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Real Hypersurfaces with *-Ricci Solitons of Non-flat Complex Space Forms

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Abstract. Kaimakamis and Panagiotidou in [11] introduced the notion of *-Ricci soliton and studied the real hypersurfaces of a non-flat complex space form admitting a *-Ricci soliton whose potential vector field is the structure vector field. In this article, we consider a real hypersurface of a non-flat complex space form which admits a *-Ricci soliton whose potential vector field belongs to the principal curvature space and the holomorphic distribution.

1. Introduction

An n-dimensional complex space form is an n-dimensional Kähler manifold with constant sectional curvature c. A complete and simple connected complex space form with $c \neq 0$ (i.e., a complex projective space $\mathbb{C}P^n$ or a complex hyperbolic space $\mathbb{C}H^n$) is called a *non-flat complex space form* and denoted by $\widetilde{M}^n(c)$.

Let M be a real hypersurface of $\widetilde{M}^n(c)$. Then there exists an almost contact structure (ϕ, η, ξ, g) on M induced from $\widetilde{M}^n(c)$. The study of real hypersurfaces in a non-flat complex space form is a very interesting and active field in recent decades and many results of the classification of real hypersurfaces in non-flat complex space forms were achieved (see [1, 13, 17, 18, 20]). In particular, if ξ is an eigenvector of the shape operator A then M is called a *Hopf hypersurface*, and we note that the following conclusion is due to Kimura and Takagi for $\mathbb{C}P^n$ and Berndt for $\mathbb{C}H^n$.

THEOREM 1 ([1, 12, 19]). *Let M be a Hopf hypersurface in non-flat complex space form $\widetilde{M}^n(c), n \geq 2$. If M has constant principal curvatures, then the classification is as follows*:

- *In case of $\mathbb{C}P^n$, M is locally congruent to one of the following*:
  1. A_1: *Geodesic hyperspheres.*
  2. A_2: *Tubes over a totally geodesic complex projective space $\mathbb{C}P^k$ for $1 \leq k \leq n-2$.*
  3. B: *Tubes over a complex quadric Q_{n-1} and $\mathbb{R}P^n$.*

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4. C: *Tubes over Segre embedding of* $\mathbb{C}P^1 \times \mathbb{C}P^{\frac{n-1}{2}}$, *n is odd and* $n \geq 5$.
5. D: *Tubes over Plücker embedding of the complex Grassmannian manifold* $G_{2,5}$. *This occurs only for* $n = 9$.
6. E: *Tubes over the canonical embedding Hermitian symmetry space* $SO(10)/U(5)$. *This occurs only for* $n = 15$.

• *In case of* $\mathbb{C}H^n$, M *is locally congruent to one of the following*:

1. A_1: *Geodesic hyperspheres* (*Type* A_{11}) *and tubes over totally geodesic complex hyperbolic hyperplanes* (*Type* A_{12}).
2. A_2: *Tubes over totally geodesic* $\mathbb{C}H^k \subset \mathbb{C}H^n$ *for some* $k \in \{1, \ldots, n-2\}$.
3. B: *Tubes over a totally geodesic real hyperbolic space* $\mathbb{R}H^n \subset \mathbb{C}H^n$.
4. N: *Horospheres.*

In particular, if M has two distinct constant principal curvatures, the classification is as follows:

THEOREM 2 ([17], Corollary 2 in [3]). *Let* M *be a hypersurface in non-flat complex space form* $\widetilde{M}^n(c)$ *with two distinct constant principal curvatures and* $n \geq 2$. *Then*
• *in case of* $\mathbb{C}P^n$, M *is locally congruent geodesic hyperspheres in* $\mathbb{C}P^n$(*Type* A_1);
• *in case of* $\mathbb{C}H^n$, M *is locally congruent to one of the following*:

1. A_{11}: *Geodesic hyperspheres in* $\mathbb{C}H^n$.
2. A_2: *Tubes around a totally geodesic* $\mathbb{C}H^{n-1} \subset \mathbb{C}H^n$.
3. B: *Tubes of radius* $r = \ln(2+\sqrt{3})$ *around a totally geodesic real hyperbolic space* $\mathbb{R}H^n \subset \mathbb{C}H^n$.
4. N: *Horospheres in* $\mathbb{C}H^n$.

Since there are no Einstein real hypersurfaces in $\widetilde{M}^n(c)$ (see [4] and [14]), Cho and Kimura in [5] considered a real hypersurface in $\widetilde{M}^n(c)$ admitting a Ricci soliton. The notion of Ricci soliton, introduced firstly by Hamilton in [7], is the generalization of Einstein metric, that is, a Riemannian metric g satisfying

$$\frac{1}{2}\mathcal{L}_W g + \mathrm{Ric} - \lambda g = 0\,,$$

where λ is a constant and Ric is the Ricci tensor of M. The vector field W is called *potential vector field*. Moreover, the Ricci soliton is called shrinking, steady, and expanding according as λ is positive, zero, and negative, respectively. In [5], it is proved that there does not admit a Ricci soliton on M when the potential vector field is the structure field ξ. At the same time, by introducing a so-called η-Ricci soliton (η, g) on M, which satisfies

$$\frac{1}{2}\mathcal{L}_W g + \mathrm{Ric} - \lambda g - \mu\eta\otimes\eta = 0\,,$$

for constants λ, μ, they gave a classification of a real hypersurface admitting an η-Ricci soliton whose potential vector is the structure field ξ. In [6], Cho and Kimura also proved that a compact real hypersurface of contact-type in a complex number space admitting a Ricci soliton is a sphere and a compact Hopf hypersurface in a non-flat complex space form does not admit a Ricci soliton.

As the corresponding of Ricci tensor, in [8] Hamada defined the *-Ricci tensor Ric* in real hypersurfaces of complex space form as

$$\mathrm{Ric}^*(X,Y) = \frac{1}{2}(\mathrm{trace}\{\phi \circ R(X,\phi Y)\}), \quad \text{for all } X, Y \in TM\,,$$

and if the *-Ricci tensor is a constant multiple of $g(X,Y)$ for all X, Y orthogonal to ξ, then M is said to be a **-Einstein manifold*. Furthermore, Hamada gave the following result of the *-Einstein Hopf hypersurfaces in non-flat space forms.

THEOREM 3 ([8]). *Let M be a *-Einstein Hopf hypersurface in non-flat complex space form $\widetilde{M}^n(c)$, $n \geq 2$.*

• *In case of $\mathbb{C}P^n$, M is an open part of one of the following*:

1. A_1: *a geodesic hypersphere*;
2. A_2: *a tube over a totally geodesic complex projective space $\mathbb{C}P^k$ of radius $\frac{\pi r}{4}$ for $1 \leq k \leq n-2$, where $r = \frac{2}{\sqrt{c}}$*;
3. B: *a tube over a complex quadric Q_{n-1} and $\mathbb{R}P^n$.*

• *In case of $\mathbb{C}H^n$, M is an open part of one of the following*:

1. A_{11}: *a geodesic hypersphere*;
2. A_{12} : *a tube around a totally geodesic complex hyperbolic hyperplane*;
3. B: *a tube around a totally geodesic real hyperbolic space $\mathbb{R}H^n$*;
4. N: *a horosphere.*

Motivated by the works in [5, 6, 8], Kaimakamis and Panagiotidou in [11] introduced a so-called *-Ricci soliton, that is, a Riemannian metric g on M satisfying

$$\frac{1}{2}\mathcal{L}_W g + \mathrm{Ric}^* - \lambda g = 0\,, \tag{1}$$

where λ is constant and Ric* is the *-Ricci tensor of M. They considered the case where W is the structure field ξ and obtained that a real hypersurface in complex projective space does not admit a *-Ricci soliton and a real hypersurface in complex hyperbolic space admitting a *-Ricci soltion is locally congruent to a geodesic hypersphere.

It is well-known that the tangent bundle TM can be decomposed as $TM = \mathbb{R}\xi \oplus \mathcal{D}$, where $\mathcal{D} = \{X \in TM, \eta(X) = 0\}$ is called *holomorphic distribution*. In the last part of [11], they proposed two open problems:

Problem 1 Are there real hypersurfaces admitting a *-Ricci soliton whose potential vector field is a principal vector field of the real hypersurface?

Problem 2 Are there real hypersurfaces admitting a *-Ricci soliton whose potential vector field belongs to the holomorphic distribution $\mathcal{D}$?

In the present paper, we shall consider the above two problems. For Problem 1, we consider the case of 2-dimensional non-flat complex space forms. Denote by T_χ the distribution on M formed by principal curvature spaces of χ and $\Gamma(T_\chi)$ by the all smooth sections of T_χ. We obtain the following conclusions:

THEOREM 4. *Let M be a hypersurface of non-flat complex space form $\widetilde{M}^2(c)$ with a *-Ricci soliton whose potential vector field $W \in \Gamma(T_\chi)$, $\chi \neq 0$. If the principal curvatures are constant along ξ and $A\xi$ then*
• *in case of $\mathbb{C}P^2$, M is an open part of a tube around the complex quadric, or a geodesic hypersphere;*
• *in case of $\mathbb{C}H^2$, M is an open part of*

(1) *a geodesic hypersphere, or*
(2) *a tube around a totally geodesic $\mathbb{C}H^1$, or*
(3) *a tube around a totally geodesic real hyperbolic space $\mathbb{R}H^2$, or*
(4) *a horosphere.*

THEOREM 5. *Let M be a hypersurface of complex projective space $\mathbb{C}P^2$, admitting a *-Ricci soliton whose potential vector field $W \in \Gamma(T_0)$. Then M is an open part of a tube around the complex quadric.*

For Problem 2, we first obtain the following result:

THEOREM 6. *Let M be a hypersurface of complex projective space $\mathbb{C}P^2$ with a *-Ricci soliton whose potential vector field $W \in \mathcal{D}$. If the principal curvatures are constant along ξ and $A\xi$, then M is locally congruent to a geodesic hypersphere in $\mathbb{C}P^2$. Moreover, if $g(A\xi, \xi) = 0$ then W is Killing.*

Furthermore, due to the decomposition $TM = \mathbb{R}\xi \oplus \mathcal{D}$, we have $A\xi = a\xi + V$, where $V \in \mathcal{D}$ and a is a smooth function on M. The following conclusion is obtained:

THEOREM 7. *Let M^{2n-1} be a hypersurface of complex space form $\widetilde{M}^n(c)$ and $n \geq 2$. Then*
• *in case of $\mathbb{C}P^n$ there are no real hypersurfaces admitting a *-Ricci soliton with potential vector field $W = V$;*
• *in case of $\mathbb{C}H^n$, if M admits a *-Ricci soliton with potential vector field $W = V$, it is locally congruent to a geodesic hypersphere.*

This paper is organized as follows. In Section 2, some basic concepts and formulas are presented. To prove M is Hopf under the assumptions of theorems, in Section 3 we give some formulas for the non-Hopf hypersurfaces with *-Ricci solitons, and the proofs of theorems are given in Sections 4, 5, and 6, respectively.

2. Preliminaries

Let $(\widetilde{M}^n, \widetilde{g})$ be a complex n-dimensional Kähler manifold and M be an immersed real hypersurface of $\widetilde{M}^n$ with induced metric g. We denote by J the complex structure on $\widetilde{M}^n$. There exists a local defined unit normal vector field N on M and we write $\xi := -JN$ by the structure vector field of M. An induced one-form η is defined by $\eta(\cdot) = \widetilde{g}(J\cdot, N)$, which is dual to ξ. For any vector field X on M the tangent part of JX is denoted by $\phi X = JX - \eta(X)N$. Moreover, the following identities hold:

$$\phi^2 = -Id + \eta \otimes \xi, \quad \eta \circ \phi = 0, \quad \phi \circ \xi = 0, \quad \eta(\xi) = 1\,, \tag{2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)\,, \tag{3}$$

$$g(X, \xi) = \eta(X)\,, \tag{4}$$

where $X, Y \in \mathfrak{X}(M)$. By (2)–(4), we know that (ϕ, η, ξ, g) is an almost contact metric structure on M.

Denote by ∇, A the induced Riemannian connection and the shape operator on M, respectively. Then the Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \widetilde{\nabla}_X N = -AX\,, \tag{5}$$

where $\widetilde{\nabla}$ is the connection on $\widetilde{M}^n$ with respect to $\widetilde{g}$. Also, we have

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX\,. \tag{6}$$

M is said to be a *Hopf hypersurface* if the structure vector field ξ is an eigenvector of A.

From now on we always assume that the sectional curvature of $\widetilde{M}^n$ is constant $c \neq 0$, i.e., $\widetilde{M}^n$ is a non-flat complex space form, denoted by $\widetilde{M}^n(c)$, then the curvature tensor R of M is given by

$$\begin{aligned} R(X, Y)Z = \frac{c}{4}\Big(&g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &+ 2g(X, \phi Y)\phi Z\Big) + g(AY, Z)AX - g(AX, Z)AY\,, \end{aligned} \tag{7}$$

and the shape operator A satisfies

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\Big(\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\Big)\,, \tag{8}$$

for any vector fields X, Y, Z on M.

Recall that the *-Ricci operator Q^* of M is defined by

$$g(Q^* X, Y) = \mathrm{Ric}^*(X, Y) = \frac{1}{2}\mathrm{trace}\{\phi \circ R(X, \phi Y)\}, \quad \text{for all } X, Y \in TM\,.$$

By (7), it is proved in Theorem 2 of [9] that the *-Ricci operator is expressed as

$$Q^* = -\Big[\frac{cn}{2}\phi^2 + (\phi A)^2\Big]. \tag{9}$$

In particular, if $Q^* = 0$ then M is said to be a **-Ricci flat hypersurface*. Due to (2) *-Ricci Soliton Equation (1) becomes

$$\begin{aligned} g(\nabla_X W, Y) + g(X, \nabla_Y W) + ncg(X, Y) - nc\eta(X)\eta(Y) \\ + 2g(\phi AX, A\phi Y) - 2\lambda g(X, Y) = 0\,, \end{aligned} \tag{10}$$

for any vector fields X, Y on M.

3. Non-Hopf hypersurfaces with *-Ricci solitons

In this section we assume that M is a non-Hopf hypersurface in $\widetilde{M}^2(c)$ with a *-Ricci soliton. Since M is not Hopf, due to the decomposition $TM = \mathbb{R}\xi \oplus \mathcal{D}$, we can write $A\xi$ as

$$A\xi = \alpha\xi + \beta U\,, \tag{11}$$

where $\alpha = \eta(A\xi)$, $\beta = |\phi\nabla_\xi\xi|$ are the smooth functions on M and $U = -\frac{1}{\beta}\phi\nabla_\xi\xi \in \mathcal{D}$ is a unit vector field with $\beta \neq 0$. Write

$$\mathcal{N} := \{p \in M : \beta \neq 0 \quad \text{in a neighbourhood of } p\}\,.$$

LEMMA 1. *On $\mathcal{N}$, we have $A\phi U = 0$.*

PROOF. In view of *-Ricci Soliton Equation (1), we know $\mathrm{Ric}^*(X, Y) = \mathrm{Ric}^*(Y, X)$ for every vector fields $X, Y \in TM$. That means that for every vector field X,

$$\phi A\phi AX = A\phi A\phi X\,. \tag{12}$$

On the other hand, we have

$$\begin{aligned} \phi^2 A\phi AX &= -A\phi AX + \eta(A\phi AX)\xi \\ &= -A\phi AX + g(\alpha\xi + \beta U, \phi AX)\xi \\ &= -A\phi AX - \beta g(\phi U, AX)\xi \end{aligned}$$

and

$$\begin{aligned} \phi A\phi A\phi X &= A\phi A\phi^2 X \\ &= -A\phi AX + \eta(X)A\phi A\xi \\ &= -A\phi AX + \beta\eta(X)A\phi U\,. \end{aligned}$$

Since $\beta \neq 0$ on $\mathcal{N}$, we get from (12) that $-g(\phi U, AX)\xi = \eta(X)A\phi U$. Taking $X = \xi$ in this formula, we obtain the desired result. □

Since $\{\xi, U, \phi U\}$ is a locally orthonormal frame on $\mathcal{N}$, there are smooth functions γ, μ, δ such that

$$AU = \beta\xi + \gamma U + \delta\phi U, \qquad A\phi U = \delta U + \mu\phi U\,. \tag{13}$$

By Lemma 1, we have $\delta = \mu = 0$. Moreover, in [16] the following lemma was proved:

LEMMA 2. *With respect to the orthonormal basis* $\{\xi, U, \phi U\}$, *we have*

$$\nabla_U\xi = \gamma\phi U, \qquad \nabla_{\phi U}\xi = 0, \quad \nabla_\xi\xi = \beta\phi U\,,$$

$$\nabla_U U = k_1\phi U, \qquad \nabla_{\phi U} U = k_2\phi U, \quad \nabla_\xi U = k_3\phi U\,,$$

$$\nabla_U\phi U = -k_1U - \gamma\xi, \quad \nabla_{\phi U}\phi U = -k_2U, \quad \nabla_\xi\phi U = -k_3U - \beta\xi\,,$$

where k_1, k_2, k_3 *are smooth functions on* M.

Applying Lemma 2, we have the following.

PROPOSITION 1. *The following formulas on* $\mathcal{N}$ *are valid*:

$$k_3\beta + \alpha\beta - \phi U(\alpha) = 0, \quad k_2 = 0\,, \tag{14}$$

$$k_3\gamma + \beta^2 - \phi U(\beta) = -\frac{c}{4}\,, \tag{15}$$

$$\xi(\beta) = U(\alpha), \quad \xi(\gamma) = U(\beta)\,, \tag{16}$$

$$\beta^2 + k_3\gamma - \alpha\gamma - \beta k_1 = \frac{c}{4}\,, \tag{17}$$

$$k_1\beta + \alpha\gamma - \phi U(\beta) = -\frac{c}{2}\,. \tag{18}$$

PROOF. By taking $X = \xi$ and $Y = \phi U$ in Relation (8), we obtain

$$(\nabla_\xi A)\phi U - (\nabla_{\phi U}A)\xi = -\frac{c}{4}U\,.$$

In view of (13) and Lemma 2, the above formula leads to $k_2 = 0$ since $\beta \neq 0$. Also (14) and Formula (15) are attained. By a straightforward computation, Relation (8) for $X = \xi$ and $Y = U$ implies (16) and (17). Moreover Relation (8) for $X = U$ and $Y = \phi U$ gives (18). □

Let us assume that W is an eigenvector of A, namely, there is a smooth function χ such that $AW = \chi W$ holds. On $\mathcal{N}$, in the basis of $\{\xi, U, \phi U\}$ the potential vector W may be expressed as

$$W = f_1\xi + f_2U + f_3\phi U\,,$$

where f_1, f_2, f_3 are the smooth functions on $\mathcal{N}$.

In view of Lemma 2, by a direct computation, we have

$$\nabla_\xi W = (\xi(f_1) - f_3\beta)\xi + (\xi(f_2) - f_3k_3)U + (f_1\beta + f_2k_3 + \xi(f_3))\phi U\,, \tag{19}$$

$$\nabla_U W = (U(f_1) - f_3\gamma)\xi + (U(f_2) - f_3k_1)U + (f_1\gamma + f_2k_1 + U(f_3))\phi U\,, \tag{20}$$

$$\nabla_{\phi U} W = \phi U(f_1)\xi + \phi U(f_2)U + \phi U(f_3)\phi U\,. \tag{21}$$

Inserting $X = Y = \xi$ into Formula (10), by (19) we find

$$\xi(f_1) - f_3\beta = \lambda\,. \tag{22}$$

Furthermore, inserting $X = Y = U$ and $X = Y = \phi U$ into Formula (10) respectively, we get from (20) and (21) that

$$U(f_2) - f_3k_3 + c - \lambda = 0\,, \tag{23}$$

$$\phi U(f_3) + c - \lambda = 0\,. \tag{24}$$

Also, when X and Y are taken as the different vectors of ξ, U, and ϕU in Formula (10), a similar computation leads to

$$\begin{cases} \xi(f_2) - f_3k_3 + U(f_1) - f_3\gamma = 0\,, \\ f_1\beta + f_2k_3 + \xi(f_3) + \phi U(f_1) = 0\,, \\ f_1\gamma + f_2k_1 + U(f_3) + \phi U(f_2) = 0\,. \end{cases} \tag{25}$$

Actually, Lemma 1 shows that at every point of $\mathcal{N}$ there exists a principal curvature 0 and ϕU is the corresponding principal vector. It turns out that there are at least two distinct principal curvatures in non-flat complex space forms (see [15, Theorem 1.5]).

Let λ_i be the principal curvatures for $i = 1, 2, 3$, where $\lambda_3 = 0$. We may assume that $e_1 = \cos\theta\xi + \sin\theta U$, $e_2 = \sin\theta\xi - \cos\theta U$ are the unit principal vectors corresponding to λ_1 and λ_2, respectively, where θ is the angle between principal vector e_1 and ξ. It is clear that $\{e_1, e_2, e_3 = \phi U\}$ is also an orthonormal frame. Namely,

$$A(e_1, e_2, e_3) = (e_1, e_2, e_3)\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & 0 \end{pmatrix}.$$

Denote by

$$B = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the transformation matrix of two frames, i.e.,

$$(e_1, e_2, e_3) = (\xi, U, \phi U)B\,.$$

Moreover, since

$$A(\xi, U, \phi U) = (\xi, U, \phi U)\begin{pmatrix} \alpha & \beta & 0 \\ \beta & \gamma & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

we get

$$\begin{pmatrix} \alpha & \beta & 0 \\ \beta & \gamma & 0 \\ 0 & 0 & 0 \end{pmatrix} = B \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & 0 \end{pmatrix} B^T .$$

A straightforward calculation leads to

$$\alpha = \lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta, \quad \beta = \frac{1}{2}(\lambda_1 - \lambda_2)\sin 2\theta, \quad \gamma = \lambda_1 \sin^2\theta + \lambda_2 \cos^2\theta . \tag{26}$$

If M has only two distinct principal curvatures at any point $p \in \mathcal{N}$, then either $\lambda_1 = \lambda_2 \neq 0$, or one of λ_1 and λ_2 vanishes. However, the second of (26) will come to $\beta = 0$ if $\lambda_1 = \lambda_2$, thus it is impossible. Without loss generality, we set $\lambda_1 = 0$ and $\lambda_2 \neq 0$. In terms of [10, Theorem 4], α, β and γ satisfy

$$\xi(\alpha) = \xi(\beta) = \xi(\gamma) = 0 ,$$
$$U(\alpha) = \beta(\alpha + \gamma) .$$

Using (16), we thus derive $\alpha + \gamma = 0$ because $\beta \neq 0$. This shows $\lambda_2 = 0$ from the first and third of (26). It is a contradiction. Therefore on $\mathcal{N}$ there are three distinct principal curvatures, i.e., λ_1, λ_2 are not zero and $\lambda_1 \neq \lambda_2$.

Using (16) again, we derive from (26) that

$$U(\lambda_1)\cos^2\theta + U(\lambda_2)\sin^2\theta - (\lambda_1 - \lambda_2)\sin 2\theta U(\theta)$$
$$= \frac{1}{2}\xi(\lambda_1 - \lambda_2)\sin 2\theta + (\lambda_1 - \lambda_2)\cos 2\theta \xi(\theta) ,$$
$$\xi(\lambda_1)\sin^2\theta + \xi(\lambda_2)\cos^2\theta + (\lambda_1 - \lambda_2)\sin 2\theta \xi(\theta)$$
$$= \frac{1}{2}U(\lambda_1 - \lambda_2)\sin 2\theta + (\lambda_1 - \lambda_2)\cos 2\theta U(\theta) .$$

From which we arrive at

$$\xi(\theta) = \frac{U(\lambda_1 - \lambda_2) + U(\lambda_1 + \lambda_2)\cos 2\theta - \xi(\lambda_1 + \lambda_2)\sin 2\theta}{2(\lambda_1 - \lambda_2)} ,$$
$$U(\theta) = \frac{-\xi(\lambda_1 - \lambda_2) + \xi(\lambda_1 + \lambda_2)\cos 2\theta + U(\lambda_1 + \lambda_2)\sin 2\theta}{2(\lambda_1 - \lambda_2)} .$$

Thus we obtain

PROPOSITION 2. *If on $\mathcal{N}$ the principal curvatures are constant along ξ and $A\xi$, then the following equations hold*:

$$\xi(\theta) = U(\theta) = 0 , \tag{27}$$

$$\xi(\beta) = U(\alpha) = \xi(\gamma) = U(\beta) = 0 . \tag{28}$$

4. Proofs of Theorems 4 and 5

In order to prove our theorems, we first prove the following two conclusions.

PROPOSITION 3. *Let M be a real hypersurface in $\widetilde{M}^2(c)$ with a *-Ricci soliton whose potential vector field $W \in \Gamma(T_\chi)$, $\chi \neq 0$. If the principal curvatures are constant along ξ and $A\xi$ then M is Hopf.*

PROOF. Suppose that M is not Hopf, then $\mathcal{N}$ is not empty. Write $W = a_1 e_1 + a_2 e_2 + a_3 e_3$, where a_1, a_2, a_3 are the smooth functions on $\mathcal{N}$. Since $\chi \neq 0$, $a_3 = 0$ and $\chi = \lambda_1$ or λ_2. Since a_1, a_2 are not all zero, without loss of generality, we may assume $a_1 \neq 0$, then

$$AW = \chi W \Rightarrow \chi = \lambda_1 \quad \text{and} \quad a_2 = 0 \quad \text{since} \quad \lambda_1 \neq \lambda_2 .$$

Thus the potential vector field can be written as

$$W = a_1 \cos\theta \xi + a_1 \sin\theta U .$$

Replacing f_1 in Formula (22) and f_2 in (23) by $a_1 \cos\theta$ and $a_1 \sin\theta$, respectively, we have

$$\xi(a_1 \cos\theta) = \lambda, \quad U(a_1 \sin\theta) = 0 \tag{29}$$

because $c = \lambda$ followed from (24). Similarly, in view of the first equation of (25), we obtain

$$\xi(a_1 \sin\theta) + U(a_1 \cos\theta) = 0 . \tag{30}$$

With the help of (29) and (30), we further obtain

$$a_1(\sin\theta\xi(\theta) - \cos\theta U(\theta)) = -\lambda \sin^2\theta .$$

By (27), $\lambda \sin^2\theta = 0$. If $\sin\theta \neq 0$ then $\lambda = 0$. This leads to a contradiction because $\lambda = c \neq 0$. If $\sin\theta = 0$ then $W = a_1 \cos\theta\xi$, i.e., ξ is a principal vector, which is also a contradiction. Therefore we complete the proof. □

PROPOSITION 4. *A real hypersurface in $\mathbb{C}P^2$, admitting a *-Ricci soliton whose potential vector field $W \in \Gamma(T_0)$, is Hopf.*

PROOF. Suppose that M is not Hopf, then $\mathcal{N}$ is not empty. We may write $W = b_1 e_1 + b_2 e_2 + b_3 e_3$ in the basis $\{e_1, e_2, e_3\}$, where b_1, b_2, b_3 are smooth functions on $\mathcal{N}$. By Lemma 1, $A\phi U = 0$, so $AW = 0$ implies $b_1 = b_2 = 0$, i.e., $W = b_3 \phi U$ with $b_3 \neq 0$. Hence (25) becomes

$$k_3 = -\gamma, \quad \xi(b_3) = 0, \quad U(b_3) = 0 . \tag{31}$$

And (23) becomes

$$-b_3\gamma = c - \lambda . \tag{32}$$

Since $b_1 = 0$, Formula (22) becomes

$$-b_3\beta = \lambda . \tag{33}$$

So by taking the differentiation of (32) along ϕU, we derive from (24) that

$$b_3\phi U(\beta) = (c - \lambda)\beta\,. \tag{34}$$

On the other hand, it follows from (32) and (33) that

$$\frac{\gamma}{\beta} = \frac{c}{\lambda} - 1\,. \tag{35}$$

If $c = \lambda$, then Equation (35) shows $\gamma = 0$. Further, in view of (24) we find $\phi U(b_3) = \lambda - c = 0$, which means that b_3 is constant since $\xi(b_3) = U(b_3) = 0$. Now we derive from (33) that β is constant. Hence together (17) with (18), we obtain $\beta^2 = -\frac{c}{4}$. It is impossible.

Next we assume $c \neq \lambda$. Thus Equation (35) follows $\gamma \neq 0$ and Formula (15) follows from (31)

$$\phi U(\beta) = \beta^2 - \gamma^2 + \frac{c}{4}\,.$$

Substituting this into (34), we get from (32) that

$$\Big(\beta^2 - \gamma^2 + \frac{c}{4}\Big)\frac{1}{\gamma} = -\beta \quad \Rightarrow \quad 1 - \Big(\frac{\gamma}{\beta}\Big)^2 + \frac{c}{4\beta^2} = -\frac{\gamma}{\beta}\,,$$

which reduces from Equation (35) that β is constant. Finally we derive a contradiction from (34). Hence we complete the proof of proposition. □

PROOF OF THEOREM 4. Under the hypothesis of Theorem 4, by Proposition 3, M is a Hopf hypersurface of $\widetilde{M}^2(c)$, i.e., $A\xi = \alpha\xi$. Due to [15, Theorem 2.1], α is constant. We consider a point $p \in M$ and a unit vector field $e \in \mathcal{D}_p$ such that $Ae = \kappa e$ and $A\phi e = \nu\phi e$, where κ, ν are smooth functions on M. Then $\{\xi, e, \phi e\}$ is a local orthonormal basis of M. By Corollary 2.3 in [15],

$$\kappa\nu = \frac{\kappa + \nu}{2}\alpha + \frac{c}{4}\,. \tag{36}$$

Moreover, by a straightforward computation, we have the following lemma.

LEMMA 3. *With respect to* $\{\xi, e, \phi e\}$ *the Levi-Civita connection is given by*

$$\nabla_e\xi = \kappa\phi e, \qquad \nabla_{\phi e}\xi = -\nu e, \quad \nabla_\xi\xi = 0\,,$$
$$\nabla_e e = a_1\phi e, \qquad \nabla_{\phi e} e = \nu\xi + a_2\phi e, \quad \nabla_\xi e = a_3\phi e\,,$$
$$\nabla_e\phi e = -a_1 e - \kappa\xi, \quad \nabla_{\phi e}\phi e = -a_2 e, \quad \nabla_\xi\phi e = -a_3 e\,,$$

where $a_1 = g(\nabla_e e, \phi e)$, $a_2 = g(\nabla_{\phi e} e, \phi e)$, $a_3 = g(\nabla_\xi e, \phi e)$ *are smooth functions on* M.

Under the orthonormal basis $\{\xi, e, \phi e\}$ we may assume that there are smooth functions g_1, g_2, g_3 such that the potential vector filed W can be written as

$$W = g_1\xi + g_2 e + g_3\phi e\,.$$

Since $AW = \chi W$ with $\chi \neq 0$, we get $\alpha g_1 = \chi g_1$, $\kappa g_2 = \chi g_2$ and $\nu g_3 = \chi g_3$.

Next we consider the following cases:

• Case I: g_1, g_2, g_3 are not equal to zero.
Then $\kappa = \nu = \alpha$, which leads to $c = 0$ from Equation (36). This is a contradiction.

• Case II: Only one of g_1, g_2, g_3 is equal to zero.
If $g_1 = 0$, then $\kappa = \nu$. Equation (36) yields $(\kappa - \frac{\alpha}{2})^2 = \frac{\alpha^2+c}{4}$, which shows $\kappa = \nu = const.$ and $\alpha \neq \kappa$; If $g_2 = 0$, then $\alpha = \nu$, Equation (36) implies $\kappa = \frac{c+2\alpha^2}{2\alpha}$ with $\kappa \neq \alpha$; If $g_3 = 0$, then $\kappa = \alpha$, which implies $\nu = \frac{c+2\alpha^2}{2\alpha}$, $\nu \neq \alpha$ by Equation (36).

• Case III: Two of g_1, g_2, g_3 are equal to zero.
When $g_1 = g_2 = 0$. Formula (10) for $X = \xi$ and $Y = e$ implies

$$g(\nabla_\xi W, e) + g(\xi, \nabla_e W) = 0\,.$$

In view of Lemma 3, a simple calculation leads to $\kappa = -a_3$. On the other hand, Relation (8) for $X = e$ and $Y = \xi$ yields $(\nabla_e A)\xi - (\nabla_\xi A)e = -\frac{c}{4}\phi e$. By Lemma 3, we find

$$\alpha\kappa - \kappa\nu - \kappa a_3 + a_3\nu = -\frac{c}{4}\,. \tag{37}$$

A similar computation using Relation (8) for $X = \phi e$, $Y = \xi$ yields

$$-\alpha\nu + \kappa\nu - \kappa a_3 + a_3\nu = \frac{c}{4}\,. \tag{38}$$

Moreover, inserting $\kappa = -a_3$ into the above equation gives

$$\kappa^2 - \alpha\nu = \frac{c}{4}\,. \tag{39}$$

The combination of (37) and (38) leads to $(\kappa - \nu)(2\kappa + \alpha) = 0$ because $a_3 = -\kappa$. If $\nu = \kappa$ then $\alpha \neq \kappa$, otherwise, Formula (39) will lead to $c = 0$. If $\nu \neq \kappa$ then $\kappa = -\frac{\alpha}{2}$ and $\nu = \frac{\alpha^2-c}{4\alpha}$.

When $g_1 = g_3 = 0$, we put $X = \xi$, $Y = \phi e$ in Formula (10). By Lemma 3, $a_3 = -\nu$, so we get $(\kappa - \nu)(2\nu + \alpha) = 0$ from (37) and (38). If $\kappa = \nu$ then $\alpha \neq \nu$ as before. If $\kappa \neq \nu$ then $\nu = -\frac{\alpha}{2}$ and $\kappa = \frac{\alpha^2-c}{4\alpha}$.

When $g_2 = g_3 = 0$, Relation (8) for $X = e$, $Y = \phi e$ leads to $c = 0$ by Lemma 3, which is a contradiction.

In a word we have proved that there are two or three distinct constant principal curvatures on M. For the case of $\mathbb{C}P^2$, by Theorem 2 and [20, Theorem 4.1], M is an open part of a hypersphere, or a tube around the complex quadric.

For the case of $\mathbb{C}H^2$, if M has three distinct principal curvatures, by the proof of [2, Theorem 1.1], we know that the ruled real hypersurfaces cannot be Hopf, which is a contradiction with Proposition 3. Thus in this case M has only two distinct constant principle curvatures. In view of Theorem 2, the real hypersurface M is one of Type A_{11}, A_2, B and N.

This finishes the proof of Theorem 4. □

PROOF OF THEOREM 5. Under the assumption of Theorem 5, by Proposition 4 we know that M is a Hopf hypersurface of $\mathbb{C}P^2$. Hence Equation (36) and Lemma 3 are valid. We adopt the same notations as the proof of Theorem 4.

Since $AW = 0$, we have $\alpha g_1 = \kappa g_2 = \nu g_3 = 0$. If $\alpha = 0$ then it follows from Equation (36) that $\kappa\nu = \frac{c}{4}$, which means that κ, ν are non-zero. So we get $g_2 = g_3 = 0$. From the Case III in the proof of Theorem 4, we know it is impossible.

In the following we assume $\alpha \neq 0$, then $g_1 = 0$. If g_2 is also equal to zero, then g_3 must be non-zero, and further we obtain $\nu = 0$ and $\kappa = -\frac{\alpha}{2} \neq 0$ from the Case III in the proof of Theorem 4. If g_2 is non-zero then $\kappa = 0$. Equation (36) implies $\alpha\nu = -\frac{c}{2}$, that shows ν is a non-zero constant. Further we know $\alpha \neq \nu$ since $c > 0$.

Summarizing the above discussion, we have proved that there are three distinct constant principal curvatures in M. Therefore we complete the proof of Theorem 5 by [20, Theorem 4.1]. □

5. Proof of Theorem 6

In this section we suppose that M is a real hypersurface of $\mathbb{C}P^2$ with a *-Ricci soliton whose potential vector field W belongs to the holomorphic distribution $\mathcal{D}$. First we prove the following result:

PROPOSITION 5. *Let M be a real hypersurface in $\mathbb{C}P^2$ with a *-Ricci soliton whose potential vector field $W \in \mathcal{D}$. If the principal curvatures are constant along ξ and $A\xi$ then M is Hopf.*

PROOF. If M is not Hopf then $\mathcal{N}$ is not empty. Let $W = c_1e_1 + c_2e_2 + c_3e_3 \in \mathcal{D}$, where c_i are smooth functions on $\mathcal{N}$, then

$$c_1 \cos\theta + c_2 \sin\theta = 0\,. \tag{40}$$

Formula (22) becomes

$$-c_3\beta = \lambda\,. \tag{41}$$

And by Proposition 2, (23)–(25) accordingly become

$$U(c_1)\sin\theta - U(c_2)\cos\theta - c_3k_3 + c - \lambda = 0\,, \tag{42}$$

$$\phi U(c_3) + c - \lambda = 0\,, \tag{43}$$

and

$$\begin{cases} \xi(c_1)\sin\theta - \xi(c_2)\cos\theta - c_3k_3 - c_3\gamma = 0\,, \\ (c_1\sin\theta - c_2\cos\theta)k_3 + \xi(c_3) = 0\,, \\ (c_1\sin\theta - c_2\cos\theta)k_1 + U(c_3) + \phi U(c_1\sin\theta - c_2\cos\theta) = 0\,. \end{cases} \tag{44}$$

If $c_3 = 0$, then (41) and (43) show $c = \lambda = 0$. It is impossible. Thus $c_3 \neq 0$, which further implies $\lambda \neq 0$ from (41). By (43) and Formula (15), differentiating (41) along ϕU gives

$$k_3\gamma + \beta^2\frac{c}{\lambda} + \frac{c}{4} = 0\,. \tag{45}$$

When $\gamma = 0$, this shows β is constant. So it follows from Formula (15) that $\beta^2 = -\frac{c}{4}$, which is impossible because $c > 0$. Hence $\gamma \neq 0$ and we get from (45) that

$$k_3 = -\frac{\beta^2\frac{c}{\lambda} + \frac{c}{4}}{\gamma}\,.$$

If $c_1 = c_2 = 0$, as the proof of Proposition 4, by using (41)–(44), we arrive at a contradiction. Thus one of c_1, c_2 must be not zero.

Without loss of generality we set $c_1 \neq 0$. Taking the differentiation of (41) along ξ and U, respectively, we obtain from (28) that $\xi(c_3) = U(c_3) = 0$ since $\beta \neq 0$. In view of the second equation of (44) and (40), we find $k_3 = 0$, that is,

$$\beta^2\frac{c}{\lambda} + \frac{c}{4} = 0\,,$$

thus β is constant. As before from Formula (15) we have $\beta^2 = -\frac{c}{4}$, which is impossible. This completes the proof. □

PROOF OF THEOREM 6. Under the hypothesis of Theorem 6, by Proposition 5 we know that M is a Hopf hypersurface of $\mathbb{C}P^2$. That means that the structure vector field ξ is a principal vector field, i.e., $A\xi = a\xi$ and a is constant as before.

For any point $p \in M$ we consider a unit vector $Z \in \mathcal{D}_p$ such that $AZ = \mu Z$, then the following relation holds (see [15, Corollary 2.3]):

$$\Big(\mu - \frac{a}{2}\Big)A\phi Z = \Big(\frac{\mu a}{2} + \frac{c}{4}\Big)\phi Z\,.$$

If $\mu = \frac{a}{2}$ the above equation implies $\frac{\mu a}{2} + \frac{c}{4} = 0$, i.e., $\mu^2 + \frac{c}{4} = 0$, that is impossible. Hence $\mu \neq \frac{a}{2}$, which means that ϕZ is a principal vector with principal curvature ν satisfying

$$\mu\nu = \frac{\mu + \nu}{2}a + \frac{c}{4}\,. \tag{46}$$

Now we know that $\mathrm{Span}\{Z, \phi Z\} = \mathcal{D}_p$ and $\{\xi, Z, \phi Z\}$ is an orthonormal basis of T_pM. By a straightforward computation, we have

$$\nabla_Z\phi Z = -g(\nabla_Z Z, \phi Z)Z - \mu\xi, \quad \nabla_{\phi Z}Z = \nu\xi + g(\nabla_{\phi Z}Z, \phi Z)\phi Z\,.$$

Taking $X = Z$ and $Y = \phi Z$ in Relation (8) and using the above formulas, we get

$$\mu\nu - \nu a = \frac{c}{4}\,.$$

Next we distinguish into two cases.

Case 1. If $a \neq 0$ then it follows $\mu = \nu$ by combining with (46) and further μ, ν are constant. Furthermore, we find $\mu = \nu \neq a$, otherwise, the above formula will lead to $c = 0$. By Theorem 2 we get that M is of Type A_1.

Case 2. We assume $a = 0$, then $\mu\nu = \frac{c}{4}$. In this case M is a *-Einstein hypersurface (see [9, Remark 1]). *-Ricci Soliton Equation (1) shows W is a conformal Killing vector field, i.e., $\mathcal{L}_W g = 2(\lambda - 5c)g$. From (7), we calculate the Ricci operator

$$QX = \frac{c}{4}\{5X - 3\eta(X)\xi\} + hAX - A^2X, \quad \text{for all } X \in TM\,,$$

where $h = \text{trace}(A)$. Hence by a direct computation we can get that the scalar curvature $r = 3c + 2\mu\nu$.

Notice that on an n-dimensional Riemannian manifold a conformal Killing vector field X, i.e., $\mathcal{L}_X g = 2\rho g$, satisfies

$$\mathcal{L}_X r = 2(n-1)\Delta\rho - 2\rho r\,,$$

where r is the scalar curvature (see [21, Eq. (5.38)]). Since $\mu\nu = \frac{c}{4}$, the scalar curvature $r = \frac{7c}{2} \neq 0$. Using the above formula we find that W is a Killing vector field.

Moreover, since M is *-Einstein, we derive from Theorem 3 that M is one of Type A_1, A_2, and B. But according to the list of principal curvatures of Type A_1, A_2 and B hypersurfaces (see [15, Theorems 3.13–3.15]), we find that in this case only Type A_1 is satisfied.

Therefore we complete the proof of Theorem 6. □

6. Proof of Theorem 7

Since the tangent bundle TM can be decomposed as $TM = \mathbb{R}\xi \oplus \mathcal{D}$, where $\mathcal{D} = \{X \in TM : \eta(X) = 0\}$. Then $A\xi$ can be written as

$$A\xi = a\xi + V\,, \tag{47}$$

where $V \in \mathcal{D}$ and a is a smooth function on M. In this section we assume that the hypersurface M of $\widetilde{M}^n(c)$ is equipped with a *-Ricci soliton such that the potential vector field $W = V$.

LEMMA 4. *On M the following equation is valid*:

$$(\nabla_\xi A)\xi = Da + 2A\phi V\,, \tag{48}$$

where Da denotes the gradient vector field of a.

PROOF. By (6) and (47), for any vector field X

$$\begin{aligned}(\nabla_X A)\xi &= \nabla_X(A\xi) - A\nabla_X\xi \\ &= X(a)\xi + a\nabla_X\xi + \nabla_X V - A\phi AX\,.\end{aligned} \tag{49}$$

Thus

$$\begin{aligned}g((\nabla_X A)\xi, \xi) &= X(a) + g(\nabla_X V, \xi) - g(A\phi AX, \xi) \\ &= X(a) - g(V, \nabla_X\xi) - g(\phi AX, A\xi) \\ &= X(a) + 2g(AX, \phi V)\,.\end{aligned}$$

From the well-known relation $g((\nabla_X A)\xi, \xi) = g((\nabla_\xi A)\xi, X)$ (see [15, Corollary 2.1]), we arrive at (48). □

Next it follows from (49) and Relation (8) that

$$\begin{aligned}\nabla_X V &= (\nabla_X A)\xi - X(a)\xi - a\nabla_X\xi + A\phi AX \\ &= (\nabla_\xi A)X - \tfrac{c}{4}\phi X - X(a)\xi - a\phi AX + A\phi AX\,.\end{aligned} \tag{50}$$

Therefore, by Lemma 4 we have

$$\begin{aligned}\nabla_\xi V =&(\nabla_\xi A)\xi - \xi(a)\xi - a\phi A\xi + A\phi A\xi \\ =& - a\phi V + Da - \xi(a)\xi + 3A\phi V\,.\end{aligned} \tag{51}$$

Since $\eta(V) = 0$, differentiating this along any vector X, we have

$$g(\nabla_X V, \xi) + g(V, \phi AX) = 0\,. \tag{52}$$

In particular, by taking $X = \xi$ in (52), we find $g(\nabla_\xi V, \xi) = 0$ because of $\nabla_\xi\xi = \phi V$. Hence, taking into account $X = Y = \xi$ in Formula (10), we conclude that $\lambda = 0$.

Take $X = \xi$ and $Y = \xi$ respectively in Formula (10), and it follows from (51) and (52) that

$$-a\phi V + Da - \xi(a)\xi + 4A\phi V - 2\phi A\phi V = 0\,,$$

$$-a\phi V + Da - \xi(a)\xi + 4A\phi V = 0\,.$$

Hence $\phi A\phi V = 0$, which implies $A\phi V = 0$ because of (3) and (47). Differentiating $A\phi V = 0$ along vector field ξ and using the first equation of (6), (50), and (51), we get

$$\begin{aligned}0 = \nabla_\xi(A\phi V) &= (\nabla_\xi A)\phi V + A(\nabla_\xi\phi)V + A\phi(\nabla_\xi V) \\ &= \nabla_{\phi V}V - \frac{c}{4}V + (\phi V)(a)\xi + aAV + A\phi(Da)\,.\end{aligned}$$

Therefore

$$\nabla_{\phi V}V = \frac{c}{4}V - (\phi V)(a)\xi - aAV - A\phi(Da)\,. \tag{53}$$

If we put $X = Y = \phi V$ in Formula (10), then Equation (53) leads to $nc|V|^2 = 0$, i.e., V is a zero vector field. Since $\lambda = 0$, the following proposition is proved:

PROPOSITION 6. *Every real hypersurface in a non-flat complex space form $\widetilde{M}^n(c)$, $n \geq 2$, admitting a *-Ricci soliton with potential vector field V, is a *-Ricci flat Hopf hypersurface.*

PROOF OF THEOREM 7. Let M be a *-Ricci flat Hopf hypersurface, namely, $A\xi = a\xi$ and $Q^*X = 0$ for all X, where a is constant. In view of (9), we have $\frac{cn}{2}\phi^2 X + (\phi A)^2 X = 0$ for all X, which further implies

$$\frac{cn}{2}\phi X + A\phi A X = 0\,. \tag{54}$$

For any point $p \in M$, let $Z \in \mathcal{D}_p$ is a principal vector, namely, there is a certain function μ_1 such that $AZ = \mu_1 Z$, then it follows from (54)

$$\mu_1 A\phi Z = -\frac{cn}{2}\phi Z\,,$$

which shows that ϕZ is also a principal curvature vector, i.e., $A\phi Z = \nu\phi Z$ with $\nu = -\frac{cn}{2\mu_1}$. On the other hand, as before we know that the following relation is also valid:

$$\Big(\mu_1 - \frac{a}{2}\Big)A\phi Z = \Big(\frac{\mu_1 a}{2} + \frac{c}{4}\Big)\phi Z\,. \tag{55}$$

In the following we divide into two cases.

• Case I: $a^2 + c \neq 0$.
If $\mu_1 = \frac{a}{2}$ then $\frac{\mu_1 a}{2} + \frac{c}{4} = 0$, which is a contradiction. Hence $\mu_1 \neq \frac{a}{2}$ and from (55) we find that the principal curvature ν is also equal $(\frac{\mu_1 a}{2} + \frac{c}{4})\Big/(\mu_1 - \frac{a}{2})$. Hence we obtain that μ_1 satisfies

$$2a\mu_1^2 + (1+2n)c\mu_1 - acn = 0\,, \tag{56}$$

from which we can see that μ_1 is constant. Thus M has constant principal curvatures. However, since M is *-Ricci flat, in view of Theorem 1 and Section 3 in [8], we find that there are no hypersurfaces in $\mathbb{C}P^n$ satisfying this case.

For the case of $\mathbb{C}H^n$, in terms of Section 3 in [8], only Type A_{11} and A_{12} hypersurfaces may be *-Ricci flat. But for the Type A_{12}, we further get $2n = \tanh^2(u)$, which is impossible since $0 < \tanh(u) < 1$.

• Case II: $a^2 + c = 0$.
In this case the ambient space is $\mathbb{C}H^n$, since $c = -a^2 < 0$, $a \neq 0$. If $\mu_1 \neq \frac{a}{2}$, by (56), we get $\mu_1 = na$ and $\nu = \frac{a}{2}$. If $\mu_1 = \frac{a}{2}$ then $\nu = na$. Hence it is proved that there are three distinct constant principal curvatures for all $p \in M$.

However, since M is a Hopf, in terms of Theorem 1 and the analysis of Section 3 in [8], we know that the Type A_2 hypersurfaces cannot be *-Einstein, and the Type B and Type N hyersurfaces cannot be *-Ricci flat.

Summarizing this two cases, we complete the proof of Theorem 7. □

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