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On the Unique Solvability of Nonlinear Fuchsian Partial Differential Equations

Dennis B. BACANI, Jose Ernie C. LOPE* and Hidetoshi TAHARA

*Sophia University and *University of the Philippines Diliman*

Abstract. We consider a singular nonlinear partial differential equation of the form

$$(t\partial_t)^m u = F\Big(t, x, \big\{(t\partial_t)^j \partial_x^\alpha u\big\}_{(j,\alpha)\in I_m}\Big)$$

with arbitrary order m and $I_m = \{(j,\alpha) \in \mathbb{N}\times\mathbb{N}^n\,;\, j+|\alpha| \le m, j < m\}$ under the condition that $F(t,x,\{z_{j,\alpha}\}_{(j,\alpha)\in I_m})$ is continuous in t and holomorphic in the other variables, and it satisfies $F(0,x,0)\equiv 0$ and $(\partial F/\partial z_{j,\alpha})(0,x,0)\equiv 0$ for any $(j,\alpha)\in I_m\cap\{|\alpha|>0\}$. In this case, the equation is said to be a nonlinear Fuchsian partial differential equation. We show that if $F(t,x,0)$ vanishes at a certain order as t tends to 0 then the equation has a unique solution with the same decay order.

1. Introduction

We denote by $(t,x) = (t,x_1,\dots,x_n)$ the variables in $\mathbb{R}_t\times\mathbb{C}_x^n$. Let $\mathbb{N} = \{0,1,\dots\}$ and $\mathbb{N}^* = \{1,2,\dots\}$. For $\alpha = (\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n$ we write $|\alpha| = \alpha_1+\cdots+\alpha_n$ and $\partial_x^\alpha = (\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}$. For $m\in\mathbb{N}^*$ we define $I_m = \{(j,\alpha)\in\mathbb{N}\times\mathbb{N}^n\,;\, j+|\alpha|\le m, j<m\}$, $I_m^+ = \{(j,\alpha)\in I_m\,;\,|\alpha|>0\}$, and $N = \#I_m$. For $R>0$ and $\rho>0$, we set $D_R = \{x = (x_1,\dots,x_n)\in\mathbb{C}^n\,;\,|x_i|<R \text{ for all } 1\le i\le n\}$ and $D_\rho^N = \{z = \{z_{j,\alpha}\}_{(j,\alpha)\in I_m}\in\mathbb{C}^N\,;\,|z_{j,\alpha}|<\rho \text{ for all } (j,\alpha)\in I_m\}$.

Let $T_0>0$, $R_0>0$, $\rho_0>0$, and let $F(t,x,z)$ be a function on $[0,T_0]\times D_{R_0}\times D_{\rho_0}^N$. In this paper, we consider the singular nonlinear partial differential equation

$$(t\partial_t)^m u = F\Big(t, x, \big\{(t\partial_t)^j \partial_x^\alpha u\big\}_{(j,\alpha)\in I_m}\Big) \tag{1.1}$$

under the following assumptions:

(A_1) $F(t,x,z)$ is continuous in t and holomorphic in (x,z);
(A_2) $F(0,x,0)\equiv 0$ on D_{R_0};
(A_3) $(\partial F/\partial z_{j,\alpha})(0,x,0)\equiv 0$ on D_{R_0} for any $(j,\alpha)\in I_m^+$.

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In this situation, the equation (1.1) is called a nonlinear Fuchsian (also known as Gérard-Tahara type) partial differential equation.

In the case where $F(t, x, z)$ is a holomorphic function in all the variables (t, x, z), the equation (1.1) was studied quite well by Gérard and Tahara [3], Tahara and Yamane [10], and Tahara and Yamazawa [11].

Some results can be found in the case where $F(t, x, z)$ is holomorphic in (x, z) but only continuous in t. The linear case was studied by Baouendi and Goulaouic [1], and Lope [4, 5]. In the nonlinear case, Baouendi and Goulaouic [2] considered (1.1) under some particular conditions, while Lope, Roque and Tahara [6] investigated the first order equation in a more general setting.

To the best of the authors' knowledge, the general case for arbitrary order m had not been solved yet. Thus the purpose of this paper is to solve the equation (1.1) in a completely general setting.

2. Main result

Let us consider the equation (1.1) under the conditions (A_1), (A_2), and (A_3). By writing $F(t, x, z)$ into its Taylor series expansion in z, (1.1) may be expressed in the form

$$(t\partial_t)^m u = a(t,x) + \sum_{(j,\alpha)\in I_m} b_{j,\alpha}(t,x)(t\partial_t)^j \partial_x^\alpha u + \mathcal{R}_2\Big(t, x, \big\{(t\partial_t)^j \partial_x^\alpha u\big\}_{(j,\alpha)\in I_m}\Big) \quad (2.1)$$

where $a(t, x) = F(t, x, 0)$, $b_{j,\alpha}(t, x) = (\partial F/\partial z_{j,\alpha})(t, x, 0)$, and $\mathcal{R}_2(t, x, z)$ represents the sum of all the remaining terms, each of which has a degree at least two with respect to z.

It is clear from (A_2) and (A_3) that we have $a(0, x) \equiv 0$ and $b_{j,\alpha}(0, x) \equiv 0$ for any $(j, \alpha) \in I_m^+$. In order to describe the decay order of the functions $a(t, x)$ and $b_{j,\alpha}(t, x)$ as t tends to 0, we introduce a *weight function* $\mu(t)$ on $(0, T_0]$, which is defined as a continuous, nonnegative and increasing function on $(0, T_0]$ that satisfies

$$\int_0^{T_0} \frac{\mu(s)}{s} ds < \infty .$$

By this definition, we have $\lim_{t\to 0} \mu(t) = 0$, and the function

$$\varphi(t) = \int_0^t \frac{\mu(s)}{s} ds$$

is well defined on $(0, T_0]$. Moreover, we have $\lim_{t\to 0} \varphi(t) = 0$ and $\varphi'(t) = \mu(t)/t$ on $(0, T_0)$. Typical examples of such functions are t^ε and $1/(-\log t)^{\varepsilon+1}$ with $\varepsilon > 0$.

Let $\mu(t)$ be a weight function on $(0, T_0]$. We suppose that

$$a(t, x) = O(\mu(t)^m) \ \text{(as } t \longrightarrow 0\text{) uniformly on } D_{R_0} , \quad (2.2)$$

$$b_{j,\alpha}(t,x) = O(\mu(t)^{|\alpha|}) \ \text{(as } t \longrightarrow 0\text{) uniformly on } D_{R_0} \tag{2.3}$$

for any $(j,\alpha) \in I_m^+$.

The characteristic polynomial associated with the equation (2.1) is given by

$$\mathcal{C}(\lambda, x) = \lambda^m - \sum_{j<m} b_{j,0}(0,x)\lambda^j$$

and the roots $\lambda_1(x), \dots, \lambda_m(x)$ of the equation $\mathcal{C}(\lambda, x) = 0$ are called the characteristic exponents of (2.1). As usual, we assume that

$$\operatorname{Re}\lambda_j(0) < 0, \quad j = 1, \dots, m. \tag{2.4}$$

Let $0 < T \le T_0$. For $r > 0$ and $R > 0$, we define the region W_r by

$$W_r = \{(t,x) \in [0,T] \times \mathbb{C}\,;\ |x| + \varphi(t)/r < R\}.$$

Note here that even though the region W_r also depends on T and R, this is not indicated in our notation for the sake of simplicity. We then define two function spaces on the region W, which can either be W_r or $[0,T] \times D_R$.

DEFINITION 2.1. A function $w(t,x)$ is said to belong to the space $\mathscr{X}_0(W)$ if $w(t,x) \in C^0(W)$ and is holomorphic in x for any fixed t. In addition, if $w(t,x) \in C^m(W \cap \{(t,x)\,;\ t > 0\})$ and $(t\partial_t)^j w(t,x) \in \mathscr{X}_0(W)$ for $j = 1, \dots, m$, then $w(t,x)$ is said to belong to the space $\mathscr{X}_m(W)$.

The following is our main result.

THEOREM 2.2. *Suppose that* $(A_1)-(A_3)$, (2.2), (2.3), *and* (2.4) *hold. Then there exist* $r > 0$, $R > 0$, $T > 0$, *and* $M > 0$ *with* $M\mu(T)^m < \rho_0$ *such that the equation* (1.1) *has a unique solution* $u(t,x)$ *in* $\mathscr{X}_m(W_r)$ *that satisfies the estimates*

$$|(t\partial_t)^j \partial_x^\alpha u(t,x)| \le M\mu(t)^m \quad \text{on } W_r \text{ for all } (j,\alpha) \in I_m.$$

By setting

$$\mathcal{P} = (t\partial_t)^m - \sum_{j<m} b_{j,0}(t,x)(t\partial_t)^j$$

and

$$\Phi[u] = \sum_{(j,\alpha)\in I_m^+} b_{j,\alpha}(t,x)(t\partial_t)^j \partial_x^\alpha u + \mathcal{R}_2\Big(t, x, \big\{(t\partial_t)^j \partial_x^\alpha u\big\}_{(j,\alpha)\in I_m}\Big),$$

we can write the equation (1.1) in the simple form

$$\mathcal{P}u = a(t,x) + \Phi[u]. \tag{2.5}$$

In the next section, we discuss some properties of the unique solution of the equation $\mathcal{P}w = g$ and other fundamental tools that are needed to prove our main result. In Section 4, we employ the technique of Nirenberg [8] and Nishida [9] to prove Theorem 2.2 under the condition

$$a(t,x) = O(\mu(t)^{2m}) \ \ (\text{as } t \longrightarrow 0) \text{ uniformly on } D_{R_0}\,, \tag{2.6}$$

and then we prove it in the general case (2.2) in Section 5. In the last section, we give a slight generalization of our main result.

3. Basic tools

In this section, we present some known results that are essential for our proofs. The first lemma, which is used to estimate the derivative of a holomorphic function, is due to Nagumo [7] (see also Walter [12]).

LEMMA 3.1. *Let $u(t,x) \in \mathscr{X}_0(W_r)$, $a \geq 0$, and $K \geq 0$. If the function $u(t,x)$ satisfies the estimate*

$$|u(t,x)| \leq \frac{K}{(R-|x|-\varphi(t)/r)^a} \quad \text{on } W_r\,,$$

then we have

$$\Big|\frac{\partial u}{\partial x_i}(t,x)\Big| \leq \frac{Ke(a+1)}{(R-|x|-\varphi(t)/r)^{a+1}} \quad \text{on } W_r \text{ for any } i = 1,\dots,n\,.$$

The next lemma is very important to estimating some integral expressions involving the weight function $\mu(t)$.

LEMMA 3.2. *The following estimates hold for any $L \geq 1$ and $k \geq 1$:*

$$\int_0^t \frac{\mu(\tau)^L}{(R-|x|-\varphi(\tau)/r)^k}\frac{d\tau}{\tau} \leq \frac{\mu(t)^{L-1}\varphi(t)}{(R-|x|-\varphi(t)/r)^k}\,, \tag{3.1}$$

$$\int_0^t \frac{\mu(\tau)^L}{(R-|x|-\varphi(\tau)/r)^{k+1}}\frac{d\tau}{\tau} \leq \frac{\mu(t)^{L-1}r}{k(R-|x|-\varphi(t)/r)^k}\,. \tag{3.2}$$

PROOF. The first estimate (3.1) immediately follows from the definition of $\varphi(t)$ and the inequality

$$\int_0^t \frac{\mu(\tau)^L}{(R-|x|-\varphi(\tau)/r)^k}\frac{d\tau}{\tau} \leq \frac{\mu(t)^{L-1}}{(R-|x|-\varphi(t)/r)^k}\int_0^t \frac{\mu(\tau)}{\tau}d\tau\,.$$

Similarly, we obtain

$$\begin{aligned}\int_0^t \frac{\mu(\tau)^L}{(R-|x|-\varphi(\tau)/r)^{k+1}}\frac{d\tau}{\tau} &\leq \mu(t)^{L-1}\int_0^t \frac{\varphi'(\tau)}{(R-|x|-\varphi(\tau)/r)^{k+1}}d\tau \\ &= \frac{\mu(t)^{L-1}}{k}\Big[\frac{r}{(R-|x|-\varphi(\tau)/r)^k}\Big]_0^t\end{aligned}$$

$$\leq \frac{\mu(t)^{L-1} r}{k(R - |x| - \varphi(t)/r)^k},$$

which is the second estimate (3.2). □

Now let $\mathcal{P}$ be as in (2.5) and consider the equation

$$\mathcal{P}w = g(t, x). \tag{3.3}$$

We have the following result which is due to Baouendi and Goulaouic [1], and Lope [5, Proposition 1].

PROPOSITION 3.3. *Suppose that* (2.4) *holds. Let $T > 0$ and $R > 0$ be sufficiently small, and set W be either* $[0, T] \times D_R$ *or* W_r (*with arbitrary* $r > 0$). *Then for any* $g(t, x) \in \mathscr{X}_0(W)$ *the equation* (3.3) *has a unique solution* $w(t, x) \in \mathscr{X}_m(W)$. *Moreover, if* $|g(t, x)| \leq K\psi(t)$ *on W for some $K > 0$ and for some nondecreasing nonnegative function $\psi(t)$, then we have*

$$|(t\partial_t)^j w(t, x)| \leq \Lambda_1 K \psi(t) \quad \text{on } W \text{ for any } j = 0, 1, \ldots, m,$$

where $\Lambda_1 > 0$ is a constant independent of $g(t, x)$.

For any $\alpha \in \mathbb{N}^n$, we set

$$d(\alpha) = \begin{cases} 1 & \text{if } \alpha = 0, \\ |\alpha| & \text{if } |\alpha| > 0. \end{cases}$$

The next proposition plays an essential role in the proof of our main result, which makes use of the method of Nirenberg [8] and Nishida [9].

PROPOSITION 3.4. *Suppose that* (2.4) *holds. Let $T > 0$ and $R > 0$ be sufficiently small, and let $w(t, x) \in \mathscr{X}_m(W_r)$ be the unique solution of* (3.3) *for a given $g(t, x) \in \mathscr{X}_0(W_r)$. Then there is a constant $\Lambda > 0$, which is independent of $g(t, x)$, such that the following estimates hold for any $K > 0$ and for any nondecreasing nonnegative function $\psi(t)$*:

(1) *If $|g(t, x)| \leq K\psi(t)\mu(t)^m$ on W_r, then for any $(j, \alpha) \in I_m$ we have*

$$|(t\partial_t)^j \partial_x^\alpha w(t, x)| \leq \frac{\Lambda K \psi(t) \mu(t)^{m-d(\alpha)} r^{d(\alpha)-1} \varphi(t)}{R - |x| - \varphi(t)/r} \quad \text{on } W_r. \tag{3.4}$$

(2) *Similarly, if*

$$|g(t, x)| \leq \frac{K\psi(t)\mu(t)^m}{R - |x| - \varphi(t)/r} \quad \text{on } W_r,$$

then for any $(j, \alpha) \in I_m$ we have

$$|(t\partial_t)^j \partial_x^\alpha w(t, x)| \leq \frac{\Lambda K \psi(t) \mu(t)^{m-d(\alpha)} r^{d(\alpha)}}{R - |x| - \varphi(t)/r} \quad \text{on } W_r. \tag{3.5}$$

PROOF. Let us show (1). Since $R > 0$ is sufficiently small, we may assume that $0 < R \leq 1$, which implies that $(R - |x| - \varphi(t)/r)^{-1} \geq 1$ on W_r. By Proposition 3.3, we know that the unique solution $w(t, x)$ satisfies

$$|(t\partial_t)^i w(t,x)| \leq \Lambda_1 K \psi(t)\mu(t)^m \quad \text{on } W_r \text{ for any } i = 0, 1, \ldots, m\,. \tag{3.6}$$

Let us first show (3.4) for any $(j, 0) \in I_m$. Set

$$f_{j,0}(t,x) = (t\partial_t + 1)(t\partial_t)^j w(t,x)\,. \tag{3.7}$$

By (3.6) we have the estimate

$$|f_{j,0}(t,x)| \leq A_{j,0}\Lambda_1 K\psi(t)\mu(t)^m \leq \frac{A_{j,0}\Lambda_1 K\psi(t)\mu(t)^m}{R - |x| - \varphi(t)/r}$$

for some $A_{j,0} > 0$. Since (3.7) is equivalent to the integral equation

$$(t\partial_t)^j w(t,x) = \int_0^t \Big(\frac{\tau}{t}\Big) f_{j,0}(\tau, x)\frac{d\tau}{\tau}\,,$$

by (1) of Lemma 3.2, we obtain

$$\begin{aligned} |(t\partial_t)^j w(t,x)| &\leq \int_0^t |f_{j,0}(\tau,x)|\frac{d\tau}{\tau} \leq \int_0^t \frac{A_{j,0}\Lambda_1 K\psi(\tau)\mu(\tau)^m}{R - |x| - \varphi(\tau)/r}\frac{d\tau}{\tau} \\ &\leq \frac{A_{j,0}\Lambda_1 K\psi(t)\mu(t)^{m-1}\varphi(t)}{R - |x| - \varphi(t)/r} \quad \text{on } W_r\,, \end{aligned}$$

which proves (3.4) for any $(j, 0) \in I_m$.

Now let us show (3.4) for any $(j, \alpha) \in I_m^+$. Set

$$f_{j,\alpha}(t,x) = (t\partial_t + 1)^{|\alpha|}(t\partial_t)^j \partial_x^\alpha w(t,x)\,. \tag{3.8}$$

By (3.6) and Lemma 3.1, we have the estimate

$$|f_{j,\alpha}(t,x)| \leq \frac{A_{j,\alpha}\Lambda_1 K\psi(t)\mu(t)^m}{(R - |x| - \varphi(t)/r)^{|\alpha|}}$$

for some $A_{j,\alpha} > 0$. Since the equation (3.8) is equivalent to the integral equation

$$(t\partial_t)^j \partial_x^\alpha w(t,x) = \int_0^t \cdots \int_0^{\tau_2} \Big(\frac{\tau_{|\alpha|}}{t}\Big)\cdots\Big(\frac{\tau_1}{\tau_2}\Big) f_{j,\alpha}(\tau_1, x)\frac{d\tau_1}{\tau_1}\cdots\frac{d\tau_{|\alpha|}}{\tau_{|\alpha|}}\,,$$

by (1) and (2) of Lemma 3.2, we obtain

$$\begin{aligned} |(t\partial_t)^j \partial_x^\alpha w(t,x)| &\leq \int_0^t \cdots \int_0^{\tau_2} |f_{j,\alpha}(\tau_1,x)|\frac{d\tau_1}{\tau_1}\cdots\frac{d\tau_{|\alpha|}}{\tau_{|\alpha|}} \\ &\leq \int_0^t \cdots \int_0^{\tau_2} \frac{A_{j,\alpha}\Lambda_1 K\psi(\tau_1)\mu(\tau_1)^m}{(R - |x| - \varphi(\tau_1)/r)^{|\alpha|}}\frac{d\tau_1}{\tau_1}\cdots\frac{d\tau_{|\alpha|}}{\tau_{|\alpha|}} \end{aligned}$$

$$\leq \frac{A_{j,\alpha}\Lambda_1 K\psi(t)\mu(t)^{m-|\alpha|}r^{|\alpha|-1}\varphi(t)}{(|\alpha|-1)! \times (R-|x|-\varphi(t)/r)} \quad \text{on } W_r\,.$$

This proves (3.4) for any $(j,\alpha) \in I_m^+$.

From the above computations, it is clear that a suitable $\Lambda > 0$ exists so that (3.4) holds for any $(j,\alpha) \in I_m$. The estimate in (2) can be proved in the same way, noting that $\mu(t) \leq \mu(T) \leq r$ for sufficiently small $T > 0$. □

4. **Proof of Theorem 2.2 under** (2.6)

In this section, we solve the equation (1.1) under the following assumptions:

$$|a(t,x)| \leq A\mu(t)^{2m} \quad \text{on } [0,T_1] \times D_{R_1}\,, \tag{4.1}$$

$$|b_{j,\alpha}(t,x)| \leq B_{j,\alpha}\mu(t)^{|\alpha|} \quad \text{on } [0,T_1] \times D_{R_1}\,, \tag{4.2}$$

for some $A > 0$, $B_{j,\alpha} > 0$ ($(j,\alpha) \in I_m^+$), $0 < T_1 < T_0$, and $0 < R_1 < R_0$.

Take any $0 < \rho_1 < \rho_0$. We have the following result.

THEOREM 4.1. *Suppose that* (4.1) *and* (4.2) *hold. Then there exist* $T > 0$, $R > 0$, $r > 0$, *and* $M > 0$ *with* $M\mu(T)^m < \rho_1$ *such that the equation* (1.1) *has a unique solution* $u(t,x) \in \mathscr{X}_m(W_r)$ *that satisfies the estimates*

$$|(t\partial_t)^j \partial_x^\alpha u(t,x)| \leq M\mu(t)^{2m-d(\alpha)} \quad \text{on } W_r \text{ for all } (j,\alpha) \in I_m\,.$$

We prove this result by the method of Nirenberg [8] and Nishida [9] with a slight modification so that it can be applied to the case $m \geq 2$ without reducing the equation to first order systems.

Set $\Delta_1 = [0,T_1] \times D_{R_1} \times D_{\rho_1}^N$ and

$$C_{\mathcal{R}} = \sup_{\Delta_1}\left\{\left|\frac{\partial^2 \mathcal{R}_2}{\partial z_{j,\alpha}\partial z_{i,\beta}}(t,x,z)\right|;\ (j,\alpha),(i,\beta) \in I_m\right\}.$$

Let $0 < T \leq T_1$ and $0 < R \leq R_1$. As in the previous sections, we set W to be either $[0,T] \times D_R$ or W_r (with arbitrary $r > 0$).

LEMMA 4.2. *Let* $w_i(t,x) \in \mathscr{X}_m(W)$ $(i = 1,2)$. *If* $|(t\partial_t)^j \partial_x^\alpha w_i| \leq M\mu(t)^m \leq \rho_1$ $(i = 1,2)$ *on* W *for any* $(j,\alpha) \in I_m$, *then* $\Phi[w_i] \in \mathscr{X}_0(W)$ $(i = 1,2)$ *and we have*

$$|\Phi[w_1] - \Phi[w_2]| \leq \sum_{(j,\alpha)\in I_m} (B_{j,\alpha} + C_{\mathcal{R}}NM)\mu(t)^{d(\alpha)}|(t\partial_t)^j \partial_x^\alpha (w_1 - w_2)|$$

on W, *where* $B_{j,0} = 0$ (*for* $j < m$).

Before constructing approximate solutions $u_k(t,x)$ $(k = 0,1,2,\ldots)$ for the equation (1.1), we introduce a decreasing sequence of positive numbers $\{r_k\}_{k=0}^\infty$ that will play an important role in the proof of their convergence. We take any $M \geq A$ and then choose a sufficiently small $T > 0$ so that $M\mu(T)^m \leq \rho_1$. We also take a sufficiently small $R > 0$ so that

Propositions 3.3 and 3.4 are valid on W. Take a constant $c_0 > 0$ such that

$$c_0 \geq \max\Big\{1,\ C_{\mathcal{R}} N^2 M + \sum_{(j,\alpha)\in I_m^+} B_{j,\alpha}\Big\} \tag{4.3}$$

and let Λ be the same constant as in Proposition 3.4. We choose $r_0 > 0$ sufficiently small so that $0 < 2\Lambda c_0 r_0 < 1$ and $0 < r_0 \leq 1$, and define the decreasing sequence $r_0 > r_1 > r_2 > \cdots$ by

$$r_k = r_0 \times \prod_{p=1}^{k} \Big(1 - (2\Lambda c_0 r_0)^p\Big), \quad k = 1, 2, \ldots .$$

This is a sequence of positive numbers converging to a positive number r_∞. Moreover, we have

$$\frac{(\Lambda c_0 r_0)^k}{1 - r_k/r_{k-1}} = \Big(\frac{1}{2}\Big)^k, \quad k = 1, 2, \ldots . \tag{4.4}$$

In addition to the condition set on $T > 0$, we also require it to satisfy $\mu(T) \leq r_\infty$.

Now let us solve the equation (1.1). As seen in (2.5), we can consider the equation in the form

$$\mathcal{P}u = a(t,x) + \Phi[u] . \tag{4.5}$$

We solve (4.5) by the method of successive approximations. Set $u_0(t,x) \equiv 0$ and define the approximate solutions $u_k(t,x) \in \mathscr{X}_m(W_{r_{k-1}})$ $(k = 1, 2, \ldots)$ by

$$\mathcal{P}u_k = a(t,x) + \Phi[u_{k-1}] . \tag{4.6}$$

Since $r_k > r_{k+1}$ for any $k \geq 0$, we have

$$W_{r_0} \supset W_{r_1} \supset W_{r_2} \supset \cdots \supset W_{r_k} \supset \cdots \supset W_{r_\infty} .$$

LEMMA 4.3. *For any $k \geq 1$, the equation* (4.6) *has a unique solution $u_k(t,x) \in \mathscr{X}_m(W_{r_{k-1}})$ which satisfies the following estimates for any $(j,\alpha) \in I_m$:*

$$|(t\partial_t)^j \partial_x^\alpha (u_k - u_{k-1})| \leq \frac{\Lambda c_0 (\Lambda c_0 r_0)^{k-1} M \mu(t)^{2m-d(\alpha)} \varphi(t)}{R - |x| - \varphi(t)/r_{k-1}} \ \textit{on}\ W_{r_{k-1}} , \tag{4.7}$$

$$|(t\partial_t)^j \partial_x^\alpha u_k| \leq \sum_{i=1}^{k} (1/2)^i M \mu(t)^{2m-d(\alpha)} \ \textit{on}\ W_{r_k} . \tag{4.8}$$

Before proceeding to the proof of Lemma 4.3, we note that in the equation (4.6), if $u_{k-1}(t,x) \in \mathscr{X}_m(W_{r_{k-1}})$ is already known and on $W_{r_{k-1}}$, it satisfies $|(t\partial_t)^j \partial_x^\alpha u_{k-1}(t,x)| \leq M\mu(t)^m$ for any $(j,\alpha) \in I_m$, then the right-hand side of the equation makes sense and we can consider it as an equation with unknown function u_k.

PROOF OF LEMMA 4.3. We prove by mathematical induction.

(*Base case*) By (1) of Proposition 3.4, we see that the equation (4.6) ($k = 1$) has a unique solution $u_1(t, x) \in \mathscr{X}_m(W_{r_0})$ which satisfies

$$|(t\partial_t)^j \partial_x^\alpha u_1| \le \frac{\Lambda A \mu(t)^{2m-d(\alpha)} r_0^{d(\alpha)-1} \varphi(t)}{R - |x| - \varphi(t)/r_0} .$$

This leads to (4.7) ($k = 1$) since $M \ge A$, $c_0 \ge 1$, and $0 < r_0 \le 1$. Consequently, since $R - |x| - \varphi(t)/r_0 \ge \varphi(t)(1/r_1 - 1/r_0)$ on W_{r_1}, we obtain

$$\begin{aligned} |(t\partial_t)^j \partial_x^\alpha u_1| &\le \frac{\Lambda c_0 M \mu(t)^{2m-d(\alpha)} \varphi(t)}{\varphi(t)(1/r_1 - 1/r_0)} = \frac{\Lambda c_0 r_1 M \mu(t)^{2m-d(\alpha)}}{1 - r_1/r_0} \\ &\le \frac{\Lambda c_0 r_0 M \mu(t)^{2m-d(\alpha)}}{1 - r_1/r_0} = (M/2)\mu(t)^{2m-d(\alpha)} \quad \text{on } W_{r_1} . \end{aligned}$$

The last equality follows from (4.4) with $k = 1$. This proves (4.8) ($k = 1$).

(*Inductive step*) Suppose that for each $k = 1, 2, \dots, p$, the equation (4.6) has a unique solution $u_k(t, x) \in \mathscr{X}_m(W_{r_{k-1}})$ satisfying the estimates (4.7) and (4.8). Let us consider the equation (4.6) ($k = p + 1$) on W_{r_p}, that is,

$$\mathcal{P} u_{p+1} = a(t, x) + \Phi[u_p] \quad \text{on } W_{r_p} . \tag{4.9}$$

By Lemma 4.2 we have $\Phi[u_p] \in \mathscr{X}_0(W_{r_p})$ and so by Proposition 3.3, (4.9) has a unique solution $u_{p+1}(t, x) \in \mathscr{X}_m(W_{r_p})$.

Let us show (4.7) ($k = p + 1$). We know that $(u_{p+1} - u_p)(t, x)$ is the unique solution of the equation

$$\mathcal{P}(u_{p+1} - u_p) = \Phi[u_p] - \Phi[u_{p-1}] \quad \text{on } W_{r_p} . \tag{4.10}$$

By (4.7) ($k = p$), (4.8) ($k = p - 1, p$), and the fact that $r_p < r_{p-1}$, $W_{r_p} \subset W_{r_{p-1}}$, and $\mu(t) \le \mu(T) \le r_\infty < 1$, the following estimates hold on W_{r_p} for any $(j, \alpha) \in I_m$:

$$\begin{aligned} &|(t\partial_t)^j \partial_x^\alpha (u_p - u_{p-1})| \le \frac{\Lambda c_0 (\Lambda c_0 r_0)^{p-1} M \mu(t)^{2m-d(\alpha)} \varphi(t)}{R - |x| - \varphi(t)/r_p} , \\ &|(t\partial_t)^j \partial_x^\alpha u_p| \le M \mu(t)^{2m-d(\alpha)} \le M \mu(t)^m , \\ &|(t\partial_t)^j \partial_x^\alpha u_{p-1}| \le M \mu(t)^{2m-d(\alpha)} \le M \mu(t)^m . \end{aligned}$$

Thus, by Lemma 4.2 with $w_1 = u_p$ and $w_2 = u_{p-1}$, we obtain

$$\begin{aligned} &|\Phi[u_p] - \Phi[u_{p-1}]| \\ &\le \sum_{(j,\alpha)\in I_m} (B_{j,\alpha} + C_{\mathcal{R}} N M) \mu(t)^{d(\alpha)} \times \frac{\Lambda c_0 (\Lambda c_0 r_0)^{p-1} M \mu(t)^{2m-d(\alpha)} \varphi(t)}{R - |x| - \varphi(t)/r_p} \\ &\le \frac{\Lambda c_0^2 (\Lambda c_0 r_0)^{p-1} M \mu(t)^{2m} \varphi(t)}{R - |x| - \varphi(t)/r_p} \quad \text{on } W_{r_p} . \end{aligned}$$

Applying (2) of Proposition 3.4 with $\psi(t) = \mu(t)^m \varphi(t)$ to the equation (4.10), we arrive at the estimate

$$|(t\partial_t)^j \partial_x^\alpha (u_{p+1} - u_p)| \le \Lambda \times \frac{\Lambda c_0^2 (\Lambda c_0 r_0)^{p-1} M \mu(t)^{2m-d(\alpha)} r_p^{d(\alpha)} \varphi(t)}{R - |x| - \varphi(t)/r_p}$$
$$\le \frac{\Lambda c_0 (\Lambda c_0 r_0)^p M \mu(t)^{2m-d(\alpha)} \varphi(t)}{R - |x| - \varphi(t)/r_p} \quad \text{on } W_{r_p}$$

for any $(j, \alpha) \in I_m$, where the second inequality follows from the fact that $r_p^{d(\alpha)} < r_0$. This proves (4.7) $(k = p + 1)$.

Now let us show (4.8) $(k = p + 1)$. Since $R - |x| - \varphi(t)/r_p \ge \varphi(t)(1/r_{p+1} - 1/r_p)$ on $W_{r_{p+1}}$, by (4.7) $(k = p + 1)$ we have

$$|(t\partial_t)^j \partial_x^\alpha (u_{p+1} - u_p)| \le \frac{\Lambda c_0 (\Lambda c_0 r_0)^p M \mu(t)^{2m-d(\alpha)} \varphi(t)}{\varphi(t)(1/r_{p+1} - 1/r_p)}$$
$$\le \frac{(\Lambda c_0 r_0)^{p+1} M \mu(t)^{2m-d(\alpha)}}{(1 - r_{p+1}/r_p)}$$
$$= \left(\frac{1}{2}\right)^{p+1} M \mu(t)^{2m-d(\alpha)} \quad \text{on } W_{r_{p+1}} .$$

This estimate together with (4.8) $(k = p)$ yields (4.8) $(k = p+1)$ when we apply the Triangle Inequality. This concludes the inductive step and the proof of Lemma 4.3. □

Let us show the existence of a solution of (4.5) by using the approximate solutions $u_k(t, x) \in \mathscr{X}_m(W_{r_k})$ $(k = 1, 2, \ldots)$ constructed in Lemma 4.3. Since $r_0 > r_1 > r_2 > \cdots > r_\infty > 0$, by restricting the domain of $u_k(t, x)$ $(k = 1, 2, \ldots)$ to W_{r_∞}, we have $u_k(t, x) \in \mathscr{X}_m(W_{r_\infty})$ and

$$u_k(t, x) = \sum_{i=1}^{k} (u_i(t, x) - u_{i-1}(t, x)) \quad \text{on } W_{r_\infty} .$$

As can be seen in the proof of Lemma 4.3, the estimates (4.7) and (4.8) imply that our approximate solutions converge to a function $u(t, x) \in \mathscr{X}_m(W_{r_\infty})$ satisfying $|(t\partial_t)^j \partial_x^\alpha u| \le M\mu(t)^{2m-d(\alpha)}$ on W_{r_∞} for any $(j, \alpha) \in I_m$. This proves the existence of a solution of (1.1).

The uniqueness of the solution can be proved in the same way. This completes the proof of Theorem 4.1.

5. Proof of Theorem 2.2 in the general case

In this section, we prove Theorem 2.2 in the general case, that is, we solve the equation (1.1) under the following assumptions:

$$|a(t, x)| \le A\mu(t)^m \quad \text{on } [0, T_1] \times D_{R_1} , \tag{5.1}$$

$$|b_{j,\alpha}(t,x)| \le B_{j,\alpha}\mu(t)^{|\alpha|} \quad \text{on } [0,T_1]\times D_{R_1}\,, \tag{5.2}$$

for some $A > 0$, $B_{j,\alpha} > 0$ ($(j,\alpha) \in I_m^+$), $0 < T_1 < T_0$ and $0 < R_1 < R_0$.

Take any $0 < \rho_1 < \rho_0$. We have the following result:

PROPOSITION 5.1. *Suppose that* (5.1) *and* (5.2) *hold. Then there exist* $T > 0$, $R > 0$, $M > 0$ *with* $M\mu(T)^m < \rho_1$, $A^\star > 0$, *and* $w(t,x) \in \mathscr{X}_m([0,T]\times D_R)$ *that satisfy the following conditions*:

$$|(t\partial_t)^j \partial_x^\alpha w(t,x)| \le M\mu(t)^m \quad \textit{on } [0,T]\times D_R \textit{ for any } (j,\alpha)\in I_m\,, \tag{5.3}$$

$$|a(t,x) + \Phi[w] - \mathcal{P}w| \le A^\star \mu(t)^{2m} \quad \textit{on } [0,T]\times D_R\,. \tag{5.4}$$

Let us assume Proposition 5.1 for a while. Using this result, we can reduce our problem to Theorem 4.1 in the following way. By setting

$$a^\star(t,x) = a(t,x) + \Phi[w] - \mathcal{P}w\,,$$
$$u(t,x) = w(t,x) + V(t,x)\,,$$

the equation (1.1) with respect to $u(t,x)$ can be reduced to the equation

$$\mathcal{P}V = a^\star(t,x) + \Phi[w+V] - \Phi[w] \tag{5.5}$$

with unknown function $V(t,x)$. It is easy to see that the equation (5.5) may be expressed in the form

$$(t\partial_t)^m V = a^\star(t,x) + \sum_{(j,\alpha)\in I_m} b^\star_{j,\alpha}(t,x)(t\partial_t)^j\partial_x^\alpha V + \mathcal{R}_2^\star\Big(t,x,\big\{(t\partial_t)^j\partial_x^\alpha V\big\}_{(j,\alpha)\in I_m}\Big) \tag{5.6}$$

for some $b^\star_{j,\alpha}(t,x)$ and $\mathcal{R}_2^\star(t,x,z)$, which are continuous functions in t and holomorphic in the other variables. Moreover, we have

$$\begin{aligned} b^\star_{j,\alpha}(t,x) &= b_{j,\alpha}(t,x) + \frac{\partial\mathcal{R}_2}{\partial z_{j,\alpha}}\Big(t,x,\big\{(t\partial_t)^i\partial_x^\beta w\big\}_{(i,\beta)\in I_m}\Big) \\ &= O(\mu(t)^{|\alpha|}) + O(\mu(t)^m) = O(\mu(t)^{|\alpha|}) \quad (\text{as } t \longrightarrow 0) \end{aligned}$$

uniformly on D_R for any $(j,\alpha)\in I_m^+$. Thus, by applying Theorem 4.1 to the equation (5.6), we obtain a solution $V(t,x) \in \mathscr{X}_m(W_r)$ of (5.5) for some $r > 0$. This in turn gives the desired solution $u(t,x) = w(t,x) + V(t,x) \in \mathscr{X}_m(W_r)$ in Theorem 2.2, the uniqueness of which can be proved by the same reduction technique above.

Thus, to complete the proof of Theorem 2.2, it is sufficient to show Proposition 5.1.

PROOF OF PROPOSITION 5.1. Set $w_0(t,x) \equiv 0$ and define the approximate solutions $w_i(t,x)$ ($i = 1,\dots,m$) for the equation (1.1) by

$$\mathcal{P}w_i = a(t,x) + \Phi[w_{i-1}]\,. \tag{5.7}$$

Set $w(t,x) = w_m(t,x)$. Let us show that this $w(t,x)$ satisfies the conditions (5.3) and (5.4).

For simplicity, we take $T_1 > 0$ and $R_1 > 0$ sufficiently small so that Proposition 3.3 is valid for $W = [0,T] \times D_R$ for any $0 < T \le T_1$ and $0 < R \le R_1$. We also take a constant $c_0 > 0$ satisfying (4.3), a sequence $0 < R_{m+1} < R_m < \cdots < R_2 < R_1$, and then choose a sequence $\{M_i; i = 0, 1, \dots, m\}$ such that $M_0 = A$ (the constant in (5.3)) and

$$\frac{m!\Lambda_1 c_0 M_{i-1}}{(R_i - R_{i+1})^m} \le M_i \quad \text{for } i = 1, \dots, m\,. \tag{5.8}$$

Finally, we take $T > 0$ sufficiently small so that $\mu(T) \le 1$ and

$$(M_1 + \cdots + M_m)\mu(T)^m \le \rho_1\,.$$

LEMMA 5.2. *For any* $1 \le i \le m$, *there exists a unique solution* $w_i(t,x) \in \mathscr{X}_m([0,T] \times D_{R_i})$ *of the equation* (5.7) *that satisfies the following estimates for any* $(j,\alpha) \in I_m$:

$$|(t\partial_t)^j \partial_x^\alpha (w_i - w_{i-1})| \le M_i \mu(t)^{m+i-1} \quad \text{on } [0,T] \times D_{R_{i+1}}\,, \tag{5.9}$$

$$|(t\partial_t)^j \partial_x^\alpha w_i| \le (M_1 + \cdots + M_i)\mu(t)^m \quad \text{on } [0,T] \times D_{R_{i+1}}\,. \tag{5.10}$$

PROOF. Since $w_0(t,x) \equiv 0$, by (5.1) and Proposition 3.3, the equation (5.7) ($i = 1$) has a unique solution $w_1(t,x) \in \mathscr{X}_m([0,T] \times D_{R_1})$ and it satisfies $|(t\partial_t)^j w_1| \le \Lambda_1 A\mu(t)^m$ on $[0,T] \times D_{R_1}$ for any $j = 0, 1, \dots, m$. Then, by Cauchy's estimate, we obtain

$$|(t\partial_t)^j \partial_x^\alpha w_1| \le \frac{|\alpha|!\Lambda_1 A\mu(t)^m}{(R_1 - R_2)^{|\alpha|}} \le M_1 \mu(t)^m \quad \text{on } [0,T] \times D_{R_2}$$

for any $(j,\alpha) \in I_m$. This proves that (5.9) and (5.10) hold for $i = 1$.

We now proceed to the inductive step. Suppose that for each $i = 1, \dots, p$, the equation (5.7) has a unique solution $w_i(t,x) \in \mathscr{X}_m([0,T] \times D_{R_i})$ that satisfies the estimates (5.9) and (5.10). Let us now consider the equation (5.7) ($i = p+1$) on $[0,T] \times D_{R_{p+1}}$, that is,

$$\mathcal{P} w_{p+1} = a(t,x) + \Phi[w_p] \quad \text{on } [0,T] \times D_{R_{p+1}}\,. \tag{5.11}$$

By Lemma 4.2 we have $\Phi[w_p] \in \mathscr{X}_0([0,T] \times D_{R_{p+1}})$ and so by Proposition 3.3, (5.11) has a unique solution $w_{p+1}(t,x) \in \mathscr{X}_m([0,T] \times D_{R_{p+1}})$.

Let us show (5.9) ($i = p+1$). We know that $(w_{p+1} - w_p)(t,x)$ is the unique solution of the equation

$$\mathcal{P}(w_{p+1} - w_p) = \Phi[w_p] - \Phi[w_{p-1}] \quad \text{on } [0,T] \times D_{R_{p+1}}\,. \tag{5.12}$$

By Lemma 4.2 and the induction hypothesis, we have

$$\begin{aligned} &|\Phi[w_p] - \Phi[w_{p-1}]| \\ &\le \sum_{(j,\alpha) \in I_m} (B_{j,\alpha} + C_{\mathcal{R}} NM)\mu(t)^{d(\alpha)} \times M_p \mu(t)^{m+p-1} \end{aligned}$$

$$\leq c_0 M_p \mu(t)^{m+p} \quad \text{on } [0,T] \times D_{R_{p+1}} .$$

Therefore, by applying Proposition 3.3 to the equation (5.12), we arrive at the estimate $|(t\partial_t)^j(w_{p+1} - w_p)| \leq \Lambda_1 c_0 M_p \mu(t)^{m+p}$ on $[0,T] \times D_{R_{p+1}}$ for any $j = 0, 1, \ldots, m$. Consequently, when we apply Cauchy's estimates and (5.8), we obtain

$$|(t\partial_t)^j \partial_x^\alpha (w_{p+1} - w_p)| \leq \frac{|\alpha|!\Lambda_1 c_0 M_p \mu(t)^{m+p}}{(R_{p+1} - R_{p+2})^{|\alpha|}} \leq M_{p+1}\mu(t)^{m+p}$$

on $[0,T] \times D_{R_{p+2}}$ for any $(j,\alpha) \in I_m$. This proves (5.9) $(i = p+1)$.

The estimate (5.10) $(i = p+1)$ follows immediately from (5.9) $(i = p+1)$ and (5.10) $(i = p)$ when we apply the Triangle Inequality. This completes the proof of Lemma 5.2. □

Now, let us complete the proof of Proposition 5.1. Set $R = R_{m+1}$ and $M = M_1 + \cdots + M_m$. Then it follows that the function $w(t,x) = w_m(t,x)$ belongs to $\mathscr{X}_m([0,T] \times D_R)$ and by (5.10) $(i = m)$, it satisfies the estimate (5.3). Moreover, by Lemma 4.2 we see that this function also satisfies the estimate

$$\begin{aligned}|a(t,x) + \Phi[w] - \mathcal{P}w| &= |\Phi[w_m] - \Phi[w_{m-1}]| \\ &\leq \sum_{(j,\alpha)\in I_m} (B_{j,\alpha} + C_{\mathcal{R}} NM)\mu(t)^{d(\alpha)} \times M_m \mu(t)^{2m-1} \\ &\leq c_0 M_m \mu(t)^{2m} \leq A^\star \mu(t)^{2m} \quad \text{on } [0,T] \times D_R ,\end{aligned}$$

for some $A^\star > 0$. □

6. A generalization

We finish off by giving a slight generalization of our main result. It is clear that Theorem 2.2 holds when $a(t,x) = O(\mu(t)^q)$ for some $q \in [m, \infty)$. Now, let $0 < q < m$, and let us consider the case

$$a(t,x) = O(\mu(t)^q) \quad \text{uniformly on } D_{R_0} \text{ (as } t \longrightarrow 0) . \tag{6.1}$$

In this case, it seems difficult to solve the equation (1.1) restricted only to the conditions $(A_1) - (A_3)$, (2.3), and (2.4). However, we can obtain a unique solvability result by imposing the following additional assumption on the second-order partial derivatives of $\mathcal{R}_2$:

$$\frac{\partial^2 \mathcal{R}_2}{\partial z_{j,\alpha} \partial z_{i,\beta}}(t,x,z) = O(\mu(t)^{\max\{|\alpha|,|\beta|\}-q}) \quad \text{uniformly on } D_{R_0} \times D_{\rho_0}^N$$
$$\text{(as } t \longrightarrow 0) \text{ for any } (j,\alpha), (i,\beta) \in I_m . \tag{6.2}$$

Note that (6.2) is trivial if $q \geq m$.

THEOREM 6.1. *Suppose that* $(A_1) - (A_3)$, (2.3), (2.4), (6.1), *and* (6.2) *hold. Then there exist* $r > 0$, $R > 0$, $T > 0$, *and* $M > 0$ *with* $M\mu(T)^q < \rho_0$ *such that the equation* (1.1) *has a unique solution* $u(t,x)$ *in* $\mathscr{X}_m(W_r)$ *that satisfies the estimates*

$$|(t\partial_t)^j \partial_x^\alpha u(t,x)| \le M\mu(t)^q \quad \text{on } W_r \text{ for all } (j,\alpha) \in I_m\,.$$

Set $\delta = \min\{q, 1\} > 0$. The following lemma shows that Theorem 6.1 can be proved in a similar way as Theorem 2.2.

LEMMA 6.2. *The following statements hold for sufficiently small* $T_1 > 0$ *and* $R_1 > 0$:

(1) *Let* $0 < T \le T_1$ *and* $0 < R \le R_1$. *Let* $w_i(t,x) \in \mathscr{X}_m(W)$ $(i = 1,2)$, *where* W *can be either* $[0,T] \times D_R$ *or* W_r *(for any* $r > 0$*). If* $|(t\partial_t)^j \partial_x^\alpha w_i| \le M\mu(t)^q$ $(i = 1,2)$ *on* W *for any* $(j,\alpha) \in I_m$, *then* $\Phi[w_i] \in \mathscr{X}_0(W)$ $(i = 1,2)$ *and we have*

$$|\Phi[w_1] - \Phi[w_2]| \le \sum_{(j,\alpha)\in I_m} (B_{j,\alpha} + C_{\mathcal{R}} NM)\mu(t)^\delta |(t\partial_t)^j \partial_x^\alpha (w_1 - w_2)|$$

on W, *where* $B_{j,0} = 0$ *(for* $j < m$*).*

(2) *Let* $w(t,x) \in \mathscr{X}_m([0,T_1] \times D_{R_1})$. *If* $|(t\partial_t)^k \partial_x^\gamma w| \le M\mu(t)^q$ *on* $[0,T_1] \times D_{R_1}$ *for any* $(k,\gamma) \in I_m$, *then there exists* $C_{j,\alpha} > 0$ *such that*

$$\left| \frac{\partial \mathcal{R}_2}{\partial z_{j,\alpha}} \Big(t, x, \big\{ (t\partial_t)^k \partial_x^\gamma w \big\}_{(k,\gamma)\in I_m} \Big) \right| \le C_{j,\alpha} \mu(t)^{|\alpha|} \quad \text{on } [0,T_1] \times D_{R_1}$$

for any $(j,\alpha) \in I_m^+$.

PROOF. The first result (1) of this lemma can be verified in the same way as Lemma 4.2.

Let us show the second result. Under the additional assumption (6.2), we see that

$$\begin{aligned}
&\frac{\partial \mathcal{R}_2}{\partial z_{j,\alpha}} \Big(t, x, \big\{ (t\partial_t)^k \partial_x^\gamma w \big\}_{(k,\gamma)\in I_m} \Big) \\
&= \sum_{(i,\beta)\in I_m} \int_0^1 \frac{\partial^2 \mathcal{R}_2}{\partial z_{i,\beta} \partial z_{j,\alpha}} \Big(t, x, \big\{ s(t\partial_t)^k \partial_x^\gamma w \big\}_{(k,\gamma)\in I_m} \Big) ds \times (t\partial_t)^i \partial_x^\beta w \\
&= \sum_{(i,\beta)\in I_m} O(\mu(t)^{\max\{|\alpha|,|\beta|\}-q}) \times O(\mu(t)^q) \\
&= O(\mu(t)^{|\alpha|}) \quad \text{uniformly on } D_{R_1} \text{ (as } t \longrightarrow 0)\,.
\end{aligned}$$

This gives the desired estimate in (2). □

By using (1) of Lemma 6.2, we can find a $w(t,x) \in \mathscr{X}_m([0,T] \times D_R)$ such that the conditions (5.3) and (5.4) are satisfied. Following the computations in Section 5, by using (2) of Lemma 6.2, we can reduce our problem to the same situation in Theorem 4.1. This completes the proof of Theorem 6.1.

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References

[1] M. S. BAOUENDI and C. GOULAOUIC, Cauchy problems with characteristic initial hypersurface, Comm. Pure Appl. Math. **26** (1973), 455–475.

[2] M. S. BAOUENDI and C. GOULAOUIC, Singular nonlinear Cauchy problems, J. Differential Equations **22** (1976), no. 2, 268–291.

[3] R. GÉRARD and H. TAHARA, Solutions holomorphes et singulières d'équations aux dérivées partielles singulières non linéaires, Publ. Res. Inst. Math. Sci. **29** (1993), 121–151.

[4] J. E. C. LOPE, A sharp existence and uniqueness theorem for linear Fuchsian partial differential equations, Tokyo J. Math. **24** (2001), 477–486.

[5] J. E. C. LOPE, Existence and uniqueness theorems for a class of linear Fuchsian partial differential equations, J. Math. Sci. Univ. Tokyo **6** (1999), 527–538.

[6] J. E. C. LOPE, M. P. ROQUE and H. TAHARA, On the unique solvability of certain nonlinear singular partial differential equations, Z. Anal. Anwend. **31** (2012), no. 3, 291–305.

[7] M. NAGUMO, *Über* das Anfangswertproblem partieller Differentialgleichungen, Japan. J. Math. **18** (1942), 41–47.

[8] L. NIRENBERG, An abstract form of the nonlinear Cauchy-Kowalewski theorem, J. Differential Geometry **6** (1972), 561–576.

[9] T. NISHIDA, A note on a theorem of Nirenberg, J. Differential Geometry **12** (1977), no. 4, 629–633.

[10] H. TAHARA and H. YAMANE, Logarithmic singularities of solutions to nonlinear partial differential equations, J. Math. Soc. Japan **60** (2008), no. 2, 603–630.

[11] H. TAHARA and H. YAMAZAWA, Structure of solutions of nonlinear partial differential equations of Gerard-Tahara type, Publ. Res. Inst. Math. Sci. **41** (2005), 339–373.

[12] W. WALTER, An elementary proof of the Cauchy-Kowalevsky theorem, Amer. Math. Monthly **92** (1985), no. 2, 115–126.

Present Addresses:
DENNIS B. BACANI
FACULTY OF SCIENCE AND TECHNOLOGY,
SOPHIA UNIVERSITY,
7–1 KIOI-CHO, CHIYODA-KU, TOKYO 102–8554, JAPAN.
e-mail: dennis.bacani@sophia.ac.jp

JOSE ERNIE C. LOPE
INSTITUTE OF MATHEMATICS,
UNIVERSITY OF THE PHILIPPINES DILIMAN,
C.P. GARCIA ST., U.P. CAMPUS, QUEZON CITY 1101, PHILIPPINES.
e-mail: ernie@math.upd.edu.ph

HIDETOSHI TAHARA
DEPARTMENT OF INFORMATION AND COMMUNICATION SCIENCES,
SOPHIA UNIVERSITY,
7–1 KIOI-CHO, CHIYODA-KU, TOKYO 102–8554, JAPAN.
e-mail: h-tahara@sophia.ac.jp