

Mean Values of the Barnes Double Zeta-function

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Abstract. In the study of order estimation of the Riemann zeta-function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, solving Lindelöf hypothesis is an important theme. As one of the relationships, asymptotic behavior of mean values has been studied. Furthermore, the theory of the mean values is also noted in the double zeta-functions, and the mean values of the Euler-Zagier type of double zeta-function and Mordell-Tornheim type of double zeta-function were studied. In this paper, we prove asymptotic formulas for mean square values of the Barnes double zeta-function $\zeta_2(s, \alpha; v, w) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\alpha + vm + wn)^{-s}$ with respect to $\text{Im}(s)$ as $\text{Im}(s) \rightarrow +\infty$.

1. Introduction and the statement of results

The Barnes double zeta-function was first introduced by Barnes [1] in the course of developing his theory of double gamma functions, and the double series of the form as

$$\zeta_2(s, \alpha; v, w) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(\alpha + vm + wn)^s} \quad (1)$$

was introduced and studied in [2]. As a subsequent research, multiple series of similar form as (1) was introduced in connection with the theory of multiple gamma functions by Barnes [3].

Let r be a positive integer, $s = \sigma + it$ a complex variable, α a real parameter, and w_j ($j = 1, \dots, r$) complex parameters which are located on one of the complex half-plane divided by a straight line through the origin. The Barnes multiple zeta-function $\zeta_r(s, \alpha; w_1, \dots, w_r)$ is defined by

$$\zeta_r(s, \alpha; w_1, \dots, w_r) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} \frac{1}{(\alpha + w_1 m_1 + \cdots + w_r m_r)^s} \quad (2)$$

where the series on the right-hand side is absolutely convergent for $\text{Re}(s) > r$, and is continued meromorphically to the complex s -plane, and its only singularities are the simple poles located at $s = j$ ($j = 1, 2, \dots, r$).

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In this paper, we focus on the case $r = 2$ and $(w_1, w_2) = (v, w)$ for any $v, w > 0$ of (2), which is the Barnes double zeta-function (1) and study the asymptotic behavior of

$$\int_1^T |\zeta_2(\sigma + it, \alpha; v, w)|^2 dt$$

as $T \rightarrow +\infty$.

Let

$$\zeta_2^{[2]}(s_1, s_2, \alpha; v, w) = \sum_{\substack{m_1, n_1, m_2, n_2 \geq 0 \\ vm_1 + wn_1 = vm_2 + wn_2}} \frac{1}{(\alpha + vm_1 + wn_1)^{s_1} (\alpha + vm_2 + wn_2)^{s_2}},$$

which is absolutely convergent for $\operatorname{Re}(s_1 + s_2) > 2$. If v, w are linearly independent over \mathbf{Q} , then $vm_1 + wn_1 = vm_2 + wn_2$ is equivalent to $(m_1, n_1) = (m_2, n_2)$, and hence we have

$$\zeta_2^{[2]}(s_1, s_2, \alpha; v, w) = \zeta_2(s_1 + s_2, \alpha; v, w).$$

THEOREM 1. For $s = \sigma + it \in \mathbf{C}$ with $\sigma > 2$, we have

$$\int_1^T |\zeta_2(s, \alpha; v, w)|^2 dt = \zeta_2^{[2]}(\sigma, \sigma, \alpha; v, w)T + O(1)$$

as $T \rightarrow +\infty$.

THEOREM 2. For $s = \sigma + it \in \mathbf{C}$ with $3/2 < \sigma \leq 2$, we have

$$\begin{aligned} \int_1^T |\zeta_2(s, \alpha; v, w)|^2 dt \\ = \zeta_2^{[2]}(\sigma, \sigma, \alpha; v, w)T + \begin{cases} O(T^{4-2\sigma} \log T) & (3/2 < \sigma \leq 7/4) \\ O(T^{1/2}) & (7/4 < \sigma \leq 2) \end{cases} \end{aligned}$$

as $T \rightarrow +\infty$.

REMARK 1. We mention here some recent results on mean values of double zeta-functions. Matsumoto-Tsumura [6] treated the Euler double zeta-function

$$\zeta_2(s_1, s_2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^{s_1} (m+n)^{s_2}}$$

and gave some formulas which imply

$$\int_2^T |\zeta_2(\sigma_1 + it_1, \sigma_2 + it_2)|^2 dt_2 \sim \zeta_2^{[2]}(\sigma_1 + it_1, 2\sigma_2)T \quad (T \rightarrow \infty) \quad (3)$$

in some subsets in a region for $\sigma_1 + \sigma_2 > 3/2$, see [6] for detail. Here, $\zeta_2^{[2]}(\sigma_1 + it_1, 2\sigma_2)$ is defined by

$$\zeta_2^{[2]}(s_1, s_2) = \sum_{k=2}^{\infty} \left| \sum_{m=1}^{k-1} \frac{1}{m^{s_1}} \right|^2 \frac{1}{k^{s_2}}$$

which is absolutely convergent for $\text{Re}(s_2) > 1/2$ and $\text{Re}(s_1 + s_2) > 3/2$. Ikeda-Matsuoka-Nagata [4] extended the region of results of Matsumoto-Tsumura [6], and further they gave some asymptotic formulas which imply

$$\int_2^T |\zeta_2(\sigma_1 + it_1, \sigma_2 + it_2)|^2 dt_2 \asymp T \log T \quad (T \rightarrow \infty)$$

on polygonal line $\{(\sigma_1, \sigma_2) \mid \sigma_1 + \sigma_2 = 3/2 \text{ and } \sigma_2 > 1/2\} \cup \{(\sigma_1, \sigma_2) \mid \sigma_1 \geq 1 \text{ and } \sigma_2 = 1/2\}$. Also, they gave similar results on

$$\int_2^T |\zeta_2(\sigma + it, s_2)|^2 dt, \quad \int_2^T |\zeta_2(\sigma_1 + it, \sigma_2 + it)|^2 dt,$$

see [4] for detail. On the other hand, for the Mordell-Tornheim double zeta-function

$$\zeta_{MT,2}(s_1, s_2; s_3) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^{s_1} n^{s_2} (m+n)^{s_3}},$$

Okamoto-Onozuka [7] obtained some results on the mean square values which imply

$$\int_2^T |\zeta_{MT,2}(s_1, s_2; \sigma + it)|^2 dt \sim \zeta_{MT,2}^{[2]}(s_1, s_2; 2\sigma) T \quad (T \rightarrow \infty) \tag{4}$$

in some subset in the region for $\sigma_1 + \sigma_2 + \sigma > 3/2$, here $\zeta_{MT,2}^{[2]}(s_1, s_2; 2\sigma)$ is defined by

$$\zeta_{MT,2}^{[2]}(s_1, s_2; s) = \sum_{k=2}^{\infty} \left| \sum_{m=1}^{k-1} \frac{1}{m^{s_1} (k-m)^{s_2}} \right|^2 \frac{1}{k^s},$$

which is absolutely convergent for $2\text{Re}(s_1) + \text{Re}(s) > 1$, $2\text{Re}(s_2) + \text{Re}(s) > 1$ and $2\text{Re}(s_1) + 2\text{Re}(s_2) + \text{Re}(s) > 3$. Theorem 1 and Theorem 2 are the results corresponding to (3), (4) of the Barnes double zeta-function (1) version.

2. Proof of Theorem 1

In this section, we give a proof of Theorem 1.

PROOF OF THEOREM 1. Let $\sigma + it \in \mathbf{C}$ with $\sigma > 2$. We first calculate $|\zeta_2(s, \alpha; v, w)|^2$. We have

$$|\zeta_2(s, \alpha; v, w)|^2 = \zeta_2(s, \alpha; v, w) \overline{\zeta_2(s, \alpha; v, w)}$$

$$\begin{aligned}
&= \sum_{m_1=0}^{\infty} \sum_{n_1=0}^{\infty} \frac{1}{(\alpha + vm_1 + wn_1)^{\sigma+it}} \sum_{m_2=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{1}{(\alpha + vm_2 + wn_2)^{\sigma-it}} \\
&= \sum_{\substack{m_1, n_1, m_2, n_2 \geq 0 \\ vm_1 + wn_1 = vm_2 + wn_2}} \frac{1}{(\alpha + vm_1 + wn_1)^{\sigma} (\alpha + vm_2 + wn_2)^{\sigma}} \\
&\quad + \sum_{\substack{m_1, n_1, m_2, n_2 \geq 0 \\ vm_1 + wn_1 \neq vm_2 + wn_2}} \frac{1}{(\alpha + vm_1 + wn_1)^{\sigma} (\alpha + vm_2 + wn_2)^{\sigma}} \left(\frac{\alpha + vm_2 + wn_2}{\alpha + vm_1 + wn_1} \right)^{it} \\
&= \zeta_2^{[2]}(\sigma, \sigma, \alpha; v, w) \\
&\quad + \sum_{\substack{m_1, n_1, m_2, n_2 \geq 0 \\ vm_1 + wn_1 \neq vm_2 + wn_2}} \frac{1}{(\alpha + vm_1 + wn_1)^{\sigma} (\alpha + vm_2 + wn_2)^{\sigma}} \left(\frac{\alpha + vm_2 + wn_2}{\alpha + vm_1 + wn_1} \right)^{it}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\int_1^T |\zeta_2(s, \alpha; v, w)|^2 dt = \zeta_2^{[2]}(\sigma, \sigma, \alpha; v, w)(T-1) \\
&\quad + \sum_{\substack{m_1, n_1, m_2, n_2 \geq 0 \\ vm_1 + wn_1 \neq vm_2 + wn_2}} \frac{1}{(\alpha + vm_1 + wn_1)^{\sigma} (\alpha + vm_2 + wn_2)^{\sigma}} \\
&\quad \times \frac{e^{iT \log\{(\alpha + vm_2 + wn_2)/(\alpha + vm_1 + wn_1)\}} - e^{i \log\{(\alpha + vm_2 + wn_2)/(\alpha + vm_1 + wn_1)\}}}{i \log\{(\alpha + vm_2 + wn_2)/(\alpha + vm_1 + wn_1)\}}.
\end{aligned}$$

The second term on the right-hand side is

$$\begin{aligned}
&\ll \sum_{\substack{m_1, n_1, m_2, n_2 \geq 0 \\ vm_1 + wn_1 < vm_2 + wn_2}} \frac{1}{(\alpha + vm_1 + wn_1)^{\sigma} (\alpha + vm_2 + wn_2)^{\sigma}} \\
&\quad \times \frac{1}{\log\{(\alpha + vm_2 + wn_2)/(\alpha + vm_1 + wn_1)\}} \\
&= \left(\sum_{\substack{m_1, n_1, m_2, n_2 \geq 0 \\ \alpha + vm_1 + wn_1 < \alpha + vm_2 + wn_2 < 2(\alpha + vm_1 + wn_1)}} + \sum_{\substack{m_1, n_1, m_2, n_2 \geq 0 \\ \alpha + vm_2 + wn_2 \geq 2(\alpha + vm_1 + wn_1)}} \right) \\
&\quad \frac{1}{(\alpha + vm_1 + wn_1)^{\sigma} (\alpha + vm_2 + wn_2)^{\sigma} \log\{(\alpha + vm_2 + wn_2)/(\alpha + vm_1 + wn_1)\}}.
\end{aligned}$$

We denote the right-hand side by $V_1 + V_2$. Then we have

$$\begin{aligned} V_2 &\ll \sum_{\substack{m_1, n_1, m_2, n_2 \geq 0 \\ \alpha + vm_2 + wn_2 \geq 2(\alpha + vm_1 + wn_1)}} \frac{1}{(\alpha + vm_1 + wn_1)^\sigma (\alpha + vm_2 + wn_2)^\sigma} \\ &\leq \sum_{m_1 \geq 0} \sum_{n_1 \geq 0} \frac{1}{(\alpha + vm_1 + wn_1)^\sigma} \sum_{m_2 \geq 0} \sum_{n_2 \geq 0} \frac{1}{(\alpha + vm_2 + wn_2)^\sigma} = O(1). \end{aligned}$$

Next we consider the order of V_1 . The range of n_2 satisfying the inequalities $\alpha + vm_1 + wn_1 < \alpha + vm_2 + wn_2 < 2(\alpha + vm_1 + wn_1)$ of the condition on the sum V_1 is

$$\frac{v}{w}(m_1 - m_2) + n_1 < n_2 < \frac{\alpha}{w} + \frac{v}{w}(2m_1 - m_2) + 2n_1.$$

Let $\varepsilon = \varepsilon(m_1, m_2, n_1), \delta = \delta(m_1, m_2, n_1)$ be the quantities satisfying $0 \leq \varepsilon, \delta < 1$ and $(v/w)(m_1 - m_2) + n_1 + \varepsilon \in \mathbf{Z}$ and $\alpha/w + (v/w)(2m_1 - m_2) + 2n_1 - \delta \in \mathbf{Z}$. Then $K = \alpha/w + (v/w)m_1 + n_1 - \varepsilon - \delta$ is an integer, and n_2 can be rewritten as

$$n_2 = \frac{v}{w}(m_1 - m_2) + n_1 + \varepsilon + k \quad (\text{for some } k = 0, 1, 2, \dots, K).$$

Since $K \asymp \alpha + vm_1 + wn_1 \asymp 1 + m_1 + n_1$, we have

$$\log \frac{\alpha + vm_2 + wn_2}{\alpha + vm_1 + wn_1} = \log \left(1 + \frac{w\varepsilon + wk}{\alpha + vm_1 + wn_1} \right) \asymp \frac{w\varepsilon + wk}{\alpha + vm_1 + wn_1},$$

and hence

$$\begin{aligned} V_1 &\ll \sum_{m_1 \geq 0} \sum_{n_1 \geq 0} \sum_{0 \leq m_2 \ll K} \sum_{0 \leq k \leq K} \frac{1}{(\alpha + vm_1 + wn_1)^\sigma} \\ &\quad \times \frac{1}{(\alpha + vm_1 + wn_1 + wk + w\varepsilon)^\sigma} \times \frac{\alpha + vm_1 + wn_1}{wk + w\varepsilon} \\ &\ll \sum_{m_1 \geq 0} \sum_{n_1 \geq 0} \sum_{0 \leq m_2 \ll K} \frac{\log K}{(\alpha + vm_1 + wn_1)^{2\sigma-1}} \\ &\ll \sum_{m_1 \geq 0} \sum_{n_1 \geq 0} \frac{\log(\alpha + vm_1 + wn_1)}{(\alpha + vm_1 + wn_1)^{2\sigma-2}} \\ &\ll \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{\log(2 + m + n)}{(1 + m + n)^{2\sigma-2}} \ll 1, \end{aligned}$$

provided that $\sigma > 2$. Therefore the proof of Theorem 1 is complete. □

3. The approximation theorem

Let $\sigma_1 > 0$, $x \geq 1$ and $C > 1$. Suppose $s = \sigma + it \in \mathbf{C}$ with $\sigma \geq \sigma_1$ and $|t| \leq 2\pi x/C$. Then

$$\zeta(s) = \sum_{1 \leq n \leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma}) \quad (x \rightarrow \infty). \quad (5)$$

This asymptotic formula has been proved by Hardy and Littlewood (see Theorem 4.11 in Titchmarsh [8]). Here we prove an analogue of (5) for the case of the Barnes double zeta-functions as follows.

THEOREM 3. *Let $1 < \sigma_1 < \sigma_2$, $x \geq 1$ and $C > 1$. Suppose $s = \sigma + it \in \mathbf{C}$ with $\sigma_1 < \sigma < \sigma_2$ and $|t| \leq 2\pi x/C$. Then*

$$\begin{aligned} \zeta_2(s, \alpha; v, w) &= \sum_{0 \leq m \leq x} \sum_{0 \leq n \leq x} \frac{1}{(\alpha + vm + wn)^s} \\ &\quad + \frac{(\alpha + vx)^{2-s} + (\alpha + wx)^{2-s} - (\alpha + vx + wx)^{2-s}}{vw(s-1)(s-2)} + O(x^{1-\sigma}) \end{aligned} \quad (6)$$

as $x \rightarrow \infty$.

LEMMA 1 (Lemma 4.10 in [8]). *Let $f(\xi)$ be a real function with a continuous and steadily decreasing derivative $f'(\xi)$ in (a, b) , and let $f'(b) = c$, $f'(a) = d$. Let $g(\xi)$ be a real positive decreasing function with a continuous derivative $g'(\xi)$, satisfying that $|g'(\xi)|$ is steadily decreasing. Then*

$$\begin{aligned} \sum_{a < n \leq b} g(n)e^{2\pi i f(n)} &= \sum_{\substack{v \in \mathbf{Z} \\ c-\varepsilon < v < d+\varepsilon}} \int_a^b g(\xi)e^{2\pi i(f(\xi)-v\xi)} d\xi \\ &\quad + O(g(a) \log(d-c+2)) + O(|g'(a)|) \end{aligned} \quad (7)$$

for an arbitrary $\varepsilon \in (0, 1)$.

PROOF OF THEOREM 3. Let $N \in \mathbf{N}$ be sufficiently large. Then we have

$$\begin{aligned} &\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(\alpha + vm + wn)^s} \\ &= \left(\sum_{m=0}^N \sum_{n=0}^N + \sum_{m=0}^N \sum_{n=N+1}^{\infty} + \sum_{m=N+1}^{\infty} \sum_{n=0}^N + \sum_{m=N+1}^{\infty} \sum_{n=N+1}^{\infty} \right) \frac{1}{(\alpha + vm + wn)^s}. \end{aligned}$$

We denote the second, the third and the fourth term on the right-hand side by A_1 , A_2 and A_3 , respectively. By the Euler-Maclaurin summation formula (see Equation (2.1.2) in [8]), we

have for any $a, b \in \mathbf{Z}$ with $0 < a < b$,

$$\begin{aligned} \sum_{m=a+1}^b \frac{1}{(\alpha + vm + wn)^s} &= \frac{(\alpha + vb + wn)^{1-s} - (\alpha + va + wn)^{1-s}}{v(1-s)} \\ &\quad -vs \int_a^b \frac{x - [x] - 1/2}{(\alpha + vx + wn)^{s+1}} dx + \frac{1}{2} \{(\alpha + vb + wn)^{-s} - (\alpha + va + wn)^{-s}\}, \\ \sum_{n=a+1}^b \frac{1}{(\alpha + vm + wn)^s} &= \frac{(\alpha + vm + wb)^{1-s} - (\alpha + vm + wa)^{1-s}}{w(1-s)} \\ &\quad -ws \int_a^b \frac{x - [x] - 1/2}{(\alpha + vm + wx)^{s+1}} dx + \frac{1}{2} \{(\alpha + vm + wb)^{-s} - (\alpha + vn + wa)^{-s}\}. \end{aligned}$$

If we take $a = N$ and let $b \rightarrow \infty$, we have

$$\begin{aligned} \sum_{n=N+1}^{\infty} \frac{1}{(\alpha + vm + wn)^s} &= \frac{1}{w(s-1)} \cdot \frac{1}{(\alpha + vm + wN)^{s-1}} \\ &\quad -ws \int_N^{\infty} \frac{x - [x] - 1/2}{(\alpha + vm + wx)^{s+1}} dx - \frac{1}{2} \cdot \frac{1}{(\alpha + vm + wN)^s} \\ &= \frac{1}{w(s-1)} \cdot \frac{1}{(\alpha + vm + wN)^{s-1}} - \frac{1}{2} \cdot \frac{1}{(\alpha + vm + wN)^s} + O(N^{-\sigma}), \end{aligned}$$

for $\sigma > 1$, uniformly in $m = 0, 1, \dots$. Therefore we have

$$\begin{aligned} A_1 &= \sum_{m=0}^N \sum_{n=N+1}^{\infty} \frac{1}{(\alpha + vm + wn)^s} \\ &= \sum_{m=0}^N \left\{ \frac{1}{w(s-1)} \cdot \frac{1}{(\alpha + vm + wN)^{s-1}} - \frac{1}{2} \cdot \frac{1}{(\alpha + vm + wN)^s} + O(N^{-\sigma}) \right\} \\ &= \frac{1}{w(s-1)} \sum_{m=1}^N \frac{1}{(\alpha + vm + wN)^{s-1}} - \frac{1}{2} \sum_{m=1}^N \frac{1}{(\alpha + vm + wN)^s} \\ &\quad + \frac{1}{w(s-1)} \cdot \frac{1}{(\alpha + wN)^{s-1}} - \frac{1}{2} \cdot \frac{1}{(\alpha + wN)^s} + O(N^{1-\sigma}). \end{aligned}$$

Applying again the formula (2.1.2) in [8] to the first term and the second term on the right-hand side of the above, we obtain

$$A_1 = \frac{1}{vw(s-1)(s-2)} \left\{ \frac{1}{(\alpha + wN)^{s-2}} - \frac{1}{(\alpha + vN + wN)^{s-2}} \right\}$$

$$\begin{aligned}
& -\frac{v}{w} \int_0^N \frac{x - [x] - 1/2}{(\alpha + vx + wN)^s} dx \\
& -\frac{1}{2w(s-1)} \left\{ \frac{1}{(\alpha + wN)^{s-1}} - \frac{1}{(\alpha + vN + wN)^{s-1}} \right\} \\
& -\frac{1}{2v(s-1)} \left\{ \frac{1}{(\alpha + wN)^{s-1}} - \frac{1}{(\alpha + vN + wN)^{s-1}} \right\} \\
& + \frac{vs}{2} \int_0^N \frac{x - [x] - 1/2}{(\alpha + vx + wN)^{s+1}} dx + \frac{1}{4} \left\{ \frac{1}{(\alpha + wN)^s} - \frac{1}{(\alpha + vN + wN)^s} \right\} \\
& + \frac{1}{w(s-1)} \cdot \frac{1}{(\alpha + wN)^{s-1}} - \frac{1}{2} \cdot \frac{1}{(\alpha + wN)^s} + O(N^{1-\sigma}) \\
& = \frac{(\alpha + wN)^{2-s} - (\alpha + vN + wN)^{2-s}}{vw(s-1)(s-2)} + O(N^{1-\sigma}).
\end{aligned}$$

Applying the same method to A_2 and A_3 , we obtain

$$A_2 = \frac{(\alpha + vN)^{2-s} - (\alpha + vN + wN)^{2-s}}{vw(s-1)(s-2)} + O(N^{1-\sigma}) \quad (\sigma > 1)$$

$$A_3 = \frac{(\alpha + vN + wN)^{2-s}}{vw(s-1)(s-2)} + O(N^{1-\sigma}) \quad (\sigma > 1).$$

Therefore we have

$$\begin{aligned}
\zeta_2(s, \alpha; v, w) &= \sum_{m=0}^N \sum_{n=0}^N \frac{1}{(\alpha + vm + wn)^s} \\
&+ \frac{(\alpha + vN)^{2-s} + (\alpha + wN)^{2-s} - (\alpha + vN + wN)^{2-s}}{vw(s-1)(s-2)} + O(N^{1-\sigma}) \quad (8)
\end{aligned}$$

for $\sigma > 1$. Next we consider the double sum on the right-hand side of (8). First we divide the sum as follows:

$$\begin{aligned}
& \sum_{m=0}^N \sum_{n=0}^N \frac{1}{(\alpha + vm + wn)^s} \\
&= \left(\sum_{m \leq x} \sum_{n \leq x} + \sum_{m \leq x} \sum_{x < n \leq N} + \sum_{n \leq x} \sum_{x < m \leq N} + \sum_{x < m \leq N} \sum_{x < n \leq N} \right) \frac{1}{(\alpha + vm + wn)^s}.
\end{aligned}$$

We denote the second, the third and the fourth term on the right-hand side by B_1 , B_2 and B_3 , respectively. Fix $m \in \mathbf{N}$, set

$$f(\xi) = \frac{t}{2\pi} \log(\alpha + vm + w\xi), \quad g(\xi) = (\alpha + vm + w\xi)^{-\sigma}$$

and take $(a, b) = (x, N)$ in Lemma 1. Then we have

$$(c, d) = \left(\frac{tw}{2\pi(\alpha + vm + wN)}, \frac{tw}{2\pi(\alpha + vm + wx)} \right).$$

We see that

$$|f'(x)| = \left| \frac{tw}{2\pi(\alpha + vm + wx)} \right| \leq \frac{1}{2\pi} \left| \frac{2\pi x}{C} \cdot \frac{w}{\alpha + vm + wx} \right| \leq \frac{1}{C} < 1.$$

When $\sigma > 0$, the function $g(\xi)$ is decreasing, and hence Lemma 1 can be applied. For sufficiently large N , we can take ε such that $c - \varepsilon < 0 < d + \varepsilon < 1$, by which only the term with $v = 0$ appears in the sum on the right-hand side of (7). We obtain from (7) that

$$\sum_{x < n \leq N} \frac{e^{it \log(\alpha + vm + wn)}}{(\alpha + vm + wn)^{-\sigma}} = \int_x^N (\alpha + vm + w\xi)^{-\sigma + it} d\xi + O((m + x)^{-\sigma}).$$

Taking complex conjugates on the both sides, we have

$$\begin{aligned} \sum_{x < n \leq N} \frac{1}{(\alpha + vm + wn)^s} &= \int_x^N (\alpha + vm + w\xi)^{-s} d\xi + O((m + x)^{-\sigma}) \\ &= \frac{(\alpha + vm + wN)^{1-s} - (\alpha + vm + wx)^{1-s}}{w(1-s)} + O((m + x)^{-\sigma}). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} B_1 &= \sum_{m \leq x} \sum_{x < n \leq N} \frac{1}{(\alpha + vm + wn)^s} \\ &= \sum_{m \leq x} \left\{ \frac{(\alpha + vm + wN)^{1-s} - (\alpha + vm + wx)^{1-s}}{w(1-s)} + O((m + x)^{-\sigma}) \right\} \\ &= \frac{1}{w(1-s)} \left\{ \sum_{m \leq x} \frac{1}{(\alpha + vm + wN)^{s-1}} - \sum_{m \leq x} \frac{1}{(\alpha + vm + wx)^{s-1}} \right\} + O(x^{1-\sigma}). \end{aligned}$$

We denote the first and the second term on the right-hand side by $(1/w(1-s))(B_{11} - B_{12})$, and apply Lemma 1 for B_{11} and B_{12} . For B_{11} set

$$f(\xi) = \frac{t}{2\pi} \log(\alpha + v\xi + wN), \quad g(\xi) = (\alpha + v\xi + wN)^{1-\sigma}$$

and on taking $(a, b) = (0, x)$ in Lemma 1. We can treat B_{12} similarly, where Lemma 1 is applied on replacing the variable ξ by η , on setting

$$f(\eta) = \frac{t}{2\pi} \log(\alpha + v\eta + wx), \quad g(x) = (\alpha + v\eta + wx)^{1-\sigma}$$

and $(a, b) = (0, x)$. Then we have

$$B_{11} = \frac{(\alpha + vx + wN)^{2-s} - (\alpha + wN)^{2-s}}{v(2-s)} + O(N^{1-\sigma}),$$

$$B_{12} = \frac{(\alpha + vx + wx)^{2-s} - (\alpha + wx)^{2-s}}{v(2-s)} + O(x^{1-\sigma}).$$

Therefore, we obtain

$$B_1 = \frac{(\alpha + vx + wN)^{2-s} - (\alpha + vx + wx)^{2-s} - (\alpha + wN)^{2-s} + (\alpha + wx)^{2-s}}{vw(s-1)(s-2)} + O(N^{1-\sigma}) + O(x^{1-\sigma}).$$

By the argument similar to the treatment of B_1 , we obtain

$$B_2 = \frac{(\alpha + vN + wx)^{2-s} - (\alpha + vx + wx)^{2-s} - (\alpha + vN)^{2-s} + (\alpha + vx)^{2-s}}{vw(s-1)(s-2)} + O(N^{1-\sigma}) + O(x^{1-\sigma})$$

and

$$B_3 = \frac{(\alpha + vN + wN)^{2-s} - (\alpha + vx + wN)^{2-s} - (\alpha + vN + wx)^{2-s} + (\alpha + vx + wx)^{2-s}}{vw(s-1)(s-2)} + O(N^{1-\sigma}) + O(x^{1-\sigma}).$$

Summing up the results above, we obtain

$$\begin{aligned} & \sum_{m=0}^N \sum_{n=0}^N \frac{1}{(\alpha + vm + wn)^s} \\ &= \sum_{m \leq x} \sum_{n \leq x} \frac{1}{(\alpha + vm + wn)^s} + \frac{(\alpha + vx)^{2-s} + (\alpha + wx)^{2-s} - (\alpha + vx + wx)^{2-s}}{vw(s-1)(s-2)} \\ & \quad - \frac{(\alpha + vN)^{2-s} + (\alpha + wN)^{2-s} - (\alpha + vN + wN)^{2-s}}{vw(s-1)(s-2)} + O(x^{1-\sigma}) + O(N^{1-\sigma}), \end{aligned}$$

and by (8), we conclude that

$$\begin{aligned} & \zeta_2(s, \alpha; v, w) \\ &= \sum_{m \leq x} \sum_{n \leq x} \frac{1}{(\alpha + vm + wn)^s} + \frac{(\alpha + vx)^{2-s} + (\alpha + wx)^{2-s} - (\alpha + vx + wx)^{2-s}}{vw(s-1)(s-2)} \\ & \quad + O(x^{1-\sigma}) + O(N^{1-\sigma}) \end{aligned}$$

in the region $\sigma > 1$. Letting $N \rightarrow \infty$, we obtain the proof of Theorem 3. \square

4. Proof of Theorem 2

In this section, we prove Theorem 2 from Theorem 3.

PROOF OF THEOREM 2. Setting $C = 2\pi$ and $x = t$ in (6), we easily see that the second term on the right-hand side is $O(t^{-\sigma})$, hence we have

$$\zeta_2(s, \alpha; v, w) = \sum_{m \leq t} \sum_{n \leq t} \frac{1}{(\alpha + vm + wn)^s} + O(t^{1-\sigma}). \tag{9}$$

We denote the first term on the right-hand side by $\Sigma(s)$. Then

$$\begin{aligned} & \int_1^T |\Sigma(s)|^2 dt \\ &= \int_1^T \sum_{m_1 \leq t} \sum_{n_1 \leq t} \frac{1}{(\alpha + vm_1 + wn_1)^{\sigma+it}} \sum_{m_2 \leq t} \sum_{n_2 \leq t} \frac{1}{(\alpha + vm_2 + wn_2)^{\sigma-it}} dt. \end{aligned}$$

Now we change the order of summation and integration. First we note that $1 \leq m_1, n_1, m_2, n_2 \leq T$. Let us fix one such (m_1, n_1, m_2, n_2) . Then from the condition $m_1 \leq t, n_1 \leq t, m_2 \leq t, n_2 \leq t$, we find that the range of t is $M = \max\{m_1, n_1, m_2, n_2\} \leq t \leq T$. Therefore

$$\begin{aligned} & \int_1^T |\Sigma(s)|^2 dt \\ &= \sum_{m_1, n_1 \leq T} \frac{1}{(\alpha + vm_1 + wn_1)^\sigma} \sum_{m_2, n_2 \leq T} \frac{1}{(\alpha + vm_2 + wn_2)^\sigma} \\ & \quad \times \int_M^T \left(\frac{\alpha + vm_1 + wn_1}{\alpha + vm_2 + wn_2} \right)^{it} dt \\ &= \sum_{\substack{0 \leq m_1, n_1, m_2, n_2 \leq T \\ vm_1 + wn_1 = vm_2 + wn_2}} \frac{1}{(\alpha + vm_1 + wn_1)^\sigma (\alpha + vm_2 + wn_2)^\sigma} \times (T - M) \\ & \quad + \sum_{\substack{m_1, n_1, m_2, n_2 \geq 0 \\ vm_1 + wn_1 \neq vm_2 + wn_2}} \frac{1}{(\alpha + vm_1 + wn_1)^\sigma (\alpha + vm_2 + wn_2)^\sigma} \\ & \quad \times \frac{e^{iT \log\{(\alpha + vm_2 + wn_2)/(\alpha + vm_1 + wn_1)\}} - e^{iM \log\{(\alpha + vm_2 + wn_2)/(\alpha + vm_1 + wn_1)\}}}{i \log\{(\alpha + vm_2 + wn_2)/(\alpha + vm_1 + wn_1)\}}. \end{aligned}$$

We denote the first and the second term on the right-hand side by S_1 and S_2 respectively. As for S_1 , we have

$$S_1 = T \{ \zeta_2^{[2]}(\sigma, \sigma, \alpha, v, w) - (U_1 + U_2 + U_3 + U_4) \}$$

$$- \sum_{\substack{0 \leq m_1, n_1, m_2, n_2 \leq T \\ vm_1 + wn_1 = vm_2 + wn_2}} \frac{M(m_1, n_1, m_2, n_2)}{(\alpha + vm_1 + wn_1)^\sigma (\alpha + vm_2 + wn_2)^\sigma},$$

where

$$U_1 = \left(\begin{array}{cc} \sum_{\substack{m_1 > T \\ 0 \leq n_1, m_2, n_2 \leq T \\ vm_1 + wn_1 = vm_2 + wn_2}} & + \sum_{\substack{n_1 > T \\ 0 \leq m_1, m_2, n_2 \leq T \\ vm_1 + wn_1 = vm_2 + wn_2}} \\ \\ + \sum_{\substack{m_2 > T \\ 0 \leq m_1, n_1, n_2 \leq T \\ vm_1 + wn_1 = vm_2 + wn_2}} & + \sum_{\substack{n_2 > T \\ 0 \leq m_1, n_1, m_2 \leq T \\ vm_1 + wn_1 = vm_2 + wn_2}} \end{array} \right) \frac{1}{(\alpha + vm_1 + wn_1)^\sigma (\alpha + vm_2 + wn_2)^\sigma},$$

$$U_2 = \left(\begin{array}{cc} \sum_{\substack{m_1, m_2 > T \\ 0 \leq n_1, n_2 \leq T \\ vm_1 + wn_1 = vm_2 + wn_2}} & + \sum_{\substack{n_1, n_2 > T \\ 0 \leq m_1, m_2 \leq T \\ vm_1 + wn_1 = vm_2 + wn_2}} \end{array} \right) \frac{1}{(\alpha + vm_1 + wn_1)^\sigma (\alpha + vm_2 + wn_2)^\sigma},$$

$$U_3 = \left(\begin{array}{cc} \sum_{\substack{m_1, n_2 > T \\ 0 \leq n_1, m_2 \leq T \\ vm_1 + wn_1 = vm_2 + wn_2}} & + \sum_{\substack{n_1, m_2 > T \\ 0 \leq m_1, n_2 \leq T \\ vm_1 + wn_1 = vm_2 + wn_2}} \end{array} \right) \frac{1}{(\alpha + vm_1 + wn_1)^\sigma (\alpha + vm_2 + wn_2)^\sigma},$$

$$U_4 = \left(\begin{array}{cc} \sum_{\substack{m_1, m_2, n_1 > T \\ 0 \leq n_2 \leq T \\ vm_1 + wn_1 = vm_2 + wn_2}} & + \sum_{\substack{m_1, m_2, n_2 > T \\ 0 \leq n_1 \leq T \\ vm_1 + wn_1 = vm_2 + wn_2}} \\ \\ + \sum_{\substack{m_1, n_1, n_2 > T \\ 0 \leq m_2 \leq T \\ vm_1 + wn_1 = vm_2 + wn_2}} & + \sum_{\substack{m_2, n_1, n_2 > T \\ 0 \leq m_1 \leq T \\ vm_1 + wn_1 = vm_2 + wn_2}} \end{array} \right) \frac{1}{(\alpha + vm_1 + wn_1)^\sigma (\alpha + vm_2 + wn_2)^\sigma}.$$

We can estimate U_1 as follows. Since $\alpha + vm + wn \asymp 1 + m + n$ we have

$$U_1 \ll \sum_{\substack{k > T \\ 0 \leq l, m, n \leq T \\ k+l > m+n}} \frac{1}{(1+k+l)^\sigma (1+m+n)^\sigma}.$$

Setting $j = k + l$, since $T + 1 < j \asymp m + n \leq 2T \ll T$ and $m \ll j$, we obtain

$$\begin{aligned} U_1 &\ll \sum_{T+1 \leq j \ll T} \sum_{T < k \leq j} \sum_{0 \leq m \ll j} \frac{1}{(1+j)^{2\sigma}} \\ &\ll \sum_{T+1 \leq j \ll T} \sum_{T < j \leq j} \frac{1}{(1+j)^{2\sigma-1}} = \sum_{T+1 \leq j \ll T} \frac{j-T}{(1+j)^{2\sigma-1}} \\ &\ll \sum_{T+1 \leq j \ll T} \frac{1}{(1+j)^{2\sigma-2}} \asymp T^{3-2\sigma} \quad \left(\sigma \geq \frac{3}{2}\right). \end{aligned}$$

Similarly we obtain $U_2, U_3, U_4 \ll T^{3-2\sigma}$. The sum involving $M(m_1, n_1, m_2, n_2)$ in S_1 is estimated, since $M(k, l, m, n) \ll k + l + m + n$ in this case, as

$$\begin{aligned} &\ll \sum_{\substack{0 \leq k, l, m, n \leq T \\ k+l > m+n}} \frac{k+l+m+n}{(1+k+l)^\sigma (1+m+n)^\sigma} \ll \sum_{0 \leq j \leq 2T} \sum_{0 \leq k \leq j} \sum_{0 \leq m \ll j} \frac{1}{(1+j)^{2\sigma-1}} \\ &\ll \sum_{0 \leq j \leq 2T} \frac{1}{(1+j)^{2\sigma-3}} \ll \begin{cases} T^{4-2\sigma} & (3/2 \leq \sigma < 2), \\ \log T & (\sigma = 2). \end{cases} \end{aligned}$$

Therefore, we have

$$S_1 = \zeta_2^{[2]}(\sigma, \sigma, \alpha, v, w)T + \begin{cases} O(T^{4-2\sigma}) & (3/2 \leq \sigma < 2), \\ O(\log T) & (\sigma = 2). \end{cases} \tag{10}$$

Next, as for S_2 , we have

$$\begin{aligned} S_2 &\ll \sum_{\substack{0 \leq m_1, n_1, m_2, n_2 \leq T \\ vm_1 + wn_1 < vm_2 + wn_2}} \frac{1}{(\alpha + vm_1 + wn_1)^\sigma (\alpha + vm_2 + wn_2)^\sigma} \\ &\quad \times \frac{1}{\log\{(\alpha + vm_2 + wn_2)/(\alpha + vm_1 + wn_1)\}} \\ &= \left(\sum_{\substack{0 \leq m_1, n_1, m_2, n_2 \leq T \\ \alpha + vm_1 + wn_1 < \alpha + vm_2 + wn_2 < 2(\alpha + vm_1 + wn_1)}} + \sum_{\substack{0 \leq m_1, n_1, m_2, n_2 \leq T \\ \alpha + vm_2 + wn_2 \geq 2(\alpha + vm_1 + wn_1)}} \right) \\ &\quad \frac{1}{(\alpha + vm_1 + wn_1)^\sigma (\alpha + vm_2 + wn_2)^\sigma \log\{(\alpha + vm_2 + wn_2)/(\alpha + vm_1 + wn_1)\}}. \end{aligned}$$

We denote the first and the second term on the right-hand side by W_1 and W_2 , respectively.

As for W_2 , we have

$$\begin{aligned} W_2 &\ll \sum_{\substack{0 \leq m_1, n_1, m_2, n_2 \leq T \\ \alpha + vm_2 + wn_2 \geq 2(\alpha + vm_1 + wn_1)}} \frac{1}{(\alpha + vm_1 + wn_1)^\sigma (\alpha + vm_2 + wn_2)^\sigma} \\ &\ll \left\{ \sum_{0 \leq m, n \leq T} \frac{1}{(1 + m + n)^\sigma} \right\}^2 \\ &= \left\{ \sum_{0 \leq j \leq 2T} \sum_{m=0}^j \frac{1}{(1 + j)^\sigma} \right\}^2 = \left\{ \sum_{0 \leq j \leq 2T} \frac{1}{(1 + j)^{\sigma-1}} \right\}^2 \\ &\ll \begin{cases} T^{4-2\sigma} & (1 < \sigma < 2), \\ (\log T)^2 & (\sigma = 2). \end{cases} \end{aligned}$$

Next we consider the order of W_1 . The range of n_2 in the inequalities $\alpha + vm_1 + wn_1 < \alpha + vm_2 + wn_2 < 2(\alpha + vm_1 + wn_1)$ of the summation condition on W_1 is

$$\frac{v}{w}(m_1 - m_2) + n_1 < n_2 < \min \left\{ \frac{\alpha}{w} + \frac{v}{w}(2m_1 - m_2) + 2n_1, [T] + 1 \right\}.$$

Let $\varepsilon = \varepsilon(m_1, m_2, n_1)$, $\delta = \delta(m_1, m_2, n_1)$ be constants satisfying $0 \leq \varepsilon, \delta < 1$ and $(v/w)(m_1 - m_2) + n_1 + \varepsilon \in \mathbf{Z}$ and $\alpha/w + (v/w)(2m_1 - m_2) + 2n_1 - \delta \in \mathbf{Z}$. Then

$$K = \min \left\{ \frac{\alpha}{w} + \frac{v}{w} \cdot m_1 + n_1 - \varepsilon - \delta, [T] - \frac{v}{w}(m_1 - m_2) - n_1 - \varepsilon \right\}$$

is an integer, and n_2 can be rewritten as

$$n_2 = \frac{v}{w}(m_1 - m_2) + n_1 + \varepsilon + k \quad (\text{for some } k = 0, 1, 2, \dots, K).$$

Since

$$\log \frac{\alpha + vm_2 + wn_2}{\alpha + vm_1 + wn_1} = \log \left(1 + \frac{wk + w\varepsilon}{\alpha + vm_1 + wn_1} \right) \asymp \frac{wk + w\varepsilon}{\alpha + vm_1 + wn_1},$$

we obtain

$$\begin{aligned} W_1 &\ll \sum_{0 \leq m_1 \leq T} \sum_{0 \leq n_1 \leq T} \sum_{0 \leq m_2 \ll K} \sum_{0 \leq k \leq K} \frac{1}{(\alpha + vm_1 + wn_1)^\sigma} \\ &\quad \times \frac{1}{(\alpha + vm_1 + wn_1 + wk + w\varepsilon)^\sigma} \times \frac{\alpha + vm_1 + wn_1}{wk + w\varepsilon} \\ &\ll \sum_{0 \leq m_1 \leq T} \sum_{0 \leq n_1 \leq T} \sum_{0 \leq m_2 \ll K} \frac{\log K}{(\alpha + vm_1 + wn_1)^{2\sigma-1}} \\ &\ll \sum_{0 \leq m_1 \leq T} \sum_{0 \leq n_1 \leq T} \frac{\log(\alpha + vm_1 + wn_1)}{(\alpha + vm_1 + wn_1)^{2\sigma-2}} \end{aligned}$$

$$\begin{aligned} &\ll \sum_{0 \leq m \leq T} \sum_{0 \leq n \leq T} \frac{\log(2+m+n)}{(1+m+n)^{2\sigma-2}} \\ &\ll \int_0^T \int_0^T \frac{\log(2+x+y)}{(1+x+y)^{2\sigma-2}} dx dy \\ &= \begin{cases} O(T^{4-2\sigma} \log T) & (1 < \sigma < 3/2, 3/2 < \sigma < 2), \\ O(T(\log T)^2) & (\sigma = 3/2), \\ O((\log T)^2) & (\sigma = 2). \end{cases} \end{aligned}$$

Then, we have

$$\begin{aligned} S_2 &= \begin{cases} O(T^{4-2\sigma}) & (1 < \sigma < 2) \\ O((\log T)^2) & (\sigma = 2) \end{cases} + \begin{cases} O(T^{4-2\sigma} \log T) & (1 < \sigma < 3/2, 3/2 < \sigma < 2) \\ O(T(\log T)^2) & (\sigma = 3/2) \\ O((\log T)^2) & (\sigma = 2) \end{cases} \\ &= \begin{cases} O(T^{4-2\sigma} \log T) & (1 < \sigma < 3/2, 3/2 < \sigma < 2), \\ O(T(\log T)^2) & (\sigma = 3/2), \\ O((\log T)^2) & (\sigma = 2). \end{cases} \end{aligned}$$

By (10), we have

$$\begin{aligned} &\int_1^T |\Sigma(s)|^2 dt \\ &= \zeta_2^{[2]}(\sigma, \sigma; \alpha, v, w) T + \begin{cases} O(T^{4-2\sigma} \log T) & (1 < \sigma < 3/2, 3/2 < \sigma < 2), \\ O(T(\log T)^2) & (\sigma = 3/2), \\ O((\log T)^2) & (\sigma = 2). \end{cases} \end{aligned}$$

Furthermore, we obtain from (9) that

$$\begin{aligned} &\int_1^T |\zeta_2(\sigma + it, \alpha; v, w)|^2 dt \\ &= \int_1^T |\Sigma(s) + O(t^{1-\sigma})|^2 dt \\ &= \int_1^T |\Sigma(s)|^2 dt + O\left(\int_1^T |\Sigma(s)| t^{1-\sigma} dt\right) + O\left(\int_1^T t^{2-2\sigma} dt\right). \end{aligned}$$

We see that the third term on the right-hand side is estimated as

$$\begin{cases} O(T^{3-2\sigma}) & (1 < \sigma < 3/2), \\ O(\log T) & (\sigma = 3/2), \\ O(1) & (3/2 < \sigma \leq 2). \end{cases} \tag{11}$$

Also, using the Cauchy-Schwarz inequality for the second term on the right-hand side, we see that

$$\begin{aligned} \int_1^T |\Sigma(s)|t^{1-\sigma} dt &\leq \left(\int_1^T |\Sigma(s)|^2 dt \right)^{1/2} \left(\int_1^T t^{2-2\sigma} dt \right)^{1/2} \\ &= \begin{cases} O(T^{7/2-2\sigma}(\log T)^{1/2}) & (1 < \sigma < 3/2) , \\ O(T^{1/2}(\log T)^{3/2}) & (\sigma = 3/2) , \\ O(T^{1/2}) & (3/2 < \sigma \leq 2) . \end{cases} \end{aligned} \quad (12)$$

Therefore, since (11) and (12) we have

$$\int_1^T |\zeta_2(s, \alpha; v, w)|^2 dt = \zeta_2^{[2]}(\sigma, \alpha; v, w)T + \begin{cases} O(T^{4-2\sigma} \log T) & (3/2 < \sigma \leq 7/4) , \\ O(T^{1/2}) & (7/4 < \sigma \leq 2) , \end{cases}$$

and hence the proof of Theorem 2 is complete. \square

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