\phi Y, Z) \\ &= 2g(X, Y + hY)\eta(Z) - 2g(X, Z + hZ)\eta(Y) \\ &+ 2g(X, h((\nabla_Y\phi)Z - (\nabla_Z\phi)Y))\,. \end{aligned} \tag{22}$$

PROOF. Differentiating $\nabla_Z \xi = -\phi Z - \phi h Z$, yields

$$R_{YZ}\xi = -(\nabla_Y \phi)Z + (\nabla_Z \phi)Y - (\nabla_Y \phi h)Z + (\nabla_Z \phi h)Y .$$

Thus (21) is obtained. Now we set

$$\begin{aligned} A(X, Y, Z) = &(\nabla_Y \Phi)(Z, X) - (\nabla_{\phi Y} \Phi)(\phi Z, X) \\ &+ (\nabla_Y \Phi)(\phi Z, \phi X) + (\nabla_{\phi Y} \Phi)(Z, \phi X) \end{aligned}$$

and

$$\begin{aligned} B(X, Y, Z) = &- g(X, (\nabla_Y \phi h)Z) + g(X, (\nabla_{\phi Y} \phi h)\phi Z) \\ &- g(\phi X, (\nabla_Y \phi h)\phi Z) - g(\phi X, (\nabla_{\phi Y} \phi h)Z) . \end{aligned}$$

By using (21), the left side of (22) is equal to

$$A(X, Y, Z) - A(X, Z, Y) + B(X, Y, Z) - B(X, Z, Y) . \tag{23}$$

On the other hand by (18) and (19) we have

$$A(X, Y, Z) = 2\eta(Z)g(X, Y) - 2\eta(X)g(hY, Z) - 2\eta(X)\eta(Y)\eta(Z) ,$$

and thus

$$\begin{aligned} A(X, Y, Z) - A(X, Z, Y) = &2\eta(Z)g(X, Y) - 2\eta(Y)g(X, Z) \\ &- 2\eta(X)[g(hY, Z) - g(hZ, Y)] , \end{aligned} \tag{24}$$

and from (20) and (1) we obtain

$$B(X, Y, Z) = 2\eta(Z)g(X, hY) - 2\eta(X)g(Y, hZ) + 2g(X, h(\nabla_Y \phi)Z) ,$$

and thus

$$\begin{aligned} B(X, Y, Z) - B(X, Z, Y) = &2\eta(Z)g(X, hY) - 2\eta(Y)g(X, hZ) \\ &- 2\eta(X)[g(Y, hZ) - g(Z, hY)] \\ &+ 2g(X, h((\nabla_Y \phi)Z - (\nabla_Z \phi)Y) . \end{aligned} \tag{25}$$

Finally, substitution of (24) and (25) into (23) yields (22). □

Recall that, for a local orthogonal basis $\{e_1, ..., e_{2n+1}\}$, we defined $\|\nabla\phi\|^2$ and $\|d\Phi\|^2$ as

$$\|\nabla\phi\|^2 = \sum_{i,j,k=1}^{2n+1} g((\nabla_{e_i}\phi)e_j, e_k)^2 , \qquad \|d\Phi\|^2 = \sum_{i,j,k=1}^{2n+1} d\Phi(e_i, e_j, e_k)^2 .$$

Also, if S and S^* are the scalar curvature and the $*$-scalar curvature, respectively, then

$$S = \sum_{i,j=1}^{2n+1} g(R_{e_i e_j} e_j, e_i) , \qquad S^* = \sum_{i,j=1}^{2n+1} g(R_{e_i e_j} \phi e_j, \phi e_i) .$$

LEMMA 3. *Let $M = (M, \phi, \xi, \eta, g)$ be a $(2n+1)$-dimensional quasi contact metric manifold and $\{e_1, ..., e_{2n+1}\}$ be an orthonormal and covariant constant frame at $m \in M$. Then we have*

$$\sum_{i,j=1}^{2n+1} g((R_{e_j e_i}\phi)e_i, \phi e_j) = \sum_{i,j=1}^{2n+1} g((\nabla_{e_i}\phi)e_j, (\nabla_{e_j}\phi)e_i) - 4n^2 . \tag{26}$$

PROOF. From (13), (6) and (12), we have

$$\begin{aligned}\sum_{i,j=1}^{2n+1} g((\nabla_{e_j}\nabla_{e_i}\phi)e_i, \phi e_j) &= \sum_{i,j=1}^{2n+1} g(\nabla_{e_j}(\nabla_{e_i}\phi)e_i - (\nabla_{e_i}\phi)\nabla_{e_j}e_i, \phi e_j)\\ &= 2n\sum_{j=1}^{2n+1} g(\nabla_{e_j}\xi, \phi e_j) + \sum_{i,j=1}^{2n+1} g((\nabla_{e_i}\phi)\phi e_j, \nabla_{e_j}e_i)\\ &= -2n\sum_{j=1}^{2n+1} g(\phi e_j + \phi h e_j, \phi e_j) = -4n^2\end{aligned}$$

and from (14) we obtain

$$\begin{aligned}\sum_{i,j=1}^{2n+1} g((\nabla_{e_i}\nabla_{e_j}\phi)e_i, \phi e_j) &= \sum_{i,j=1}^{2n+1} g(\nabla_{e_i}(\nabla_{e_j}\phi)e_i - (\nabla_{e_j}\phi)\nabla_{e_i}e_i, \phi e_j)\\ &= \sum_{i,j=1}^{2n+1} \left\{e_i g((\nabla_{e_j}\phi)e_i, \phi e_j) - g((\nabla_{e_j}\phi)e_i, (\nabla_{e_i}\phi)e_j)\right\}\\ &= -\sum_{i,j=1}^{2n+1} g((\nabla_{e_j}\phi)e_i, (\nabla_{e_i}\phi)e_j) .\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{i,j=1}^{2n+1} g((R_{e_j e_i}\phi)e_i, \phi e_j) &= \sum_{i,j=1}^{2n+1} \left\{g((\nabla_{e_j}\nabla_{e_i}\phi)e_i, \phi e_j) - g((\nabla_{e_i}\nabla_{e_j}\phi)e_i, \phi e_j)\right\}\\ &= -4n^2 + \sum_{i,j=1}^{2n+1} g((\nabla_{e_j}\phi)e_i, (\nabla_{e_i}\phi)e_j) .\end{aligned}$$

□

Olszak in [6] established a relationship among S, S^* and $\|\nabla\phi\|^2$ on a contact metric manifold such that

$$S^* - S + 4n^2 = \frac{1}{2}\|\nabla\phi\|^2 + \mathrm{tr}h^2 - 2n .$$

In the following we construct a similar relation for a quasi contact metric manifold which is used in the proof of the Main Theorem.

PROPOSITION 1. *On a quasi contact metric manifold M^{2n+1} we have*

$$S^* - S + 4n^2 = \frac{1}{2}\|\nabla\phi\|^2 - \frac{3}{2}\|d\Phi\|^2 + \mathrm{tr}h^2 - 2n\,. \tag{27}$$

PROOF. Let $\{e_1, ..., e_{2n+1}\}$ be an orthonormal basis in T_mM. By the same letters, we denote the local extension vector fields of this frame which are orthonormal and covariant constant at $m \in M$. Now by Lemma 3 we have

$$\sum_{i,j=1}^{2n+1} g((R_{e_je_i}\phi)e_i, \phi e_j) = \sum_{i,j=1}^{2n+1} g((\nabla_{e_i}\phi)e_j, (\nabla_{e_j}\phi)e_i) - 4n^2\,.$$

We set

$$\begin{aligned} D &= \sum_{i,j=1}^{2n+1} g((\nabla_{e_i}\phi)e_j, (\nabla_{e_j}\phi)e_i) \\ &= \sum_{i,j,k=1}^{2n+1} g((\nabla_{e_i}\phi)e_j, e_k)g((\nabla_{e_j}\phi)e_i, e_k) \end{aligned} \tag{28}$$

to write this in the form

$$\sum_{i,j=1}^{2n+1} g((R_{e_je_i}\phi)e_i, \phi e_j) = D - 4n^2\,.$$

On the other hand, computation of the left hand side of the last equation yields

$$\begin{aligned} \sum_{i,j=1}^{2n+1} g((R_{e_je_i}\phi)e_i, \phi e_j) &= -\sum_{i,j=1}^{2n+1} \left\{g(R_{e_ie_j}\phi e_i, \phi e_j) - g(\phi R_{e_ie_j}e_i, \phi e_j)\right\} \\ &= S^* - S + \sum_{i,j=1}^{2n+1} \eta(e_j)\eta(R_{e_ie_j}e_i)\,, \end{aligned}$$

which implies

$$S^* - S + Ric(\xi) = D - 4n^2\,. \tag{29}$$

Furthermore by (28) and $g(e_i, (\nabla_{e_j}\phi)e_k) = -g(e_k, (\nabla_{e_j}\phi)e_i)$, we have

$$9\|d\Phi\|^2 = 9\sum_{i,j,k=1}^{2n+1} d\Phi(e_i, e_j, e_k)^2$$

$$= \sum_{i,j,k=1}^{2n+1} \Big\{(\nabla_{e_i}\Phi)(e_j, e_k) + (\nabla_{e_j}\Phi)(e_k, e_i) + (\nabla_{e_k}\Phi)(e_i, e_j)\Big\}^2$$

$$= \sum_{i,j,k=1}^{2n+1} \Big\{g(e_j, (\nabla_{e_i}\phi)e_k) + g(e_k, (\nabla_{e_j}\phi)e_i) + g(e_i, (\nabla_{e_k}\phi)e_j)\Big\}^2$$

$$= \sum_{i,j,k=1}^{2n+1} \Big\{g(e_j, (\nabla_{e_i}\phi)e_k)^2 + g(e_k, (\nabla_{e_j}\phi)e_i)^2 + g(e_i, (\nabla_{e_k}\phi)e_j)^2\Big\}$$

$$+ 2\sum_{i,j,k=1}^{2n+1} \Big\{g(e_j, (\nabla_{e_i}\phi)e_k)g(e_k, (\nabla_{e_j}\phi)e_i)$$

$$+ g(e_j, (\nabla_{e_i}\phi)e_k)g(e_i, (\nabla_{e_k}\phi)e_j) + g(e_k, (\nabla_{e_j}\phi)e_i)g(e_i, (\nabla_{e_k}\phi)e_j)\Big\}$$

$$= \sum_{i,j,k=1}^{2n+1} \Big\{3g((\nabla_{e_i}\phi)e_j, e_k)^2 - 6g((\nabla_{e_i}\phi)e_j, e_k)g((\nabla_{e_j}\phi)e_i, e_k)\Big\}$$

$$= 3\|\nabla\phi\|^2 - 6D\,.$$

Therefore

$$D = \frac{1}{2}\|\nabla\phi\|^2 - \frac{3}{2}\|d\Phi\|^2\,. \tag{30}$$

Substituting the above equation and (17) in (29), we obtain the proposition. □

4. Quasi contact metric manifolds with constant sectional curvature

Now we can prove Main Theorem that describes a quasi contact metric manifold with constant sectional curvature.

PROOF OF THE MAIN THEOREM. Since M has constant sectional curvature c we have

$$R_{XY}Z = c\{g(Y, Z)X - g(X, Z)Y\}\,.$$

This and (16) give $h^2X = (c-1)\phi^2X$, and thus $\mathrm{tr}h^2 = 2n(1-c)$. On the other hand in virtue of (22) we have

$$(c-1)\big\{\eta(Z)Y - \eta(Y)Z\big\} = h\big\{\eta(Z)Y - \eta(Y)Z + (\nabla_Y\phi)Z - (\nabla_Z\phi)Y\big\}\,.$$

Applying h and using $h^2 = (c-1)\phi^2$, we get

$$(c-1)\big\{\eta(Z)hY - \eta(Y)hZ\big\} = (c-1)\big\{\eta(Y)Z - \eta(Z)Y + (\nabla_Z\phi)Y - (\nabla_Y\phi)Z$$
$$+ g(Z, hY)\xi - g(Y, hZ)\xi\big\}\,.$$

If $c \neq 1$, then

$$(\nabla_Y \phi)Z - (\nabla_Z \phi)Y = \eta(Y)(Z + hZ) - \eta(Z)(Y + hY) + g(Z, hY)\xi - g(Y, hZ)\xi \,, \tag{31}$$

and thus

$$g((\nabla_Y \phi)Z, Y) = g(Y + hY, \eta(Y)Z - \eta(Z)Y) \,.$$

Therefore the substitution $Y \mapsto X + Y$ in the last equation and (31) yield

$$g((\nabla_X \phi)Y + (\nabla_Y \phi)X, Z) = g(X + hX, \eta(Z)Y - \eta(Y)Z) + g(Y + hY, \eta(Z)X - \eta(X)Z) \,,$$

and

$$g((\nabla_X \phi)Y - (\nabla_Y \phi)X, Z) = \eta(Z)[g(Y, hX) - g(X, hY)] + \eta(X)g(Y + hY, Z) - \eta(Y)g(X + hX, Z) \,, \tag{32}$$

respectively. Adding last two equations, we get

$$g((\nabla_X \phi)Y, Z) = -\eta(Y)g(X + hX, Z) + \eta(Z)g(Y, X + hX) \,, \tag{33}$$

which implies

$$\|\nabla \phi\|^2 = 4n + 2\|h\|^2 \,. \tag{34}$$

Putting $Z = \xi$ in (33), we obtain

$$\eta((\nabla_X \phi)Y) = g(X + hX, Y) - \eta(X)\eta(Y) \,. \tag{35}$$

Also, from (32) we have $g((\nabla_{\phi X} \phi)\phi Y - (\nabla_{\phi Y} \phi)\phi X, \phi Z) = 0$, which by taking a ϕ-basis $\{e_i\}_{i=1}^{2n+1}$, is written in the form

$$\begin{aligned}
0 &= \sum_{i,j,k=1}^{2n+1} g((\nabla_{\phi e_i} \phi)\phi e_j - (\nabla_{\phi e_j} \phi)\phi e_i, \phi e_k)^2 \\
&= 2 \sum_{i,j,k=1}^{2n+1} \left\{ g((\nabla_{\phi e_i} \phi)\phi e_j, \phi e_k)^2 - g((\nabla_{\phi e_i} \phi)\phi e_j, \phi e_k) g((\nabla_{\phi e_j} \phi)\phi e_i, \phi e_k) \right\} \\
&= 2 \sum_{i,j,k=1}^{2n+1} g((\nabla_{e_i} \phi)e_j, e_k)^2 - 4 \sum_{i,j=1}^{2n+1} \eta((\nabla_{e_i} \phi)e_j)^2 \\
&\quad - 2 \sum_{i,j,k=1}^{2n+1} g((\nabla_{e_i} \phi)e_j, e_k) g((\nabla_{e_j} \phi)e_i, e_k) + 2 \sum_{i,j=1}^{2n+1} \eta((\nabla_{e_i} \phi)e_j)\eta((\nabla_{e_j} \phi)e_i)
\end{aligned}$$

$$= 2\|\nabla\phi\|^2 - 2D - 4\sum_{i,j=1}^{2n+1}\eta((\nabla_{e_i}\phi)e_j)^2 + 2\sum_{i,j=1}^{2n+1}\eta((\nabla_{e_i}\phi)e_j)\eta((\nabla_{e_j}\phi)e_i)\,. \qquad (36)$$

Now (35) and straightforward computation show that

$$\sum_{i,j=1}^{2n+1}\eta((\nabla_{e_i}\phi)e_j)^2 = 2n + \|h\|^2, \qquad \sum_{i,j=1}^{2n+1}\eta((\nabla_{e_i}\phi e_j)\eta((\nabla_{e_j}\phi)e_i) = 2n + \mathrm{tr}h^2\,.$$

Substituting (34) and the above two equations in (36), it yields $D = 2n + \mathrm{tr}h^2$. This and $\mathrm{tr}h^2 = 2n(1-c)$, give

$$D = 4n - 2nc\,. \qquad (37)$$

Also it can be checked that for this manifold we have $S = 2n(2n+1)c$ and $S^* = 2nc$, thus

$$S^* - S = -4n^2c\,. \qquad (38)$$

Equations (27), (30), (37), (38) and $\mathrm{tr}h^2 = 2n(1-c)$, give $(4n^2-4n)(c-1) = 0$. Since $n > 2$, it contradicts $c \neq 1$. Therefore, $c = 1$ and $S^* - S = -4n^2$ and $\|\nabla\phi\|^2 - 4n = 3\|d\Phi\|^2$. Since $c = 1$, we have $R_{X\xi}\xi = X$ for every X orthogonal to ξ, therefore, if ξ is Killing, then by Proposition 7.4 of [1] M is a K-contact manifold. Thus it is a special case of Theorem 1, and we conclude that M is Sasakian.

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