

Weak-type Estimates in Morrey Spaces for Maximal Commutator and Commutator of Maximal Function

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Abstract. In this paper it is shown that the Hardy-Littlewood maximal operator M is not bounded on Zygmund-Morrey space $\mathcal{M}_{L(\log L),\lambda}$, $0 < \lambda < n$, but M is still bounded on $\mathcal{M}_{L(\log L),\lambda}$ for radially decreasing functions. The boundedness of the iterated maximal operator M^2 from $\mathcal{M}_{L(\log L),\lambda}$ to weak Zygmund-Morrey space $\mathcal{WM}_{L(\log L),\lambda}$ is proved. The class of functions for which the maximal commutator C_b is bounded from $\mathcal{M}_{L(\log L),\lambda}$ to $\mathcal{WM}_{L(\log L),\lambda}$ are characterized. It is proved that the commutator of the Hardy-Littlewood maximal operator M with function $b \in \text{BMO}(\mathbf{R}^n)$ such that $b^- \in L_\infty(\mathbf{R}^n)$ is bounded from $\mathcal{M}_{L(\log L),\lambda}$ to $\mathcal{WM}_{L(\log L),\lambda}$. New pointwise characterizations of $M_\alpha M$ by means of norm of Hardy-Littlewood maximal function in classical Morrey spaces are given.

1. Introduction

Given a locally integrable function f on \mathbf{R}^n and $0 \leq \alpha < n$, the fractional maximal function $M_\alpha f$ of f is defined by

$$M_\alpha f(x) := \sup_{Q \ni x} |Q|^{\frac{\alpha-n}{n}} \int_Q |f(y)| dy, \quad (x \in \mathbf{R}^n),$$

where the supremum is taken over all cubes Q containing x . The operator $M_\alpha : f \rightarrow M_\alpha f$ is called the fractional maximal operator. $M := M_0$ is the classical Hardy-Littlewood maximal operator.

The study of maximal operators is one of the most important topics in harmonic analysis. These significant non-linear operators, whose behavior are very informative in particular in

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differentiation theory, provided the understanding and the inspiration for the development of the general class of singular and potential operators (see, for instance, [8, 12–14, 31–33]).

Let $f \in L_1^{\text{loc}}(\mathbf{R}^n)$. Then f is said to be in $\text{BMO}(\mathbf{R}^n)$ if the seminorm given by

$$\|f\|_* := \sup_Q \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy$$

is finite.

DEFINITION 1.1. Given a measurable function b the maximal commutator is defined by

$$C_b(f)(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |b(x) - b(y)| |f(y)| dy$$

for all $x \in \mathbf{R}^n$.

This operator plays an important role in the study of commutators of singular integral operators with BMO symbols (see, for instance, [9, 22, 28, 29]). The maximal operator C_b has been studied intensively and there exist plenty of results about it. Garcia-Cuerva et al. [9] proved the following statement.

THEOREM 1.2. *Let $1 < p < \infty$. C_b is bounded on $L_p(\mathbf{R}^n)$ if and only if $b \in \text{BMO}(\mathbf{R}^n)$.*

DEFINITION 1.3. Given a measurable function b the commutator of the Hardy-Littlewood maximal operator M and b is defined by

$$[M, b]f(x) := M(bf)(x) - b(x)Mf(x)$$

for all $x \in \mathbf{R}^n$.

The operator $[M, b]$ was studied by Milman et al. in [23] and [2]. This operator arises, for example, when one tries to give a meaning to the product of a function in H^1 and a function in BMO (which may not be a locally integrable function, see, for instance, [5]). Using real interpolation techniques, in [23], Milman and Schonbek proved the L_p -boundedness of the operator $[M, b]$. Bastero, Milman and Ruiz proved the next theorem in [2].

THEOREM 1.4. *Let $1 < p < \infty$. Then the following assertions are equivalent:*

- (i) $[M, b]$ is bounded on $L_p(\mathbf{R}^n)$;
- (ii) $b \in \text{BMO}(\mathbf{R}^n)$ and $b^- \in L_\infty(\mathbf{R}^n)$.¹

The operators C_b and $[M, b]$ enjoy weak-type $L(1 + \log^+ L)$ estimate.

THEOREM 1.5 ([1, Theorem 1.5], see also [15] and [16]). *The following assertions are equivalent:*

¹Denote by $b^+(x) = \max\{b(x), 0\}$ and $b^-(x) = -\min\{b(x), 0\}$, consequently $b = b^+ - b^-$ and $|b| = b^+ + b^-$.

(i) *There exists a positive constant c such that for each $\lambda > 0$, inequality*

$$(1.1) \quad |\{x \in \mathbf{R}^n : C_b(f)(x) > \lambda\}| \leq c \int_{\mathbf{R}^n} \frac{|f(x)|}{\lambda} \left(1 + \log^+ \left(\frac{|f(x)|}{\lambda}\right)\right) dx$$

holds for all $f \in L(1 + \log^+ L)(\mathbf{R}^n)$.

(ii) $b \in \text{BMO}(\mathbf{R}^n)$.

THEOREM 1.6 ([1, Theorem 1.6]). *Let $b \in \text{BMO}(\mathbf{R}^n)$ such that $b^- \in L_\infty(\mathbf{R}^n)$. Then there exists a positive constant c such that*

$$(1.2) \quad |\{x \in \mathbf{R}^n : |[M, b]f(x)| > \lambda\}| \leq cc_0 (1 + \log^+ c_0) \int_{\mathbf{R}^n} \frac{|f(x)|}{\lambda} \left(1 + \log^+ \left(\frac{|f(x)|}{\lambda}\right)\right) dx,$$

for all $f \in L(1 + \log^+ L)(\mathbf{R}^n)$ and $\lambda > 0$, where $c_0 = \|b^+\|_ + \|b^-\|_\infty$.*

Operators C_b and $[M, b]$ essentially differ from each other. For example, C_b is a positive and sublinear operator, but $[M, b]$ is neither positive nor sublinear. However, if b satisfies some additional conditions, then operator C_b controls $[M, b]$.

LEMMA 1.7 ([1, Lemma 3.1 and 3.2]). *Let b be any non-negative locally integrable function. Then*

$$(1.3) \quad |[M, b]f(x)| \leq C_b(f)(x) \quad (x \in \mathbf{R}^n)$$

holds for all $f \in L_1^{\text{loc}}(\mathbf{R}^n)$.

If b is any locally integrable function on \mathbf{R}^n , then

$$(1.4) \quad |[M, b]f(x)| \leq C_b(f)(x) + 2b^-(x)Mf(x) \quad (x \in \mathbf{R}^n)$$

holds for all $f \in L_1^{\text{loc}}(\mathbf{R}^n)$.

We recall the following statement from [1].

THEOREM 1.8 ([1, Theorem 1.13]). *Let $b \in \text{BMO}(\mathbf{R}^n)$. Suppose that X is a Banach space of measurable functions defined on \mathbf{R}^n . Moreover, assume that X satisfies the lattice property, that is,*

$$0 \leq g \leq f \quad \Rightarrow \quad \|g\|_X \lesssim \|f\|_X.$$

Assume that M is bounded on X . Then the operator C_b is bounded on X , and the inequality

$$\|C_b f\|_X \leq c \|b\|_* \|f\|_X$$

holds with constant c independent of f .

Moreover, if $b^- \in L_\infty(\mathbf{R}^n)$, then the operator $[M, b]$ is bounded on X , and the inequality

$$\|[M, b]f\|_X \leq c(\|b^+\|_* + \|b^-\|_\infty) \|f\|_X$$

holds with constant c independent of f .

The proof of previous theorem is based on the following inequalities.

THEOREM 1.9 ([1, Corollary 1.11 and 1.12]). *Let $b \in \text{BMO}(\mathbf{R}^n)$. Then, there exists a positive constant c such that the inequality*

$$(1.5) \quad C_b(f)(x) \leq c \|b\|_* M^2 f(x) \quad (x \in \mathbf{R}^n)$$

holds for all $f \in L_1^{\text{loc}}(\mathbf{R}^n)$.

Moreover, if $b^- \in L_\infty(\mathbf{R}^n)$, then, there exists a positive constant c such that the inequality

$$(1.6) \quad |[M, b]f(x)| \leq c (\|b^+\|_* + \|b^-\|_\infty) M^2 f(x) \quad (x \in \mathbf{R}^n)$$

holds for all $f \in L_1^{\text{loc}}(\mathbf{R}^n)$.

The classical Morrey spaces $\mathcal{M}_{p,\lambda} \equiv \mathcal{M}_{p,\lambda}(\mathbf{R}^n)$, were introduced by C. Morrey in [24] in order to study regularity questions which appear in the Calculus of Variations, and defined as follows: for $0 \leq \lambda \leq n$ and $1 \leq p \leq \infty$,

$$\mathcal{M}_{p,\lambda} := \left\{ f \in L_p^{\text{loc}}(\mathbf{R}^n) : \|f\|_{\mathcal{M}_{p,\lambda}} := \sup_{x \in \mathbf{R}^n, r > 0} r^{\frac{\lambda-n}{p}} \|f\|_{L_p(B(x,r))} < \infty \right\},$$

where $B(x, r)$ is the open ball centered at x of radius r .

Note that $\mathcal{M}_{p,0}(\mathbf{R}^n) = L_\infty(\mathbf{R}^n)$ and $\mathcal{M}_{p,n}(\mathbf{R}^n) = L_p(\mathbf{R}^n)$.

These spaces describe local regularity more precisely than Lebesgue spaces and appeared to be quite useful in the study of the local behavior of solutions to partial differential equations, a priori estimates and other topics in PDE (cf. [10]).

The boundedness of the Hardy-Littlewood maximal operator M in Morrey spaces $\mathcal{M}_{p,\lambda}$ was proved by F. Chiarenza and M. Frasca in [7]: It was shown that Mf is a.e. finite if $f \in \mathcal{M}_{p,\lambda}$ and an estimate

$$(1.7) \quad \|Mf\|_{\mathcal{M}_{p,\lambda}} \leq c \|f\|_{\mathcal{M}_{p,\lambda}}$$

holds if $1 < p < \infty$ and $0 < \lambda < n$, and a weak type estimate (1.7) replaces for $p = 1$, that is, the inequality

$$(1.8) \quad t |\{Mf > t\} \cap B(x, r)| \leq cr^{n-\lambda} \|f\|_{\mathcal{M}_{1,\lambda}}$$

holds with constant c independent of x, r, t and f .

In [11], it is proved that the Hardy-Littlewood maximal operator M is bounded on $\mathcal{M}_{1,\lambda}$, $0 \leq \lambda < n$, for radially decreasing functions, that is, the inequality

$$(1.9) \quad \|Mf\|_{\mathcal{M}_{1,\lambda}} \lesssim \|f\|_{\mathcal{M}_{1,\lambda}}, \quad f \in \mathfrak{M}^{\text{rad},\downarrow}$$

holds with constant independent of f , and an example which shows that M is not bounded on $\mathcal{M}_{1,\lambda}$, $0 < \lambda < n$ is given.

Combining Theorem 1.9 with inequalities (1.7) and (1.9), it is easy to generalize Theorems 1.2 and 1.4 to Morrey spaces (see Theorems 3.1 and 3.3).

In this paper the Zygmund-Morrey and the weak Zygmund-Morrey spaces are defined. In order to investigate the boundedness of the maximal commutator C_b and the commutator of maximal function $[M, b]$ on Zygmund-Morrey spaces we start to study the boundedness properties of the Hardy-Littlewood maximal operator on these spaces. It is shown that the Hardy-Littlewood maximal operator M is not bounded on Zygmund-Morrey spaces $\mathcal{M}_{L(\log L), \lambda}$, but M is still bounded on $\mathcal{M}_{L(\log L), \lambda}$ for radially decreasing functions. The boundedness of the iterated maximal operator M^2 from Zygmund-Morrey spaces $\mathcal{M}_{L(\log L), \lambda}$ to weak Zygmund-Morrey spaces $\mathcal{WM}_{L(\log L), \lambda}$ is proved.

Our main results about weak-type estimates for maximal commutator and commutator of maximal function in Zygmund-Morrey spaces are following.

THEOREM 1.10. *Let $0 < \lambda < n$. The following assertions are equivalent:*

- (i) $b \in \text{BMO}(\mathbf{R}^n)$;
- (ii) *The operator C_b is bounded from $\mathcal{M}_{L(1+\log^+ L), \lambda}$ to $\mathcal{WM}_{L(1+\log^+ L), \lambda}$.*

THEOREM 1.11. *Let $0 < \lambda < n$. Assume that $b \in \text{BMO}(\mathbf{R}^n)$. Then the operator C_b is bounded on $\mathcal{M}_{L(1+\log^+ L), \lambda}$ for radially decreasing functions.*

THEOREM 1.12. *Let $0 < \lambda < n$ and b is in $\text{BMO}(\mathbf{R}^n)$ such that $b^- \in L_\infty(\mathbf{R}^n)$. Then the operator $[M, b]$ is bounded from $\mathcal{M}_{L(1+\log^+ L), \lambda}$ to $\mathcal{WM}_{L(1+\log^+ L), \lambda}$ and the inequality*

$$\|[M, b]f\|_{\mathcal{WM}_{L(1+\log^+ L), \lambda}} \leq c (\|b^+\|_* + \|b^-\|_{L_\infty}) \|f\|_{\mathcal{M}_{L(1+\log^+ L), \lambda}}$$

holds with positive constant c independent of f .

THEOREM 1.13. *Let $0 < \lambda < n$ and b is in $\text{BMO}(\mathbf{R}^n)$ such that $b^- \in L_\infty(\mathbf{R}^n)$. Then the operator $[M, b]$ is bounded on $\mathcal{M}_{L(1+\log^+ L), \lambda}$ for radially decreasing functions, and the inequality*

$$\|[M, b]f\|_{\mathcal{M}_{L(1+\log^+ L), \lambda}} \leq c (\|b^+\|_* + \|b^-\|_{L_\infty}) \|f\|_{\mathcal{M}_{L(1+\log^+ L), \lambda}}, \quad f \in \mathfrak{M}^{\text{rad}, \downarrow}$$

holds with positive constant c independent of f .

The paper is organized as follows. In Section 2 notations and preliminary results are given. Boundedness of maximal commutator and commutator of maximal function in Morrey spaces are investigated in Section 3. New characterizations of $M_\alpha M$ are obtained in Section 4. In Section 5 it is shown that the Hardy-Littlewood maximal operator M is not bounded on Zygmund-Morrey spaces $\mathcal{M}_{L(\log L), \lambda}$, but M is still bounded on $\mathcal{M}_{L(\log L), \lambda}$ for radially decreasing functions. The boundedness of the iterated maximal operator from $\mathcal{M}_{L(\log L), \lambda}$ to $\mathcal{WM}_{L(\log L), \lambda}$ is proved in Section 6. In Section 7 we give proofs of Theorems 1.10–1.13.

2. Notations and Preliminaries

Now we make some conventions. Throughout the paper, we always denote by c a positive constant, which is independent of main parameters, but it may vary from line to line. However a constant with subscript such as c_1 does not change in different occurrences. By $A \lesssim B$ we mean that $A \leq cB$ with some positive constant c independent of appropriate quantities involved in A and B . For a measurable set E , χ_E denotes the characteristic function of E . Throughout this paper cubes will be assumed to have their sides parallel to the coordinate axes. Given $\lambda > 0$ and a cube Q , λQ denotes the cube with the same center as Q and whose side is λ times that of Q . For a fixed p with $p \in [1, \infty)$, p' denotes the dual exponent of p , namely, $p' = p/(p - 1)$. For any measurable set E and any integrable function f on E , we denote by f_E the mean value of f over E , that is, $f_E = (1/|E|) \int_E f(x) dx$. Unless a special remark is made, the differential element dx is omitted when the integrals under consideration are the Lebesgue integrals.

For the sake of completeness we recall the definitions and some properties of the spaces we are going to use.

Let Ω be any measurable subset of \mathbf{R}^n , $n \geq 1$. Let $\mathfrak{M}(\Omega)$ denote the set of all measurable functions on Ω and $\mathfrak{M}_0(\Omega)$ the class of functions in $\mathfrak{M}(\Omega)$ that are finite a.e.

For $p \in (0, \infty]$, we define the functional $\|\cdot\|_{p,\Omega}$ on $\mathfrak{M}(\Omega)$ by

$$\|f\|_{p,\Omega} := \begin{cases} \left(\int_{\Omega} |f(x)|^p dx \right)^{1/p} & \text{if } p < \infty, \\ \text{ess sup}_{\Omega} |f(x)| & \text{if } p = \infty. \end{cases}$$

The Lebesgue space $L_p(\Omega)$ is given by

$$L_p(\Omega) := \{f \in \mathfrak{M}(\Omega) : \|f\|_{p,\Omega} < \infty\}$$

and it is equipped with the quasi-norm $\|\cdot\|_{p,\Omega}$.

Denote by $\mathfrak{M}^{\text{rad},\downarrow} = \mathfrak{M}^{\text{rad},\downarrow}(\mathbf{R}^n)$ the set of all measurable, radially decreasing functions on \mathbf{R}^n , that is,

$$\mathfrak{M}^{\text{rad},\downarrow} := \{f \in \mathfrak{M}(\mathbf{R}^n) : f(x) = \varphi(|x|), x \in \mathbf{R}^n \text{ with } \varphi \in \mathfrak{M}^{\downarrow}(0, \infty)\}$$

where $\mathfrak{M}^{\downarrow}(0, \infty)$ is the subset of those functions from $\mathfrak{M}(0, \infty)$ which are non-increasing on $(0, \infty)$.

Recall that $Mf \approx Hf$, $f \in \mathfrak{M}^{\text{rad},\downarrow}$, where

$$Hf(x) := \frac{1}{|B(0, |x|)|} \int_{B(0, |x|)} |f(y)| dy$$

is n -dimensional Hardy operator. Obviously, $Hf \in \mathfrak{M}^{\text{rad},\downarrow}$, when $f \in \mathfrak{M}^{\text{rad},\downarrow}$.

The non-increasing rearrangement (see, e.g., [4, p. 39]) of a function $f \in \mathfrak{M}_0(\mathbf{R}^n)$ is defined by

$$f^*(t) := \inf \{ \lambda > 0 : |\{x \in \mathbf{R}^n : |f(x)| > \lambda\}| \leq t \} \quad (0 < t < \infty).$$

Then f^{**} will denote the maximal function of f^* defined by

$$f^{**}(t) := \frac{1}{t} \int_0^t f^*(s) ds, \quad (t > 0).$$

The Zygmund class $L(\log^+ L)(\Omega)$ is the set of all $f \in \mathfrak{M}(\Omega)$ such that

$$\int_{\Omega} |f(x)|(\log^+ |f(x)|) dx < \infty,$$

where $\log^+ t = \max\{\log t, 0\}$, $t > 0$. Generally, this is not a linear set. Nevertheless, considering the class

$$\begin{aligned} &L(1 + \log^+ L)(\Omega) \\ &= \left\{ f \in \mathfrak{M}(\Omega) : \|f\|_{L(1+\log^+ L)(\Omega)} := \int_{\Omega} |f(x)| (1 + \log^+ |f(x)|) dx < \infty \right\}, \end{aligned}$$

we obtain a linear set, the Zygmund space.

The size of M^2 is given by the following inequality.

LEMMA 2.1 ([25, Lemma 1.6]). *There exists a positive constant c such that for any function f and for all $\lambda > 0$,*

$$(2.1) \quad |\{x \in \mathbf{R}^n : M^2 f(x) > \lambda\}| \leq c \int_{\mathbf{R}^n} \frac{|f(x)|}{\lambda} \left(1 + \log^+ \left(\frac{|f(x)|}{\lambda} \right) \right) dx.$$

The following important result regarding BMO is true.

LEMMA 2.2 ([18] and [3]). *For $p \in (0, \infty)$, $\text{BMO}_p(\mathbf{R}^n) = \text{BMO}(\mathbf{R}^n)$, with equivalent norms, where*

$$\|f\|_{\text{BMO}_p(\mathbf{R}^n)} := \sup_Q \left(\frac{1}{|Q|} \int_Q |f(y) - f_Q|^p dy \right)^{\frac{1}{p}}.$$

A continuously increasing function on $[0, \infty]$, say $\Psi : [0, \infty] \rightarrow [0, \infty]$ such that $\Psi(0) = 0$, $\Psi(1) = 1$ and $\Psi(\infty) = \infty$, will be referred to as an Orlicz function. If Ψ is an Orlicz function, then

$$\Phi(t) = \sup\{ts - \Psi(s); s \in [0, \infty]\}$$

is the complementary Orlicz function to Ψ .

The Orlicz space denoted by $L^\Psi = L^\Psi(\mathbf{R}^n)$ consists of all measurable functions $g : \mathbf{R}^n \rightarrow \mathbf{R}$ such that

$$\int_{\mathbf{R}^n} \Psi \left(\frac{|g(x)|}{\alpha} \right) dx < \infty$$

for some $\alpha > 0$.

When Ψ is a Young function, i.e. a convex Orlicz function, the quantity

$$\|f\|_{\Psi} = \inf \left\{ \alpha > 0 : \int_{\mathbf{R}^n} \Psi \left(\frac{|f(y)|}{\alpha} \right) dy \leq 1 \right\}$$

is well known Luxemburg norm in the space L^{Ψ} (see [26]).

A Young function Ψ is said to satisfy the ∇_2 -condition, denoted $\Psi \in \nabla_2$, if for some $K > 1$

$$\Psi(t) \leq \frac{1}{2K} \Psi(Kt) \text{ for all } t > 0.$$

It should be noted that $\Psi(t) \equiv t$ fails the ∇_2 -condition.

THEOREM 2.3 ([19]). *The Hardy-Littlewood maximal operator is bounded on L^{Ψ} , provided that $\Psi \in \nabla_2$.*

Combining Theorem 2.3 and 1.8, we obtain the following statement.

THEOREM 2.4. *Let $b \in \text{BMO}(\mathbf{R}^n)$ and $\Psi \in \nabla_2$. Then the operator C_b is bounded on L^{Ψ} , and the inequality*

$$\|C_b f\|_{L^{\Psi}} \leq c \|b\|_* \|f\|_{L^{\Psi}}$$

holds with constant c independent of f .

Moreover, if $b^- \in L_{\infty}(\mathbf{R}^n)$, then the operator $[M, b]$ is bounded on L^{Ψ} , and the inequality

$$\|[M, b]f\|_{L^{\Psi}} \leq c (\|b^+\|_* + \|b^-\|_{\infty}) \|f\|_{L^{\Psi}}$$

holds with constant c independent of f .

Let us define the Ψ -average of g over a cube Q of \mathbf{R}^n by

$$\|g\|_{\Psi, Q} = \inf \left\{ \alpha > 0 : \frac{1}{|Q|} \int_Q \Psi \left(\frac{|g(x)|}{\alpha} \right) dx \leq 1 \right\}.$$

The generalized Hölder’s inequality

$$(2.2) \quad \frac{1}{|Q|} \int_Q |f(y)g(y)| dy \leq 2 \|f\|_{\Phi, Q} \|g\|_{\Psi, Q},$$

where Ψ is the complementary Young function associated to Φ , holds.

If $f \in L^{\Psi}(\mathbf{R}^n)$, the Orlicz maximal function of f with respect to Ψ is defined by setting

$$M_{\Psi} f(x) = \sup_{x \in Q} \|f\|_{\Psi, Q},$$

where the supremum is taken over all cubes Q of \mathbf{R}^n containing x .

The main example that we are going to use is $\Psi(t) = t(1 + \log^+ t)$ with maximal function defined by $M_{L(1+\log^+ L)}$. The complementary Young function is given by $\Phi(t) \approx e^t$ with the corresponding maximal function denoted by $M_{\exp L}$.

We define the weak $L(1 + \log^+ L)$ -average of g over a cube Q of \mathbf{R}^n analogously by

$$\|g\|_{WL(1+\log^+ L), Q} = \inf \left\{ \alpha > 0 : \sup_{t>0} \frac{1}{|Q|} \frac{|\{x \in Q : |g(x)| > \alpha t\}|}{\frac{1}{t} (1 + \log^+ \frac{1}{t})} \leq 1 \right\}.$$

Let $0 < \lambda < n$. The Zygmund-Morrey spaces $\mathcal{M}_{L(\log L), \lambda}(\mathbf{R}^n) \equiv \mathcal{M}_{L(1+\log^+ L), \lambda}(\mathbf{R}^n)$ and the weak Zygmund-Morrey spaces $\mathcal{WM}_{L(\log L), \lambda}(\mathbf{R}^n) \equiv \mathcal{WM}_{L(1+\log^+ L), \lambda}(\mathbf{R}^n)$ are defined as follows:

$$\mathcal{M}_{L(1+\log^+ L), \lambda}(\mathbf{R}^n)$$

$$:= \left\{ f \in \mathfrak{M}(\mathbf{R}^n) : \|f\|_{\mathcal{M}_{L(1+\log^+ L), \lambda}} := \sup_Q |Q|^{\frac{\lambda}{n}} \|f\|_{L(1+\log^+ L), Q} < \infty \right\},$$

$$\mathcal{WM}_{L(1+\log^+ L), \lambda}(\mathbf{R}^n)$$

$$:= \left\{ f \in \mathfrak{M}(\mathbf{R}^n) : \|f\|_{\mathcal{WM}_{L(1+\log^+ L), \lambda}} := \sup_Q |Q|^{\frac{\lambda}{n}} \|f\|_{WL(1+\log^+ L), Q} < \infty \right\},$$

respectively. Note that $\mathcal{M}_{L(1+\log^+ L), \lambda}$ is a special case of Orlicz-Morrey spaces $\mathcal{L}^{\Phi, \phi}$ (with $\Phi(t) = t(1 + \log^+ t)$ and $\phi(t) = t^\lambda, t > 0$) defined in [27, Definitions 2.3]. As we know, a weak version has not been defined yet in such form.

3. Boundedness of maximal commutator and commutator of maximal function in Morrey spaces

In this section we investigate boundedness of maximal commutator and commutator of maximal function in Morrey spaces.

THEOREM 3.1. *Let $1 < p < \infty, 0 \leq \lambda \leq n$. The following assertions are equivalent:*

- (i) $b \in \text{BMO}(\mathbf{R}^n)$;
- (ii) *The operator C_b is bounded on $\mathcal{M}_{p, \lambda}$.*

PROOF. (i) \Rightarrow (ii). Suppose that $b \in \text{BMO}(\mathbf{R}^n)$. By Theorem 1.9 and inequality (1.7) it follows that C_b is bounded in Morrey space $\mathcal{M}_{p, \lambda}$ and the following inequality holds:

$$\|C_b(f)\|_{\mathcal{M}_{p, \lambda}} \lesssim \|b\|_* \|f\|_{\mathcal{M}_{p, \lambda}}.$$

- (ii) \Rightarrow (i). Assume that there exists $c > 0$ such that

$$\|C_b(f)\|_{\mathcal{M}_{p, \lambda}} \leq c \|f\|_{\mathcal{M}_{p, \lambda}}$$

for all $f \in \mathcal{M}_{p, \lambda}$.

Obviously,

$$\|f\|_{\mathcal{M}_{p,\lambda}} \approx \sup_{Q'} \left(|Q'|^{\frac{\lambda-n}{n}} \int_{Q'} |f(y)|^p dy \right)^{\frac{1}{p}}.$$

Let Q be a fixed cube. We consider $f = \chi_Q$. It is easy to compute that

$$\begin{aligned} (3.1) \quad \|\chi_Q\|_{\mathcal{M}_{p,\lambda}} &\approx \sup_{Q'} \left(|Q'|^{\frac{\lambda-n}{n}} \int_{Q'} \chi_Q(y) dy \right)^{\frac{1}{p}} = \sup_{Q'} \left(|Q' \cap Q| |Q'|^{\frac{\lambda-n}{n}} \right)^{\frac{1}{p}} \\ &= \sup_{Q' \subseteq Q} \left(|Q'| |Q'|^{\frac{\lambda-n}{n}} \right)^{\frac{1}{p}} = |Q|^{\frac{\lambda}{np}}. \end{aligned}$$

On the other hand, since

$$C_b(\chi_Q)(x) \gtrsim \frac{1}{|Q|} \int_Q |b(y) - b_Q| dy \quad \text{for all } x \in Q,$$

then

$$\begin{aligned} (3.2) \quad \|C_b(\chi_Q)\|_{\mathcal{M}_{p,\lambda}} &\approx \sup_{Q'} \left(|Q'|^{\frac{\lambda-n}{n}} \int_{Q'} |C_b(\chi_Q)(y)|^p dy \right)^{\frac{1}{p}} \\ &\gtrsim |Q|^{\frac{\lambda}{np}} \frac{1}{|Q|} \int_Q |b(y) - b_Q| dy. \end{aligned}$$

Since by assumption

$$\|C_b(\chi_Q)\|_{\mathcal{M}_{p,\lambda}} \lesssim \|\chi_Q\|_{\mathcal{M}_{p,\lambda}},$$

by (3.1) and (3.2), we get that

$$\frac{1}{|Q|} \int_Q |b(y) - b_Q| dy \lesssim c.$$

□

Combining Theorem 1.9 with inequality (1.9), we get the following statement.

THEOREM 3.2. *Let $0 < \lambda < n$. Assume that $b \in \text{BMO}(\mathbf{R}^n)$. Then the operator C_b is bounded on $\mathcal{M}_{1,\lambda}$ for radially decreasing functions.*

The following theorem was proved in [34].

THEOREM 3.3. *Let $1 < p < \infty$, $0 \leq \lambda \leq n$. Suppose that b is a real valued, locally integrable function in \mathbf{R}^n . The following assertions are equivalent:*

- (i) $b \in \text{BMO}(\mathbf{R}^n)$ such that $b^- \in L_\infty(\mathbf{R}^n)$;
- (ii) The commutator $[M, b]$ is bounded in $\mathcal{M}_{p,\lambda}$.

REMARK 3.4. (i) \Rightarrow (ii). Assume that $b \in \mathbf{BMO}(\mathbf{R}^n)$ such that $b^- \in L_\infty(\mathbf{R}^n)$. By Theorem 1.9 and inequality (1.7) it follows that $[M, b]$ is bounded in Morrey space $\mathcal{M}_{p,\lambda}$ and the following inequality holds:

$$\|[M, b]f\|_{\mathcal{M}_{p,\lambda}} \lesssim (\|b^+\|_* + \|b^-\|_\infty) \|f\|_{\mathcal{M}_{p,\lambda}}.$$

Combining Theorem 1.9 with inequality (1.9), we obtain the following statement.

THEOREM 3.5. *Let $0 < \lambda < n$. Suppose that $b \in \mathbf{BMO}(\mathbf{R}^n)$ such that $b^- \in L_\infty(\mathbf{R}^n)$. Then $[M, b]$ is bounded on $\mathcal{M}_{1,\lambda}$ for radially decreasing functions.*

4. Some auxiliary results

To prove the theorems in the next sections we need the following results.

THEOREM 4.1. *Let $0 \leq \alpha < n$. Then*

$$\begin{aligned} M_\alpha(Mf)(x) &= \sup_{Q \ni x} |Q|^{\frac{\alpha-n}{n}} \int_Q Mf \approx \sup_{Q \ni x} |Q|^{\frac{\alpha}{n}} \|f\|_{L(1+\log^+ L), Q} \\ &\approx \sup_{Q \ni x} |Q|^{\frac{\alpha-n}{n}} \int_Q |f| \left(1 + \log^+ \frac{|f|}{|f|_Q}\right) \end{aligned}$$

holds for all $f \in L_1^{\text{loc}}(\mathbf{R}^n)$.

The statement of Theorem 4.1 follows by the following lemmas.

LEMMA 4.2. *The inequality*

$$(4.1) \quad \frac{1}{|Q|} \int_Q Mf(y) dy \lesssim \sup_{Q \subset Q'} \|f\|_{L(1+\log^+ L), Q'}$$

holds for all $f \in L_1^{\text{loc}}(\mathbf{R}^n)$ and all Q .

PROOF. Let Q be a cube in \mathbf{R}^n and $f = f_1 + f_2$, where $f_1 = f \chi_{3Q}$. Then

$$(4.2) \quad \frac{1}{|Q|} \int_Q Mf(y) dy \leq \frac{1}{|Q|} \int_Q Mf_1(y) dy + \frac{1}{|Q|} \int_Q Mf_2(y) dy.$$

We recall a simple geometric observation: for a fixed point $x \in Q$, if a cube Q' satisfies $Q' \ni x$ and $Q' \cap (3Q)^c \neq \emptyset$, then $Q \subset 3Q'$. Hence

$$Mf_2(x) = \sup_{Q' \ni x} \frac{1}{|Q'|} \int_{Q'} |f_2(y)| dy \leq \sup_{Q \subset 3Q'} \frac{1}{|Q'|} \int_{Q'} |f(y)| dy.$$

Consequently, we have that

$$(4.3) \quad \frac{1}{|Q|} \int_Q Mf_2(y) dy \lesssim \sup_{Q \subset Q'} \frac{1}{|Q'|} \int_{Q'} |f(y)| dy.$$

Since for any cube Q'

$$\frac{1}{|Q'|} \int_{Q'} |f(y)| dy \leq \|f\|_{L(1+\log^+ L), Q'},$$

we get

$$(4.4) \quad \frac{1}{|Q|} \int_Q Mf_2(y) dy \lesssim \sup_{Q \subset Q'} \|f\|_{L(1+\log^+ L), Q'}.$$

On the other hand

$$\frac{1}{|Q|} \int_Q Mf(y) dy \lesssim \|f\|_{L(1+\log^+ L), Q}$$

for all f such that $\text{supp } f \subset Q$ (see [25, p. 174]). Thus

$$(4.5) \quad \frac{1}{|Q|} \int_Q Mf_1(y) dy \lesssim \frac{1}{|3Q|} \int_{3Q} Mf_1(y) dy \lesssim \|f\|_{L(1+\log^+ L), 3Q}.$$

From (4.2), (4.4) and (4.5), it follows that

$$(4.6) \quad \begin{aligned} \frac{1}{|Q|} \int_Q Mf(y) dy &\lesssim \sup_{Q \subset Q'} \|f\|_{L(1+\log^+ L), Q'} + \|f\|_{L(1+\log^+ L), 3Q} \\ &\lesssim \sup_{Q \subset Q'} \|f\|_{L(1+\log^+ L), Q'}. \end{aligned}$$

□

We recall the following statement (see, for instance, [17, p. 175]). For the completeness we give the proof.

LEMMA 4.3. *Note that the estimation*

$$\int_Q M(f\chi_Q) \approx \int_Q |f| \left(1 + \log^+ \frac{|f|}{|f|_Q}\right)$$

holds for all $f \in L_1^{\text{loc}}(\mathbf{R}^n)$ and all Q .

PROOF. Let Q be a cube in \mathbf{R}^n . We are going to use weak type estimates (see [30], for instance): there exist positive constants $c_1 < 1$ and $c_2 > 1$ such that for every $f \in L_1^{\text{loc}}(\mathbf{R}^n)$ and for every $t > 1/|Q| \int_Q |f|$

$$c_1 \int_{\{x \in Q : |f(x)| > t\}} \frac{|f(x)|}{t} dx \leq |\{x \in Q : M(f\chi_Q)(x) > t\}| \leq c_2 \int_{\{x \in Q : |f(x)| > t/2\}} \frac{|f(x)|}{t} dx$$

holds.

We have that

$$\int_Q M(f\chi_Q) = \int_0^\infty |\{x \in Q : M(f\chi_Q)(x) > \lambda\}| d\lambda$$

$$\begin{aligned}
&= \int_0^{|f|_Q} |\{x \in Q : M(f\chi_Q)(x) > \lambda\}| d\lambda \\
&\quad + \int_{|f|_Q}^\infty |\{x \in Q : M(f\chi_Q)(x) > \lambda\}| d\lambda \\
&= |Q||f|_Q + \int_{|f|_Q}^\infty |\{x \in Q : M(f\chi_Q)(x) > \lambda\}| d\lambda \\
&\geq |Q||f|_Q + c_1 \int_{|f|_Q}^\infty \left(\int_{\{x \in Q : |f(x)| > \lambda\}} |f(x)| dx \right) \frac{d\lambda}{\lambda} \\
&= |Q||f|_Q + c_1 \int_{\{x \in Q : |f(x)| > |f|_Q\}} \left(\int_{|f|_Q}^{|f(x)|} \frac{d\lambda}{\lambda} \right) |f(x)| dx \\
&= |Q||f|_Q + c_1 \int_{\{x \in Q : |f(x)| > |f|_Q\}} |f(x)| \log \left(\frac{|f(x)|}{|f|_Q} \right) dx \\
&\gtrsim \int_Q |f| \left(1 + \log^+ \frac{|f|}{|f|_Q} \right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_Q M(f\chi_Q) &= \int_0^\infty |\{x \in Q : M(f\chi_Q)(x) > \lambda\}| d\lambda \\
&\approx \int_0^\infty |\{x \in Q : M(f\chi_Q)(x) > 2\lambda\}| d\lambda \\
&= \int_0^{|f|_Q} |\{x \in Q : M(f\chi_Q)(x) > 2\lambda\}| d\lambda \\
&\quad + \int_{|f|_Q}^\infty |\{x \in Q : M(f\chi_Q)(x) > 2\lambda\}| d\lambda \\
&\leq |Q||f|_Q + c_2 \int_{|f|_Q}^\infty \left(\int_{\{x \in Q : |f(x)| > \lambda\}} |f(x)| dx \right) \frac{d\lambda}{\lambda} \\
&= |Q||f|_Q + c_2 \int_{\{x \in Q : |f(x)| > |f|_Q\}} |f(x)| \log \left(\frac{|f(x)|}{|f|_Q} \right) dx \\
&\lesssim \int_Q |f| \left(1 + \log^+ \frac{|f|}{|f|_Q} \right).
\end{aligned}$$

□

LEMMA 4.4. *Inequalities*

$$\frac{1}{|Q|} \int_Q |f| \left(1 + \log^+ \frac{|f|}{|f|_Q} \right) \approx \|f\|_{L(1+\log^+ L), Q}$$

hold for all $f \in L_1^{\text{loc}}(\mathbf{R}^n)$ and all Q .

PROOF. Since

$$1 \leq \frac{1}{|Q|} \int_Q \frac{|f|}{|f|_Q} \left(1 + \log^+ \frac{|f|}{|f|_Q} \right),$$

then

$$|f|_Q \leq \|f\|_{L(1+\log^+ L), Q}.$$

Using the inequality $\log^+(ab) \leq \log^+ a + \log^+ b$, $a, b \in \mathbf{R}^+$, we have that

$$\begin{aligned} & \frac{1}{|Q|} \int_Q |f| \left(1 + \log^+ \frac{|f|}{|f|_Q} \right) \\ &= \frac{1}{|Q|} \int_Q |f| \left(1 + \log^+ \left(\frac{|f|}{\|f\|_{L(1+\log^+ L), Q}} \frac{\|f\|_{L(1+\log^+ L), Q}}{|f|_Q} \right) \right) \\ &\leq \frac{1}{|Q|} \int_Q |f| \left(1 + \log^+ \frac{|f|}{\|f\|_{L(1+\log^+ L), Q}} \right) \\ &\quad + \frac{1}{|Q|} \int_Q |f| \log^+ \frac{\|f\|_{L(1+\log^+ L), Q}}{|f|_Q} \\ &\leq \|f\|_{L(1+\log^+ L), Q} + |f|_Q \log^+ \frac{\|f\|_{L(1+\log^+ L), Q}}{|f|_Q}. \end{aligned}$$

In view of inequalities $\frac{\|f\|_{L(1+\log^+ L), Q}}{|f|_Q} \geq 1$ and $\log t \leq t$, $t \geq 1$, we get that

$$\frac{1}{|Q|} \int_Q |f| \left(1 + \log^+ \frac{|f|}{|f|_Q} \right) \leq 2\|f\|_{L(1+\log^+ L), Q}.$$

On the other hand, since

$$\|f\|_{L(1+\log^+ L), Q} = \frac{1}{|Q|} \int_Q |f| \left(1 + \log^+ \frac{|f|}{\|f\|_{L(1+\log^+ L), Q}} \right),$$

on using $|f|_Q \leq \|f\|_{L(1+\log^+ L), Q}$, we arrive at

$$\|f\|_{L(1+\log^+ L), Q} \lesssim \frac{1}{|Q|} \int_Q |f| \left(1 + \log^+ \frac{|f|}{|f|_Q} \right).$$

□

PROOF OF THEOREM 4.1. By Lemma 4.2, we get that

$$\sup_{Q \ni x} |Q|^{\frac{\alpha-n}{n}} \int_Q Mf(y) dy \lesssim \sup_{Q \ni x} |Q|^{\frac{\alpha}{n}} \sup_{Q \subset Q'} \|f\|_{L(1+\log^+ L), Q'} \leq \sup_{Q \ni x} |Q|^{\frac{\alpha}{n}} \|f\|_{L(1+\log^+ L), Q}.$$

The equivalence

$$\sup_{Q \ni x} |Q|^{\frac{\alpha}{n}} \|f\|_{L(1+\log^+ L), Q} \approx \sup_{Q \ni x} |Q|^{\frac{\alpha-n}{n}} \int_Q |f| \left(1 + \log^+ \frac{|f|}{|f|_Q}\right)$$

is obvious in view of Lemma 4.4.

By Lemma 4.3, we have that

$$\sup_{Q \ni x} |Q|^{\frac{\alpha-n}{n}} \int_Q |f| \left(1 + \log^+ \frac{|f|}{|f|_Q}\right) \lesssim \sup_{Q \ni x} |Q|^{\frac{\alpha-n}{n}} \int_Q Mf.$$

□

The following corollaries follow from Theorem 4.1.

COROLLARY 4.5. *Inequalities*

$$(4.7) \quad M^2 f(x) \approx M_{L(1+\log^+ L)} f(x) \approx \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f| \left(1 + \log^+ \frac{|f|}{|f|_Q}\right)$$

holds for all $x \in \mathbf{R}^n$ and $f \in L_1^{\text{loc}}(\mathbf{R}^n)$.

COROLLARY 4.6. *Let $0 < \lambda < n$. The equivalencies*

$$\|Mf\|_{\mathcal{M}_{1,\lambda}} \approx \|f\|_{\mathcal{M}_{L(1+\log^+ L),\lambda}} \approx \sup_Q |Q|^{\frac{\lambda-n}{n}} \int_Q |f| \left(1 + \log^+ \frac{|f|}{|f|_Q}\right)$$

hold.

Note that $M^2 f \approx M_{L(1+\log^+ L)} f$ was proved in [25] (see, also [13, p. 159]). For the second part of (4.7) see [6], [20], [21] and [25]. The equivalence $\|Mf\|_{\mathcal{M}_{1,\lambda}} \approx \|f\|_{\mathcal{M}_{L(1+\log^+ L),\lambda}}$ is a special case of [27, Lemma 3.5].

5. A note on the boundedness of the maximal function on Zygmund-Morrey spaces

In this section we prove that the Hardy-Littlewood maximal operator M is bounded on $\mathcal{M}_{L(1+\log^+ L),\lambda}$, $0 < \lambda < n$, for radially decreasing functions, and we give an example which shows that M is not bounded on $\mathcal{M}_{L(1+\log^+ L),\lambda}$, $0 < \lambda < n$.

In order to prove the main result of this section we need the following auxiliary lemmas.

LEMMA 5.1. *Assume that $0 < \lambda < n$. Let $f \in \mathfrak{M}^{\text{rad},\downarrow}(\mathbf{R}^n)$ with $f(z) = \varphi(|z|)$, $z \in \mathbf{R}^n$. The equivalency*

$$\|f\|_{\mathcal{M}_{L(1+\log^+ L),\lambda}} \approx \sup_{x>0} x^{\lambda-n} \int_0^x \frac{1}{t} \int_0^t |\varphi(\rho)| \rho^{n-1} d\rho dt$$

holds.

PROOF. Recall that

$$\|f\|_{\mathcal{M}_{L(1+\log^+ L), \lambda}} \approx \sup_B |B|^{\frac{\lambda-n}{n}} \int_B Mf = \|M_\lambda(Mf)\|_\infty, \quad f \in \mathfrak{M}(\mathbf{R}^n).$$

Since $M_\lambda(f)(y) \gtrsim \frac{1}{|B(0, |y|)|^{1-\lambda/n}} \int_{B(0, |y|)} |f(z)| dz$, in view of $Mf \approx Hf$, $f \in \mathfrak{M}^{\text{rad}, \downarrow}$, switching to polar coordinates, we have that

$$\begin{aligned} M_\lambda(Mf)(y) &\gtrsim \frac{1}{|B(0, |y|)|^{1-\lambda/n}} \int_{B(0, |y|)} Mf(z) dz \\ &\approx \frac{1}{|B(0, |y|)|^{1-\lambda/n}} \int_{B(0, |y|)} |Hf(z)| dz \\ &= \frac{1}{|B(0, |y|)|^{1-\lambda/n}} \int_{B(0, |y|)} \frac{1}{|B(0, |z|)|} \int_{B(0, |z|)} |f(w)| dw dz \\ &\approx \frac{1}{|B(0, |y|)|^{1-\lambda/n}} \int_{B(0, |y|)} |z|^{-n} \int_0^{|z|} |\varphi(\rho)| \rho^{n-1} d\rho dz \\ &\approx |y|^{\lambda-n} \int_0^{|y|} \frac{1}{t} \int_0^t |\varphi(\rho)| \rho^{n-1} d\rho dt. \end{aligned}$$

Consequently,

$$\begin{aligned} \|f\|_{\mathcal{M}_{L(1+\log^+ L), \lambda}} &\gtrsim \text{ess sup}_{y \in \mathbf{R}^n} |y|^{\lambda-n} \int_0^{|y|} \frac{1}{t} \int_0^t |\varphi(\rho)| \rho^{n-1} d\rho dt \\ &= \sup_{x>0} x^{\lambda-n} \int_0^x \frac{1}{t} \int_0^t |\varphi(\rho)| \rho^{n-1} d\rho dt, \end{aligned}$$

where $f(\cdot) = \varphi(|\cdot|)$.

On the other hand,

$$\begin{aligned} \|f\|_{\mathcal{M}_{L(1+\log^+ L), \lambda}} &\lesssim \sup_B |B|^{\frac{\lambda-n}{n}} \int_0^{|B|} (Mf)^*(t) dt \\ &\approx \sup_B |B|^{\frac{\lambda-n}{n}} \int_0^{|B|} f^{**}(t) dt \\ &= \sup_B |B|^{\frac{\lambda-n}{n}} \int_0^{|B|} \frac{1}{t} \int_0^t f^*(s) ds dt \\ &= \sup_B |B|^{\frac{\lambda-n}{n}} \int_0^{|B|} \frac{1}{t} \int_0^t |\varphi(s^{\frac{1}{n}})| ds dt \\ &\approx \sup_B |B|^{\frac{\lambda-n}{n}} \int_0^{|B|} \frac{1}{t} \int_0^{t^{\frac{1}{n}}} |\varphi(\rho)| \rho^{n-1} d\rho dt \end{aligned}$$

$$\begin{aligned} &\approx \sup_B |B|^{\frac{\lambda-n}{n}} \int_0^{|B|^{\frac{1}{n}}} \frac{1}{x} \int_0^x |\varphi(\rho)| \rho^{n-1} d\rho dx \\ &= \sup_{x>0} x^{\lambda-n} \int_0^x \frac{1}{t} \int_0^t |\varphi(\rho)| \rho^{n-1} d\rho dt, \end{aligned}$$

where $f(\cdot) = \varphi(|\cdot|)$. □

COROLLARY 5.2. *Assume that $0 < \lambda < n$. Let $f \in \mathfrak{M}^{\text{rad},\downarrow}(\mathbf{R}^n)$ with $f(z) = \varphi(|z|)$, $z \in \mathbf{R}^n$. The equivalency*

$$\|Mf\|_{\mathcal{M}_{L(1+\log^+ L),\lambda}} \approx \sup_{x>0} x^{\lambda-n} \int_0^x \frac{1}{y} \int_0^y \frac{1}{t} \int_0^t \varphi(\rho) \rho^{n-1} d\rho dt dy$$

holds.

PROOF. Let $f \in \mathfrak{M}^{\text{rad},\downarrow}$ with $f(z) = \varphi(|z|)$, $z \in \mathbf{R}^n$. Since $Mf \approx Hf$ and $Hf \in \mathfrak{M}^{\text{rad},\downarrow}$, by Lemma 5.1, switching to polar coordinates, we have that

$$\begin{aligned} \|Mf\|_{\mathcal{M}_{L(1+\log^+ L),\lambda}} &\approx \sup_{x>0} x^{\lambda-n} \int_0^x \frac{1}{y} \int_0^y \left(\frac{1}{|B(0,t)|} \int_{B(0,t)} |f(y)| dy \right) t^{n-1} dt dy \\ &\approx \sup_{x>0} x^{\lambda-n} \int_0^x \frac{1}{y} \int_0^y \frac{1}{t} \int_0^t \varphi(\rho) \rho^{n-1} d\rho dt dy. \end{aligned}$$

□

LEMMA 5.3. *Assume that $0 < \lambda < n$. Let $f \in \mathfrak{M}^{\text{rad},\downarrow}$ with $f(z) = \varphi(|z|)$, $z \in \mathbf{R}^n$. The inequality*

$$\|Mf\|_{\mathcal{M}_{L(1+\log^+ L),\lambda}} \lesssim \|f\|_{\mathcal{M}_{L(1+\log^+ L),\lambda}}, \quad f \in \mathfrak{M}^{\text{rad},\downarrow}$$

holds if and only if the inequality

$$\begin{aligned} &\sup_{x>0} x^{\lambda-n} \int_0^x \frac{1}{y} \int_0^y \frac{1}{t} \int_0^t \varphi(\rho) \rho^{n-1} d\rho dt dy \\ &\lesssim \sup_{x>0} x^{\lambda-n} \int_0^x \frac{1}{t} \int_0^t \varphi(\rho) \rho^{n-1} d\rho dt, \quad \varphi \in \mathfrak{M}^{+,\downarrow}(0, \infty) \end{aligned}$$

holds true.

PROOF. The statement immediately follows from Lemma 5.1 and Corollary 5.2. □

LEMMA 5.4. *Let $0 < \lambda < n$. Then inequality*

$$(5.1) \quad \sup_{x>0} x^{\lambda-n} \int_0^x \frac{1}{y} \int_0^y \frac{1}{t} \int_0^t \varphi(\rho) \rho^{n-1} d\rho dt dy \lesssim \sup_{x>0} x^{\lambda-n} \int_0^x \frac{1}{t} \int_0^t \varphi(\rho) \rho^{n-1} d\rho dt$$

holds for all $\varphi \in \mathfrak{M}^{+,\downarrow}(0, \infty)$.

PROOF. Indeed:

$$\begin{aligned}
& \sup_{x>0} x^{\lambda-n} \int_0^x \frac{1}{y} \int_0^y \frac{1}{t} \int_0^t \varphi(\rho) \rho^{n-1} d\rho dt dy \\
&= \sup_{x>0} x^{\lambda-n} \int_0^x y^{n-\lambda-1} y^{\lambda-n} \int_0^y \frac{1}{t} \int_0^t \varphi(\rho) \rho^{n-1} d\rho dt dy \\
&\leq \sup_{y>0} y^{\lambda-n} \int_0^y \frac{1}{t} \int_0^t \varphi(\rho) \rho^{n-1} d\rho dt \cdot \left(\sup_{x>0} x^{\lambda-n} \int_0^x y^{n-\lambda-1} dy \right) \\
&\approx \sup_{y>0} y^{\lambda-n} \int_0^y \frac{1}{t} \int_0^t \varphi(\rho) \rho^{n-1} d\rho dt.
\end{aligned}$$

□

THEOREM 5.5. Assume that $0 < \lambda < n$. The inequality

$$\|Mf\|_{\mathcal{M}_{L(1+\log^+ L), \lambda}} \lesssim \|f\|_{\mathcal{M}_{L(1+\log^+ L), \lambda}}$$

holds for all $f \in \mathfrak{M}^{\text{rad}, \downarrow}$.

PROOF. The statement follows by Lemmas 5.3 and 5.4. □

EXAMPLE 5.6. We give an example which shows that M is not bounded on $\mathcal{M}_{L(1+\log^+ L), \lambda}$, $0 < \lambda < n$. For simplicity let $n = 1$ and $\lambda = 1/2$. Consider even function f defined as follows:

$$f(x) = \sum_{k=0}^{\infty} \chi_{[k^2 \ln^2(k+e), k^2 \ln^2(k+e)+1]}(x), \quad x \geq 0.$$

It is easy to see that Mf and $M^2 f$ are even functions. Obviously,

$$\begin{aligned}
Mf(x) &\approx \sum_{k=0}^{\infty} \chi_{[k^2 \ln^2(k+e), k^2 \ln^2(k+e)+1]}(x) \\
&+ \sum_{k=0}^{\infty} \frac{1}{x - k^2 \ln^2(k+e)} \chi_{[k^2 \ln^2(k+e)+1, k^2 \ln^2(k+e)+1+m_k]}(x) \\
&+ \sum_{k=0}^{\infty} \frac{1}{(k+1)^2 \ln^2(k+1+e) + 1 - x} \chi_{[k^2 \ln^2(k+e)+1+m_k, (k+1)^2 \ln^2(k+1+e)]}(x), \quad x \geq 0,
\end{aligned}$$

where

$$m_k = \frac{(k+1)^2 \ln^2(k+1+e) - k^2 \ln^2(k+e) - 1}{2}, \quad k = 0, 1, 2, \dots$$

Then

$$\begin{aligned} \|f\|_{\mathcal{M}_{L(1+\log^+ L), 1/2}(\mathbf{R})} &\approx \|Mf\|_{\mathcal{M}_{1, 1/2}(\mathbf{R})} = \sup_I |I|^{-1/2} \int_I Mf \\ &\leq \sup_{I: |I| \leq 1} |I|^{-1/2} \int_I Mf + \sup_{I: |I| > 1} |I|^{-1/2} \int_I Mf. \end{aligned}$$

It is clear that

$$\sup_{I: |I| \leq 1} |I|^{-1/2} \int_I Mf \leq \sup_{I: |I| \leq 1} |I|^{1/2} \leq 1.$$

Since

$$\int_{j^2 \ln^2(j+e)}^{(j+1)^2 \ln^2(j+e+1)} Mf(x) dx \approx (1 + 2 \ln(1 + m_j)), \quad j = 0, 1, 2, \dots,$$

we have that

$$\begin{aligned} \sup_{I: |I| > 1} |I|^{-1/2} \int_I Mf(x) dx &= \sup_{m \geq 2} \sup_{I: m-1 < |I| \leq m} |I|^{-1/2} \int_I Mf(x) dx \\ &\lesssim \sup_{m \geq 2} m^{-1/2} \int_0^m Mf(x) dx \\ &\leq \sup_{m \geq 2} m^{-1/2} \sum_{i^2 \ln^2(j+e) < m} \int_{j^2 \ln^2(j+e)}^{(j+1)^2 \ln^2(j+e+1)} Mf(x) dx \\ &\approx \sup_{m \geq 2} m^{-1/2} \sum_{i^2 \ln^2(j+e) < m} (1 + 2 \ln(1 + m_j)) \\ &\lesssim \sup_{m \geq 2} m^{-1/2} \sum_{i^2 \ln^2(j+e) < m} \ln(j + e) \\ &\lesssim \sup_{m \geq 2} m^{-1/2} m^{1/2} = 1, \end{aligned}$$

therefore

$$\|f\|_{\mathcal{M}_{L(1+\log^+ L), 1/2}(\mathbf{R})} \lesssim 1 + 1 = 2.$$

On the other hand, it is easy to see that

$$\begin{aligned} M^2 f(x) &\geq \frac{1}{x - (k^2 \ln^2(k+e) + 1)} \int_{k^2 \ln^2(k+e)+1}^x \frac{dt}{t - k^2 \ln^2(k+e)} \\ &= \frac{\ln(x - k^2 \ln^2(k+e))}{x - (k^2 \ln^2(k+e) + 1)} \\ &\geq \frac{\ln(x - k^2 \ln^2(k+e))}{x - k^2 \ln^2(k+e)} \end{aligned}$$

for any $x \in [k^2 \ln^2(k+e) + e, k^2 \ln^2(k+e) + m_k]$.

Thus

$$M^2 f(x) \geq \sum_{k=0}^{\infty} \frac{\ln(x - k^2 \ln^2(k+e))}{x - k^2 \ln^2(k+e)} \chi_{[k^2 \ln^2(k+e)+e, k^2 \ln^2(k+e)+m_k]}(x).$$

Finally,

$$\begin{aligned} \|Mf\|_{\mathcal{M}_{L(1+\log^+ L), 1/2}(\mathbf{R})} &\approx \|M^2 f\|_{\mathcal{M}_{1, 1/2}(\mathbf{R})} \\ &\gtrsim \sup_k (k \ln(k+e))^{-1} \int_0^{k^2 \ln^2(k+e)} M^2 f(x) dx \\ &\geq \sup_k (k \ln(k+e))^{-1} \sum_{j=1}^{k-1} \int_{j^2 \ln^2(j+e)+e}^{j^2 \ln^2(j+e)+m_j} M^2 f(x) dx \\ &\geq \sup_k (k \ln(k+e))^{-1} \sum_{j=1}^{k-1} \int_{j^2 \ln^2(j+e)+e}^{j^2 \ln^2(j+e)+m_j} \frac{\ln(x - k^2 \ln^2(k+e))}{x - k^2 \ln^2(k+e)} dx \\ &= \sup_k (k \ln(k+e))^{-1} \sum_{j=1}^{k-1} \int_e^{m_j} \frac{\ln x}{x} dx \\ &\gtrsim \sup_k (k \ln(k+e))^{-1} \sum_{j=1}^{k-1} \ln^2 m_j \\ &\gtrsim \sup_k (k \ln(k+e))^{-1} \sum_{j=1}^{k-1} \ln^2(j+e) \\ &\gtrsim \sup_k (k \ln(k+e))^{-1} k \ln^2(k+e) \\ &= \sup_k \ln(k+e) = \infty. \end{aligned}$$

6. Weak-type estimates in Morrey spaces for the iterated maximal function

In this section the boundedness of the iterated maximal operator M^2 from Zygmund-Morrey spaces $\mathcal{M}_{L(1+\log^+ L), \lambda}$ to weak Zygmund-Morrey spaces $\mathcal{WM}_{L(1+\log^+ L), \lambda}$ is proved.

THEOREM 6.1. *Let $0 < \lambda < n$. Then the operator M^2 is bounded from $\mathcal{M}_{L(1+\log^+ L), \lambda}$ to $\mathcal{WM}_{L(1+\log^+ L), \lambda}$ and the inequality*

$$(6.1) \quad \|M^2 f\|_{\mathcal{WM}_{L(1+\log^+ L), \lambda}} \leq c \|f\|_{\mathcal{M}_{L(1+\log^+ L), \lambda}}$$

holds with positive constant c independent of f .

PROOF. Let Q be any cube in \mathbf{R}^n and let $f = f_1 + f_2$, where $f_1 = f\chi_{4Q}$. By subadditivity of M^2 we get

$$M^2 f \leq M^2 f_1 + M^2 f_2.$$

Since for any cube Q' conditions $z \in 2Q \cap Q'$ and $Q' \cap \{\mathbf{R}^n \setminus 4Q\} \neq \emptyset$ imply $Q \subset 4Q'$, we have

$$(6.2) \quad Mf_2(z) = M(f\chi_{\mathbf{R}^n \setminus 4Q})(z) \leq \sup_{Q \subset 4Q'} \frac{1}{|Q'|} \int_{Q'} |f|$$

for any $z \in 2Q$. Thus for any $z \in \mathbf{R}^n$

$$(6.3) \quad Mf_2(z) \leq \chi_{2Q}(z) \sup_{Q \subset 4Q'} \frac{1}{|Q'|} \int_{Q'} |f| + \chi_{\mathbf{R}^n \setminus 2Q}(z) Mf(z).$$

Applying the operator M to both sides of the inequality (6.3), we get that

$$(6.4) \quad M^2 f_2(y) \leq M(\chi_{2Q})(y) \sup_{Q \subset 4Q'} \frac{1}{|Q'|} \int_{Q'} |f| + M(\chi_{\mathbf{R}^n \setminus 2Q} Mf)(y)$$

for any $y \in Q$.

Since $M(\chi_{2Q})(y) = 1$, $y \in Q$, by inequality (6.2), we arrive at

$$(6.5) \quad M^2 f_2(y) \leq \sup_{Q \subset 4Q'} \frac{1}{|Q'|} \int_{Q'} |f| + \sup_{Q \subset 2Q'} \frac{1}{|Q'|} \int_{Q'} Mf \lesssim \sup_{Q \subset Q'} \frac{1}{|Q'|} \int_{Q'} Mf.$$

Consequently, for $y \in Q$

$$(6.6) \quad M^2 f(y) \lesssim M^2(f\chi_{4Q})(y) + \sup_{Q \subset Q'} \frac{1}{|Q'|} \int_{Q'} Mf.$$

Since

$$(6.7) \quad 1 + \log^+(ab) \leq (1 + \log^+ a)(1 + \log^+ b),$$

by Lemma 2.1, for any $\alpha > 0$ and $t > 0$, we have

$$\begin{aligned} & \left| \left\{ x \in Q : M^2(f\chi_{4Q})(x) > \alpha t \right\} \right| \\ & \leq \left| \left\{ x \in \mathbf{R}^n : M^2(f\chi_{4Q})(x) > \alpha t \right\} \right| \\ & \leq c \int_{\mathbf{R}^n} \frac{|(f\chi_{4Q})(x)|}{\alpha t} \left(1 + \log^+ \left(\frac{|(f\chi_{4Q})(x)|}{\alpha t} \right) \right) dx \\ & \leq c \frac{1}{\alpha} \left(1 + \log^+ \frac{1}{\alpha} \right) \int_{4Q} \frac{|f(x)|}{t} \left(1 + \log^+ \left(\frac{|f(x)|}{t} \right) \right) dx. \end{aligned}$$

Therefore

$$\frac{|\{x \in Q : M^2(f\chi_{4Q})(x) > \alpha t\}|}{\frac{1}{\alpha} \left(1 + \log^+ \frac{1}{\alpha}\right)} \leq c \int_{4Q} \frac{|f(x)|}{t} \left(1 + \log^+ \left(\frac{|f(x)|}{t}\right)\right) dx.$$

Consequently,

$$\sup_{\alpha > 0} \frac{1}{|Q|} \frac{|\{x \in Q : M^2(f\chi_{4Q})(x) > \alpha t\}|}{\frac{1}{\alpha} \left(1 + \log^+ \frac{1}{\alpha}\right)} \leq c \frac{1}{|4Q|} \int_{4Q} \frac{|f(x)|}{t} \left(1 + \log^+ \left(\frac{|f(x)|}{t}\right)\right) dx.$$

Thus

$$\begin{aligned} \inf \left\{ t > 0 : \sup_{\alpha > 0} \frac{1}{|Q|} \frac{|\{x \in Q : M^2(f\chi_{4Q})(x) > \alpha t\}|}{\frac{1}{\alpha} \left(1 + \log^+ \frac{1}{\alpha}\right)} \leq 1 \right\} \\ \leq \inf \left\{ t > 0 : \frac{1}{|4Q|} \int_{4Q} \frac{c|f(x)|}{t} \left(1 + \log^+ \left(\frac{|f(x)|}{t}\right)\right) dx \leq 1 \right\} \\ \leq \inf \left\{ t > 0 : \frac{1}{|4Q|} \int_{4Q} \frac{c|f(x)|}{t} \left(1 + \log^+ \left(\frac{c|f(x)|}{t}\right)\right) dx \leq 1 \right\} \end{aligned}$$

that is,

$$(6.8) \quad \|M^2(f\chi_{4Q})\|_{WL(1+\log^+ L), Q} \leq \|cf\|_{L(1+\log^+ L), 4Q} = c\|f\|_{L(1+\log^+ L), 4Q}.$$

For the second summand on the right hand side of inequality (6.6), applying the inequality (4.6) we obtain

$$(6.9) \quad \left\| \sup_{Q \subset Q'} \frac{1}{|Q'|} \int_{Q'} Mf \right\|_{WL(1+\log^+ L), Q} \lesssim \sup_{Q \subset Q'} \frac{1}{|Q'|} \int_{Q'} Mf \\ \lesssim \sup_{Q \subset Q'} \|f\|_{L(1+\log^+ L), Q'}.$$

By inequalities (6.6), (6.8) and (6.9) we get

$$(6.10) \quad \|M^2 f\|_{WL(1+\log^+ L), Q} \leq c \sup_{Q \subset 4Q'} \|f\|_{L(1+\log^+ L), Q'}.$$

Therefore

$$\begin{aligned} \sup_Q |Q|^{\frac{\lambda}{n}} \|M^2 f\|_{WL(1+\log^+ L), Q} &\leq c \sup_Q |Q|^{\frac{\lambda}{n}} \sup_{Q \subset 4Q'} \|f\|_{L(1+\log^+ L), Q'} \\ &\leq c \left(\sup_Q |Q|^{\frac{\lambda}{n}} \sup_{Q \subset 4Q'} |Q'|^{-\frac{\lambda}{n}} \right) \sup_Q |Q|^{\frac{\lambda}{n}} \|f\|_{L(1+\log^+ L), Q} \\ &\approx \sup_Q |Q|^{\frac{\lambda}{n}} \|f\|_{L(1+\log^+ L), Q}, \end{aligned}$$

that is,

$$\|M^2 f\|_{\mathcal{WM}_{L(1+\log^+ L),\lambda}} \leq c \|f\|_{\mathcal{M}_{L(1+\log^+ L),\lambda}}.$$

□

7. Proofs of main results

In this section we prove main results.

PROOF OF THEOREM 1.10. (i) \Rightarrow (ii). Assume that $b \in \text{BMO}(\mathbf{R}^n)$. By Theorems 1.9 and 6.1 operator C_b is bounded from $\mathcal{M}_{L(1+\log^+ L),\lambda}$ to $\mathcal{WM}_{L(1+\log^+ L),\lambda}$ and the inequality

$$(7.1) \quad \|C_b(f)\|_{\mathcal{WM}_{L(1+\log^+ L),\lambda}} \leq c \|b\|_* \|f\|_{\mathcal{M}_{L(1+\log^+ L),\lambda}}$$

holds with positive constant c independent of f .

(ii) \Rightarrow (i). Assume that the inequality

$$(7.2) \quad \|C_b(f)\|_{\mathcal{WM}_{L(1+\log^+ L),\lambda}} \leq c \|f\|_{\mathcal{M}_{L(1+\log^+ L),\lambda}}$$

holds with positive constant c independent of f . Let Q_0 be any cube in \mathbf{R}^n and let $f = \chi_{Q_0}$. By Theorem 4.1,

$$\begin{aligned} \|\chi_{Q_0}\|_{\mathcal{M}_{L(1+\log^+ L),\lambda}} &\approx \sup_Q |Q|^{\frac{\lambda-n}{n}} \int_Q \chi_{Q_0} \left(1 + \log^+ \frac{\chi_{Q_0}}{(\chi_{Q_0})_Q}\right) \\ &= \sup_{Q: Q \cap Q_0 \neq \emptyset} |Q|^{\frac{\lambda}{n}} \frac{|Q \cap Q_0|}{|Q|} \left(1 + \log \frac{|Q|}{|Q \cap Q_0|}\right). \end{aligned}$$

Obviously,

$$\|\chi_{Q_0}\|_{\mathcal{M}_{L(1+\log^+ L),\lambda}} \gtrsim |Q_0|^{\frac{\lambda}{n}}.$$

Let $\varepsilon \in (0, 1 - \lambda/n)$. Since the function $(1 + \log t)/t^\varepsilon$ is bounded on the interval $[1, \infty)$, we get

$$\begin{aligned} \|\chi_{Q_0}\|_{\mathcal{M}_{L(1+\log^+ L),\lambda}} &\lesssim \sup_{Q: Q \cap Q_0 \neq \emptyset} |Q|^{\frac{\lambda}{n}} \frac{|Q \cap Q_0|}{|Q|} \left(\frac{|Q|}{|Q \cap Q_0|}\right)^\varepsilon \\ &= \sup_{Q: Q \cap Q_0 \neq \emptyset} |Q|^{\frac{\lambda}{n} + \varepsilon - 1} |Q \cap Q_0|^{1-\varepsilon} \\ &= \sup_{Q \subseteq Q_0} |Q|^{\frac{\lambda}{n} + \varepsilon - 1} |Q \cap Q_0|^{1-\varepsilon} = |Q_0|^{\frac{\lambda}{n}}. \end{aligned}$$

Thus

$$(7.3) \quad \|\chi_{Q_0}\|_{\mathcal{M}_{L(1+\log^+ L),\lambda}} \approx |Q_0|^{\frac{\lambda}{n}}.$$

On the other hand

$$\begin{aligned} \|C_b(\chi_{Q_0})\|_{W\mathcal{M}_{L(1+\log^+ L),\lambda}} &= \sup_Q |Q|^{\frac{\lambda}{n}} \|C_b(\chi_{Q_0})\|_{WL(1+\log^+ L),Q} \\ &\geq |Q_0|^{\frac{\lambda}{n}} \|C_b(\chi_{Q_0})\|_{WL(1+\log^+ L),Q_0}. \end{aligned}$$

Note that

$$\begin{aligned} \|C_b(\chi_{Q_0})\|_{WL(1+\log^+ L),Q_0} &= \inf \left\{ \lambda > 0 : \sup_{t>0} \frac{1}{|Q_0|} \frac{|\{x \in Q_0 : |C_b(\chi_{Q_0})(x)| > \lambda t\}|}{\frac{1}{t} \left(1 + \log^+ \frac{1}{t}\right)} \leq 1 \right\} \\ &\geq \inf \left\{ \lambda > 0 : \frac{2}{|Q_0|} |\{x \in Q_0 : |C_b(\chi_{Q_0})(x)| > 2\lambda\}| \leq 1 \right\}. \end{aligned}$$

Since for any $x \in Q_0$

$$C_b(\chi_{Q_0})(x) \geq \frac{1}{|Q_0|} \int_{Q_0} |b(x) - b(y)| dy \geq \frac{1}{2|Q_0|} \int_{Q_0} |b(y) - b_{Q_0}| dy,$$

then

$$\frac{2}{|Q_0|} \left| \left\{ x \in Q_0 : C_b(\chi_{Q_0})(x) > 2 \frac{1}{4|Q_0|} \int_{Q_0} |b(y) - b_{Q_0}| dy \right\} \right| = 2.$$

Thus

$$\|C_b(\chi_{Q_0})\|_{WL(1+\log^+ L),Q_0} \geq \frac{1}{4|Q_0|} \int_{Q_0} |b(y) - b_{Q_0}| dy.$$

Consequently,

$$(7.4) \quad \|C_b(\chi_{Q_0})\|_{W\mathcal{M}_{L(1+\log^+ L),\lambda}} \gtrsim |Q_0|^{\frac{\lambda}{n}} \frac{1}{|Q_0|} \int_{Q_0} |b(y) - b_{Q_0}| dy.$$

By (7.2), (7.3) and (7.4) we arrive at

$$\frac{1}{|Q_0|} \int_{Q_0} |b(y) - b_{Q_0}| dy \lesssim c.$$

□

PROOF OF THEOREM 1.11. Combining Theorems 1.9 and 5.5, we get the proof of Theorem 1.11. □

PROOF OF THEOREM 1.12. The statement of Theorem 1.12 follows by Theorems 1.9 and 6.1. □

PROOF OF THEOREM 1.13. The proof of Theorem 1.13 follows by Theorems 1.9 and 5.5. □

References

- [1] M. AGCAYAZI, A. GOGATISHVILI, K. KOCA and R. MUSTAFAYEV, A note on maximal commutators and commutators of maximal functions, *J. Math. Soc. Japan.* **67** (2015), no. 2, 581–593.
- [2] J. BASTERO, M. MILMAN and F. J. RUIZ, Commutators for the maximal and sharp functions, *Proc. Amer. Math. Soc.* **128** (2000), no. 11, 3329–3334 (electronic). MR1777580 (2001i:42027)
- [3] C. BENNETT and R. SHARPLEY, Weak-type inequalities for H^p and BMO, *Harmonic analysis in Euclidean spaces* (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978), Part 1, Proc. Sympos. Pure Math., XXXV, Part, Amer. Math. Soc., Providence, R.I., 1979, pp. 201–229. MR545259 (80j:46044)
- [4] C. BENNETT and R. SHARPLEY, *Interpolation of operators*, Pure and Applied Mathematics, vol. 129, Academic Press Inc., Boston, MA, 1988. xiv+469. 0-12-088730-4, MR928802 (89e:46001)
- [5] A. BONAMI, T. IWANIEC, P. JONES and M. ZINSMEISTER, On the product of functions in BMO and H^1 , *Ann. Inst. Fourier (Grenoble)* **57** (2007), no. 5, 1405–1439. MR2364134 (2009d:42054)
- [6] M. CAROZZA and A. PASSARELLI DI NAPOLI, Composition of maximal operators, *Publ. Mat.* **40** (1996), no. 2, 397–409. MR1425627 (98f:42013)
- [7] F. CHIARENZA and M. FRASCA, Morrey spaces and Hardy-Littlewood maximal function, *Rend. Mat. Appl. (7)* **7** (1987), no. 3–4, 273–279 (1988). MR985999 (90f:42017)
- [8] J. GARCIA-CUERVA and J. L. RUBIO DE FRANCIA, *Weighted norm inequalities and related topics*, North-Holland Mathematics Studies, vol. 116, North-Holland Publishing Co., Amsterdam, 1985. Notas de Matemática [Mathematical Notes], 104.
- [9] J. GARCIA-CUERVA, E. HARBOURE, C. SEGOVIA and J. L. TORREA, Weighted norm inequalities for commutators of strongly singular integrals, *Indiana Univ. Math. J.* **40** (1991), no. 4, 1397–1420. MR1142721 (93f:42031)
- [10] D. GILBARG and N. S. TRUDINGER, *Elliptic partial differential equations of second order*, 2nd ed., Springer-Verlag, Berlin, 1983.
- [11] A. GOGATISHVILI and R. CH. MUSTAFAYEV, A note on boundedness of the Hardy-Littlewood maximal operator on Morrey spaces, *Mediterr. J. Math.* **13** (2016), no. 4, 1885–1891.
- [12] L. GRAFAKOS, *Classical Fourier analysis*, 2nd ed., Graduate Texts in Mathematics, vol. 249, Springer, New York, 2008. MR2445437 (2011c:42001)
- [13] L. GRAFAKOS, *Modern Fourier analysis*, 2nd ed., Graduate Texts in Mathematics, vol. 250, Springer, New York, 2009. MR2463316 (2011d:42001)
- [14] M. DE GUZMÁN, *Differentiation of integrals in R^n* , Lecture Notes in Mathematics, Vol. 481, Springer-Verlag, Berlin-New York, 1975. With appendices by Antonio Córdoba, and Robert Fefferman, and two by Roberto Moriyón.
- [15] G. HU, H. LIN and D. YANG, Commutators of the Hardy-Littlewood maximal operator with BMO symbols on spaces of homogeneous type, *Abstr. Appl. Anal.* (2008), Art. ID 237937, 21. MR2393116 (2009d:42046)
- [16] G. HU and D. YANG, Maximal commutators of BMO functions and singular integral operators with non-smooth kernels on spaces of homogeneous type, *J. Math. Anal. Appl.* **354** (2009), no. 1, 249–262. MR2510436 (2010c:43018)
- [17] T. IWANIEC and G. MARTIN, *Geometric function theory and non-linear analysis*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2001. MR1859913
- [18] F. JOHN and L. NIRENBERG, On functions of bounded mean oscillation, *Comm. Pure Appl. Math.* **14** (1961), 415–426. MR0131498 (24 #A1348)
- [19] H. KITA, On maximal functions in Orlicz spaces, *Proc. Amer. Math. Soc.* **124** (1996), no. 10, 3019–3025. MR1376993 (97b:42031)
- [20] M. A. LECKBAND and C. J. NEUGEBAUER, A general maximal operator and the A_p -condition, *Trans. Amer. Math. Soc.* **275** (1983), no. 2, 821–831, DOI 10.2307/1999056. MR682735 (84c:42029)
- [21] M. A. LECKBAND, A note on maximal operators and reversible weak type inequalities, *Proc. Amer. Math.*

- Soc. **92** (1984), no. 1, 19–26, DOI 10.2307/2045145. MR749882 (86i:42009)
- [22] D. LI, G. HU and X. SHI, Weighted norm inequalities for the maximal commutators of singular integral operators, *J. Math. Anal. Appl.* **319** (2006), no. 2, 509–521. MR2227920 (2007a:42041)
- [23] M. MILMAN and T. SCHONBEK, Second order estimates in interpolation theory and applications, *Proc. Amer. Math. Soc.* **110** (1990), no. 4, 961–969. MR1075187 (91k:46088)
- [24] C. B. MORREY, On the solutions of quasi-linear elliptic partial differential equations, *Trans. Amer. Math. Soc.* **43** (1938), no. 1, 126–166, DOI 10.2307/1989904. MR1501936
- [25] C. PEREZ, Endpoint estimates for commutators of singular integral operators, *J. Funct. Anal.* **128** (1995), no. 1, 163–185. MR1317714 (95j:42011)
- [26] M. M. RAO and Z. D. REN, *Theory of Orlicz spaces*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 146, Marcel Dekker Inc., New York, 1991. MR1113700 (92e:46059)
- [27] Y. SAWANO, S. SUGANO and H. TANAKA, Orlicz-Morrey spaces and fractional operators, *Potential Anal.* **36** (2012), no. 4, 517–556, DOI 10.1007/s11118-011-9239-8. MR2904632
- [28] C. SEGOVIA and J. L. TORREA, Weighted inequalities for commutators of fractional and singular integrals, *Publ. Mat.* **35** (1991), no. 1, 209–235. Conference on Mathematical Analysis (El Escorial, 1989). MR1103616 (93f:42035)
- [29] C. SEGOVIA and J. L. TORREA, Higher order commutators for vector-valued Calderón-Zygmund operators, *Trans. Amer. Math. Soc.* **336** (1993), no. 2, 537–556. MR1074151 (93f:42036)
- [30] E. M. STEIN, Note on the class $L \log$, *Studia Math.* **32** (1969), 305–310. MR0247534 (40 #799)
- [31] E. M. STEIN, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970. MR0290095 (44 #7280)
- [32] E. M. STEIN, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy; Monographs in Harmonic Analysis, III. MR1232192 (95c:42002)
- [33] A. TORCHINSKY, *Real-variable methods in harmonic analysis*, Pure and Applied Mathematics, vol. 123, Academic Press, Inc., Orlando, FL, 1986. MR869816 (88e:42001)
- [34] C. P. XIE, Some estimates of commutators, *Real Anal. Exchange* **36** (2010/11), no. 2, 405–415. MR3016724

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