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Fundamental Solutions of the Knizhnik-Zamolodchikov Equation of One Variable and the Riemann-Hilbert Problem

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Abstract. In this article, we show that the generalized inversion formulas of the multiple polylogarithms of one variable, which are generalizations of the inversion formula of the dilogarithm, characterize uniquely the multiple polylogarithms under certain conditions. This means that the multiple polylogarithms are constructed from the multiple zeta values. We call such a problem of determining certain functions a recursive Riemann-Hilbert problem of additive type. Furthermore we show that the fundamental solutions of the KZ equation of one variable are uniquely characterized by the connection relation between the fundamental solutions of the KZ equation normalized at $z = 0$ and $z = 1$ under some assumptions. Namely the fundamental solutions of the KZ equation are constructed from the Drinfel'd associator. We call this problem a Riemann-Hilbert problem of multiplicative type.

1. Introduction

The polylogarithms

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}$$

and the multiple polylogarithms

$$\mathrm{Li}_{2,\underbrace{1,\dots,1}_{(r-1)\text{-fold}}}(z) = \sum_{n_1 > n_2 > \cdots > n_r > 0} \frac{z^{n_1}}{n_1^2 n_2 \dots n_r}$$

satisfy the inversion formula of the polylogarithms [OkU]

$$\mathrm{Li}_k(z) + \sum_{j=1}^{k-1} \frac{(-\log z)^j}{j!} \mathrm{Li}_{k-j}(z) + \mathrm{Li}_{2,\underbrace{1,\dots,1}_{(k-2)\text{-fold}}}(1-z) = \zeta(k)\,. \tag{1}$$

The special case of $k = 2$ of the formula is known as Euler's inversion formula of the dilogarithm.

Conversely, in [OiU2], we showed the following theorem:

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THEOREM 1. *Let $D^{(+)}$, $D^{(-)}$ be domains of* $\mathbf{C}$ *defined by*

$$D^{(+)} = \{z = x + yi \mid x < 1,\ -\infty < y < \infty\},$$
$$D^{(-)} = \{z = x + yi \mid 0 < x,\ -\infty < y < \infty\}.$$

Put $f_1^{(+)}(z) = \mathrm{Li}_1(z)$. For $k \geq 2$, we assume that $f_k^{(\pm)}(z)$ are holomorphic functions on $D^{(\pm)}$ satisfying the functional relation

$$f_k^{(+)}(z) + \sum_{j=1}^{k-1} \frac{(-\log z)^j}{j!} f_{k-j}^{(+)}(z) + f_k^{(-)}(z) = \zeta(k) \quad (z \in D^{(+)} \cap D^{(-)}),$$

here $\log z$ *is the principal value of the logarithmic function, the asymptotic conditions*

$$\frac{d}{dz} f_k^{(\pm)}(z) \to 0 \quad (z \to \infty,\ z \in D^{(\pm)}),$$

and the normalizing condition

$$f_k^{(+)}(0) = 0.$$

Then we have

$$f_k^{(+)}(z) = \mathrm{Li}_k(z), \qquad f_k^{(-)}(z) = \mathrm{Li}_{2,\underbrace{1,\ldots,1}_{(k-2)\text{-fold}}}(1-z) \qquad (k \geq 2).$$

This theorem says that the polylogarithms $\mathrm{Li}_k(z)$ are characterized as the unique holomorphic functions on $D^{(\pm)}$ satisfying the inversion formulas of polylogarithms (1) on the intersection $D^{(+)} \cap D^{(-)}$ under asymptotic and normalizing conditions.

We call a problem of determining holomorphic functions inductively from additive relations on the overlapped domain, a recursive Riemann-Hilbert problem of additive type or a Plemelj-Birkhoff decomposition of additive type [Bi, M, P].

In this article, generalizing this scheme, we characterize the multiple polylogarithms of one variable $\mathrm{Li}_{k_1,\ldots,k_r}(z)$ from the multiple zeta values

$$\zeta(k_1, k_2, \ldots, k_r) = \sum_{n_1 > \cdots > n_r > 0} \frac{1}{n_1^{k_1} \ldots n_r^{k_r}}.$$

The multiple polylogarithms of one variable are holomorphic functions on $|z| < 1$ determined by the Taylor expansions

$$\mathrm{Li}_{k_1,\ldots,k_r}(z) = \sum_{n_1 > \cdots > n_r > 0} \frac{z^{n_1}}{n_1^{k_1} \ldots n_r^{k_r}},$$

where $r \geq 1, k_1, \ldots, k_r \geq 1$. By using iterated integrals (8), they can be expressed as

$$\mathrm{Li}_{k_1,\ldots,k_r}(z) = \int_0^z \left(\frac{dz}{z}\right)^{k_1-1} \frac{dz}{1-z} \cdots \left(\frac{dz}{z}\right)^{k_r-1} \frac{dz}{1-z},$$

and can be continued onto $\mathbf{P}^1 \setminus \{0, 1, \infty\}$ as many-valued analytic functions along the path of integration, where $\mathbf{P}^1$ denotes the Riemann sphere.

The generating function (13) of the multiple polylogarithms yields a fundamental solution of the Knizhnik-Zamolodchikov equation (the KZ equation, for short) of one variable. This is an ordinary differential equation

$$\frac{dG}{dz} = \left(\frac{X_0}{z} + \frac{X_1}{1-z}\right) G \tag{2}$$

defined on the moduli space

$$\mathcal{M}_{0,4} = \mathbf{P}^1 \setminus \{0, 1, \infty\},$$

where X_0, X_1 are generators of the free Lie algebra $\mathfrak{X} = \mathbf{C}\{X_0, X_1\}$, which is a Lie algebra derived from the lower central series of the fundamental group of $\mathcal{M}_{0,4}$ [I]. The 1-forms $\xi_0 = \frac{dz}{z}$ and $\xi_1 = \frac{dz}{1-z}$ are considered to be dual variables of X_0, X_1 and generate a shuffle algebra. This algebra describes iterated integrals of the forms ξ_0 and ξ_1.

The fundamental solutions $\mathcal{L}^{(0)}(z)$ and $\mathcal{L}^{(1)}(z)$ of (2) normalized at $z = 0$ and $z = 1$ (see Section 2, Proposition 7) are the grouplike elements in $\widetilde{\mathcal{U}}(\mathfrak{X}) = \mathbf{C}\langle\langle X_0, X_1\rangle\rangle$, which denotes the non-commutative formal power series algebra of the variables X_0, X_1. These fundamental solutions satisfy the connection relation

$$\left(\mathcal{L}^{(1)}(z)\right)^{-1} \mathcal{L}^{(0)}(z) = \Phi_{\mathrm{KZ}}. \tag{3}$$

Here the connection matrix Φ_{KZ} is known as the Drinfel'd associator [D]. The Drinfel'd associator Φ_{KZ} is expressed as the generating function (18), appears in Section 2, of the multiple zeta values $\zeta(k_1, k_2, \ldots, k_r)$.

In [OkU], it was shown that the connection relation (3) is equivalent to the system of the functional relations among extended multiple polylogarithms (defined by (10))

$$\sum_{uv=w} \mathrm{Li}(\tau(u); 1-z)\, \mathrm{Li}(v; z) = \zeta(\mathrm{reg}^{10}(w)), \tag{4}$$

referred to as "the generalized inversion formulas." Here w denotes a word of ξ_0 and ξ_1. The inversion formulas (1) of the polylogarithms are the special cases of the generalized inversion formulas (4) with $w = \xi_0^{k-1}\xi_1$.

The first purpose of this article is determining functions $f^{(0)}$ and $f^{(1)}$ satisfying the following relation

$$\sum_{uv=w} f^{(1)}(\tau(u); z) f^{(0)}(v; z) = \zeta(\mathrm{reg}^{10}(w))$$

under asymptotic and normalizing conditions (Section 4, Theorem 11). This theorem is a generalization of Theorem 1 and is also a recursive Riemann-Hilbert problem of additive type.

Furthermore we establish the existence and uniqueness of a solution $(F^{(0)}(z), F^{(1)}(z))$ to the equation

$$\left(F^{(1)}(z)\right)^{-1} F^{(0)}(z) = \Phi_{\mathrm{KZ}} \tag{5}$$

under certain assumptions (Section 5, Theorem 12). In other words, the fundamental solutions $\mathcal{L}^{(0)}(z)$ and $\mathcal{L}^{(1)}(z)$ of the KZ equation (2) are completely characterized by equation (5). To determine functions in such a way is solving a Riemann-Hilbert problem of multiplicative type, which is the second purpose of this article.

This article is organized as follows: In Section 2, we prepare some terminologies on free Lie algebras and shuffle algebras, and survey the connection problem of the KZ equation of one variable due to [OkU] and [OiU1]. In Section 3, we consider in details the generalized inversion formulas. We prove the generalized inversion formulas directly. In Section 4, we derive the multiple polylogarithms from the multiple zeta values as holomorphic functions satisfying the generalized inversion formulas. Finally, in Section 5, we solve equation (5) and characterize the fundamental solutions of the KZ equation of one variable.

2. The connection problem of the KZ equation of one variable

Let $S = S(\xi_0, \xi_1)$ be a shuffle algebra generated by 1-forms

$$\xi_0 = \frac{dz}{z}, \qquad \xi_1 = \frac{dz}{1-z}.$$

This is the non-commutative polynomial algebra $\mathbf{C}\langle\xi_0, \xi_1\rangle$ generated by ξ_0, ξ_1 with the shuffle product ш. The shuffle product is defined recursively as

$$w \sqcup\!\sqcup \mathbf{1} = \mathbf{1} \sqcup\!\sqcup w = w,$$

$$(\xi_{i_1} w_1) \sqcup\!\sqcup (\xi_{i_2} w_2) = \xi_{i_1}(w_1 \sqcup\!\sqcup (\xi_{i_2} w_2)) + \xi_{i_2}((\xi_{i_1} w_1) \sqcup\!\sqcup w_2),$$

where $\mathbf{1}$ is the unit of $\mathbf{C}\langle\xi_0, \xi_1\rangle$ (that is, $\mathbf{1}$ stands for the empty word) and w, w_1, w_2 are words of $\mathbf{C}\langle\xi_0, \xi_1\rangle$.

By virtue of Reutenauer [R], S is an associative commutative algebra and has a Hopf algebra structure by the coproduct

$$\Delta^*(\xi_{i_1}\dots\xi_{i_r}) = \sum_{k=0}^{r} \xi_{i_1}\dots\xi_{i_k}\otimes\xi_{i_{k+1}}\dots\xi_{i_r}$$

(we regard $\xi_{i_1}\dots\xi_{i_0}$ (at $k=0$) and $\xi_{i_{r+1}}\dots\xi_{i_r}$ (at $k=r$) as $\mathbf{1}$), the counit $\varepsilon^*(\xi_i)=0$ and the antipode $\rho^*(\xi_{i_1}\dots\xi_{i_r}) = (-1)^r\xi_{i_r}\dots\xi_{i_1}$.

The shuffle algebra

$$S = \bigoplus_{s=0}^{\infty} S_s$$

is a graded Hopf algebra with the grading defined by the length of words.

The dual of this Hopf algebra with respect to the pairing

$$\langle \xi_{i_1}\dots\xi_{i_r}, X_{j_1}\dots X_{j_s}\rangle = \begin{cases} 1 & (r=s,\, i_k=j_k \text{ for } 1\le k\le r)\,, \\ 0 & (\text{otherwise}) \end{cases}$$

is the non-commutative formal power series algebra $\widetilde{\mathcal{U}} = \mathbf{C}\langle\langle X_0, X_1\rangle\rangle$. The algebra $\widetilde{\mathcal{U}}$ is the completion of the non-commutative polynomial algebra $\mathcal{U} = \mathbf{C}\langle X_0, X_1\rangle$, which is the universal enveloping algebra $\mathcal{U}(\mathfrak{X})$ of the free Lie algebra $\mathfrak{X} = \mathbf{C}\{X_0, X_1\}$ generated by X_0, X_1, with respect to a grading defined by the length of words:

$$\mathcal{U} = \bigoplus_{s=0}^{\infty} \mathcal{U}_s\,.$$

The algebra $\mathcal{U}$ (respectively $\widetilde{\mathcal{U}}$) have a co-commutative Hopf algebra structure by the following coproduct Δ, the counit ε as algebra morphisms and the antipode ρ as an anti-algebra morphism:

$$\Delta(X_i) = \mathbf{I}\otimes X_i + X_i\otimes\mathbf{I}\,,$$

$$\varepsilon(X_i) = 0\,,$$

$$\rho(X_i) = -X_i\,,$$

where $\mathbf{I}$ stands for the unit (namely the empty word) of $\mathcal{U}$ (resp. $\widetilde{\mathcal{U}}$).

In what follows, we denote conveniently the sum over all words w in S by $\sum_{w\in S}$ (or similar notations), and the dual element of w by the capital letter W (that is, for $w = \xi_{i_1}\dots\xi_{i_r}\in S$, the capitalization W stands for $X_{i_1}\dots X_{i_r}\in\mathcal{U}$).

The following lemmas are basic and will be used in Section 5.

LEMMA 2. *A $\widetilde{\mathcal{U}}$-valued function*

$$F(z)=\sum_{w\in S} f(w;z)W\,,$$

which is holomorphic at $z=0$ and $F(0)=\mathbf{I}$, is grouplike if and only if $f(w;z)$ is a shuffle homomorphism, namely $f(w_1 ш w_2;z)=f(w_1;z)f(w_2;z)$ for all words $w_1,w_2\in S$.

LEMMA 3. *If a $\widetilde{\mathcal{U}}$-valued function*

$$F(z)=\sum_{w\in S} f(w;z)W$$

is grouplike, holomorphic at $z=0$ and $F(0)=\mathbf{I}$, the reciprocal of $F(z)$ is written as

$$(F(z))^{-1}=\sum_{w\in S} f(w;z)\rho(W)=\sum_{w\in S} f(\rho^*(w);z)W\,.$$

We denote by S^0 and S^{10} the subalgebras of S defined as

$$S^0=\mathbf{C1}+S\xi_1\,,$$
$$S^{10}=\mathbf{C1}+\xi_0 S\xi_1\,.$$

The algebra S has polynomial ring structures as follows:

PROPOSITION 4 ([R]). *The algebra S is a polynomial algebra of ξ_0 whose coefficients are in S^0, and is a polynomial algebra of ξ_0,ξ_1 whose coefficients are in S^{10};*

$$S=S^0[\xi_0]=S^{10}[\xi_0,\xi_1]\,.$$

That is, any word w in S can be written as

$$w=\sum_j w_j ш \xi_0^j=\sum_{i,j}\xi_1^i ш w_{ij} ш \xi_0^j \tag{6}$$

uniquely, where $w_j\in S^1$ and $w_{ij}\in S^{10}$.

We define the regularizing maps reg^0 and reg^{10} as the restrictions to the constant terms of a word with respect to the decomposition (6):

$$\mathrm{reg}^0 : S=S^0[\xi_0]\to S^0\,,\quad u=\sum w_j ш \xi_0^j \mapsto w_0\ (w_j\in S^0)\,,$$
$$\mathrm{reg}^{10} : S=S^{10}[\xi_0,\xi_1]\to S^{10}\,,\quad u=\sum \xi_1^i ш w_{ij} ш \xi_0^j \mapsto w_{00}\ (w_{ij}\in S^{10})\,.$$

The maps reg^0 and reg^{10} are shuffle homomorphisms and are calculated by the following lemma. This lemma was essentially shown in Proposition 8 and Corollary 5 of [IKZ].

LEMMA 5 ([IKZ]). 1. *For a word $u \in S^0$, we have*

$$u\xi_0^n = \sum_{j=0}^{n} \mathrm{reg}^0(u\xi_0^{n-j}) ш \xi_0^j \,,$$

$$\mathrm{reg}^0(u\xi_0^n) = \sum_{j=0}^{n} (-1)^j (u\xi_0^{n-j}) ш \xi_0^j \,.$$

2. *For a word $u \in S^{10}$, we have*

$$\xi_1^m u\xi_0^n = \sum_{i=0}^{m}\sum_{j=0}^{n} \xi_1^i ш \mathrm{reg}^{10}(\xi_1^{m-i} u\xi_0^{n-j}) ш \xi_0^j \,, \tag{7}$$

$$\mathrm{reg}^{10}(\xi_1^m u\xi_0^n) = \sum_{i=0}^{m}\sum_{j=0}^{n} (-1)^{i+j} \xi_1^i ш (\xi_1^{m-i} u\xi_0^{n-j}) ш \xi_0^j \,.$$

Let $\mathbf{D}_0$ and $\mathbf{D}_1$ be domains on $\mathbf{C}$ defined by

$$\mathbf{D}_0 = \mathbf{C} \setminus \{z = x \mid x \geq 1\} \,,$$

$$\mathbf{D}_1 = \mathbf{C} \setminus \{z = x \mid x \leq 0\} \,.$$

For a word $w = \xi_{i_1} \dots \xi_{i_r} \in S^0$ (that is, $i_r = 1$), we define an iterated integral by

$$\int_0^z w = \begin{cases} 1 & (w = \mathbf{1}) \,, \\ \int_0^z \xi_{i_1} \left(\int_0^z \xi_{i_2} \dots \xi_{i_r} \right) & \text{(otherwise)} \end{cases} \tag{8}$$

recursively and extend it to S^0 as a linear map.

For $z \in \mathbf{D}_0$ and $w = \xi_{i_1} \dots \xi_{i_r} \in S^0$, we define the multiple polylogarithms of one variable $\mathrm{Li}(w; z)$ as

$$\mathrm{Li}(w; z) = \int_0^z w = \int_0^z \xi_{i_1} \dots \xi_{i_r} \,. \tag{9}$$

These are holomorphic functions on $\mathbf{D}_0$ and have Taylor expansions

$$\mathrm{Li}(\xi_0^{k_1-1}\xi_1 \dots \xi_0^{k_r-1}\xi_1; z) = \sum_{n_1 > n_2 \cdots > n_r > 0} \frac{z^{n_1}}{n_1^{k_1} \dots n_r^{k_r}} \quad (|z| < 1) \,,$$

which coincides with $\mathrm{Li}_{k_1,\dots,k_r}(z)$. In what follows, we consider the multiple polylogarithms only of one variable, so we omit "of one variable."

We extend $\mathrm{Li}(w; z)$ to S as follows: For a word $w = \sum_j w_j ш \xi_0^j$ ($w_j \in S^0$), we set an extended multiple polylogarithm by

$$\mathrm{Li}(w; z) = \sum_j \mathrm{Li}(w_j; z)\frac{(\log z)^j}{j!}\,. \tag{10}$$

Here $\log z$ is defined as the principal value on $\mathbf{D}_1$. These extended multiple polylogarithms are holomorphic on $\mathbf{D}_0 \cap \mathbf{D}_1$, and the map

$$\mathrm{Li}(\bullet; z) \,:\, S \longrightarrow \mathbf{C}, \quad w \longmapsto \mathrm{Li}(w; z) \quad (z \in \mathbf{D}_0 \cap \mathbf{D}_1)$$

is a shuffle homomorphism.

LEMMA 6 ([HPH, Ok]). *The extended multiple polylogarithms satisfy the following recursive differential relations*:

$$\frac{d}{dz}\mathrm{Li}(\xi_0 u; z) = \frac{\mathrm{Li}(u; z)}{z}\,, \qquad \frac{d}{dz}\mathrm{Li}(\xi_1 u; z) = \frac{\mathrm{Li}(u; z)}{1 - z}\,, \tag{11}$$

where u is a word of S.

We observe that the extended multiple polylogarithms can be continued onto $\mathbf{P}^1 \setminus \{0, 1, \infty\}$ as many-valued analytic functions along the integral paths of (9).

If $w = \xi_0^{k_1-1}\xi_1 \dots \xi_0^{k_r-1}\xi_1 \in S^{10}$, the limit of $\mathrm{Li}(w; z)$ as z tends to 1 in $\mathbf{D}_0 \cap \mathbf{D}_1$ converges and defines multiple zeta values:

$$\zeta(w) = \lim_{\substack{z\to 1\\ z\in \mathbf{D}_0\cap\mathbf{D}_1}} \mathrm{Li}(w; z) = \sum_{n_1>n_2\cdots>n_r>0} \frac{1}{n_1^{k_1}\dots n_r^{k_r}}\,. \tag{12}$$

The multiple zeta values $\zeta(w)$ are denoted by $\zeta(k_1, \dots, k_r)$ as usual.

Under these notations, the specific solution of equation (2) can be written as follows:

PROPOSITION 7 ([OiU1, OkU]). *The KZ equation of one variable* (2) *has the solution $\mathcal{L}^{(0)}(z)$ which satisfies the asymptotic condition*

$$\mathcal{L}^{(0)}(z) = \widehat{\mathcal{L}}^{(0)}(z)z^{X_0}\,,$$

where $\widehat{\mathcal{L}}^{(0)}(z)$ is holomorphic at $z = 0$ and $\widehat{\mathcal{L}}^{(0)}(0) = \mathbf{I}$.

The solution $\mathcal{L}^{(0)}(z)$ is uniquely characterized by this condition and is a grouplike element of $\widetilde{\mathcal{U}}$.

Furthermore the solution $\mathcal{L}^{(0)}(z)$ is expressed as

$$\mathcal{L}^{(0)}(z) = \sum_{w\in S} \mathrm{Li}(w; z)W \tag{13}$$

$$= \Big(\sum_{w\in S} \mathrm{Li}(\mathrm{reg}^0(w); z)W\Big)z^{X_0} \tag{14}$$

$$= (1-z)^{-X_1}\Bigl(\sum_{w\in S}\mathrm{Li}(\mathrm{reg}^{10}(w);z)W\Bigr)z^{X_0}\,. \tag{15}$$

We call the solution $\mathcal{L}^{(0)}(z)$ the fundamental solution of (2) normalized at $z = 0$. We also refer to the solution

$$\mathcal{L}^{(1)}(z) = \widehat{\mathcal{L}}^{(1)}(z)(1-z)^{-X_1}\,,$$

where $\widehat{\mathcal{L}}^{(1)}(z)$ is holomorphic at $z = 1$, $\widehat{\mathcal{L}}^{(1)}(1) = \mathbf{I}$, as the fundamental solution normalized at $z = 1$.

With respect to the transformation $t : z \mapsto 1 - z$, we introduce the automorphism t^* on S by

$$t^*(\xi_0) = -\xi_1\,, \qquad t^*(\xi_1) = -\xi_0\,,$$

which is the pull back induced from t, and we also introduce the automorphism t_* on $\mathcal{U}$ by

$$t_*(X_0) = -X_1\,, \qquad t_*(X_1) = -X_0\,,$$

which is the dual map of t^*. Let $\tau : S \to S$ be an anti-automorphism defined by $\tau = t^* \circ \rho^*$, that is,

$$\tau(\xi_0) = \xi_1\,, \qquad \tau(\xi_1) = \xi_0\,.$$

Furthermore put $T = t_* \circ \rho$ which is an anti-automorphism on $\mathcal{U}$ satisfying

$$T(X_0) = X_1\,, \qquad T(X_1) = X_0\,.$$

Using the transformation t and the automorphism t_*, the fundamental solution $\mathcal{L}^{(1)}$ of the KZ equation (2) normalized at $z = 1$ is written as

$$\mathcal{L}^{(1)}(z) = \sum_{w\in S}\mathrm{Li}(w;1-z)t_*(W)$$

$$= \Bigl(\sum_{w\in S}\mathrm{Li}(\mathrm{reg}^{0}(w);1-z)t_*(W)\Bigr)(1-z)^{-X_1} \tag{16}$$

$$= z^{X_0}\Bigl(\sum_{w\in S}\mathrm{Li}(\mathrm{reg}^{10}(w);1-z)t_*(W)\Bigr)(1-z)^{-X_1}\,. \tag{17}$$

The connection relation between $\mathcal{L}^{(0)}$ and $\mathcal{L}^{(1)}$ is described as follows:

PROPOSITION 8 ([D, OkU]). 1. *The connection matrix between $\mathcal{L}^{(0)}(z)$ and $\mathcal{L}^{(1)}(z)$ is given by the Drinfel'd associator*

$$\Phi_{\mathrm{KZ}} = \Phi_{\mathrm{KZ}}(X_0, X_1) = \sum_{w\in S}\zeta(\mathrm{reg}^{10}(w))W\,. \tag{18}$$

That is, the connection formula reads

$$\mathcal{L}^{(0)}(z) = \mathcal{L}^{(1)}(z)\Phi_{\mathrm{KZ}} \,. \tag{19}$$

2. *The connection formula* (19) *is equivalent to the system of relations*

$$\sum_{uv=w} \mathrm{Li}(\tau(u);\, 1-z)\,\mathrm{Li}(v;\, z) = \zeta(\mathrm{reg}^{10}(w)) \tag{20}$$

for all words $w \in S$.

We call the relations (20) the generalized inversion formulas for extended multiple polylogarithms.

This proposition follows from the representation (15), (17), Lemma 3, and the definition (12) of the multiple zeta values.

3. The generalized inversion formulas for the extended multiple polylogarithms

In the previous section, we derived the generalized inversion formulas (20) from the connection formula (19) of the KZ equation of one variable. However the generalized inversion formulas can be proved directly as follows.

PROPOSITION 9. *For any word* $w \in S$, *the generalized inversion formula*

$$\sum_{uv=w} \mathrm{Li}(\tau(u);\, 1-z)\,\mathrm{Li}(v;\, z) = \zeta(\mathrm{reg}^{10}(w)) \tag{21}$$

holds.

To prove this, it is enough to show the following lemma. This lemma also plays a key role to prove Theorem 11 in Section 4.

LEMMA 10. 1. *For any word* w *in* S, *we have*

$$\frac{d}{dz}\left(\sum_{uv=w} \mathrm{Li}(\tau(u);\, 1-z)\,\mathrm{Li}(v;\, z)\right) = 0 \,.$$

2. *For any word* w *in* S, *we have*

$$\lim_{\substack{z\to 1,\\ z\in \mathbf{D}_0\cap\mathbf{D}_1}} \sum_{uv=w} \mathrm{Li}(\tau(u);\, 1-z)\,\mathrm{Li}(v;\, z) = \zeta(\mathrm{reg}^{10}(w)) \,, \tag{22}$$

$$\lim_{\substack{z\to 0,\\ z\in \mathbf{D}_0\cap\mathbf{D}_1}} \sum_{uv=w} \mathrm{Li}(\tau(u);\, 1-z)\,\mathrm{Li}(v;\, z) = \zeta(\mathrm{reg}^{10}(\tau(w))) \,. \tag{23}$$

PROOF. 1. We represent the differential recursive relations (11) in terms of the exterior derivative with respect to the variable z;

$$d\,\mathrm{Li}(\xi_i w;\, z) = \xi_i\,\mathrm{Li}(w;\, z) \qquad (i = 0, 1) \,.$$

From this, it follows that

$$d\,\mathrm{Li}(\tau(\xi_i)w;\,1-z) = -\xi_i\,\mathrm{Li}(w;\,z) \qquad (i=0,1)\,.$$

Hence, for a word $w = \xi_{i_1}\ldots\xi_{i_r}$, we have

$$\begin{aligned}
&d\left(\sum_{uv=w}\mathrm{Li}(\tau(u);\,1-z)\,\mathrm{Li}(v;\,z)\right)\\
&= d\left(\sum_{k=0}^{r}\mathrm{Li}(\tau(\xi_{i_1}\ldots\xi_{i_k});\,1-z)\,\mathrm{Li}(\xi_{i_{k+1}}\ldots\xi_{i_r};\,z)\right)\\
&= \xi_{i_1}\,\mathrm{Li}(\xi_{i_2}\ldots\xi_{i_r};\,z)\\
&\quad+\sum_{k=1}^{r-1}\Big(-\xi_{i_k}\,\mathrm{Li}(\tau(\xi_{i_1}\ldots\xi_{i_{k-1}});\,1-z)\,\mathrm{Li}(\xi_{i_{k+1}}\ldots\xi_{i_r};\,z)\\
&\qquad\qquad+\xi_{i_{k+1}}\,\mathrm{Li}(\tau(\xi_{i_1}\ldots\xi_{i_k});\,1-z)\,\mathrm{Li}(\xi_{i_{k+2}}\ldots\xi_{i_r};\,z)\Big)\\
&\quad-\xi_{i_r}\,\mathrm{Li}(\tau(\xi_{i_1}\ldots\xi_{i_{r-1}});\,1-z)\\
&= \xi_{i_1}\,\mathrm{Li}(\xi_{i_2}\ldots\xi_{i_r};\,z)-\xi_{i_1}\,\mathrm{Li}(\xi_{i_2}\ldots\xi_{i_r};\,z)+\xi_{i_r}\,\mathrm{Li}(\tau(\xi_{i_1}\ldots\xi_{i_{r-1}});\,1-z)\\
&\quad-\xi_{i_r}\,\mathrm{Li}(\tau(\xi_{i_1}\ldots\xi_{i_{r-1}});\,1-z)\\
&= 0\,.
\end{aligned}$$

2. For a word $w=\xi_0^r$ or $w=\xi_1^r$, the both sides of (22) and (23) are trivially zero. For a word $w=\xi_1^k w'\xi_0^l,\ w'\in S^{10}$, we will prove (22). One can similarly prove (23).

For $w=\xi_1^k w'\xi_0^l,\ w'\in S^{10}$, we have

$$\begin{aligned}
&\sum_{uv=w}\mathrm{Li}(\tau(u);\,1-z)\,\mathrm{Li}(v;\,z)\\
&=\sum_{i=0}^{k}\mathrm{Li}(\tau(\xi_1^i);\,1-z)\,\mathrm{Li}(\xi_1^{k-i}w'\xi_0^l;\,z)+\sum_{\substack{uv=w'\xi_0^l\\ u\neq\mathbf{1}}}\mathrm{Li}(\tau(\xi_1^k u);\,1-z)\,\mathrm{Li}(v;\,z)\,.
\end{aligned}\tag{24}$$

For the second term of the right hand side of (24), by putting $u=\xi_0 u'$, we obtain

$$\begin{aligned}
\mathrm{Li}(\tau(\xi_1^k u);\,1-z)\,\mathrm{Li}(v;\,z) &= \mathrm{Li}(\tau(\xi_1^k\xi_0 u');\,1-z)\,\mathrm{Li}(v;\,z)\\
&= \mathrm{Li}(\tau(u')\xi_1\xi_0^k);\,1-z)\,\mathrm{Li}(v;\,z)\\
&= \sum_{s=0}^{k}\mathrm{Li}(\mathrm{reg}^0(\tau(u')\xi_1\xi_0^{k-s}));\,1-z)\,\frac{(\log(1-z))^s}{s!}\,\mathrm{Li}(v;\,z)\,.
\end{aligned}$$

Since $\mathrm{reg}^0(\tau(u')\xi_1\xi_0^{k-s}) \neq \mathbf{1}$ so that

$$\mathrm{Li}(\mathrm{reg}^0(\tau(u')\xi_1\xi_0^{k-s}));1-z) = O(1-z) \quad (z\to 1)\,,$$

and since $\mathrm{Li}(v;z)$ diverges at most of logarithmic order as $z\to 1$, we have

$$\mathrm{Li}(\tau(\xi_1^k u);1-z)\,\mathrm{Li}(v;z)\to 0 \quad (z\to 1)\,.$$

Next, we consider the first term of the right hand side of (24). By using Lemma 5, we have

$$\begin{aligned}
&\sum_{i=0}^{k}\mathrm{Li}(\tau(\xi_1^i);1-z)\,\mathrm{Li}(\xi_1^{k-i}w'\xi_0^l;z)\\
&=\sum_{i=0}^{k}\mathrm{Li}(\xi_0^i;1-z)\,\mathrm{Li}(\xi_1^{k-i}w'\xi_0^l;z)\\
&=\sum_{i=0}^{k}\sum_{p=0}^{k-i}\sum_{q=0}^{l}\mathrm{Li}(\xi_0^i;1-z)\,\mathrm{Li}(\xi_1^p;z)\,\mathrm{Li}(\mathrm{reg}^{10}(\xi_1^{k-i-p}w'\xi_0^{l-q});z)\,\mathrm{Li}(\xi_0^q;z)\\
&=\sum_{i=0}^{k}\sum_{p=0}^{k-i}\sum_{q=0}^{l}\frac{(\log(1-z))^i}{i!}\frac{(-\log(1-z))^p}{p!}\,\mathrm{Li}(\mathrm{reg}^{10}(\xi_1^{k-i-p}w'\xi_0^{l-q});z)\frac{(\log z)^q}{q!}\\
&=\sum_{r=0}^{k}\sum_{q=0}^{l}\left(\sum_{i+p=r}\frac{1}{i!}\frac{(-1)^p}{p!}\right)(\log(1-z))^r\,\mathrm{Li}(\mathrm{reg}^{10}(\xi_1^{k-r}w'\xi_0^{l-q});z)\frac{(\log z)^q}{q!}\,.
\end{aligned}\tag{25}$$

From the identity

$$\sum_{i+p=r}\frac{1}{i!}\frac{(-1)^p}{p!} = \begin{cases}1 & (r=0)\,,\\ 0 & (r\neq 0)\,,\end{cases}$$

it follows that

$$\begin{aligned}
(25) &= \sum_{q=0}^{l}\mathrm{Li}(\mathrm{reg}^{10}(\xi_1^k w'\xi_0^{l-q});z)\frac{(\log z)^q}{q!}\\
&\to \mathrm{Li}(\mathrm{reg}^{10}(\xi_1^k w'\xi_0^l);1) = \zeta(\mathrm{reg}^{10}(\xi_1^k w'\xi_0^l)) \qquad (z\to 1)\,.
\end{aligned}$$

□

We should observe that some relations among the multiple zeta values are derived from the generalized inversion formulas. For instance, in the relation obtained by replacing w to $\tau(w)$ in (21), we have

$$\sum_{uv=\tau(w)}\mathrm{Li}(\tau(u);1-z)\,\mathrm{Li}(v;z) = \zeta(\mathrm{reg}^{10}(\tau(w)))\,. \tag{26}$$

In (26), replacing z to $1-z$ and put $u=\tau(v')$, $v=\tau(u')$, we have

$$\sum_{u'v'=w} \mathrm{Li}(v';z)\,\mathrm{Li}(\tau(u');1-z)=\zeta(\mathrm{reg}^{10}(\tau(w)))\,. \tag{27}$$

The left hand side of (27) coincides with the left hand side of (21), so that we have the duality relations for multiple zeta values

$$\zeta(\mathrm{reg}^{10}(w))=\zeta(\mathrm{reg}^{10}(\tau(w))) \tag{28}$$

for any word w in S.

We note that the duality relation of the multiple zeta values (28) can be obtained immediately by comparing equations (22) and (23).

4. The Riemann-Hilbert problem of additive type for multiple polylogarithms

In this section, we give a characterization of the multiple polylogarithms using the generalized inversion formulas. The generalized inversion formulas (21) for a word $w \in S^{10}$ read

$$\mathrm{Li}(\tau(w);1-z)+\mathrm{Li}(w;z)=\zeta(w)-\sum_{\substack{uv=w\\ u,v\neq \mathbf{1}}} \mathrm{Li}(\tau(u);1-z)\,\mathrm{Li}(v;z)\,. \tag{29}$$

Equation (29) says that the right hand side, which is holomorphic on $\mathbf{D}_0\cap\mathbf{D}_1$, decomposes to the sum of $\mathrm{Li}(\tau(w);1-z)$ and $\mathrm{Li}(w;z)$, which are holomorphic on $\mathbf{D}_0$ and $\mathbf{D}_1$ respectively. Moreover the length of words appeared in the right hand side are less than the length of the word w. Hence the characterization of the multiple polylogarithms by using this decomposition is considered as a recursive Riemann-Hilbert problem of additive type. This problem is formulated as follows.

THEOREM 11. *There exist unique* ш*-homomorphisms* $f^{(0)}(\bullet;z)$, $f^{(1)}(\bullet;z): S\to\mathbf{C}$, *which satisfy*

$$f^{(0)}(\xi_0;z)=\log z\,,\quad f^{(1)}(\xi_0;z)=\log(1-z)\,,$$

and the following three conditions:

1. *For any word* $w\in S$, $f^{(0)}(w;z)$ *and* $f^{(1)}(w;z)$ *enjoy the functional equations*

$$\sum_{uv=w} f^{(1)}(\tau(u);z)f^{(0)}(v;z)=\zeta(\mathrm{reg}^{10}(w))\quad (z\in\mathbf{D}_0\cap\mathbf{D}_1)\,. \tag{30}$$

2. *For any word* $w\in S^{10}$, $f^{(0)}(w;z)$ *and* $f^{(1)}(w;z)$ *are holomorphic on* $\mathbf{D}_0$ *and* $\mathbf{D}_1$ *respectively and satisfy the asymptotic conditions*

$$\frac{d}{dz}f^{(i)}\big(w;z\big)\to 0\quad (z\to\infty,\ z\in\mathbf{D}_i)\,. \tag{31}$$

3. *For any word* $w \in S^{10}$, $f^{(0)}(w;z)$ *satisfies the normalizing conditions*

$$f^{(0)}\big(w;0\big)=0\,. \tag{32}$$

The solutions $f^{(i)}(\bullet;z)$ *are expressed in terms of extended multiple polylogarithms as follows*:

$$f^{(0)}(w;z)=\mathrm{Li}(w;z)\,,$$
$$f^{(1)}(w;z)=\mathrm{Li}(w;1-z)\,.$$

PROOF. By Proposition 9, the functions $f^{(0)}(w;z)=\mathrm{Li}(w;z)$ and $f^{(1)}(w;z)=\mathrm{Li}(w;1-z)$ satisfy all of the previous conditions.

We show that $f^{(i)}(w;z)$ are uniquely determined by using induction on the length of a word w.

First, in the case of $w=\xi_0$, equation (30) reads

$$f^{(1)}(\xi_1;z)+f^{(0)}(\xi_0;z)=0\,.$$

Therefore we obtain

$$f^{(1)}(\xi_1;z)=-f^{(0)}(\xi_0;z)=-\log z=\mathrm{Li}(\xi_1;1-z)\,.$$

In a similar fashion, we have

$$f^{(0)}(\xi_1;z)=-f^{(1)}(\xi_0;z)=-\log(1-z)=\mathrm{Li}(\xi_1;z)$$

in the case of $w=\xi_1$.

Next, we assume that $f^{(0)}(w';z)=\mathrm{Li}(w';z)$ and $f^{(1)}(w';z)=\mathrm{Li}(w';1-z)$ for words w' whose length is less than r.

Now, if the result $f^{(0)}=\mathrm{Li}(w;z)$ and $f^{(1)}=\mathrm{Li}(w;1-z)$ hold for all words $w\in S^{10}$ of length r, we obtain

$$\begin{aligned}
f^{(0)}(w;z)&=f\Big(\sum_{i,j}\xi_1^i ш w_{ij} ш \xi_0^j;z\Big)\\
&=\sum_{i,j}\frac{f(\xi_1;z)^i}{i!}f(w_{ij};z)\frac{f(\xi_0;z)^j}{j!}\\
&=\sum_{i,j}\frac{\mathrm{Li}(\xi_1;z)^i}{i!}\mathrm{Li}(w_{ij};z)\frac{\mathrm{Li}(\xi_0;z)^j}{j!}\\
&=\mathrm{Li}\Big(\sum_{i,j}\xi_1^i ш w_{ij} ш \xi_0^j;z\Big)=\mathrm{Li}(w;z)
\end{aligned}$$

for any word $w \in S$ of length r, since $f^{(0)}(w;z)$ and $\mathrm{Li}(w;z)$ are both shuffle homomorphisms and the word w has the unique decomposition (6). In the similar way, we have also $f^{(1)}(w;z) = \mathrm{Li}(w;1-z)$ for any word $w \in S$.

Therefore it suffice to show that $f^{(0)}(w;z) = \mathrm{Li}(w;z)$ and $f^{(0)}(w;z) = \mathrm{Li}(w;1-z)$ for any word $w \in S^{10}$ of length r.

Let $w = \xi_{i_1} \dots \xi_{i_r}$ be a word in S^{10} of length r. Under the assumption of induction, equation (30) becomes

$$\begin{aligned} & f^{(1)}(\tau(w);z) + \sum_{k=1}^{r-1} \mathrm{Li}(\tau(\xi_{i_1} \dots \xi_{i_k});1-z)\,\mathrm{Li}(\xi_{i_{k+1}} \dots \xi_{i_r};z) + f^{(0)}(w;z) \\ =& \zeta(\mathrm{reg}^{10}(w)) = \zeta(w)\,. \end{aligned} \tag{33}$$

Now we show $f^{(1)}(w;z) = \mathrm{Li}(w;1-z)$ and $f^{(0)}(w;z) = \mathrm{Li}(w;z)$ by using (31), (32) and (33). According to Lemma 10.1, we have

$$\begin{aligned} & d\left(\sum_{k=1}^{r-1} \mathrm{Li}(\tau(\xi_{i_1} \dots \xi_{i_k});1-z)\,\mathrm{Li}(\xi_{i_{k+1}} \dots \xi_{i_r};z)\right) \\ &\qquad = -d\,\mathrm{Li}(\xi_{i_1} \dots \xi_{i_r};z) - d\,\mathrm{Li}(\tau(\xi_{i_1} \dots \xi_{i_r});1-z)\,. \end{aligned}$$

Thus the differentiation of (33) leads to the equation

$$df^{(0)}(w;z) - d\,\mathrm{Li}(w;z) = -df^{(1)}(\tau(w);z) + d\,\mathrm{Li}(\tau(w);1-z)\,. \tag{34}$$

Here we notice that both w and $\tau(w)$ are words in S^{10}.

Since the left hand side (resp. right hand side) of (34) is holomorphic on $\mathbf{D}_0$ (resp. $\mathbf{D}_1$), the both side of (34) are entire functions. By (31), due to Liouville's theorem, we obtain

$$df^{(0)}(w;z) - d\,\mathrm{Li}(w;z) = 0\,,$$

$$df^{(1)}(\tau(w);z) - d\,\mathrm{Li}(\tau(w);1-z) = 0\,.$$

Thus the functions $f^{(0)}(w;z)$ and $f^{(1)}(w;z)$ are determined as

$$f^{(0)}(w;z) = \mathrm{Li}(w;z) + c^{(0)}(w)\,, \tag{35}$$

$$f^{(1)}(\tau(w);z) = \mathrm{Li}(\tau(w);1-z) + c^{(1)}(w)\,, \tag{36}$$

where $c^{(0)}(w)$ and $c^{(1)}(w)$ are integration constants.

Finally, we determine the integral constants $c^{(0)}(w)$ and $c^{(1)}(w)$. By (32), $c^{(0)}(w) = 0$ is clear. Substituting (35) and (36) to (30), we have

$$\sum_{uv=w} \mathrm{Li}(\tau(u);1-z)\,\mathrm{Li}(v;z) + c^{(1)}(w) = \zeta(w)\,.$$

In this relation, letting $z \to 1 (z \in \mathbf{D}_0 \cap \mathbf{D}_1)$, we obtain, from Lemma 10 (22),

$$\zeta(w) + c^{(1)}(w) = \zeta(w) .$$

Thus we have

$$c^{(1)}(w) = 0$$

and have completed the proof of this theorem. □

5. The Riemann-Hilbert problem corresponding to the KZ equation of one variable

In this section, we show that the fundamental solutions $\mathcal{L}^{(0)}(z)$ and $\mathcal{L}^{(1)}(z)$ of the KZ equation (2) are determined by the equation

$$\left(\mathcal{L}^{(1)}(z)\right)^{-1} \mathcal{L}^{(0)}(z) = \Phi_{\mathrm{KZ}} ,$$

which is a transformation of the connection formula (19).

Namely we find $\widetilde{\mathcal{U}}$-valued functions $F^{(0)}(z)$ and $F^{(1)}(z)$ which are grouplike, satisfy the relation

$$\left(F^{(1)}(z)\right)^{-1} F^{(0)}(z) = \Phi_{\mathrm{KZ}} \tag{37}$$

in $\mathbf{D}_0 \cap \mathbf{D}_1$ and some conditions. This equation can be interpreted as a decomposition of the holomorphic function $\Phi_{\mathrm{KZ}} \in \mathbf{D}_0 \cap \mathbf{D}_1$ to the multiplication of two functions $\left(F^{(1)}(z)\right)^{-1}$ and $F^{(0)}(z)$ holomorphic on $\mathbf{D}_1$ and $\mathbf{D}_0$ respectively, so that the problem determining the solution of equation (37) is a Riemann-Hilbert problem of multiplicative type.

We can solve this problem by associating with Theorem 11 as follows.

THEOREM 12. *There exist unique $\widetilde{\mathcal{U}}$-valued functions $F^{(0)}(z)$ and $F^{(1)}(z)$ denoted by*

$$F^{(0)} = (1-z)^{-X_1} \tilde{F}^{(0)}(z) z^{X_0}, \qquad \tilde{F}^{(0)}(z) = \sum_{w \in S} \tilde{f}^{(0)}(\mathrm{reg}^{10}(w); z) W, \tag{38}$$

$$F^{(1)} = z^{X_0} \tilde{F}^{(1)}(z) (1-z)^{-X_1}, \qquad \tilde{F}^{(1)}(z) = \sum_{w \in S} \tilde{f}^{(1)}(\mathrm{reg}^{10}(w); z) W , \tag{39}$$

which are grouplike, and enjoy the following conditions:

1. *$F^{(0)}(z)$ and $F^{(1)}(z)$ satisfy the functional equation*

$$\left(F^{(1)}(z)\right)^{-1} F^{(0)}(z) = \Phi_{\mathrm{KZ}} . \tag{40}$$

2. $\tilde{F}^{(0)}(z)$ *and* $\tilde{F}^{(1)}(z)$ *are holomorphic on* $\mathbf{D}_0$ *and* $\mathbf{D}_1$ *respectively and satisfy the asymptotic condition*

$$\frac{d}{dz}\tilde{F}^{(i)}(z) \to \mathbf{I} \qquad (z \to \infty, z \in \mathbf{D}_i)\,. \tag{41}$$

3. $\tilde{F}^{(0)}(z)$ *satisfies the normalizing condition*

$$\tilde{F}^{(0)}(0) = \mathbf{I}\,. \tag{42}$$

Then the functions $F^{(0)}(z)$ *and* $F^{(1)}(z)$ *give the fundamental solutions of the KZ equation of one variable normalized at* $z = 0$ *and* 1 *respectively.*

PROOF. We reduce this problem to Theorem 11. Put

$$F^{(0)}(z) = \sum_{w\in S} f^{(0)}(w; z)W\,,$$
$$F^{(1)}(z) = \sum_{w\in S} f^{(1)}(t^*(w); z)W\,.$$

Since $F^{(0)}(z)$ and $F^{(1)}(z)$ are grouplike, by virtue of Lemmas 2 and 3, the functions $f^{(i)}(\bullet; z)$ are regarded as shuffle homomorphisms from S to $\mathbf{C}$, and the reciprocal of $F^{(1)}(z)$ is given by

$$\left(F^{(1)}(z)\right)^{-1} = \sum_{w\in S} f^{(1)}(t^* \circ \rho^*(w); z)W = \sum_{w\in S} f^{(1)}(\tau(w); z)W\,.$$

Under these notation, equation (40) can be written as

$$\left(\sum_{u\in S} f^{(1)}(\tau(u); z)U\right)\left(\sum_{v\in S} f^{(0)}(v; z)V\right) = \Phi_{\mathrm{KZ}} = \sum_{w\in S} \zeta(\mathrm{reg}^{10}(w))W\,.$$

The coefficient of W of this equation is equation (30).

Next, since $f^{(0)}(w; z)$ is a shuffle homomorphism and equation (7) holds, we have

$$\begin{aligned}
F^{(0)}(z) &= \sum_{w} f^{(0)}(w; z)W \\
&= \sum_{i,j}\sum_{w\in S^{10}} f^{(0)}(\xi_1^i w \xi_0^j; z)X_1^i W X_0^j \\
&= \sum_{i,j}\sum_{w\in S^{10}}\sum_{s=0}^{i}\sum_{t=0}^{j} f^{(0)}(\xi_1^s; z) f^{(0)}(\mathrm{reg}^{10}(\xi_1^{i-s} w \xi_0^{j-t}); z) f^{(0)}(\xi_0^t; z)X_1^i W X_0^j \\
&= \left(\sum_i \frac{f(\xi_1; z)^i}{i!}X_1^i\right)\left(\sum_{w\in S} f^{(0)}(\mathrm{reg}^{10}(w); z)W\right)\left(\sum_j \frac{f(\xi_0; z)^j}{j!}X_0^j\right)\,.
\end{aligned}$$

Comparing this formula and (38), we have

$$f^{(0)}(\xi_0; z) = \log z$$

as the coefficient of X_0,

$$f^{(0)}(\xi_1; z) = -\log(1-z)$$

as the coefficient of X_1, and

$$f^{(0)}(w; z) = \tilde{f}^{(0)}(w; z) \qquad (w \in S^{10} : \text{word})$$

as the coefficient of $W = X_0 W' X_1$. Thus the asymptotic condition (41) says that

$$f^{(0)}(w; z) = \tilde{f}^{(0)}(w; z) \to 0 \qquad (z \in \mathbf{D}_0 \to \infty)$$

for any word w in S^{10} and the normalizing condition (42)

$$f^{(0)}(w; 0) = \tilde{f}^{(0)}(w; 0) = 0$$

for any word w in S^{10}.

In the similar way, comparing the equation

$$\begin{aligned} F^{(1)}(z) &= \left(\sum_i \frac{f^{(1)}(t^*(\xi_0); z)^i}{i!} X_1^i\right) \left(\sum_{w \in S} f^{(1)}(\mathrm{reg}^{10}(t^*(w)); z) W\right) \\ &\qquad \times \left(\sum_j \frac{f^{(1)}(t^*(\xi_1); z)^j}{j!} X_0^j\right) \\ &= \left(\sum_i \frac{(-f^{(1)}(\xi_1; z))^i}{i!} X_1^i\right) \left(\sum_{w \in S} f^{(1)}(\mathrm{reg}^{10}(t^*(w)); z) W\right) \\ &\qquad \times \left(\sum_j \frac{(-f^{(1)}(\xi_0; z))^j}{j!} X_0^j\right) \end{aligned}$$

and (39), we have

$$\begin{aligned} f^{(1)}(\xi_0; z) &= \log(1-z), \\ f^{(1)}(\xi_1; z) &= -\log z, \\ f^{(1)}(w; z) &= \tilde{f}^{(1)}(t^*(w); z) \qquad (w \in S^{10}). \end{aligned}$$

Thus the asymptotic condition (41) reads

$$f^{(1)}(w; z) = \tilde{f}^{(1)}(t^*(w); z) \to 0 \qquad (z \in \mathbf{D}_1 \to \infty)$$

for any words w in S^{10}.

Therefore the functions $f^{(i)}(w; z)$ satisfy the assumptions of Theorem 11, and so we have

$$f^{(0)}(w; z) = \mathrm{Li}(w; z)\,,$$
$$f^{(1)}(w; z) = \mathrm{Li}(w; 1 - z)\,.$$

Consequently,

$$F^{(0)}(z) = \sum_{w \in S} \mathrm{Li}(w; z) W\,,$$
$$F^{(1)}(z) = \sum_{w \in S} \mathrm{Li}(t^*(w); 1 - z) W = \sum_{w \in S} \mathrm{Li}(w; 1 - z) t_*(W)$$

are the unique solutions to this Riemann-Hilbert problem. The last claim follows from (14) and (16). □

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