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L^1 and L^∞-boundedness of Wave Operators for Three Dimensional Schrödinger Operators with Threshold Singularities

Kenji YAJIMA

Gakushuin University

Abstract. It is known that wave operators for three dimensional Schrödinger operators $-\Delta+V$ with threshold singularities are bounded in $L^p(\mathbf{R}^3)$ for $1<p<3$ in general and, for $1<p<\infty$ if and only if zero energy resonances are absent and all zero energy eigenfunctions ϕ of $-\Delta+V$ satisfy $\int V(x)x^\alpha\phi(x)dx=0$ for $|\alpha|\le 1$. We prove here that they are bounded in $L^1(\mathbf{R}^3)$ if and only if zero energy resonances are absent. We also show that they are bounded in $L^\infty(\mathbf{R}^3)$ if no resonances are present and all zero energy eigenfunctions $\phi(x)$ satisfy $\int_{\mathbf{R}^3} x^\alpha V(x)\phi(x)dx=0$ for $0\le|\alpha|\le 2$. This fills the unknown parts of the L^p-boundedness problem for wave operators of three dimensional Schrödinger operators.

1. Introduction

Let $H_0:=-\Delta$ be the free Schrödinger operator on the Hilbert space $\mathcal{H}:=L^2(\mathbf{R}^m)$ with domain $D(H_0)=\{u\in\mathcal{H}\colon \partial^\alpha u\in\mathcal{H},\ |\alpha|\le 2\}$ and $H:=H_0+V$, V being the multiplication with real measurable function $V(x)$ such that $|V(x)|\le C\langle x\rangle^{-\delta}$ for some $\delta>2$, $\langle x\rangle=(1+|x|^2)^{\frac12}$. Then, H and H_0 are selfadjoint in $\mathcal{H}$, the spectrum of H consists of absolutely continuous part $[0,\infty)$ and a finite number of non-positive eigenvalues of finite multiplicities and, wave operators $W_\pm$ defined by the strong limits

$$W_\pm:=\lim_{t\to\pm\infty} e^{itH}e^{-itH_0} \tag{1.1}$$

exist and complete: they are unitary from $\mathcal{H}$ to the absolutely continuous subspace $\mathcal{H}_{ac}(H)$ of $\mathcal{H}$ for H (see e.g. [16]). They enjoy the intertwining property and

$$f(H)P_{ac}(H)=W_\pm f(H_0)W_\pm^* \tag{1.2}$$

for any Borel functions f on $\mathbf{R}^1$, where $P_{ac}(H)$ is the orthogonal projection onto $\mathcal{H}_{ac}(H)$. The intertwining property reduces the mapping properties of $f(H)P_{ac}(H)$ to those of $f(H_0)$ provided that corresponding properties of $W_\pm$ are established. Thus, the L^p-boundedness

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of $W_\pm$ has attracted interest of various authors and following results have been obtained under various conditions on V. We remark that that the L^p-boundedness almost automatically implies the $W^{k,p}$-boundedness for $0 \le k \le 2$, see e.g. [24].

We write $L^2_\sigma(\mathbf{R}^m)\colon = L^2(\mathbf{R}^m, \langle x\rangle^{2\sigma}dx)$ for $\sigma \in \mathbf{R}$ and define for $1/2 < s < \delta - 1/2$ that

$$\mathcal{N}\colon = \{u \in L^2_{-s}(\mathbf{R}^m)\colon -\Delta u + Vu = 0\}\,.$$

The space $\mathcal{N}$ is independent of s, $\mathcal{N} \cap L^2(\mathbf{R}^m)$ is the eigenspace of H with eigenvalue 0 and, $u \in \mathcal{N} \setminus L^2(\mathbf{R}^m)$ is called (threshold) resonance of H. If $m \ge 3$, we have $\mathcal{N} = \{u \in L^2_{-s}(\mathbf{R}^m)\colon u + (-\Delta)^{-1}Vu = 0\}$ and, $u \in \mathcal{N}$ satisfies as $|x| \to \infty$ that

$$\lim_{|x|\to\infty} |x|^{m-2}u(x) = C_m\langle V, u\rangle \text{ where } \langle V, u\rangle\colon = \int_{\mathbf{R}^m} V(x)u(x)dx\,.$$

It follows that, if $m \ge 5$, $\mathcal{N} \subset L^2(\mathbf{R}^m)$ and resonances are absent and, if $m = 3$ and $m = 4$, $u \in \mathcal{N}$ is an eigenfunction of H if and only if $\langle V, u\rangle = 0$. We say H is of *generic type* if $\mathcal{N} = \{0\}$ and of *exceptional type* otherwise. For a Banach space X, $\mathbf{B}(X)$ is the Banach space of bounded operators in X. Following results have been obtained by various authors.

(1) If H is of generic type, $W_\pm \in \mathbf{B}(L^p(\mathbf{R}^m))$ for all $1 \le p \le \infty$ if $m \ge 3$ ([20, 21, 3]), for $1 < p < \infty$ if $m = 1$ ([19, 2, 5]) and $m = 2$ ([22, 12]). If $m = 1$, $W_\pm$ are unbounded in L^1 or L^∞ ([19, 5]). It is unknown if $W_\pm$ is bounded or not in L^1 or L^∞ if $m = 2$.

(2) Suppose H is of exceptional type, then:

(2a) If $m = 1$, $W_\pm \in \mathbf{B}(L^p(\mathbf{R}^m))$ for all $1 < p < \infty$ but not for $p = 1$ or $p = \infty$ ([19, 2, 5]).

(2b) If $m = 3$, $W_\pm$ are bounded in $L^p(\mathbf{R}^m)$ for $1 < p < 3$ in general and, for $1 < p < \infty$ if and only if all $u \in \mathcal{N}$ satisfy $\langle V, x^\alpha u\rangle = 0$ for $|\alpha| \le 1$ ([24]).

(2c) If $m = 4$ and if all $u \in \mathcal{N}$ satisfy $\langle V, u\rangle = 0$, viz. if resonances are absent, then $W_\pm \in \mathbf{B}(L^p(\mathbf{R}^m))$ for $1 \le p < 4$, for $1 \le p < \infty$ if $\langle V, x^\alpha u\rangle = 0$ for $|\alpha| \le 1$ and, also for $p = \infty$ if $\langle V, x^\alpha u\rangle = 0$ for $|\alpha| \le 2$ ([13, 9]).

(2d) If $m \ge 5$, $W_\pm \in \mathbf{B}(L^p(\mathbf{R}^m))$ for $1 \le p < m/2$ in general, for $1 \le p < m$ if and only if all $u \in \mathcal{N}$ satisfy $\langle V, u\rangle = 0$, for $1 \le p < \infty$ if and only if all $u \in \mathcal{N}$ satisfy $\langle V, x^\alpha u\rangle = 0$ for $|\alpha| \le 1$ and also for $p = \infty$ if $\langle V, x^\alpha u\rangle = 0$ for $|\alpha| \le 2$ ([8, 6, 23]).

When $m = 2$ and $\mathcal{N} \ne \{0\}$ or when $m = 4$ and resonances are present, nothing is known and the problem is still open. (See, however, Note added in proof which appears at the end of the paper.)

In this paper, when $m = 3$ and $\mathcal{N} \ne \{0\}$, we prove in particular that $W_\pm$ are bounded in $L^1(\mathbf{R}^3)$ if and only if $\mathcal{N} \subset L^2$. More precisely we prove the following theorem:

THEOREM 1.1. *Suppose that* $|V(x)| \leq C\langle x\rangle^{-7-\varepsilon}$ *and* H *is of exceptional type. Then,* $W_\pm$ *are bounded in* $L^1(\mathbf{R}^3)$ *if and only if zero energy resonances are absent from* H, *or all* $u \in \mathcal{N}$ *satisfy* $\langle V, u\rangle = 0$. *In this case* $W_\pm$ *are bounded in* $L^p(\mathbf{R}^3)$ *for* $1 \leq p < 3$ *in general and, for* $1 \leq p < \infty$ *if and only if all* $u \in \mathcal{N}$ *satisfy* $\langle V, x^\alpha u\rangle = 0$ *for* $|\alpha| \leq 1$. *If all* $u \in \mathcal{N}$ *further satisfy* $\langle V, x^\alpha u\rangle = 0$ *for* $|\alpha| = 2$, *then* $W_\pm$ *are bounded in* L^∞.

For $1 < p < \infty$, the theorem is known and we present here the proof of the if part which is different from the one given previously in [24]. We also take advantage of this occasion to correct the incomplete and partly wrong proof of Lemma 4.1 (3) and Lemma 4.4 (4) in [24] on the unboundedness of $W_\pm$ in $L^1(\mathbf{R}^3)$.

We refer readers more about the L^p boundedness of wave operators to the literature mentioned above and, jump into the proof of the theorem immediately. We think that the decay assumption on V is unnecessarily too strong, however, we do not pursue better conditions here. We shall often use Schur's lemma that the integral operator

$$Ku(x) = \int_Y K(x, y)d\nu(y)$$

is bounded from $L^p(Y, d\nu)$ to $L^p(X, d\mu)$ for all $1 \leq p \leq \infty$ if $K(x, y)$ satisfies

$$\sup_y \int_X |K(x, y)|d\mu(x) < \infty\,, \quad \sup_x \int_Y |K(x, y)|d\nu(y) < \infty\,. \tag{1.3}$$

In what follows we often identify the integral operator K with its kernel $K(x, y)$ and say K or $K(x, y)$ is *admissible* if (1.3) is satisfied. We also say that $K(x, y)$ is an L^p bounded kernel if K is bounded in $L^p(\mathbf{R}^3)$. We write $\chi(F)$ for the characteristic function of the set F and $a \leq_{|\cdot|} b$ means $|a| \leq |b|$. For $1 \leq p \leq \infty$, $\|u\|_p = \|u\|_{L^p(X)}$ for various X. We denote by C various constants whose specific values are of no importance. If the constant depends on some parameter, say Ω, it will be denoted by C_Ω. We adopt the physic convention that the inner product is linear in the second component and anti-linear in the first and denote it indistinguishably by (u, v) or $\langle u, v\rangle$. Furthermore we use this notation whenever the integral on the right of

$$(u, v) = \langle u, v\rangle :\, = \int_{\mathbf{R}^3} \overline{u(x)} v(x)dx$$

makes sense. For functions $u(x)$, $v(x)$, $|u\rangle\langle v|$ is an operator defined by

$$|u\rangle\langle v| f(x) :\, = u(x)\langle v, f\rangle\,.$$

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2. Reduction to the low energy analysis

We prove the theorem only for W_- and write $W_- = W$ in the sequel. The conjugation $\mathcal{C}u(x) = \overline{u(x)}$ changes the direction of time and results for $W_+ = C^{-1}W_-C$ follows immediately from the ones for W_-. We write $\mathbf{C}^+$ for the upper half plane and define for $\lambda \in \mathbf{C}^+$ that

$$G_0(\lambda) := (H_0 - \lambda^2)^{-1}, \quad G(\lambda) := (H - \lambda^2)^{-1}.$$

For the free resolvent $G_0(\lambda)$, we have

$$G_0(\lambda)u(x) = \frac{1}{4\pi}\int_{\mathbf{R}^3} \frac{e^{i\lambda|x-y|}}{|x-y|}u(y)dy.$$

The limiting absorption principle ([1]) and the absence of positive eigenvalues ([14]) imply that, for $\sigma > 1/2$, boundary values of $\langle x\rangle^{-\sigma}G_0(\lambda)\langle x\rangle^{-\sigma}$ and $\langle x\rangle^{-\sigma}G(\lambda)\langle x\rangle^{-\sigma}$ for $\lambda \in \mathbf{R}\setminus\{0\}$ exist in $\mathbf{B}(L^2(\mathbf{R}^3))$ and are locally Hölder continuous. When $u \in L^2_\sigma$, $\sigma > 1/2$, the stationary theory of scattering (e.g. [16]) implies that W can be represented via the boundary values of the resolvents in the form

$$Wu = u - \lim_{\varepsilon\downarrow 0, N\uparrow\infty} \frac{1}{\pi i}\int_\varepsilon^N G(\lambda)V(G_0(\lambda) - G_0(-\lambda))u\lambda d\lambda, \tag{2.1}$$

which we simply write as

$$Wu = u - \frac{1}{\pi i}\int_0^\infty G(\lambda)V(G_0(\lambda) - G_0(-\lambda))u\lambda d\lambda. \tag{2.2}$$

We decompose W into the high and the low energy parts

$$W = W_> + W_< := W\Psi(H_0) + W\Phi(H_0), \tag{2.3}$$

by using cut off functions $\Phi \in C_0^\infty(\mathbf{R})$ and $\Psi \in C^\infty(\mathbf{R})$ such that

$$\Phi(\lambda^2) + \Psi(\lambda^2) \equiv 1, \quad \Phi(\lambda^2) = 1 \text{ near } \lambda = 0 \quad \text{and} \quad \Phi(\lambda^2) = 0 \quad \text{for } |\lambda| > \lambda_0$$

for a small constant $\lambda_0 > 0$. We have proven in our previous paper [23] that, under the assumption of this paper, $W_>$ is bounded in $L^p(\mathbf{R}^3)$ for all $1 \le p \le \infty$ and we have nothing to add in this paper for $W_>$. Thus, in what follows, we shall be devoted to studying

$$W_< = \Phi(H_0) - \frac{1}{\pi i}\int_0^\infty G(\lambda)V(G_0(\lambda) - G_0(-\lambda))\lambda\Phi(H_0)d\lambda. \tag{2.4}$$

Evidently $\Phi(H_0) \in \mathbf{B}(L^p(\mathbf{R}^3))$ for all $1 \le p \le \infty$ and we have only to study the operator Z defined by the integral of (2.4), which we rewrite as

$$Zu = -\frac{1}{\pi i}\int_0^\infty G_0(\lambda)V(1 + G_0(\lambda)V)^{-1}(G_0(\lambda) - G_0(-\lambda))\lambda F(\lambda)ud\lambda \tag{2.5}$$

by using the resolvent identity $G(\lambda)V = G_0(\lambda)V(1+G_0(\lambda)V)^{-1}$ for $\lambda > 0$ and by defining $F(\lambda) = \Phi(\lambda^2)$.

We shall use the following well known results (see e.g [23]) on the behavior of $(1 + G_0(\lambda)V)^{-1}$ near the threshold $\lambda = 0$. We write $\mathcal{E} \colon = \mathcal{N} \cap L^2(\mathbf{R}^3)$. Functions $u \in \mathcal{N}$ satisfy

$$u(x) = -\frac{1}{4\pi}\int_{\mathbf{R}^3} \frac{V(y)u(y)}{|x-y|}dy$$

and, as $|x| \to \infty$,

$$u(x) = \frac{L(u)}{|x|} + O(|x|^{-2}), \quad L(u) \colon = \frac{-1}{4\pi}\int_{\mathbf{R}^3} V(x)u(x)dx\,. \tag{2.6}$$

It follows that $\mathcal{E} = \{u \in \mathcal{N} \colon L(u) = 0\}$ and $\dim \mathcal{N}/\mathcal{E} \le 1$. For $\phi \in \mathcal{E}$, we have

$$|\phi(x)| \le C\langle x\rangle^{-2}\,, \quad x \in \mathbf{R}^3\,, \tag{2.7}$$

and, for resonances u, with $L(u) \neq 0$,

$$u(x) = L(u)|x|^{-1} + O(|x|^{-2}), \quad |x| \to \infty. \tag{2.8}$$

These properties will be frequently used in what follows. Following [11], we say that H is of exceptional type of *the first kind* if $\mathcal{E} = \{0\}$, *the second* if $\mathcal{E} = \mathcal{N}$ and *the third kind* if $\{0\} \subsetneq \mathcal{E} \subsetneq \mathcal{N}$. The orthogonal projection in $\mathcal{H}$ onto $\mathcal{E}$ will be denoted by P. We let $D_0, D_1, \ldots$ be the integral operators defined by

$$D_j u(x) \colon = \frac{1}{4\pi j!}\int_{\mathbf{R}^3} |x-y|^{j-1}u(y)dy\,, \quad j = 0, 1, \ldots\,.$$

so that we have a formal Taylor expansion

$$G_0(\lambda)u(x) = \frac{1}{4\pi}\int_{\mathbf{R}^3} \frac{e^{i\lambda|x-y|}}{|x-y|}u(y)dy = \sum_{j=1}^{\infty}(i\lambda)^j D_j u\,.$$

If H is of exceptional type of the third kind, $-(Vu, u)$ is an inner product of $\mathcal{N}$ and there exists a unique $\psi \in \mathcal{N}$ such that

$$-(V\psi, u) = 0\,, \quad \forall u \in \mathcal{E}\,, \quad -(V\psi, \psi) = 1 \quad \text{and} \quad -L(\psi) > 0\,.$$

We define

$$\varphi \colon = \psi + PVD_2V\psi \in \mathcal{N} \tag{2.9}$$

and call it *the canonical resonance* ([11]). If H is of exceptional type of the first kind, then $\dim \mathcal{N} = 1$ and there is a unique $\varphi \in \mathcal{N}$ such that $-(V\varphi, \varphi) = 1$ and $-L(\varphi) > 0$ and we call this φ the canonical resonance.

PROPOSITION 2.1. *Let $m = 3$ and let V satisfy $|V(x)| \le C\langle x\rangle^{-\delta}$ for some $\delta > 3$. Suppose that H is of exceptional type of the third kind and let φ be the canonical resonance and $a = 4\pi i|\langle V, \varphi\rangle|^{-2}$. Then*:

$$(I + G_0(\lambda)V)^{-1} = \frac{PV}{\lambda^2} - \frac{PVD_3VPV}{\lambda} - \frac{a}{\lambda}|\varphi\rangle\langle\varphi|V + E(\lambda)\,, \tag{2.10}$$

where $E(\lambda)$ is the operator valued function which, when substituted for $(1 + G_0(\lambda)V)^{-1}$ in (2.5), *produces an operator which is bounded in $L^p(\mathbf{R}^3)$ for all $1 \le p \le \infty$. If H is of exceptional type of the first or the second kind*, (2.10) *still holds with $P = 0$ or $\varphi = 0$ respectively.*

3. L^1-unboundedness with resonances

If zero energy resonances are present, then Proposition 2.1 shows that their contribution to the operator Z is given via the canonical resonance φ by

$$Z_r u \colon = -\frac{ia}{\pi}\int_0^\infty G_0(\lambda)|V\varphi\rangle\langle V\varphi|(G_0(\lambda) - G_0(-\lambda))F(\lambda)u d\lambda\,, \tag{3.1}$$

where $a = 4\pi i|\langle V, \varphi\rangle|^{-2} \ne 0$. We show that Z_r is bounded in $L^p(\mathbf{R}^3)$ for $1 < p < 3$ but not for $p = 1$ or $3 \le p \le \infty$. We use the following lemma.

LEMMA 3.1. *Suppose that $K(x, y) \le_{|\cdot|} C\langle x\rangle^{-1}\langle y\rangle^{-1}\langle |x| - |y|\rangle^{-2-\varepsilon}$ for some $\varepsilon \ge 0$. Then*,

$$Ku(x) = \int_{\mathbf{R}^3} K(x, y)u(y)dy$$

is bounded in $L^p(\mathbf{R}^3)$ for any $1 \le p \le \infty$ if $\varepsilon > 0$ and, for all $1 < p < \infty$ if $\varepsilon = 0$. The same is true if $K(x, y) \le_{|\cdot|} C\langle x\rangle^{-1-\varepsilon}\langle y\rangle^{-1-\varepsilon}\langle |x| - |y|\rangle^{-2}$.

PROOF. For $\varepsilon > 0$, the lemma immediately follows from Schur's lemma. When $\varepsilon = 0$, we have

$$Ku(x) \le_{|\cdot|} C\int_0^\infty \frac{r^{2-\frac{2}{p}} r^{\frac{2}{p}}|f_u(r)|dr}{\langle x\rangle\langle r\rangle\langle |x| - r\rangle^2}\,, \quad f_u(r) = \int_{\mathbf{S}^2} u(r\omega)d\omega\,.$$

Since the right side is rotationally invariant, we have

$$\begin{aligned}\|Ku\|_p^p &\le C\int_0^\infty \left(\int_0^\infty \frac{\rho^{\frac{2}{p}} r^{2-\frac{2}{p}}(r^{\frac{2}{p}}|f_u(r)|)dr}{\langle\rho\rangle\langle r\rangle\langle\rho - r\rangle^2}\right)^p d\rho \\ &\le C\int_0^\infty \left(\int_0^\infty \frac{\langle\rho\rangle^{\frac{2}{p}-1}\langle r\rangle^{1-\frac{2}{p}}(r^{\frac{2}{p}}|f_u(r)|)dr}{\langle\rho - r\rangle^2}\right)^p d\rho\,.\end{aligned}$$

We may estimate $\langle\rho\rangle^{\frac{2}{p}-1}\langle r\rangle^{1-\frac{2}{p}}$ by $C\langle\rho-r\rangle^{\frac{2}{p}-1}$ if $p\le 2$ and by $C\langle\rho-r\rangle^{1-\frac{2}{p}}$ if $p\ge 2$. It follows that unless $p=1$ or $p=\infty$ we have a $\gamma>1$ such that

$$\|Ku\|_p \le C\left(\int_0^\infty\left(\int_0^\infty \frac{r^{\frac{2}{p}}|f_u(r)|dr}{\langle\rho-r\rangle^\gamma}\right)^p d\rho\right)^{1/p}$$

and Young's and Hölder's inequalities imply

$$\|Ku\|_p \le C\left(\int_0^\infty r^2|f_u(r)|^p dr\right)^{1/p} \le C\|u\|_p\,.$$

Since $\langle x\rangle^{-\varepsilon}\langle y\rangle^{-\varepsilon}\le C_\varepsilon\langle x-y\rangle^{-\varepsilon}$, the second statement follows from the first. This completes the proof of lemma. □

LEMMA 3.2. *Let Z_r be the operator defined by* (3.1). *Then, Z_r is bounded in $L^p(\mathbf{R}^3)$ for $1<p<3$ but not for $p=1$ nor for $3\le p\le\infty$.*

PROOF. It is known that Z_r is bounded in $L^p(\mathbf{R}^3)$ for $1<p<3$ ([24]). We give here the proof for $1<p<3$ which is different from the one given in [24]. The integral kernel of Z_r is given by

$$Z_r(x,y)=\sum_{\pm}\frac{\mp ia}{\pi}\int_0^\infty\int e^{i\lambda(|x-z|\pm|w-y|)}\frac{(V\varphi)(z)(V\varphi)(w)F(\lambda)}{16\pi^2|x-z||w-y|}dwdzd\lambda\,. \tag{3.2}$$

Since $F\in C_0^\infty([0,\infty))$, we immediately see that, with a constant $C>0$,

$$Z_r(x,y)\le_{|\cdot|}C\int\frac{|(V\varphi)(z)(V\varphi)(w)|}{|x-z||w-y|}dwdz\le\frac{C}{\langle x\rangle\langle y\rangle} \tag{3.3}$$

and $\chi(||x|-|y||\le 1)Z_r(x,y)$ and $\chi(||x|^2-|y|^2|\le 1)Z_r(x,y)$ are admissible kernels. Indeed, we have

$$\sup_y\int_{||x|-|y||\le 1}\frac{dx}{\langle x\rangle\langle y\rangle}=\sup_x\int_{||x|-|y||\le 1}\frac{dy}{\langle x\rangle\langle y\rangle}=C<\infty\,,$$

$\{(x,y)\colon ||x|^2-|y|^2|\le 1, ||x|-|y||>1\}\subset\{(x,y)\colon |x|<1,|y|<1\}$ and $Z_r(x,y)$ is obviously admissible on $\{(x,y)\colon |x|<1,|y|<1\}$. Thus *we may and do ignore the parts of $\mathbf{R}^3_x\times\mathbf{R}^3_y$ where $||x|-|y||<1$ or $||x|^2-|y|^2|<1$ in the proof.* We decompose the exponential functions $e^{i\lambda|x-z|}$ and $e^{i\lambda|w-y|}$ as

$$e^{i\lambda|x-z|}=e^{i\lambda|x|}+e^{i\lambda|x|}r(\lambda,x,z)\,,\quad r(\lambda,x,z)\colon=e^{i\lambda(|x-z|-|x|)}-1\,, \tag{3.4}$$

$$e^{i\lambda|w-y|}=e^{i\lambda|y|}+e^{i\lambda|y|}r(\lambda,y,w)\,,\quad r(\lambda,y,w)\colon=e^{i\lambda(|w-y|-|y|)}-1\,, \tag{3.5}$$

and write $Z_r(x, y)$ as a sum of four kernels: $Z_r(x, y) = \sum_{j=1}^{4} Z_j(x, y)$,

$$Z_j(x, y) := \frac{-ia}{\pi} \iint_{\mathbf{R}^6} \frac{(V\varphi)(z)(V\varphi)(w)}{16\pi^2|x-z||w-y|} F_j(x, z, w, z)dwdz$$

where $F_1, \dots, F_4$ are respectively given by

$$F_1 = F_1(x, z, w, y) := \sum_{\pm} \pm \int_0^\infty e^{i\lambda(|x|\pm|y|)} r(\lambda, x, z) r(\pm\lambda, y, w) F(\lambda) d\lambda\,,$$

$$F_2 = F_2(x, w, y) := \sum_{\pm} \pm \int_0^\infty e^{i\lambda(|x|\pm|y|)} r(\pm\lambda, y, w) F(\lambda) d\lambda\,,$$

$$F_3 = F_3(x, z, y) := \sum_{\pm} \pm \int_0^\infty e^{i\lambda(|x|\pm|y|)} r(\lambda, x, z) F(\lambda) d\lambda,$$

$$F_4 = F_4(x, y) := \sum_{\pm} \pm \int_0^\infty e^{i\lambda(|x|\pm|y|)} F(\lambda) d\lambda\,.$$

Here and hereafter the symbol $\sum_{\pm}$ means that the sum should be taken of the summands with upper signs and the ones with lower signs. We estimate $F_1, \dots, F_4$ using integration by parts. We use the following properties of $r(\lambda, x, y)$:

$$r(0, x, y) = 0, \quad \partial_\lambda(r(\pm\lambda, x, y))|_{\lambda=0} = \pm i(|x-y| - |x|), \tag{3.6}$$

$$|\partial_\lambda^k r(\lambda, x, y)| \le |y|^k, \quad k = 0, 1, \dots. \tag{3.7}$$

(1) We first show that $Z_1(x, y)$ is an admissible kernel. We apply integration by parts three times to F_1. Then, (3.6) and (3.7) imply

$$\begin{aligned} F_1(x, z, w, y) &= \sum_{\pm} \Bigg(\frac{\mp i}{(|x| \pm |y|)^3} \partial_\lambda^2 \{ r(\lambda, x, z) r(\pm\lambda, y, w) F(\lambda) \}|_{\lambda=0} \\ &\quad + \frac{\mp i}{(|x| \pm |y|)^3} \int_0^\infty e^{i\lambda(|x|\pm|y|)} \partial_\lambda^3 \{ r(\lambda, x, z) r(\pm\lambda, y, w) F(\lambda) \} d\lambda \Bigg) \\ &\le \sum_{\pm} C \frac{(1+|z|+|w|)^3}{(|x| \pm |y|)^3} \le C \frac{(1+|z|+|w|)^3}{(|x|-|y|)^3}\,. \end{aligned}$$

Recall that we are assuming $|V(x|| \le C\langle x \rangle^{-7-\varepsilon}$. It follows that

$$Z_1(x, y) \le_{|\cdot|} C \int_{\mathbf{R}^6} \frac{(1+|z|+|w|)^3 |(V\varphi)(z)(V\varphi)(w)|}{(|x|-|y|)^3 |x-z||w-y|} dwdz \le \frac{C}{(|x|-|y|)^3 \langle x \rangle \langle y \rangle}$$

and $Z_1(x, y)$ is admissible by virtue of Lemma 3.1 (recall that we are ignoring the parts where (x, y) satisfies $||x| - |y|| < 1$ or $||x|^2 - |y|^2| < 1$).

(2) We apply integration parts twice to F_2 and write it in the form

$$\sum_{\pm}\left(\frac{-i(|w-y|-|y|)}{(|x|\pm|y|)^2}\mp\int_0^\infty e^{i\lambda(|x|\pm|y|)}\frac{\partial_\lambda^2\{r(\pm\lambda,y,w)F(\lambda)\}}{(|x|\pm|y|)^2}d\lambda\right).$$

After another integration by parts we see that the integral terms are bounded by $C(1+|w|)^3(|x|\pm|y|)^{-3}$ and, when inserted into $Z_2(x,y)$, they produce admissible kernels bounded by $C\langle x\rangle^{-1}\langle y\rangle^{-1}(|x|\pm|y|)^{-3}$. Thus, modulo the admissible kernel

$$\begin{aligned} Z_2(x,y) &\equiv \sum_{\pm}\frac{-a}{\pi}\iint_{\mathbf{R}^6}\frac{(|w-y|-|y|)(V\varphi)(z)(V\varphi)(w)}{16\pi^2(|x|\pm|y|)^2\cdot|x-z||w-y|}dzdw \\ &= \sum_{\pm}\frac{a\varphi(x)}{\pi(|x|\pm|y|)^2}\int_{\mathbf{R}^3}\frac{(|w-y|-|y|)(V\varphi)(w)}{4\pi\cdot|w-y|}dw. \end{aligned} \tag{3.8}$$

Note that this is bounded in modulus by $C\langle x\rangle^{-1}\langle y\rangle^{-1}(|x|-|y|)^{-2}$ and Z_2 is bounded in $L^p(\mathbf{R}^3)$ for any $1<p<\infty$ by virtue of Lemma 3.1.

(3) For F_3, we apply integration by parts twice as in (2):

$$F_3=\sum_{\pm}\left(\mp\frac{i(|z-x|-|x|)}{(|x|\pm|y|)^2}\mp\int_0^\infty e^{i\lambda(|x|\pm|y|)}\frac{\partial_\lambda^2\{r(\lambda,x,z)F(\lambda)\}}{(|x|\pm|y|)^2}d\lambda\right).$$

By applying integration by parts once more as in (2) we see that the second terms on the right are bounded by $C(1+|z|)^3(|x|\pm|y|)^{-3}$ and their sum produces the kernel bounded by $C(|x|-|y|)^{-3}\langle x\rangle^{-1}\langle y\rangle^{-1}$ when inserted into $Z_3(x,y)$, which is admissible. Thus modulo the admissible kernel

$$\begin{aligned} Z_3(x,y) &\equiv \sum_{\pm}\frac{\mp a}{\pi}\int_{\mathbf{R}^6}\frac{(|z-x|-|x|)(V\varphi)(z)(V\varphi)(w)}{16\pi^2(|x|\pm|y|)^2|x-z||w-y|}dzdw \\ &= \frac{a}{\pi}\left(\frac{1}{(|x|+|y|)^2}-\frac{1}{(|x|-|y|)^2}\right)\varphi(y)\int_{\mathbf{R}^3}\frac{(|z-x|-|x|)(V\varphi)(z)}{4\pi|x-z|}dz \\ &\le_{|\cdot|}\frac{4a|x||y|\varphi(y)}{\pi(|x|+|y|)^2(|x|-|y|)^2}\int_{\mathbf{R}^3}\frac{|z||(V\varphi)(z)|}{4\pi|x-z|}dz\le_{|\cdot|}\frac{C}{(|x|+|y|)^2(|x|-|y|)^2}. \end{aligned}$$

Thus, $Z_3(x,y)$ is admissible.

(4) Again an integration by parts shows that

$$F_4(x,y)=\sum_{\pm}\pm\left(\frac{i}{(|x|\pm|y|)}+\frac{i}{(|x|\pm|y|)}\int_0^\infty e^{i\lambda(|x|\pm|y|)}F'(\lambda)d\lambda\right)$$

Here $F'\in C_0^\infty((0,\infty))$ and the integral terms are bounded by $C\langle|x|\pm|y|\rangle^{-N}$ for any N. It follows that the sum of the integral terms produces an admissible kernel bounded by $C\langle|x|-$

$|y|\rangle^{-N}\langle x\rangle^{-1}\langle y\rangle^{-1}$ and, modulo the admissible kernel

$$Z_4(x,y)\equiv\sum_{\pm}\frac{\pm a}{\pi(|x|\pm|y|)}\int_{\mathbf{R}^6}\frac{(V\varphi)(z)(V\varphi)(w)}{16\pi^2|x-z||w-y|}dzdw=\sum_{\pm}\frac{\pm a\varphi(x)\varphi(y)}{\pi(|x|\pm|y|)}\,.$$

(5) We prove that Z_r is unbounded in $L^1(\mathbf{R}^3)$. The combination of (1) to (4) implies that modulo admissible kernel $Z_r(x,y)$ is equal to

$$\begin{aligned}Z_{red}(x,y)=&\frac{a}{\pi}\varphi(x)\left(\frac{1}{(|x|+|y|)^2}+\frac{1}{(|x|-|y|)^2}\right)(c+|y|\varphi(y))\\&+\frac{a}{\pi}\left(\frac{\varphi(x)\varphi(y)}{|x|+|y|}-\frac{\varphi(x)\varphi(y)}{|x|-|y|}\right):\ =R_1(x,y)+R_2(x,y)\end{aligned}\tag{3.9}$$

where

$$c=\frac{1}{4\pi}\int_{\mathbf{R}^3}V(x)\varphi(x)dx=-L(\varphi)>0\,.$$

We prove that Z_{red} is unbounded in $L^1(\mathbf{R}^3)$ by contradiction. Take $u\in C_0^\infty(\mathbf{R}^3)$ such that $u(x)\ge 0$, $u(x)=0$ for $|x|\ge 1$ and $\int_{\mathbf{R}^3}u(x)dx=1$ and, define

$$u_n(x)=n^3u(nx),\quad f_n(x)=\int_{\mathbf{R}^3}Z_{red}(x,y)u_n(y)dy\,,\quad n=1,2,\dots.$$

We have $\|u_n\|_1=1$, $n=1,2,\dots$. For any $R>100$ and $100\le|x|\le R$,

$$\lim_{n\to\infty}f_n(x)=\frac{2ac}{\pi}\frac{\varphi(x)}{|x|^2}\,.$$

It follows by Fatou's lemma that

$$\begin{aligned}\frac{2|ac|}{\pi}\int_{100<|x|<R}\frac{|\varphi(x)|}{|x|^2}dx&=\int_{100<|x|<R}\lim_{n\to\infty}|f_n(x)|dx\\&\le\liminf_{n\to\infty}\int_{100<|x|<R}|f_n(x)|dx\le\|Z_r\|_{\mathbf{B}(L^1)}\,.\end{aligned}$$

Since $|\varphi(x)|\ge C|x|^{-1}$ for a constant $C>0$ for $|x|\ge 100$, this cannot happen for sufficiently large $R>0$ and Z_r is unbounded in $L^1(\mathbf{R}^3)$.

(6) We next prove that Z_r is bounded in $L^p(\mathbf{R}^3)$ for $1<p<3$. We have shown that Z_2 is bounded in $L^p(\mathbf{R}^3)$ for $1<p<\infty$ and it suffices to show that the operator $\tilde{R}_2$ defined by the kernel $R_2(x,y)\chi((|x|^2-|y|^2)\ge 1)$ is bounded in $L^p(\mathbf{R}^3)$ for $1<p<3$. We have

$$\tilde{R}_2u(x)=-\frac{2a\varphi(x)||x|}{\pi}\int_{||x|^2-|y|^2|\ge 1}\frac{|x|^{-1}}{|x|^2-|y|^2}\varphi(y)|y|u(y)dy\tag{3.10}$$

and $|\varphi(x)||x| \in L^\infty(\mathbf{R}^3)$. Hence, it suffices to show this for

$$Tu(x) = \int_{||x|^2-|y|^2|\geq 1} \frac{|x|^{-1}}{|x|^2-|y|^2} u(y)dy \, .$$

Since $Tu(x)$ is spherically symmetric, we have by using polar coordinates and by changing variables

$$\|Tu\|_p^p \leq 2^{1-p}\pi \int_0^\infty \rho^{\frac{1}{2}-\frac{p}{2}} \left(\int_{|\rho-r|\geq 1} \frac{r^{\frac{1}{2}}}{\rho - r} |M_u(\sqrt{r})| dr \right)^p d\rho$$

where $M_u(r) = \int_{\mathbf{S}^2} u(r\omega)d\omega$. Here $-1 < \frac{1}{2} - \frac{p}{2} < p-1$ if $1 < p < 3$ and $\rho^{\frac{1}{2}-\frac{p}{2}}$ is an $(A)_p$ weight on $\mathbf{R}$. It follows that by the weighted inequality (see e.g. Theorem 9.4.6 of [10]) that

$$\|Tu\|_p^p \leq C \int_0^\infty r^{\frac{1}{2}} |M_u(\sqrt{r})|^p dr \leq C \int_0^\infty r^2 |M_u(r)|^p dr \leq C\|u\|_p^p \, .$$

(7) We finally prove that Z_r is unbounded in $L^p(\mathbf{R}^3)$ for $p \geq 3$. It suffices to prove this for $\tilde{R}_2$. It follows from (3.9) that, for every compact $\Omega \subset \mathbf{R}^3$ there exists a constant $C > 0$ such that for a sufficiently large $L > 0$

$$|\tilde{R}_2(x,y)| \geq \frac{C|\varphi(x)|}{|y|^2} \, , \quad |y| \geq L \, , \quad x \in \Omega \, .$$

Since $|y|^{-2} \notin L^q(|y| \geq L)$ for any $q \leq 3/2$ and $L > 0$, the Riesz representation theorem implies that $\tilde{R}_2$ is unbounded in $L^p(\mathbf{R}^3)$ for any $p \geq 3$. This completes the proof. □

LEMMA 3.3. *Suppose $\phi \in \mathcal{E}$. Then, $R_1(x,y)$ and $R_2(x,y)$ with ϕ in place of φ are $L^p(\mathbf{R}^3)$ bounded kernels for all $1 \leq p < \infty$. If ϕ further satisfies $\langle V, x^\alpha \phi\rangle = 0$ for $|\alpha| \leq 1$, R_1 and R_2 are admissible.*

PROOF. $R_1(x,y)$ and $R_2(x,y)$ then satisfy

$$R_1(x,y) \leq_{|\cdot|} C\langle x\rangle^{-2}(|x|-|y|)^{-2}\langle y\rangle^{-1} \, , \quad R_2(x,y) \leq_{|\cdot|} C\langle x\rangle^{-2}(|x|^2-|y|^2)^{-1}\langle y\rangle^{-1} \, .$$

Thus, $R_1(x,y)$ and $R_2(x,y)$ define bounded operator in $L^p(\mathbf{R}^3)$ for all $1 < p < \infty$ by virtue of Lemma 3.1. They are bounded also in $L^1(\mathbf{R}^3)$ because

$$\sup_y \int_{||x|-|y||\geq 1} \frac{dx}{\langle x\rangle^2 \langle y\rangle (|x|-|y|)^2} \leq C \sup_y \int_0^\infty \frac{dr}{\langle y\rangle\langle r-|y|\rangle^2} < \infty \, ,$$

$$\sup_y \int_{||x|^2-|y|^2|\geq 1} \frac{dx}{\langle x\rangle^2 \langle y\rangle (|x|^2-|y|^2)} \leq 4\pi \sup_y \int_0^\infty \frac{dr}{\sqrt{r}\langle y\rangle\langle r-|y|^2\rangle} < \infty \, .$$

If $\langle V, x^\alpha\phi\rangle = 0$ for $|\alpha| \leq 1$, then $|\phi(x)| \leq C\langle x\rangle^{-2-\varepsilon}$ for an $\varepsilon > 0$ and R_1 and R_2 are admissible by virtue of Lemma 3.1. □

4. Contribution of zero-energy eigenfunctions

By virtue of Proposition 2.1 and Lemma 3.2, the following proposition shows that $W_\pm$ is bounded in $L^1(\mathbf{R}^3)$ if and only if H has no threshold resonances.

PROPOSITION 4.1. *Suppose that the threshold resonance is absent from H. Then, $W_\pm$ is bounded in $L^1(\mathbf{R}^3)$.*

If H has no resonances then (2.10) becomes

$$(I + G_0(\lambda)V)^{-1} = S(\lambda) + E(\lambda)\,, \quad S(\lambda) = \frac{PV}{\lambda^2} - \frac{PVD_3VPV}{\lambda} \tag{4.1}$$

and we need study the operator Z_s defined by

$$Z_s u \colon = \frac{i}{\pi}\int_0^\infty G_0(\lambda)VS(\lambda)V(G_0(\lambda) - G_0(-\lambda))u\rangle F(\lambda)\lambda d\lambda\,. \tag{4.2}$$

We recall that all $\phi \in \mathcal{E} = PL^2(\mathbf{R}^3)$ satisfy $\langle V, \phi\rangle = 0$ and $\phi(x) \le_{|\cdot|} C\langle x\rangle^{-2}$ for a constant $C > 0$. We take the real orthonormal basis $\{\phi_1, \dots, \phi_d\}$ of $\mathcal{E}$ and write $Z_s u = Z_{s0}u + Z_{s1}u$, where with $a_{jk} = i\pi^{-1}\langle\phi_j, VD_3V\phi_k\rangle \in \mathbf{R}$,

$$Z_{s0}u \colon = \sum_{j,k=1}^d a_{jk}\int_0^\infty G_0(\lambda)V\phi_j\langle V\phi_k, (G_0(\lambda) - G_0(-\lambda))u\rangle F(\lambda)d\lambda\,, \tag{4.3}$$

$$Z_{s1}u \colon = \sum_{j=1}^d \frac{i}{\pi}\int_0^\infty G_0(\lambda)V\phi_j\langle V\phi_j, (G_0(\lambda) - G_0(-\lambda))u\rangle F(\lambda)\frac{d\lambda}{\lambda}\,. \tag{4.4}$$

LEMMA 4.2. (1) *For any $1 \le p < \infty$, Z_{s0} is bounded in $L^p(\mathbf{R}^3)$.*
(2) *If all $\phi_1, \dots, \phi_d$ in addition satisfy $\int_{\mathbf{R}^3} x_j V(x)\phi(x)dx = 0$ for $j = 1, 2, 3$. Then Z_{s0} is bounded in $L^p(\mathbf{R}^3)$ for all $1 \le p \le \infty$.*

PROOF. The proof of Lemma 3.2 and Lemma 3.3 implies the lemma. □

In what follows, we say for operators T depending on $\phi \in \mathcal{E}$ (or the space $\mathcal{E}$) that T or $T(x, y)$ *depends generically* on ϕ (resp. $\mathcal{E}$) if T is bounded in $L^p(\mathbf{R}^3)$ for $1 \le p < 3$ in general, for $1 \le p < \infty$ if ϕ (resp. all $\phi \in \mathcal{E}$) satisfies $\langle V, x^\alpha\phi\rangle = 0$ for $|\alpha| \le 1$ and, for $1 \le p \le \infty$ if ϕ (resp. all $\phi \in \mathcal{E}$) does $\langle V, x^\alpha\phi\rangle = 0$ for $|\alpha| \le 2$.

LEMMA 4.3. *The operator Z_{s1} depends generically on $\mathcal{E}$.*

PROOF. Define for $j = 1, \dots, d$ that

$$Z_{s1,j}u \colon = \frac{i}{\pi}\int_0^\infty G_0(\lambda)|V\phi_j\langle V\phi_j, (G_0(\lambda) - G_0(-\lambda))u\rangle F(\lambda)\lambda^{-1}d\lambda\,, \tag{4.5}$$

so that $Z_{s1}u = \sum_{j=1}^{d} Z_{s1,j}u$. We prove the lemma only for $Z_{s1,1}$, which we denote by Z and ϕ_1 by ϕ for short. The proof for others is similar. As in the proof of Lemma 3.2, by splitting $e^{i\lambda|x-y|} = e^{i\lambda|x|} + e^{i\lambda|x|}r(\lambda, x, y)$ and etc., we decompose Z as

$$Zu = Z_1u + Z_2u + Z_3u + Z_4u\,.$$

Thus the integral kernels of $Z_1, \dots, Z_4$ are given respectively by

$$Z_1 \colon = \sum_{\pm} \frac{\pm i}{\pi} \int_0^\infty \int e^{i\lambda(|x|\pm|y|)} \frac{r(\lambda, x, z) r(\pm\lambda, y, w) V\phi(z) V\phi(w) F(\lambda)}{16\pi^2 |x-z||w-y|} dw dz \frac{d\lambda}{\lambda}\,,$$

$$Z_2 \colon = \sum_{\pm} \frac{\pm i}{\pi} \int_0^\infty \int e^{i\lambda(|x|\pm|y|)} \frac{r(\pm\lambda, y, w)(V\phi)(z)(V\phi)(w) F(\lambda)}{16\pi^2 |x-z||w-y|} dw dz \frac{d\lambda}{\lambda}\,,$$

$$Z_3 \colon = \sum_{\pm} \pm\frac{i}{\pi} \int_0^\infty \int e^{i\lambda(|x|\pm|y|)} \frac{r(\lambda, x, z)(V\phi)(z)(V\phi)(w) F(\lambda)}{16\pi^2 |x-z||w-y|} dw dz \frac{d\lambda}{\lambda}\,,$$

$$Z_4u \colon = \sum_{\pm} \pm\frac{i}{\pi} \int_0^\infty \int e^{i\lambda(|x|\pm|y|)} \frac{(V\phi)(z)(V\phi)(w) F(\lambda)}{16\pi^2 |x-z||w-y|} dw dz \frac{d\lambda}{\lambda}\,.$$

These operators differ from the corresponding $Z_1, \dots, Z_4$ in the proof of Lemma 3.2 only by the constant a and by ϕ and $\lambda^{-1}d\lambda$ replacing φ and $d\lambda$ respectively. We write

$$r(\lambda, x, y) = i\lambda(|x-y| - |x|) r_1(\lambda, x, y)\,, \quad r_1 \colon = \int_0^1 e^{i\lambda(|x-y|-|x|)\theta} d\theta \tag{4.6}$$

and etc. We have

$$r_1(\lambda, x, y) \le_{|\cdot|} 1\,, \quad \partial_\lambda^k r_1(\lambda, x, y) \le_{|\cdot|} |y|^k/k!\,. \tag{4.7}$$

We estimate $Z_1u, \dots, Z_4u$ individually in the following four lemmas. □

LEMMA 4.4. *Modulo an admissible kernel we have*

$$Z_1(x, y) \equiv \sum_{\pm} \frac{i}{\pi} \frac{\chi(||x| - |y|| \ge 1)|x||y|\phi(x)\phi(y)}{(|x| \pm |y|)^2} \tag{4.8}$$

and Z_1 is bounded for all $1 < p < \infty$. If ϕ satisfies $\langle V, x^\alpha \phi\rangle = 0$ for $|\alpha| \le 1$, then $Z_1(x, y)$ is an admissible kernel.

PROOF. By virtue of (4.6), we have

$$Z_1(x, y) = -\frac{i}{\pi} \int \left(\sum_{\pm} \int_0^\infty e^{i\lambda(|x|\pm|y|)} r_1(\lambda, x, z) r_1(\pm\lambda, y, w) \lambda F(\lambda) d\lambda \right) \times \frac{(|x-z| - |x|)(|w-y| - |y|)(V\phi)(z)(V\phi)(w)}{16\pi^2 |x-z||w-y|} dw dz$$

$$= -\frac{i}{\pi} \int W_1(x, z, w, y) \frac{(|x-z| - |x|)(|w-y| - |y|)(V\phi)(z)(V\phi)(w)}{16\pi^2 |x-z||w-y|} dw dz\,,$$

where the definition of W_1 should be obvious. We have $|W_1(x, z, w, y)| \le 2\|\lambda F\|_1$ and $|Z_1(x, y)| \le C\langle x\rangle^{-1}\langle y\rangle^{-1}$. Thus, the part of $Z_1(x, y)$ on $||x| - |y|| \le 1$ is admissible as remarked in the previous section and we ignore the region $\{(x, y) : ||x| - |y|| \le 1\}$ in what follows in the proof. We apply integration by parts twice to $W_1(x, z, w, y)$ and obtain

$$W_1(x, z, w, y) = \sum_{\pm} \frac{-1}{(|x| \pm |y|)^2} + Y_1(x, z, w, y)\,, \tag{4.9}$$

$$Y_1 : = \sum_{\pm} \frac{-1}{(|x| \pm |y|)^2} \int_0^\infty e^{i\lambda(|x| \pm |y|)} (r_1(\lambda, x, z) r_1(\pm\lambda, y, w)\lambda F(\lambda))'' d\lambda\,.$$

Since $\int V\phi(x)dx = 0$ and $\phi + (-\Delta)^{-1}V\phi = 0$, we have

$$\int \frac{(|x - z| - |x|)(V\phi)(z)}{4\pi|x - z|} dz = -|x| \int \frac{(V\phi)(z)}{4\pi|x - z|} dz = |x|\phi(x) \tag{4.10}$$

and the likewise for the integral involving $(V\phi)(w)$ with respect to dw. Thus, the contribution to $Z_1(x, y)$ of the boundary term in (4.9) is given by

$$\frac{i}{\pi} \sum_{\pm} \frac{|x||y|\phi(x)\phi(y)}{(|x| \pm |y|)^2}\,.$$

By virtue of Lemma 3.1 both $+$ and $-$ terms are L^p bounded kernels for all $1 < p < \infty$ and they are admissible if $\langle V, x^\alpha\phi\rangle = 0$ for $|\alpha| \le 1$. On the other hand the contribution of $Y_1(x, z, w, y)$ to $Z_1(x, y)$ produces an admissible kernel since further integration by parts and (4.7) imply

$$Y_1(x, z, w, y) \le_{|\cdot|} C \sum_{\pm} \frac{(1 + |z| + |w|)^3}{(|x| \pm |y|)^3}$$

and, its contribution to $Z_1(x, y)$ is bounded in modulus by

$$C \sum_{\pm} \int \frac{|z||w|(1 + |z| + |w|)^3 |(V\phi)(z)(V\phi)(w)|}{|x - z||w - y|(|x| \pm |y|)^3} dw dz$$

$$\le \sum_{\pm} \frac{C}{\langle x\rangle\langle y\rangle(|x| \pm |y|)^3}$$

which is an admissible kernel. This proves the lemma. □

LEMMA 4.5. *Modulo an admissible kernel*

$$Z_2(x, y) \equiv \frac{2i}{\pi} \frac{\chi(||x| - |y|| \ge 1)|x|\phi(x)|y|\phi(y)}{|x|^2 - |y|^2} \tag{4.11}$$

and Z_2 is bounded for all $1 < p < \infty$. If ϕ satisfies $\langle V, x^\alpha\phi\rangle =$ for $|\alpha| \le 1$, $Z_2(x, y)$ is admissible.

PROOF. The proof goes in parallel with that of Lemma 4.4. By virtue of the same reason as in the proof of Lemma 4.4, we ignore the part of $Z_2(x, y)$ on $\{(x, y) : ||x|-|y|| \le 1\}$. Using (4.6) and (4.10), we write $Z_2(x, y)$ as

$$\frac{1}{\pi}\int\left(\sum_{\pm}\int_0^\infty e^{i\lambda(|x|\pm|y|)} r_1(\pm\lambda, w, y)F(\lambda)d\lambda\right)\frac{(|w-y|-|y|)\phi(x)(V\phi)(w)}{4\pi|w-y|}dw\,.$$

Applying integration by parts, we have

$$W_2(x, w, y) := \sum_{\pm}\int_0^\infty e^{i\lambda(|x|\pm|y|)} r_1(\pm\lambda, w, y)F(\lambda)d\lambda = \sum_{\pm}\frac{i}{|x|\pm|y|} + Y_2\,,$$

$$Y_2 = Y_2(x, w, y) := \sum_{\pm}\frac{i}{|x|\pm|y|}\int_0^\infty e^{i\lambda(|x|\pm|y|)}(r_1(\pm\lambda, w, y)F(\lambda))' d\lambda\,.$$

The contribution of the boundary term in $W_2(x, w, y)$ to $Z_2(x, y)$ is given by virtue of (4.10) by

$$\sum_{\pm}\frac{i}{\pi}\int\frac{(|w-y|-|y|)\phi(x)(V\phi)(w)}{(|x|\pm|y|)\cdot 4\pi|w-y|}dw = \sum_{\pm}\frac{i}{\pi}\frac{\phi(x)|y|\phi(y)}{|x|\pm|y|} \tag{4.12}$$

which we put on the right of (4.11). Further integration by parts twice shows that

$$Y_2(x, w, y) = \sum_{\pm}\left(\frac{\mp(|w-y|-|y|)}{2(|x|\pm|y|)^2} + O\left(\frac{C(1+|w|)^3}{(|x|\pm|y|)^3}\right)\right)$$

and, modulo the admissible kernel produced by the second term, the contribution to $Z_2(x, y)$ of Y_2 is given by

$$\sum_{\pm}\frac{\mp 1}{(|x|\pm|y|)^2}\int_{\mathbf{R}^3}\frac{\phi(x)(|w-y|-|y|)^2(V\phi)(w)}{4\pi|w-y|}dw \le_{|\cdot|} \frac{C|\phi(x)||x||y|}{\langle y\rangle(|x|^2-|y|^2)^2}\,,$$

which is also admissible on $\{(x, y) : ||x|-|y|| \ge 1\}$. If $\langle V, x_j\phi\rangle = 0$ for $j = 1, 2, 3$, then $|\phi(x)| \le C\langle x\rangle^{-3}$ and (4.11) also becomes admissible. This proves the lemma. □

LEMMA 4.6. *Modulo an admissible kernel*

$$Z_3(x, y) \equiv \frac{-2i}{\pi}\frac{\chi(||x|-|y|| \ge 1)|x|\phi(x)|y|\phi(y)}{|x|^2-|y|^2} \tag{4.13}$$

and Z_3 is bounded for all $1 < p < \infty$. If ϕ satisfies $\langle V, x^\alpha\phi\rangle = 0$ for $|\alpha| \le 1$, then $Z_3(x, y)$ is admissible.

PROOF. The proof goes in parallel with that of Lemma 4.5 and we ignore the part of $Z_3(x, y)$ for $||x|-|y|| < 1$. By using (4.6) and (4.10) once more we write $Z_3(x, y)$ in the form

$$Z_3(x, y) = \frac{1}{\pi}\int_{\mathbf{R}^3} W_3(x, z, y)\frac{(|x-z|-|x|)(V\phi)(z)\phi(y)}{4\pi|x-z|}dz\,, \tag{4.14}$$

$$W_3(x,z,y):=\sum_{\pm}\pm\int_0^\infty e^{i\lambda(|x|\pm|y|)}r_1(\lambda,x,z)F(\lambda)d\lambda\,. \tag{4.15}$$

Application of integration by parts shows that

$$W_3(x,z,y)=\sum_{\pm}\frac{\pm i}{|x|\pm|y|}+Y_3(x,z,y), \tag{4.16}$$

$$Y_3(x,z,y):=\sum_{\pm}\frac{\pm i}{|x|\pm|y|}\int_0^\infty e^{i\lambda(|x|\pm|y|)}(r_1(\lambda,x,z)F(\lambda))'d\lambda\,. \tag{4.17}$$

The contribution of the boundary term of W_3 in (4.16) is given by virtue of (4.10) by

$$\sum_{\pm}\frac{\pm i}{\pi}\int_{\mathbf{R}^3}\frac{(|x-z|-|x|)(V\phi)(z)\phi(y)}{(|x|\pm|y|)4\pi|x-z|}dz=\sum_{\pm}\frac{\pm i}{\pi}\frac{|x|\phi(x)\phi(y)}{|x|\pm|y|}\,, \tag{4.18}$$

which is equal to the right of (4.13) for $||x|-|y||\geq 2$. This is an L^p bounded kernel for all $1<p<\infty$ and, is admissible if ϕ satisfies $\langle V,x^\alpha\phi\rangle=0$ for $|\alpha|\leq 1$ by virtue of Lemma 3.1. Further integration by parts implies

$$Y_3(x,z,y)=\sum_{\pm}\frac{\mp r_1'(0,x,z)}{(|x|\pm|y|)^2}+\sum_{\pm}\mp\int_0^\infty e^{i\lambda(|x|\pm|y|)}\frac{(r_1(\lambda,x,z)F(\lambda))''}{(|x|\pm|y|)^2}d\lambda\,.$$

Here, we have

$$r_1'(0,x,z)=\frac{i}{2}(|x-z|-|x|)$$

and the contribution of the boundary term in Y_3 to $Z_3(x,y)$ is given by purely imaginary

$$\begin{aligned}\sum_{\pm}&\frac{\mp i}{2(|x|\pm|y|)^2}\int_{\mathbf{R}^3}\frac{(|x-z|-|x|)^2(V\phi)(z)\phi(y)}{4\pi|x-z|}dz\\
&=\frac{2i|x||y|}{(|x|-|y|)^2(|x|+|y|)^2}\int_{\mathbf{R}^3}\frac{(|x-z|-|x|)^2(V\phi)(z)\phi(y)}{4\pi|x-z|}dz\\
&\leq_{|\cdot|}\frac{C|x||y||\phi(y)|}{(|x|-|y|)^2(|x|+|y|)^2\langle x\rangle}\leq\frac{C}{\langle y\rangle(|x|^2-|y|^2)^2}\end{aligned}$$

and, this is admissible. Applying integration by parts once more, we see that the integral term for $Y_3(x,z,y)$ is bounded in modulus by

$$\sum_{\pm}\frac{C\langle z\rangle^3}{(|x|\pm|y|)^3}$$

and its contribution to $Z_3(x,y)$ is bounded by

$$\sum_{\pm}\frac{C|\phi(y)|}{(|x|\pm|y|)^3\langle x\rangle}\leq\frac{C}{(|x|-|y|)^3\langle x\rangle\langle y\rangle^2}\,,$$

which is again admissible on $||x|-|y||\geq 1$. This concludes the proof of the lemma. □

For studying Z_4, we need the following lemma.

LEMMA 4.7. *Suppose $\delta > 1$ and $\kappa > 2$.*

(1) *The integral operator K whose kernel satisfies*

$$|K(x,y)| \leq \int_{-1}^{1} \langle x\rangle^{-2}\langle y\rangle^{-1}\langle |x|-\theta|y|\rangle^{-\delta} d\theta\,, \quad x,y\in \mathbf{R}^3 \tag{4.19}$$

is bounded in $L^p(\mathbf{R}^3)$ for all $1\leq p<3$.

(1a) *If* (4.19) *is satisfied with $\langle x\rangle^{-2}\langle y\rangle^{-2}$ in place of $\langle x\rangle^{-2}\langle y\rangle^{-1}$ in the integrand, then K is bounded in L^p for all $1\leq p<\infty$.*

(2) *Let K_t, $0\leq t\leq 1$ be the integral operator define by*

$$K_t(x,y) :\,= \int_{-1}^{1} \langle x\rangle^{-2t}\langle y\rangle^{-\kappa(1-t)}\langle |x|-\theta|y|\rangle^{-\delta} d\theta\,, \quad x,y\in \mathbf{R}^3\,. \tag{4.20}$$

Then, K_t is bounded in $L^{1/t}(\mathbf{R}^3)$, $0\leq t\leq 1$.

(3) *The following is an admissible kernel*:

$$\tilde{K}(x,y) :\,= \int_{-1}^{1} \langle x\rangle^{-2}\langle y\rangle^{-\kappa}\langle |x|-\theta|y|\rangle^{-\delta} d\theta\,. \tag{4.21}$$

PROOF. (1) It suffices show that K is bounded in $L^1(\mathbf{R}^3)$ and $L^p(\mathbf{R}^3)$ for $2<p<3$. By using polar coordinates, we estimate

$$\int_{\mathbf{R}^3} |K(x,y)|dx \leq C\int_0^1 \left(\int_0^\infty \frac{dr}{\langle r-\theta|y|\rangle^{\delta}}\right)\langle y\rangle^{-1} d\theta \leq C\langle y\rangle^{-1}$$

and K is bounded in $L^1(\mathbf{R}^3)$. For $2<p<3$, Minkowski's inequality implies

$$\left(\int_{\mathbf{R}^3} |K(x,y)|^p dx\right)^{1/p} \leq C\langle y\rangle^{-1}\int_0^1 G(\theta|y|)d\theta\,,$$

$$G(\theta|y|) = \left(\int_0^\infty \frac{r^2\langle r\rangle^{-2p}}{\langle r-\theta|y|\rangle^{p\delta}} dr\right)^{1/p}\,.$$

For $|y|\leq 1$ and for $|y|\geq 2$ with $\theta|y|<2$, we use the obvious estimate

$$G(\theta|y|)\leq C<\infty\,. \tag{4.22}$$

When $\theta|y|\geq 2$, split $(0,\infty)=I_1\cup I_2$, $I_1=\{r>0\colon \theta|y|/2<r<3\theta|y|/2\}$ and $I_2=\{r>0\colon |r-\theta|y||\geq \theta|y|/2\}$ and estimate by using that $2-2p\leq -2$

$$\left(\int_{I_1} \frac{r^2\langle r\rangle^{-2p}}{\langle r-\theta|y|\rangle^{p\delta}} dr\right)^{1/p} \leq C\langle \theta|y|\rangle^{\frac{2}{p}-2}\,, \tag{4.23}$$

$$\left(\int_{I_2} \frac{r^2\langle r\rangle^{-2p}}{\langle r-\theta|y|\rangle^{p\delta}} dr\right)^{1/p} \le C\langle\theta|y|\rangle^{-\delta} . \tag{4.24}$$

Combining (4.22), (4.23) and (4.24), we obtain for $|y| \ge 1$ that

$$\int_0^1 G(\theta|y|)d\theta \le C\left(\frac{1}{|y|} + \int_0^1 (\langle\theta|y|\rangle^{\frac{2}{p}-2} + \langle\theta|y|\rangle^{-\delta})d\theta\right) \le \frac{C}{|y|} .$$

Thus, for any $2 < p < 3$, we have

$$\left(\int_{\mathbf{R}^3} |K(x,y)|^p dx\right)^{1/p} \le C\langle y\rangle^{-2} \in L^{\frac{p}{p-1}}(\mathbf{R}^3) \tag{4.25}$$

and Minkowski's inequality implies

$$\|Ku\|_p \le \int_{\mathbf{R}^3}\left(\int_{\mathbf{R}^3} |K(x,y)|^p dx\right)^{1/p} |u(y)|dy \le C\|\langle y\rangle^{-2}\|_{\frac{p}{p-1}}\|u\|_p . \tag{4.26}$$

(1a) If (4.19) is satisfied with $\langle x\rangle^{-2}\langle y\rangle^{-2}$ in place of $\langle x\rangle^{-2}\langle y\rangle^{-1}$, then (4.25) is bounded by $C\langle y\rangle^{-3}$ which is in $L^{\frac{p}{p-1}}$ for all $1 \le p < \infty$ and, (4.26) is satisfied for all $2 < p < \infty$. Statement (1a) is proved.
(2) Let K_z for $z \in \mathbf{C}$ with $0 \le \Re z \le 1$ be the integral operator defined by

$$K_z(x,y) := \int_{-1}^1 \langle x\rangle^{-2z}\langle y\rangle^{-\kappa(1-z)}\langle|x|-\theta|y|\rangle^{-\delta}d\theta .$$

Then, it is obvious that K_z is an analytic family of admissible growth on $0 < \Re z < 1$ and continuous on $0 \le \Re z \le 1$ in the sense of Stein [17]. It is easy to see that

$$\sup_{y\in\mathbf{R}^3}\int_{\mathbf{R}^3}\left(\int_{-1}^1 \langle x\rangle^{-2}\langle|x|-\theta|y|\rangle^{-\delta}d\theta\right)dx \le C < \infty , \tag{4.27}$$

$$\sup_{x\in\mathbf{R}^3}\int_{\mathbf{R}^3}\left(\int_{-1}^1 \langle y\rangle^{-\kappa}\langle|x|-\theta|y|\rangle^{-\delta}d\theta\right)dy < \infty . \tag{4.28}$$

and the first part of lemma follows by Stein's interpolation theorem. Two estimates (4.27) and (4.28) show that $\tilde{K}(x,y)$ is admissible. □

LEMMA 4.8. *Modulo the kernel which depends generically on $\phi \in \mathcal{E}$,*

$$Z_4(x,y) \equiv -\frac{i}{\pi}\phi(x)\phi(y)|y| \cdot \int_{-1}^1 \frac{\chi(||x|+\theta|y|| > 1)}{|x|+\theta|y|}d\theta . \tag{4.29}$$

PROOF. Using (4.10): $(-\Delta)^{-1}V\phi = -\phi$ once again, we simplify

$$Z_4(x,y) = \frac{i}{\pi}\phi(x)\phi(y)\int_0^\infty e^{i\lambda|x|}(e^{i\lambda|y|} - e^{-i\lambda|y|})\frac{F(\lambda)}{\lambda}d\lambda$$

$$= -\frac{1}{\pi}\phi(x)\phi(y)|y| \int_{-1}^{1} \left(\int_0^\infty e^{i\lambda(|x|+\theta|y|)} F(\lambda) d\lambda \right) d\theta \, . \tag{4.30}$$

Break up $Z_4(x, y)$ into

$$Z_4(x, y) = Z_4^{\le}(x, y) + Z_4^{>}(x, y)$$

by inserting $1 = \chi(||x| + \theta|y|| \le 1) + \chi(||x| + \theta|y|| > 1)$ in front of $d\theta$ of (4.30). We clearly have

$$Z_4^{\le}(x, y) \le_{|\cdot|} \frac{2}{\pi} \int_0^1 |\phi(x)\phi(y)||y|\chi(||x| - \theta|y|| \le 1) d\theta$$

and, Lemma 4.7 imply $Z_4^{\le}(x, y)$ depends generically on $\phi \in \mathcal{E}$.

For studying $Z_4^{>}(x, y)$, we apply integration by parts:

$$\int_0^\infty e^{i\lambda(|x|+\theta|y|)} F(\lambda) d\lambda = \frac{i}{|x| + \theta|y|} + i \int_0^\infty \frac{e^{i\lambda(|x|+\theta|y|)} F'(\lambda)}{|x| + \theta|y|} d\lambda \, . \tag{4.31}$$

Since $F' \in C_0^\infty((0, \infty))$, the integral term on the right is bounded for any $N = 1, 2, \ldots$ by $C\langle |x| + \theta|y| \rangle^{-N}$ when $||x| + \theta|y|| \ge 1$ and its contribution to $Z_4^{>}(x, y)$ is bounded in modulus by

$$C \int_{-1}^{1} \frac{|\phi(x)\phi(y)||y|}{\langle |x| + \theta|y| \rangle^N} d\theta$$

which depends generically on $\phi \in \mathcal{E}$. The contribution of the first term on the right of (4.31) to $Z_4^{>}(x, y)$ is given by the right of (4.29) and the lemma is proved. □

COMPLETION OF THE PROOF OF LEMMA 4.3. We combine previous lemmas and observe that the right sides of (4.11) and (4.13) cancel each other. It follows that the lemma holds if the sum $K_0(x, y)$ of those of (4.8) and (4.29),

$$K_0(x, y) \colon = \sum_{\pm} \frac{i}{\pi} \frac{\chi(||x| - |y|| \ge 1)|x||y|\phi(x)\phi(y)}{(|x| \pm |y|)^2} \tag{4.32}$$

$$- \frac{i}{\pi}\phi(x)\phi(y)|y| \cdot \int_{-1}^{1} \frac{\chi(||x| + \theta|y|| > 1)}{|x| + \theta|y|} d\theta \tag{4.33}$$

generically depends on $\phi \in \mathcal{E}$. We write $|x| = (|x| \pm |y|) \mp |y|$ in (4.32). Then,

$$(4.32) - \sum_{\pm} \frac{i}{\pi} \frac{\chi(||x| - |y|| \ge 1)|y|\phi(x)\phi(y)}{|x| \pm |y|}$$

$$= \frac{4i\chi(||x| - |y|| \ge 1)|x||y|^3\phi(x)\phi(y)}{\pi(|x| - |y|)^2(|x| + |y|)^2}$$

$$\le_{|\cdot|} \frac{C\chi(||x| - |y|| \ge 1)|\phi(x)||y|^2\phi(y)}{(|x| - |y|)^2}$$

and it is easy to see that this depends generically on $\phi \in \mathcal{E}$. Thus, it suffices to show the same for

$$\frac{i}{\pi}\phi(x)\phi(y)|y|\left(\sum_{\pm}\frac{\chi(||x|-|y||\geq 1)}{|x|\pm|y|}-\int_{-1}^{1}\frac{\chi(||x|+\theta|y||>1)}{|x|+\theta|y|}d\theta\right).$$

We remark that

$$\frac{\phi(x)\phi(y)|y|\chi(||x|-|y||<1)}{\langle |x|\pm|y|\rangle}\leq_{|\cdot|}\frac{C\chi(||x|-|y||<1)}{\langle x\rangle\langle y\rangle} \tag{4.34}$$

is an admissible kernel and, by virtue of Lemma 3.1 and Lemma 4.7 (1),

$$\begin{aligned}
&\phi(x)\phi(y)|y|\chi(||x|-|y||\geq 1)\left(\frac{1}{|x|\pm|y|}-\frac{1}{\langle |x|\pm|y|\rangle}\right)\\
&=\frac{\phi(x)\phi(y)|y|\chi(||x|-|y||\geq 1)}{(|x|\pm|y|)\langle |x|\pm|y|\rangle\{(|x|\pm|y|)+\langle |x|\pm|y|\rangle\}}\\
&\leq_{|\cdot|}\frac{\phi(x)\phi(y)|y|\chi(||x|-|y||\geq 1)}{\langle |x|-|y|\rangle^2}
\end{aligned}$$

is L^p bounded kernel for $1\leq p<\infty$ in general and is admissible if ϕ satisfies $\langle V, x^\alpha\phi\rangle = 0$ for $|\alpha|\leq 1$. We have

$$\int_{-1}^{1}\frac{\chi(||x|+\theta|y||<1)\phi(x)\phi(y)|y|}{\langle |x|+\theta|y|\rangle}d\theta\leq C_N\int_{-1}^{1}\frac{\phi(x)\phi(y)|y|}{\langle |x|+\theta|y|\rangle^N}d\theta,\ N=1,2,\ldots$$

and

$$\begin{aligned}
&\int_{-1}^{1}\phi(x)\phi(y)|y|\chi(||x|+\theta|y||>1)\left(\frac{1}{|x|+\theta|y|}-\frac{1}{\langle |x|+\theta|y|\rangle}\right)d\theta\\
&\leq_{|\cdot|}\int_{-1}^{1}\frac{|\phi(x)\phi(y)||y|\chi(||x|+\theta|y||>1)}{\langle |x|+\theta|y|\rangle^2}d\theta
\end{aligned}$$

and Lemma 4.7 likewise implies that both depend generically on $\phi \in \mathcal{E}$. Thus, for concluding the proof of the lemma, it suffices to show that

$$\begin{aligned}
&\frac{i}{\pi}\phi(x)\phi(y)|y|\left(\sum_{\pm}\frac{1}{\langle |x|\pm|y|\rangle}-\int_{-1}^{1}\frac{d\theta}{\langle |x|+\theta|y|\rangle}\right)\\
&=\sum_{\pm}\frac{i}{\pi}\phi(x)\phi(y)|y|\int_0^1\left(\frac{1}{\langle |x|\pm|y|\rangle}-\frac{1}{\langle |x|\pm\theta|y|\rangle}\right)d\theta
\end{aligned} \tag{4.35}$$

also depends generically on $\phi \in \mathcal{E}$. But, we have

$$\frac{1}{\langle |x|\pm|y|\rangle}-\frac{1}{\langle |x|\pm\theta|y|\rangle}=\frac{\mp(1-\theta)|y|\big((|x|\pm|y|)+(|x|\pm\theta|y|)\big)}{\langle |x|\pm|y|\rangle\langle |x|\pm\theta|y|\rangle(\langle |x|\pm|y|\rangle+\langle |x|\pm\theta|y|\rangle)}$$

$$\leq_{|\cdot|} \frac{(1-\theta)|y|}{\langle |x| \pm |y|\rangle\langle |x| \pm \theta|y|\rangle} \leq \frac{(1-\theta)}{2}\left(\frac{|y|}{\langle |x| \pm |y|\rangle^2} + \frac{|y|}{\langle |x| \pm \theta|y|\rangle^2}\right)$$

and

$$(4.35) \leq_{|\cdot|} C\int_0^1 \langle x\rangle^{-2}\left(\frac{1}{\langle |x| - |y|\rangle^2} + \frac{1}{\langle |x| - \theta|y|\rangle^2}\right) d\theta\,.$$

Thus, (4.35) produces a bounded operator in $L^1(\mathbf{R}^3)$. On the other hand we have for any $0 \leq \theta \leq 1$ and $0 \leq \varepsilon \leq 1$ that $\langle |x| \pm \theta|y|\rangle^{-1} \leq \langle x\rangle^{1-\varepsilon}|y|^{\varepsilon-1}|\theta|^{\varepsilon-1}$ and

$$\frac{\phi(x)\phi(y)|y|}{\langle |x| \pm \theta|y|\rangle} \leq_{|\cdot|} \phi(x)\langle x\rangle^{1-\varepsilon} \cdot \phi(y)|y|^{-\varepsilon} \cdot \theta^{\varepsilon-1}\,. \tag{4.36}$$

Here for p_0 and p_1 sufficiently close to 3 such that $p_0 < 3 < p_1$, $\phi(x)\langle x\rangle^{1-\varepsilon} \in L^p(\mathbf{R}^3)$ for all $p \geq p_0$ and $\phi(y)|y|^{-\varepsilon} \in L^{p'}(\mathbf{R}^3)$ for all $p \leq p_1$ and, (4.35) is an L^p bounded kernel for all $p_0 \leq p < 3$, hence, by interpolation for all $1 \leq p < 3$ in general. If $\langle V, x^\alpha\phi\rangle = 0$ for $|\alpha| \leq 1$, then $\phi(y)|y|^{-\varepsilon} \in L^1$ and (4.35) is bounded in $L^\infty(\mathbf{R}^3)$ and, by interpolation, in all $L^p(\mathbf{R}^3)$, $1 \leq p \leq \infty$. This completes the proof. □

ADDED IN PROOF. After the paper was accepted, we were informed of the paper "On the L^p-boundedness of wave operators for two dimensional Schrödinger operators with threshold obstructions" by M. B. Erdoğan, M. Goldberg and W. Green where authors proves that wave operators are bounded in L^p for all $1 < p < \infty$ if there is an s-wave resonance or eigenvalue only at zero (arXiv: 1706.01530).

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Present Address:
DEPARTMENT OF MATHEMATICS,
GAKUSHUIN UNIVERSITY,
1–5–1 MEJIRO, TOSHIMA-KU,
TOKYO 171–8588, JAPAN.