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On a Variational Problem Arising from the Three-component FitzHugh-Nagumo Type Reaction-Diffusion Systems

Takashi KAJIWARA and Kazuhiro KURATA

Tokyo Metropolitan University

Abstract. We study a variational problem arising from the three-component Fitzhugh-Nagumo type reaction diffusion systems and its shadow systems.

In [15], Oshita studied the two-component systems. He revealed that a minimizer of energy corresponding to the problem oscillates under an appropriate condition and also studied its stability. Moreover, he mentioned its energy estimate without a proof.

We investigate the behavior of a minimizer corresponding to the three-component problem, its stability and its energy estimate and extend some results of Oshita to the three-component systems and its shadow systems. In particular, we give a necessary and sufficient condition that the minimizer highly oscillates as $\epsilon \to 0$. Also, we establish a precise order of the energy estimate of the minimizer as $\epsilon \to 0$. In the proof of the energy estimate, we propose a new interpolation inequality.

1. Introduction

In this paper, we study the existence and its stability of some steady states to the following reaction-diffusion systems:

$$
(1.1) \qquad \begin{cases} u_t(x,t) = \epsilon^2 \Delta u(x,t) + f(u(x,t)) - v(x,t) - w(x,t)\,, & x \in \Omega, t > 0\,, \\ \tau_1 v_t(x,t) = d_1 \Delta v(x,t) - \gamma_1 v(x,t) + \delta_1 u(x,t)\,, & x \in \Omega, t > 0\,, \\ \tau_2 w_t(x,t) = d_2 \Delta w(x,t) - \gamma_2 w(x,t) + \delta_2 u(x,t)\,, & x \in \Omega, t > 0\,, \\ \frac{\partial u}{\partial n}(x,t) = \frac{\partial v}{\partial n}(x,t) = \frac{\partial w}{\partial n}(x,t) = 0\,, & x \in \partial\Omega, t > 0\,, \end{cases}
$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with the smooth boundary $\partial\Omega$; $\frac{\partial}{\partial n}$ is a normal derivative on $\partial\Omega$; $\epsilon > 0$, $\tau_i > 0$, $d_i > 0$, $\delta_i \geq 0$ and $\gamma_i > 0$ $(i = 1, 2)$ are constants; f is a function satisfying the following conditions:

(f1) $f(t) \in C^2(\mathbb{R})$.

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(f2) There are $0 < \tau_1 < \tau_2$ such that $f(\tau_1) < 0$, $f(\tau_2) > 0$, $f'(t) < 0$ if $t \in (-\infty, \tau_1) \cup (\tau_2, \infty)$, and $f'(t) > 0$ if $t \in (\tau_1, \tau_2)$. Moreover, $\lim_{t\to\infty} \frac{f(t)}{t} = -\infty$, $\lim_{t\to-\infty} \frac{f(t)}{t} = -\infty$.

(f3) Let $I_{-1} = (-\infty, \tau_1)$, $I_0 = (\tau_1, \tau_2)$ and $I_1 = (\tau_2, \infty)$. Moreover, let $a_i \in I_i$ $(i = -1, 0, 1)$ be zero points of $f(t)$. We assume that $\int_{a_{-1}}^{a_1} f(s)\,ds > 0$.

Typical examples include $f(u) = u(1-u)(u-a)$ $(0 < a < \frac{1}{2})$ and $f(u) = u(u-\frac{1}{2})(1-u)+c$ with $c > 0$ small.

Equations (1.1) is called the FitzHugh-Nagumo type system, which is originally introduced in the field of physiology. This is also studied mathematically as a model which generates complex patterns. Oshita and Dancer-Yan studied steady state solutions of the two-component system, which corresponds to the case $\delta_2 = 0$ in (1.1), and proved that the minimizer of the energy associated with (1.1) highly oscillates between two stable values ([6, 15]). Moreover, Nishiura and Ren-Wei studied several constructions of solutions of the two-component system in the case the dimension $N = 1$ ([12, 16]).

In recent years, the three-component system (1.1) is proposed as a qualitative model of gas discharge phenomena and there are several studies on (1.1) ([1, 7, 10, 13]).

The steady state problem of (1.1) is

$$(1.2)\qquad \begin{cases} 0 = \epsilon^2 \Delta u(x) + f(u(x)) - v(x) - w(x), & x \in \Omega\,, \\ 0 = d_1 \Delta v(x) - \gamma_1 v(x) + \delta_1 u(x)\,, & x \in \Omega\,, \\ 0 = d_2 \Delta w(x) - \gamma_2 w(x) + \delta_2 u(x)\,, & x \in \Omega\,, \\ \frac{\partial u}{\partial n}(x) = \frac{\partial v}{\partial n}(x) = \frac{\partial w}{\partial n}(x) = 0\,, & x \in \partial\Omega\,. \end{cases}$$

First, we assume $d_i = 1$ $(i = 1, 2)$ for the systems (1.2).

Before we state our results, we give some notations. Let $u = h_+(v)$, $v \in f(I_1)$, be the inverse function of $v = f(u)$ restricted to I_1, and let $u = h_-(v)$, $v \in f(I_{-1})$, be the inverse function of $v = f(u)$ restricted to I_{-1}. Then there exists a unique number α_0 such that

$$\int_{h_-(\alpha_0)}^{h_+(\alpha_0)} (f(s) - \alpha_0)\,ds = 0$$

from (f3). For each $u \in H^1(\Omega)$, let $v = G_\gamma u$ be the unique solution of the following problem:

$$\begin{cases} -\Delta v(x) + \gamma v(x) = u(x)\,, & x \in \Omega\,, \\ \frac{\partial v}{\partial n}(x) = 0\,, & x \in \partial\Omega\,. \end{cases}$$

Then we see that (1.2) with $d_i = 1$ $(i = 1, 2)$ is reduced to the following nonlocal elliptic problem:

$$(1.3)\qquad \begin{cases} -\epsilon^2 \Delta u = f(u) - \delta_1 G_{\gamma_1} u - \delta_2 G_{\gamma_2} u\,, & x \in \Omega\,, \\ \frac{\partial u}{\partial n}(x) = 0\,, & x \in \partial\Omega\,. \end{cases}$$

The energy associated with (1.3) is

$$(1.4)\qquad I_\epsilon(u) = \frac{1}{2}\int_\Omega \epsilon^2|\nabla u|^2 dx + \frac{\delta_1}{2}\int_\Omega uG_{\gamma_1}u\,dx + \frac{\delta_2}{2}\int_\Omega uG_{\gamma_2}u\,dx - \int_\Omega F(u)\,dx\,,$$

where $F(t) = \int_{h_-(\alpha_0)}^t f(\tau)\,d\tau$. Then we can see the following problem has a minimizer (see Proposition 2.1):

$$(1.5)\qquad \sigma = \inf\{I_\epsilon(u) : u \in H^1(\Omega)\}\,.$$

Next, we also study the shadow system as follows:

$$(1.6)\qquad \begin{cases} u_t(x,t) = \epsilon^2\Delta u(x,t) + f(u(x,t)) - v(x,t) - \xi(t)\,, & x\in\Omega\,,\ t>0\,,\\ \tau_1 v_t(x,t) = \Delta v(x,t) - \gamma_1 v(x,t) + \delta_1 u(x,t)\,, & x\in\Omega\,,\ t>0\,,\\ \tau_2\xi_t(t) = -\gamma_2\xi(t) + \frac{\delta_2}{|\Omega|}\int_\Omega u(\zeta,t)\,d\zeta\,, & t>0\,,\\ \frac{\partial u}{\partial n}(x,t) = \frac{\partial v}{\partial n}(x,t) = 0\,, & x\in\partial\Omega\,,\ t>0\,. \end{cases}$$

Shadow systems are often used to approximate reaction-diffusion systems when one of the diffusion rates is large. (1.6) corresponds to the case that $d_1 = 1$ and d_2 is large in (1.1).

We consider the steady state of (1.6) as follows:

$$(1.7)\qquad \begin{cases} 0 = \epsilon^2\Delta u(x) + f(u(x)) - v(x) - \frac{\delta_2}{\gamma_2|\Omega|}\int_\Omega u(\zeta)\,d\zeta\,, & x\in\Omega\,,\\ 0 = \Delta v(x) - \gamma_1 v(x) + \delta_1 u(x)\,, & x\in\Omega\,,\\ \frac{\partial u}{\partial n}(x) = \frac{\partial v}{\partial n}(x) = 0\,, & x\in\partial\Omega\,. \end{cases}$$

The energy associated with (1.7) is

$$(1.8)\quad J_\epsilon(u) = \frac{1}{2}\int_\Omega \epsilon^2\,|\nabla u|^2\,dx + \frac{\delta_1}{2}\int_\Omega uG_{\gamma_1}u\,dx + \frac{\delta_2}{2\gamma_2\,|\Omega|}\left(\int_\Omega u\,dx\right)^2 - \int_\Omega F(u)dx\,.$$

Then we consider the minimizing problem:

$$(1.9)\qquad \inf\left\{J_\epsilon(u) :\ u\in H^1(\Omega)\right\}.$$

First, for the existence and properties of the minimizer of (1.5), we have the following result.

THEOREM 1.1. *Let u_ϵ be a global minimizer of* (1.5) *and let* $v_\epsilon = \delta_1 G_{\gamma_1}u_\epsilon$, $w_\epsilon = \delta_2 G_{\gamma_2}u_\epsilon$. *Moreover, let* $s_i = \frac{\delta_i}{\gamma_i}$ $(i = 1,2)$, $S = s_1 + s_2$. *Then we have*

(1) *If* $0 < S \le \frac{\alpha_0}{h_+(\alpha_0)}$, *then* $u_\epsilon \equiv h_+(c)$, $v_\epsilon \equiv \frac{s_1}{S}c$, $w_\epsilon \equiv \frac{s_2}{S}c$ *for any* $\epsilon > 0$, *where c is the constant that satisfies* $S = \frac{c}{h_+(c)}$.

(2) *If* $h_-(\alpha_0) \le 0$, $S > \frac{\alpha_0}{h_+(\alpha_0)}$, *or* $h_-(\alpha_0) > 0$, $\frac{\alpha_0}{h_-(\alpha_0)} > S > \frac{\alpha_0}{h_+(\alpha_0)}$, *then* $v_\epsilon \to \frac{s_1}{S}\alpha_0$ *in* $C^{1,\beta}(\Omega)$, $w_\epsilon \to \frac{s_2}{S}\alpha_0$ *in* $C^{1,\beta}(\Omega)$ *as* $\epsilon \to 0$ *for all* $\beta \in (0,1)$. *Moreover,* u_ϵ *has following properties*;

(a) *$u_\epsilon \to \frac{\alpha_0}{S}$ weakly in $L^2(\Omega)$. However, $(u_\epsilon)_\epsilon$ does not have any subsequence that converges strongly in $L^1(\Omega)$.*

(b) *For any $\theta > 0$ small,*

$$\lim_{\epsilon \to 0} \left| \Omega^+_{\epsilon,\theta} \cap \Omega^-_{\epsilon,\theta} \right| = 0\,,$$

where $\Omega^+_{\epsilon,\theta}$, $\Omega^-_{\epsilon,\theta}$ are defined as follows:

$$\Omega^+_{\epsilon,\theta} = \{x \in \Omega : |u_\epsilon(x) - h_+(\alpha_0)| \ge \theta\}\ ,$$

$$\Omega^-_{\epsilon,\theta} = \{x \in \Omega : |u_\epsilon(x) - h_-(\alpha_0)| \ge \theta\}\ .$$

Moreover, $(u_\epsilon)_\epsilon$ does not have any subsequence that converges in measure.

(3) *If $h_-(\alpha_0) > 0$, $S \ge \frac{\alpha_0}{h_-(\alpha_0)}$, then $u_\epsilon \equiv h_-(c)$, $v_\epsilon \equiv \frac{s_1}{S}c$, $w_\epsilon \equiv \frac{s_2}{S}c$ for any $\epsilon > 0$, where c is the constant that satisfies $S = \frac{c}{h_-(c)}$.*

REMARK 1. It is easy to see that $h_-(\alpha_0) < 0$ holds for $f(u) = u(u-a)(1-u)$ with $a \in (0, \frac{1}{2})$ and $h_+(\alpha_0) > 0$ holds for $f(u) = u(u - \frac{1}{2})(1-u) + c$ with $c > 0$ small.

REMARK 2. Oshita [15] studied the case $\delta_2 = 0$ and $h_-(\alpha_0) \le 0$ and proved almost the same results by using the notion of the Young measure.

Dancer-Yan [6] studied the both cases $h_-(\alpha_0) \le 0$ and $h_-(\alpha_0) > 0$ for the Dirichlet boundary condition.

For the minimizer of (1.9), we have the following result.

THEOREM 1.2. *Let u_ϵ be a global minimizer of (1.9) and let $v_\epsilon = \delta_1 G_{\gamma_1} u_\epsilon$. Moreover, let $s_i = \frac{\delta_i}{\gamma_i}$ $(i = 1, 2)$, $S = s_1 + s_2$. Then we have*

(1) *If $0 < S \le \frac{\alpha_0}{h_+(\alpha_0)}$, then $u_\epsilon \equiv h_+(c)$, $v_\epsilon \equiv \frac{s_1}{S}c$ for any $\epsilon > 0$, where c is the constant that satisfies $S = \frac{c}{h_+(c)}$.*

(2) *If $h_-(\alpha_0) \le 0$, $S > \frac{\alpha_0}{h_+(\alpha_0)}$, or $h_-(\alpha_0) > 0$, $\frac{\alpha_0}{h_-(\alpha_0)} > S > \frac{\alpha_0}{h_+(\alpha_0)}$, then $v_\epsilon \to \frac{s_1}{S}\alpha_0$ in $C^{1,\beta}(\Omega)$ as $\epsilon \to 0$ for all $\beta \in (0, 1)$. Moreover, u_ϵ holds (a) and (b) in Theorem 1.1.*

(3) *If $h_-(\alpha_0) > 0$, $S \ge \frac{\alpha_0}{h_-(\alpha_0)}$, then $u_\epsilon \equiv h_-(c)$, $v_\epsilon \equiv \frac{s_i}{S}c$ for any $\epsilon > 0$, where c is the constant that satisfies $S = \frac{c}{h_-(c)}$.*

For the stability of the global minimizer obtained in Theorems 1.1 and 1.2, we have the following results, which extend Theorem 1.2 in [15].

THEOREM 1.3. *Assume $\frac{\delta_1 \tau_1}{\gamma_1^2} + \frac{\delta_2 \tau_2}{\gamma_2^2} < 1$. Then the following statements hold.*

(1) *For a local minimizer* u_0 *of* $I_\epsilon(u)$ *on* $H^1(\Omega)$, (u_0, v_0, w_0) *is a stable solution to the problem* (1.1), *where* $v_0 = \delta_1 G_{\gamma_1} u_0$ *and* $w_0 = \delta_2 G_{\gamma_2} u_0$.

(2) *For a local minimizer* u_0 *of* $J_\epsilon(u)$ *on* $H^1(\Omega)$, (u_0, v_0, ξ_0) *is a stable solution to the problem* (1.6), *where* $v_0 = \delta_1 G_{\gamma_1} u_0$ *and* $\xi_0 = \frac{\delta_2}{\gamma_2 |\Omega|} \int_\Omega u_0(x)\, dx$.

We can prove Theorems 1.3 using essentially the same arguments as in the proof of the Theorem 1.2 in [15]. However, emphasizing the case of the shadow system, we give the proof for the sake of completeness in the Appendix A.

Furthermore, we give the following precise order of the energy estimates as $\epsilon \to 0$.

THEOREM 1.4. *Let* $\Omega = (0,1)^N \subset \mathbb{R}^N$ $(N \geq 1)$, ζ *be a small positive constant, and* $\bar{\delta}$ *be any positive constant. Let* u_ϵ *be a minimizer of* (1.5). *We assume* $S = s_1 + s_2$ *with* $s_i = \frac{\delta_i}{\gamma_i}$ $(i = 1, 2)$ *satisfies the following conditions*;

(i) *If* $h_-(\alpha_0) < 0$, *then* $S \in \left(\frac{\alpha_0}{h_+(\alpha_0)} + \zeta, \infty\right)$.

(ii) *If* $h_-(\alpha_0) = 0$, *then* $S \in \left(\frac{\alpha_0}{h_+(\alpha_0)} + \zeta, \frac{h_+(\alpha_0)}{\zeta}\right)$.

(iii) *If* $h_-(\alpha_0) > 0$, *then* $S \in \left(\frac{\alpha_0}{h_+(\alpha_0)} + \zeta, \frac{\alpha_0}{h_-(\alpha_0)} - \zeta\right)$.

Then there exist positive constants $\epsilon_* = \epsilon_*(\zeta, \bar{\delta})$, $l_0 = l_0(\zeta)$, L_0 *such that*

$$l_0 \epsilon^{\frac{2}{3}} \left(\delta_1^{\frac{1}{3}} + \delta_2^{\frac{1}{3}}\right) - C_* \leq I(u_\epsilon) \leq L_0 \epsilon^{\frac{2}{3}} \left(\delta_1^{\frac{1}{3}} + \delta_2^{\frac{1}{3}}\right) - C_* \tag{1.10}$$

for all $\epsilon \in (0, \epsilon_*)$, $\delta_i \in (\epsilon, \bar{\delta})$, *where*

$$C_* = \left(\frac{\alpha_0^2}{2S} - \alpha_0 h_-(\alpha_0)\right) |\Omega|. \tag{1.11}$$

THEOREM 1.5. *Let* $\Omega = (0,1)^N \subset \mathbb{R}^N$ $(N \geq 1)$, ζ *be a small positive constant, and* $\bar{\delta}$ *be any positive constant. Let* u_ϵ *be a minimizer of* (1.9). *We assume* $S = s_1 + s_2$ *with* $s_i = \frac{\delta_i}{\gamma_i}$ $(i = 1, 2)$ *satisfies the following conditions*;

(i) *If* $h_-(\alpha_0) < 0$, *then* $S \in \left(\frac{\alpha_0}{h_+(\alpha_0)} + \zeta, \infty\right)$.

(ii) *If* $h_-(\alpha_0) = 0$, *then* $S \in \left(\frac{\alpha_0}{h_+(\alpha_0)} + \zeta, \frac{h_+(\alpha_0)}{\zeta}\right)$.

(iii) *If* $h_-(\alpha_0) > 0$, *then* $S \in \left(\frac{\alpha_0}{h_+(\alpha_0)} + \zeta, \frac{\alpha_0}{h_-(\alpha_0)} - \zeta\right)$.

Then there exist positive constants ${\epsilon_*}' = {\epsilon_*}'(\zeta, \bar{\delta})$, ${l_0}' = {l_0}'(\zeta)$, L_0' *such that*

$$ {l_0}' \epsilon^{\frac{2}{3}} \delta_1^{\frac{1}{3}} - C_* \leq J_\epsilon(u_\epsilon) \leq L_0' \epsilon^{\frac{2}{3}} \delta_1^{\frac{1}{3}} - C_* \tag{1.12}$$

for all $\epsilon \in (0, {\epsilon_*}')$, $\delta_i \in (\epsilon, \bar{\delta})$, *where* C_* *is defined as* (1.11).

REMARK 3. We conjecture that the statement in Theorems 1.4 and 1.5 are also true even for general bounded domains Ω. However, to show this it will be more involved to establish technical lemmas (Lemmas 3.2 and 3.3) for general bounded domains Ω by modifying the operator T_N suitably.

REMARK 4. In this paper, we consider the 3-component model, but our proof can also be applied the following $(n+1)$-component model (one activator - n inhibitor model);

$$\begin{cases} -\epsilon^2 \Delta u(x) = f(u(x)) - \sum_{j=1}^n v_i(x)\,, & x \in \Omega\,, \\ -\Delta v_i(x) + \gamma_i v_i(x) = \delta_i u(x)\,, & x \in \Omega\,, \quad i = 1, \dots n, \\ \frac{\partial u}{\partial n}(x) = \frac{\partial v_i}{\partial n}(x) = 0\,, & x \in \partial\Omega\,, \quad i = 1, \dots n\,. \end{cases}$$

For example, if $h_-(\alpha_0) \le 0$, $S = \sum_{j=1}^n \frac{\delta_i}{\gamma_i} \in \left(\frac{\alpha_0}{h_+(\alpha_0)}, \infty\right)$, or $h_-(\alpha_0) > 0$, $S = \sum_{j=1}^n \frac{\delta_i}{\gamma_i} \in \left(\frac{\alpha_0}{h_+(\alpha_0)}, \frac{\alpha_0}{h_-(\alpha_0)}\right)$, then the minimizer u_ϵ exhibits the oscillatory behavior as in Theorem 1.1.

Roughly speaking, Theorems 1.1 and 1.2 imply that the minimizers of (1.5) and (1.9) highly oscillate between $h_-(\alpha_0)$ and $h_+(\alpha_0)$ as ϵ tends to 0 if we choose δ_1, δ_2 appropriately. This result contains a part of the main theorem of [15], which treated the case $\delta_2 = 0$ and $h_-(\alpha_0) \le 0$. In addition, we have revealed the threshold of parameters that the behavior of the minimizer changes dramatically at.

We shall note a feature of the effect of δ_2 in the shadow system in Theorem 1.2. In the case $\delta_1 = 0$, $\delta_2 > 0$ and $N = 1$ in (1.7), Suzuki and Tasaki prove the minimizer of (1.9) has only one internal layer ([17]). Moreover if δ_1 is small, $\delta_2 = 0$ and $h_-(\alpha_0) \le 0$, then minimizer of (1.9) is a constant from Theorem 1.2. However, Theorem 1.2 implies that if we take δ_2 large enough, the minimizer of (1.9) oscillates between $h_-(\alpha_0)$ and $h_+(\alpha_0)$ even for small δ_1.

Moreover, we have provided the precise order in the energy estimates of I_ϵ and J_ϵ in a special case that $\Omega = (0,1)^N$ in Theorems 1.4 and 1.5. For the two-component system, essentially the same statement was established in [15] without a proof. We also note that for the one-dimension two-component system, more precise energy estimate was obtained in [14]. Similar energy estimates have been proved for the Ohta-Kawasaki model and other related problems (see [4, 5]).

This paper is arranged as follows. In Section 2, we prove Theorems 1.1 and 1.2. Lemma 2.6 is a key lemma in the proof of the theorems. In Section 3, we prove Theorems 1.4 and 1.5. A new interpolation inequality in Lemma 3.4 is important in the proof of the lower bound estimate. We give a proof of Theorem 1.3 in Appendix A, where we consider the spectrum of the linearized problem and use the center manifold theory to show Theorem 1.3.

2. Proof of Theorems 1.1 and 1.2

2.1. Proof of Theorems 1.1. In this section, we prove a series of propositions and lemmas to prove Theorem 1.1. For simplicity, we use the notation $G_1 := G_{\gamma_1}$, $G_2 := G_{\gamma_2}$.

First, we prove the existence of a global minimizer of (1.5).

PROPOSITION 2.1. *The minimizing problem* (1.5) *has a global minimizer.*

PROOF. Let $\{u_n\}_{n\in\mathbb{N}}$ be a minimizing sequence. From (f2), there exist a constant $C > 0$ such that $-F(t) \geq -C + Ct^2$ for all $t \in \mathbb{R}$. Moreover, noting $\int_\Omega uG_i u\,dx = \|\nabla G_i u\|^2_{L^2(\Omega)} + \gamma_i \|G_i u\|^2_{L^2(\Omega)} \geq 0$ $(i = 1, 2)$ for any $u \in H^1(\Omega)$, we obtain $I_\epsilon(u) \geq \frac{\epsilon^2}{2}\|\nabla u\|^2_{L^2(\Omega)} + \frac{C}{2}\|u\|^2_{L^2(\Omega)} - C$ for any $u \in H^1(\Omega)$. Thus $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H^1(\Omega)$. Then there exists $u_\epsilon \in H^1(\Omega)$ such that $u_n \to u_\epsilon$ weakly in $H^1(\Omega)$, $u_n \to u_\epsilon$ strongly in $L^2(\Omega)$. From the weak lower semicontinuity of $L^2(\Omega)$ norm, we have $\liminf_{n\to\infty} \|\nabla u_n\|_{L^2(\Omega)} \geq \|\nabla u_\epsilon\|_{L^2(\Omega)}$. Furthermore,

$$\begin{aligned}\int_\Omega u_n G_i u_n + u_\epsilon G_i u_\epsilon\,dx &= \int_\Omega (u_n - u_\epsilon)G_i(u_n - u_\epsilon) + 2u_n G_i u_\epsilon\,dx \\ &\geq 2\int_\Omega u_n G_i u_\epsilon\,dx\,.\end{aligned}$$

Since $G_i u_\epsilon \in L^2(\Omega)$, we have $\liminf_{n\to\infty} \int_\Omega u_n G_i u_n\,dx \geq \int_\Omega u_\epsilon G_i u_\epsilon\,dx$.

Finally we show $\liminf_{n\to\infty} \int_\Omega -F(u_n)\,dx \geq \int_\Omega -F(u_\epsilon)\,dx$. We define $\bar{F}(t) = \int_{a_1}^t f(s)\,ds = F(t) - \int_{a_{-1}}^{a_1} f(s)\,ds$ so that $-\bar{F}(t) \geq 0$. Therefore we obtain

$$\liminf_{n\to\infty} \int_\Omega -\bar{F}(u_n)\,dx \geq \int_\Omega -\bar{F}(u_\epsilon)\,dx$$

from Fatou's lemma. Thus we have shown that there exists $u_\epsilon \in H^1(\Omega)$ such that $I_\epsilon(u_\epsilon) \leq \liminf_{n\to\infty} I_\epsilon(u_n) = \sigma$. □

The following lemmas are well-known and very useful (see e.g. [2, 11]).

LEMMA 2.1. *Let $f \in L^2(\Omega)$ and $u \in H^1(\Omega)$ be such that*

$$\int_\Omega \nabla u \cdot \nabla\phi\,dx + \gamma\int_\Omega u\phi\,dx = \int f\phi\,dx \quad \textit{for any } \phi \in H^1(\Omega).$$

Then we have, for a.e. $x \in \Omega$,

$$\frac{1}{\gamma}\inf_{y\in\Omega} f(y) \leq u(x) \leq \frac{1}{\gamma}\sup_{y\in\Omega} f(y)\,.$$

LEMMA 2.2. *Let $g \in C(\overline{\Omega} \times \mathbb{R})$ and $w \in C^2(\Omega \times \mathbb{R}) \cap C^1(\overline{\Omega} \times \mathbb{R})$.*

(1) *We assume w satisfies the following inequalities*:

$$\begin{cases}\Delta w(x) + g(x, w(x)) \geq 0\,, & x \in \Omega\,,\\ \frac{\partial w}{\partial n}(x) \leq 0\,, & x \in \partial\Omega\,.\end{cases}$$

If $w(x_0) = \max_{x\in\bar{\Omega}} w(x)$, then we have $g(x_0, w(x_0)) \geq 0$.

(2) *We assume w satisfies the following inequalities*:

$$\begin{cases} \Delta w(x) + g(x, w(x)) \leq 0\,, & x \in \Omega\,, \\ \frac{\partial w}{\partial n}(x) \geq 0\,, & x \in \partial\Omega\,. \end{cases}$$

If $w(x_0) = \min_{x \in \bar{\Omega}} w(x)$, then we have $g(x_0, w(x_0)) \leq 0$.

Next, we will show the boundedness of a solution of (1.2). The idea of the proof is based on [6].

LEMMA 2.3. *There is a constant $L_1 > 0$ which is independent of $\epsilon > 0$, such that for any solution $(u_\epsilon, v_\epsilon, w_\epsilon)$ of* (1.2), *we have $\|u_\epsilon\|_{L^\infty(\Omega)}, \|v_\epsilon\|_{L^\infty(\Omega)}, \|w_\epsilon\|_{L^\infty(\Omega)} < L_1$. In addition, if $\delta_i < \bar{\delta} < +\infty$, we can take L_1, which is independent of ϵ, δ_i and which satisfies $\|u_\epsilon^\delta\|_{L^\infty(\Omega)}, \|v_\epsilon^\delta\|_{L^\infty(\Omega)}, \|w_\epsilon^\delta\|_{L^\infty(\Omega)} < L_1$ for any solution of* (1.2), *where $\delta = (\delta_1, \delta_2)$.*

PROOF. We only consider the case $\delta_i \in (0, \bar{\delta})$. We prove by contradiction. Namely we suppose that there exist sequences $\{\epsilon_n\}_{n \in \mathbb{N}}$, $\{\delta_n\}_{n \in \mathbb{N}}$ such that $\|u_n\|_{L^\infty(\Omega)} \to \infty$ as $n \to \infty$, where $u_n = u_{\epsilon_n}^{\delta_n}$. Then we note

$$\frac{f(\|u_n\|_{L^\infty(\Omega)})}{\|u_n\|_{L^\infty(\Omega)}} \to -\infty \quad (n \to \infty)\,. \tag{2.1}$$

Thus there exists a subsequence $\{u_{n_j}\}_{j \in \mathbb{N}}$ which satisfies either (i) or (ii);

(i) For any $j \in \mathbb{N}$, there is a point $x_0^j \in \bar{\Omega}$ which satisfies $u_{n_j}(x_0^j) = \max_{x \in \Omega} u_{n_j}(x) = \|u_{n_j}\|_{L^\infty(\Omega)}$.

(ii) For any $j \in \mathbb{N}$, there is a point $x_1^j \in \bar{\Omega}$ which satisfies $u_{n_j}(x_1^j) = \min_{x \in \Omega} u_{n_j}(x) = -\|u_{n_j}\|_{L^\infty(\Omega)}$.

Thus we consider each case.
(i) Substituting $w = u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, $g(x, t) = f(t) - v_\epsilon(x) - w_\epsilon(x) \in C(\overline{\Omega} \times \mathbb{R})$ in Lemma 2.2(1), we can see that $f(u_\epsilon(x_0)) - v_\epsilon(x_0) - w_\epsilon(x_0) \geq 0$. Furthermore, we define $v_n = v_{\epsilon_n}^{\delta_n}$, $w_n = w_{\epsilon_n}^{\delta_n}$ for $n \in \mathbb{N}$, and then we have

$$\begin{aligned} v_{n_j}(x_0^j) + w_{n_j}(x_0^j) &\geq -\left(\frac{\delta_{1n_j}}{\gamma_1} + \frac{\delta_{2n_j}}{\gamma_2}\right) \|u_{n_j}\|_{L^\infty(\Omega)} \\ &\geq -\bar{\delta}\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right) \|u_{n_j}\|_{L^\infty(\Omega)} \end{aligned}$$

by Lemma 2.1. As a sequence, we obtain

$$\frac{f\left(\|u_{n_j}\|_{L^\infty(\Omega)}\right)}{\|u_{n_j}\|_{L^\infty(\Omega)}} \geq -\bar{\delta}\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right)$$

for all $j \in \mathbb{N}$. But this contradicts (2.1).
(ii) Repeating the proof of (i), we can easily see

$$-\frac{f\left(\|u_{n_j}\|_{L^\infty(\Omega)}\right)}{\|u_{n_j}\|_{L^\infty(\Omega)}} \leq \bar{\delta}\left(\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right)$$

for all $j \in \mathbb{N}$ and this contradicts (2.1). □

REMARK 5. From Lemma 2.3 and elliptic estimates for the second and third formulas of (1.2), we can see that v_ϵ, w_ϵ are bounded in $W^{2,p}(\Omega)$ for any $p > 1$. Especially, taking $p > N$, we have the compact embedding $W^{2,p}(\Omega) \subset C^{1,\beta}(\Omega)$ for $\beta = 1 - \frac{N}{p} > 0$. Then for any $\beta \in (0,1)$, there are $v, w \in C^{1,\beta}(\Omega)$ such that $v_\epsilon \to v$ in $C^{1,\beta}(\Omega)$, $w_\epsilon \to w$ in $C^{1,\beta}(\Omega)$ as $\epsilon \to 0$ if we take a subsequence.

We transform (1.2) in the following way.

LEMMA 2.4. *Let (u, v, w) be a function which satisfies* (1.2) *and let $s_i = \frac{\delta_i}{\gamma_i}$ $(i = 1, 2)$ and $S = s_1 + s_2$. Then we define U, V, W, H and m_0 as follows*:

$$\begin{cases} U &= \Phi(u) := au - b, \\ V &= a\left(v - \frac{s_1}{S}\alpha_0\right), \\ W &= a\left(w - \frac{s_2}{S}\alpha_0\right), \\ H(t) &= -a\int_{-1}^{t}\left(f(\Phi^{-1}(s)) - \alpha_0\right) ds, \\ m_0 &= \frac{\alpha_0}{S}a - b, \end{cases} \tag{2.2}$$

where

$$a = \frac{2}{h_+(\alpha_0) - h_-(\alpha_0)},$$

$$b = \frac{h_+(\alpha_0) + h_-(\alpha_0)}{h_+(\alpha_0) - h_-(\alpha_0)},$$

and Φ^{-1} is the inverse function of Φ, that is ,

$$\Phi^{-1}(s) = \frac{1}{a}(s + b).$$

Then (U, V, W) satisfies

$$\begin{cases} -\epsilon^2 \Delta U(x) = -H'(U(x)) - V(x) - W(x), & x \in \Omega, \\ -\Delta V(x) + \gamma_1 V(x) = \delta_1(U(x) - m_0), & x \in \Omega, \\ -\Delta W(x) + \gamma_2 W(x) = \delta_2(U(x) - m_0), & x \in \Omega, \\ \frac{\partial U}{\partial n}(x) = \frac{\partial V}{\partial n}(x) = \frac{\partial W}{\partial n}(x) = 0, & x \in \partial\Omega. \end{cases} \tag{2.3}$$

Moreover, H satisfies following conditions.

(H1) $H \in C^3(\mathbb{R})$.
(H2) $H(t) = 0$ *if* $t = \pm 1$, *and* $H(t) > 0$, *otherwise*.
(H3) $H''(\pm 1) > 0$.
(H4) *There exist positive constants* c, C' *such that* $H(t) \geq -C' + ct^2$ *for all* $t \in \mathbb{R}$.

PROOF. By (2.2), we readily see

$$\begin{cases} u = \Phi^{-1}(U) = \frac{1}{a}(U+b)\,, \\ v = \frac{1}{a}\left(V + \frac{s_1\alpha_0}{S}a\right)\,, \\ w = \frac{1}{a}\left(W + \frac{s_2\alpha_0}{S}a\right)\,. \end{cases} \tag{2.4}$$

Substituting (2.4) into (1.2), we have

$$\begin{cases} -\epsilon^2 \frac{1}{a}\Delta U = f\left(\Phi^{-1}(U)\right) - \frac{1}{a}\left(V + \frac{s_1\alpha_0}{S}a\right) - \frac{1}{a}\left(W + \frac{s_2\alpha_0}{S}a\right)\,, \\ -\frac{1}{a}\Delta V + \frac{\gamma_1}{a}\left(V + \frac{s_1\alpha_0}{S}a\right) = \frac{\delta_1}{a}(U+b)\,, \\ -\frac{1}{a}\Delta W + \frac{\gamma_2}{a}\left(W + \frac{s_2\alpha_0}{S}a\right) = \frac{\delta_2}{a}(U+b)\,. \end{cases}$$

From the definition of H, we have

$$\begin{aligned} -\epsilon^2 \Delta U &= a\left(f\left(\Phi^{-1}(U)\right) - \alpha_0\right) - V - W \\ &= -H'(U) - V - W\,. \end{aligned}$$

We also have

$$-\Delta V + \gamma_1 V = \delta_1\left(U + b - \frac{1}{s_1}\frac{s_1}{S}a\alpha_0\right) = \delta_1(U - m_0)\,,$$

$$-\Delta W + \gamma_2 W = \delta_2\left(U + b - \frac{1}{s_2}\frac{s_2}{S}a\alpha_0\right) = \delta_2(U - m_0)$$

from the definition of m_0. Moreover it is easy to check that H satisfies (H1)–(H4). □

LEMMA 2.5. *We define* $\widetilde{I}_\epsilon : H^1(\Omega) \to \mathbb{R}$ *is the energy associated with* (2.3), *that is*,

$$\begin{aligned} \widetilde{I}_\epsilon(u) = &\frac{\epsilon^2}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\delta_1}{2}\int_\Omega (u - m_0)G_1(u - m_0)\,dx \\ &+ \frac{\delta_2}{2}\int_\Omega (u - m_0)G_2(u - m_0)\,dx + \int_\Omega H(u)\,dx. \end{aligned} \tag{2.5}$$

Then $\widetilde{I}_\epsilon(u) \geq 0$ *for all* $u \in H^1(\Omega)$. *Moreover, let* $u \in H^1(\Omega)$ *and* $U = \Phi(u)$. *Then we have*

$$\widetilde{I}_\epsilon(U) = a^2\left(I_\epsilon(u) + C_*\right)\,, \tag{2.6}$$

where C_* *is the constant defined by* (1.11).

PROOF. Since $\int_\Omega uG_iu\,dx = \|\nabla u\|^2_{L^2(\Omega)} + \|G_iu\|^2_{L^2(\Omega)} \geq 0$ and $\int_\Omega H(u)\,dx \geq 0$ for all $u \in H^1(\Omega)$, it is clear $\widetilde{I}_\epsilon(u) \geq 0$ for all $u \in H^1(\Omega)$.

Moreover,

$$\begin{aligned}
\widetilde{I}_\epsilon(U) &= \widetilde{I}_\epsilon(au-b)\\
&= \frac{\epsilon^2}{2}\int_\Omega a^2\,|\nabla u|^2\,dx + \frac{\delta_1}{2}\int_\Omega (au-b-m_0)G_1(au-b-m_0)\,dx\\
&\quad+\frac{\delta_2}{2}\int_\Omega (au-b-m_0)G_2(au-b-m_0)\,dx + \int_\Omega H\left(\Phi(u)\right)\,dx\\
&:= A_1 + A_2 + A_3 + A_4\,.
\end{aligned}$$

Then we have

$$\begin{aligned}
A_2 &= \frac{\delta_1}{2}\int_\Omega a^2 uG_1u - 2auG_1(b+m_0) + (b+m_0)G_1(b+m_0)\,dx\\
&= a^2\frac{\delta_1}{2}\int_\Omega uG_1u\,dx - \delta_1 a\frac{b+m_0}{\gamma_1}\int_\Omega u\,dx + \frac{(b+m_0)^2}{\gamma_1}\frac{\delta_1}{2}\,|\Omega|\\
&= a^2\frac{\delta_1}{2}\int_\Omega uG_1u\,dx - \frac{s_1}{S}\alpha_0 a^2\int_\Omega u\,dx + \frac{s_1}{2S^2}\alpha_0^2 a^2\,|\Omega|\ .
\end{aligned}$$

Similarly,

$$A_3 = a^2\frac{\delta_2}{2}\int_\Omega uG_2u\,dx - \frac{s_2}{S}\alpha_0 a^2\int_\Omega u\,dx + \frac{s_2}{2S^2}\alpha_0^2 a^2\,|\Omega|\ .$$

Furthermore,

$$\begin{aligned}
H\left(\Phi(u)\right) &= -a\int_{-1}^{\Phi(u)}\left(f\left(\Phi^{-1}(s)\right) - \alpha_0\right)\,ds\\
&= -a^2\int_{h_-(\alpha_0)}^{u}\left(f(t) - \alpha_0\right)\,dt\\
&= -a^2F(u) + a^2\alpha_0(u - h_-(\alpha_0))\,.
\end{aligned}$$

Then we have

$$A_4 = -a^2\int_\Omega F(u)\,dx + a^2\alpha_0\int_\Omega u\,dx - a^2\alpha_0 h_-(\alpha_0)\,|\Omega|\ .$$

As a consequence, we obtain

$$\begin{aligned}
\widetilde{I}_\epsilon(U) &= A_1 + A_2 + A_3 + A_4\\
&= a^2\left(I_\epsilon(u) + \frac{\alpha_0^2}{2S}\,|\Omega| - \alpha_0 h_-(\alpha_0)\,|\Omega|\right)\\
&= a^2\left(I_\epsilon(u) + C_*\right)\ .
\end{aligned}$$

□

By Proposition 2.1 and Lemma 2.5, the following minimizing problem has a minimizer U_ϵ:

$$\inf\{\widetilde{I}_\epsilon(u) : u \in H^1(\Omega)\}. \tag{2.7}$$

Moreover U_ϵ satisfies that $U_\epsilon = \Phi(u_\epsilon)$, where u_ϵ is a minimizer of (1.5).

Then we will show Lemmas 2.6 and 2.7, which play important roles to show that the minimizer oscillates under the appropriate conditions. We mention that the ideas of these proofs are based on [15].

LEMMA 2.6. *We assume that the following condition* $(*)$:

$$(*)\ \ h_-(\alpha_0) \le 0, \quad S > \frac{\alpha_0}{h_+(\alpha_0)} \ \ or\ \ h_-(\alpha_0) > 0, \ \ \frac{\alpha_0}{h_-(\alpha_0)} > S > \frac{\alpha_0}{h_+(\alpha_0)}$$

Then we have

$$\min_{u \in H^1(\Omega)} \widetilde{I}_\epsilon(u) \to 0 \quad as\ \epsilon \to 0.$$

PROOF. From the condition $(*)$, we can easily see that $m_0 \in (-1, 1)$. Since $\widetilde{I}_\epsilon$ is nondecreasing with respect to ϵ, it suffices to consider the case $\epsilon^2 = \frac{1}{n^6}$ $(n \in \mathbb{N})$. Moreover let $u_n \in C^\infty(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ be a function satisfying $u_n(-x_1, x_2, \dots, x_N) = u_n(x_1, x_2, \dots, x_N)$, $u_n(x_1 + \frac{2}{n}, x_2, \dots, x_N) = u_n(x_1, x_2, \dots, x_N)$ and the following:

$$u_n(x) = \begin{cases} -1, & 0 \le x_1 \le a_n, \\ \chi\left(n^3(x_1 - a_n)\right), & a_n \le x_1 \le b_n, \\ 1, & b_n \le x_1 \le \frac{1}{n}, \end{cases} \tag{2.8}$$

where $\chi \in C^\infty(\mathbb{R}, [-1, 1])$, a_n, b_n are defined by

$$\chi(s) = \begin{cases} -1, & s \le 0, \\ 1, & s \ge 1, \end{cases} \tag{2.9}$$

$$a_n = \frac{1 - m_0}{2}\left(\frac{1}{n} - \frac{1}{n^3}\right),$$

$$b_n = \frac{1 - m_0}{2}\frac{1}{n} + \frac{1 + m_0}{2}\frac{1}{n^3}.$$

Then we will show that $\lim_{n\to\infty} \widetilde{I}_\epsilon(u_n) = 0$. Let Ω' be

$$\Omega' = \{x' \in \mathbb{R}^{N-1} : \text{There exists } x_1 \in \mathbb{R} \text{ such that } (x_1, x') \in \Omega\}.$$

There is $L > 0$ such that $\Omega \subset [-L, L] \times \Omega'$. Then

$$\int_\Omega |\nabla u_n|^2\, dx \le \int_{-L}^{L} \int_{\Omega'} |\nabla u_n|^2 dx' dx_1$$

$$\begin{aligned}
&\le C|\Omega'|Ln \int_{a_n}^{b_n} \left| \left\{ \chi \left(n^3(x_1 - a_n) \right) \right\}' \right|^2 dx_1 \\
&= C|\Omega'|Ln^7 \int_{a_n}^{b_n} \left| \chi' \left(n^3(x_1 - a_n) \right) \right|^2 dx_1 \\
&= C|\Omega'|Ln^4 \int_0^1 |\chi'(s)|^2 ds \\
&\le Cn^4 .
\end{aligned}$$

It follows that

$$\epsilon^2 \int_\Omega |\nabla u_n|^2 \, dx \le \frac{C}{n^2} \to 0 \quad \text{as } n \to \infty . \tag{2.10}$$

Next, we will show

$$u_n \to m_0 \text{ weakly in } L^2(\Omega) \quad \text{as } n \to \infty . \tag{2.11}$$

Let $\tilde{u}$ be a function satisfying $\tilde{u}(-x_1, x_2, \ldots, x_N) = \tilde{u}(x_1, x_2, \ldots, x_N)$, $\tilde{u}(x_1 + 2, x_2, \ldots, x_N) = \tilde{u}(x_1, x_2, \ldots, x_N)$ for all $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ and

$$\tilde{u}(x) = \begin{cases} -1 , & 0 \le x_1 \le \frac{1-m_0}{2} , \\ 1 , & \frac{1-m_0}{2} \le x_1 \le 1 . \end{cases} \tag{2.12}$$

Then we define $\tilde{u}_n(x) = \tilde{u}(nx)$. Furthermore, let

$$T_z(n) = \prod_{i=1}^{N} \left(\frac{2}{n} z_i - \frac{1}{n}, \frac{2}{n} z_i + \frac{1}{n} \right) \quad z \in \mathbb{Z}^N , \quad n \in \mathbb{N} .$$

Especially, let $T = T_0(1)$. Then $\tilde{u} \in L^2(T)$ and $\tilde{u}$ is T-periodic, that is, $\tilde{u}(x + Te_j) = \tilde{u}(x)$ for $x \in \mathbb{R}^N$, $j = 1, \ldots, N$. As a consequence, we can see

$$\tilde{u}_n \to \frac{1}{|T|} \int_T \tilde{u} \, dx = m_0 \ \text{ weakly in } \ L^2(\Omega) \ \text{ as } \ n \to \infty$$

(see [3]). In addition, there exists a constant $C > 0$ such that

$$\int_\Omega (u_n - \tilde{u}_n)^2 dx \le Cn \frac{1}{n^3} = \frac{C}{n^2}$$

for all $n \in \mathbb{N}$. Thus for any $\phi \in L^2(\Omega)$, we obtain

$$\begin{aligned}
\left| \int_\Omega (u_n - m_0) \phi \, dx \right| &\le \left| \int_\Omega (\tilde{u}_n - m_0) \phi \, dx \right| + \left| \int_\Omega (u_n - \tilde{u}_n) \phi \, dx \right| \\
&\le \left| \int_\Omega (\tilde{u}_n - m_0) \phi \, dx \right| + \|\phi\|_{L^2(\Omega)} \frac{\sqrt{C}}{n} .
\end{aligned}$$

As a consequence, we conclude (2.11).

Next, we will show

$$\lim_{n\to\infty}\int_\Omega (u_n - m_0)G_i(u_n - m_0)\,dx = 0\,. \tag{2.13}$$

Since G_i is a compact operator from $L^2(\Omega)$ to $L^2(\Omega)$, (2.11) implies

$$G_i(u_n - m_0) \to 0 \ \text{ strongly in } \ L^2(\Omega) \ \text{ as } \ n \to \infty\,.$$

Then from Hölder's inequality, we can see

$$\begin{aligned}\left|\int_\Omega (u_n - m_0)G_i(u_n - m_0)\,dx\right| &\le \|u_n - m_0\|_{L^2(\Omega)}\|G_i(u_n - m_0)\|_{L^2(\Omega)}\\ &\le C\|G_i(u_n - m_0)\|_{L^2(\Omega)}\,.\end{aligned}$$

It follows (2.13). Finally, we will show $\lim_{n\to\infty}\int_\Omega H(u_n)dx = 0$. We can see

$$\begin{aligned}\int_\Omega H(u_n)\,dx &= \int_{\{x\in\Omega : u_n(x)\neq\pm1\}} H(u_n)\,dx\\ &\le C\,|(\{x\in\Omega : u_n(x)\neq\pm1\})|\\ &\le Cn\frac{1}{n^3} \to 0 \quad (n\to\infty)\end{aligned} \tag{2.14}$$

from (H2). Combining (2.10), (2.13) and (2.14), we obtain $\lim_{\epsilon\to0}\widetilde{I}_\epsilon(u_n) = 0$ and completes the proof. □

We will show next lemma.

LEMMA 2.7. *Let u_ϵ be a minimizer of I_ϵ, $v_\epsilon = \delta_1 G_1 u_\epsilon$, $w_\epsilon = \delta_2 G_2 u_\epsilon$. Under the condition $(*)$, for any $\theta > 0$ small, we have*

$$\lim_{\epsilon\to0}\left|\Omega^+_{\epsilon,\theta}\cap\Omega^-_{\epsilon,\theta}\right| = 0\,.$$

PROOF. Let $U_\epsilon = \Phi(u_\epsilon)$. Then

$$\Omega^+_{\epsilon,\theta}\cap\Omega^-_{\epsilon,\theta} = \left\{x\in\Omega;\ |U_\epsilon(x)-1|\ge\theta'\right\}\cap\left\{x\in\Omega;\ |U_\epsilon(x)+1|\ge\theta'\right\}\,,$$

where $\theta' = \Phi(\theta)$. From Lemma 2.6, we can easily see that $0 \le \widetilde{I}_\epsilon(U_\epsilon) \le \widetilde{I}_\epsilon(u_n) \to 0$ as $\epsilon \to 0$. Then it follows $\lim_{\epsilon\to0}\int_\Omega H(U_\epsilon)\,dx = 0$. Moreover, for any $\theta' > 0$, there is a constant $c_{\theta'} > 0$ such that if $|s-1| \ge \theta'$ and $|s+1| \ge \theta'$, then $H(s) \ge c_{\theta'}$ holds by (H2). Thus we can see

$$0 < c_{\theta'}\left|\Omega^+_{\epsilon,\theta}\cap\Omega^-_{\epsilon,\theta}\right| \le \int_\Omega H(U_\epsilon)\,dx\,.$$

Finally, taking $\epsilon \to 0$, we have the conclusion. □

PROOF OF THEOREM 1.1.
(1) The case $0 < S \le \frac{\alpha_0}{h_+(\alpha_0)}$.
Let c be a constant satisfying $S = \frac{c}{h_+(c)}$. Then we have

$$\begin{aligned}
&\frac{\delta_i}{2}\int_\Omega uG_iu\,dx \\
&\quad= \frac{\delta_i}{2}\int_\Omega (u-h_+(c))G_i(u-h_+(c)) + 2uG_ih_+(c) - h_+(c)G_ih_+(c)\,dx \\
&\quad= \frac{\delta_i}{2}\int_\Omega (u-h_+(c))G_i(u-h_+(c))\,dx + \frac{\delta_i}{\gamma_i}h_+(c)\int_\Omega u\,dx - \frac{\delta_i}{2\gamma_i}h_+^2(c)\,|\Omega|\,, \\
&\qquad\int_\Omega F(u)\,dx \\
&\quad= \int_\Omega\int_{h_-(\alpha_0)}^u (f(t)-c)\,dt\,dx + c\left(\int_\Omega u\,dx - h_-(\alpha_0)\,|\Omega|\right).
\end{aligned}$$

Noting $\frac{\delta_1}{\gamma_1} + \frac{\delta_2}{\gamma_2} = S = \frac{c}{h_+(c)}$, we have

$$\begin{aligned}
I_\epsilon(u) = \int_\Omega &\left\{\frac{\epsilon^2}{2}|\nabla u|^2 + \frac{\delta_1}{2}(u-h_+(c))G_1(u-h_+(c))\right. \\
&\left. + \frac{\delta_2}{2}(u-h_+(c))G_2(u-h_+(c)) - \int_{h_-(\alpha_0)}^u (f(t)-c)\,dt\right\}dx \\
&+ \left(-\frac{Sh_+^2(c)}{2} + ch_-(\alpha_0)\right)|\Omega|\;.
\end{aligned}$$

Thus we can easily see that $u \equiv h_+(c)$ is a minimizer of (1.5). It is also clear that $v_\epsilon \equiv \frac{s_1}{S}c$, $w_\epsilon \equiv \frac{s_2}{S}c$.
(2) The case $h_-(\alpha_0) \le 0$, $S > \frac{\alpha_0}{h_+(\alpha_0)}$, or $h_-(\alpha_0) > 0$, $\frac{\alpha_0}{h_-(\alpha_0)} > S > \frac{\alpha_0}{h_+(\alpha_0)}$.
It is clear that

$$\lim_{\epsilon\to 0}\left|\Omega^+_{\epsilon,\theta}\cap\Omega^-_{\epsilon,\theta}\right| = 0$$

from Lemma 2.7. Then we will show that $u_\epsilon \to \frac{\alpha_0}{S}$ weakly in $L^2(\Omega)$. It suffices to show $U_\epsilon \to m_0$ weakly in $L^2(\Omega)$, where $U_\epsilon = \Phi(u_\epsilon)$. From Lemma 2.3, U_ϵ is bounded in $L^2(\Omega)$. Then there exists $U \in L^2(\Omega)$ such that $U_\epsilon \to U$ weakly in $L^2(\Omega)$. Therefore it follows that

$$\lim_{\epsilon\to 0}\int_\Omega (U_\epsilon - m_0)G_1(U_\epsilon - m_0)\,dx = \int_\Omega (U-m_0)G_1(U-m_0)\,dx\;.$$

In fact, noting that $G_1U_\epsilon \to G_1U$ strongly in $L^2(\Omega)$, we can see

$$\left|\int_\Omega (U_\epsilon - m_0)G_1(U_\epsilon - m_0)\,dx - \int_\Omega (U-m_0)G_1(U-m_0)\,dx\right|$$

$$\leq \left| \int_{\Omega} (U_\epsilon - m_0) G_1(U_\epsilon - m_0) - (U_\epsilon - m_0) G_1(U - m_0)\, dx \right|$$
$$+ \left| \int_{\Omega} (U_\epsilon - m_0) G_1(U - m_0) - (U - m_0) G_1(U - m_0)\, dx \right|$$
$$\leq C \| G_1(U_\epsilon - U) \|_{L^2(\Omega)} \to 0 \quad (\epsilon \to 0)\,.$$

Moreover, from Lemma 2.6, $\int_{\Omega} (U - m_0) G_1(U - m_0)\, dx = 0$. Noting that $\int_{\Omega} (U - m_0) G_1(U - m_0)\, dx = \|\nabla G_1(U - m_0)\|^2_{L^2(\Omega)} + \gamma_1 \|G_1(U - m_0)\|^2_{L^2(\Omega)}$, we conclude $\nabla G_1(U - m_0) = G_1(U - m_0) = 0$. As a result, we can get $U \equiv m_0$, that is, $U_\epsilon \to m_0$ weakly in $L^2(\Omega)$.

Next, we will show $(u_\epsilon)_\epsilon$ does not have any subsequence that converges strongly in $L^1(\Omega)$. It suffices to show $(u_\epsilon)_\epsilon$ does not have any subsequence that converges in measure. We assume $(u_\epsilon)_\epsilon$ has a subsequence $(u_n)_{n \in \mathbb{N}}$ that converges some function u_0 in measure. Then we have

$$\lim_{n \to \infty} |\{x \in \Omega : |u_n(x) - u_0(x)| \geq \theta\}| = 0$$

for any $\theta > 0$ small. We may also assume $u_n(x) \to u_0(x)$ a.e. $x \in \Omega$ (see [8]). Moreover noting u_n is uniformly bounded in $L^\infty(\Omega)$, we can see $(u_n)_{n \in \mathbb{N}}$ has subsequence $(u_{n_j})_{j \in \mathbb{N}}$ such that $u_{n_j} \to u_0$ weak* in $(L^1)^*$. In addition, $u_{n_j} \to \frac{\alpha_0}{S}$ weak* in $(L^1)^*$ since $u_{n_j} \to \frac{\alpha_0}{S}$ weakly in $L^2(\Omega)$. Thus we obtain $u_0 \equiv \frac{\alpha_0}{S}$. It follows that $\lim_{n \to \infty} |B_{n,\theta}| = 0$ for any $\theta > 0$, where $B_{n,\theta} = \{x \in \Omega;\ |u_n(x) - \frac{\alpha_0}{S}| \geq \theta\}$. As a consequence, we have

$$\lim_{n \to \infty} \left| B^c_{n,\theta} \cup (\Omega^+_{n,\theta})^c \cup (\Omega^-_{n,\theta})^c \right|$$
$$= |\Omega| - \lim_{n \to \infty} |B_{n,\theta}| + |\Omega| - \lim_{n \to \infty} \left| \Omega^+_{n,\theta} \cap \Omega^-_{n,\theta} \right| = 2\,|\Omega|$$

for sufficiently small $\theta > 0$. But it is a contradiction. Thus $(u_\epsilon)_\epsilon$ does not have any subsequence which converges in measure.

Finally, we will show that $v_\epsilon \to \frac{s_1}{S}\alpha_0$ in $C^{1,\beta}(\Omega)$ and $w_\epsilon \to \frac{s_2}{S}\alpha_0$ in $C^{1,\beta}(\Omega)$ as $\epsilon \to 0$. From Remark 5, there exist $v, w \in C^{1,\beta}(\Omega)$ such that $v_\epsilon \to v$ in $C^{1,\beta}(\Omega)$, $w_\epsilon \to w$ in $C^{1,\beta}(\Omega)$ as $\epsilon \to 0$. On the other hand, noting that $v_\epsilon = \delta_1 G_1 u_\epsilon \to \frac{s_1}{S}\alpha_0$ strongly in $L^2(\Omega)$ and $w_\epsilon = \delta_1 G_1 u_\epsilon \to \frac{s_2}{S}\alpha_0$ strongly in $L^2(\Omega)$, we can see that $v \equiv \frac{s_1}{S}\alpha_0$, $w \equiv \frac{s_2}{S}\alpha_0$.
(3) The case $h_-(\alpha_0) > 0$, $S > \frac{\alpha_0}{h_-(\alpha_0)}$.
Let c be a constant satisfying $S = \frac{c}{h_-(c)}$. Then repeating the proof of (1), we have

$$I_\epsilon(u) = \int_{\Omega} \left\{ \frac{\epsilon^2}{2} |\nabla u|^2 + \frac{\delta_1}{2}(u - h_-(c)) G_1(u - h_-(c)) \right.$$
$$\left. + \frac{\delta_2}{2}(u - h_-(c)) G_2(u - h_-(c)) - \int_{h_-(\alpha_0)}^{u} (f(t) - c)\, dt \right\} dx$$

$$+\left(-\frac{Sh_-{}^2(c)}{2}+ch_-(\alpha_0)\right)|\Omega|\ .$$

Thus we can easily deduce that $u_\epsilon \equiv h_-(c)$, $v_\epsilon \equiv \frac{s_1}{S}c$, $w_\epsilon \equiv \frac{s_2}{S}c$ for any $\epsilon > 0$. □

2.2. Proof of Theorem 1.2. We give a proof of Theorem 1.2 in this section. We just mention the differences from the proof of Theorem 1.1 although we can prove it by repeating almost the same arguments as in the proof of Theorem 1.1.

PROOF OF THEOREM 1.2. First, it is easy to check the existence of the minimizer of (1.9) since $\frac{\delta_2}{2\gamma_2|\Omega|}\left(\int_\Omega u\,dx\right)^2 \geq 0$.

For the uniform boundedness of a solution (u_ϵ, v_ϵ) of (1.7), we can also prove that there is a constant $L_1 > 0$ such that $\|u_\epsilon\|_{L^\infty(\Omega)}, \|v_\epsilon\|_{L^\infty(\Omega)} < L_1$ by applying Lemma 2.2 with $w = u_\epsilon \in C^2(\Omega)\cap C^1(\bar{\Omega})$ and $g(x,t) = f(t) - v_\epsilon(x) - \frac{\delta_2}{\gamma_2|\Omega|}\int_\Omega u_\epsilon(y)\,dy \in C(\bar{\Omega}\times\mathbb{R})$.

Next, we transform (1.7). Let (u, v) be functions which satisfy (1.7). Then we define U, V, H, m_0 in the same way as in Lemma 2.4 and we can easily see that (U, V) satisfies

$$\begin{cases} -\epsilon^2\Delta U(x) = -H'(U(x)) - V(x) - \frac{\delta_2}{\gamma_2|\Omega|}\int_\Omega (U-m_0)\,dx\,, & x\in\Omega\,,\\ -\Delta V(x)+\gamma_1 V(x) = \delta_1(U(x)-m_0)\,, & x\in\Omega\,,\\ \frac{\partial U}{\partial n}(x) = \frac{\partial V}{\partial n}(x) = 0\,, & x\in\partial\Omega\,. \end{cases} \tag{2.15}$$

Moreover, we define the energy $\widetilde{J}_\epsilon : H^1(\Omega)\to\mathbb{R}$ associated with (2.3), that is ,

$$\widetilde{J}_\epsilon(u) = \frac{\epsilon^2}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\delta_1}{2}\int_\Omega (u-m_0)G_1(u-m_0)\,dx + \frac{\delta_2}{2\gamma_2|\Omega|}\left(\int_\Omega (u-m_0)\,dx\right)^2 + \int_\Omega H(u)\,dx\,,$$

and we have

$$\widetilde{J}_\epsilon(U) = a^2\left(J_\epsilon(u)+C_*\right)\,, \tag{2.16}$$

where u, U, C_* are defined in the same way as in Lemma 2.5. Thus we can see the following minimizing problem has a minimizer U_ϵ:

$$\inf\left\{\widetilde{J}_\epsilon(u);\ u\in H^1(\Omega)\right\}\,. \tag{2.17}$$

Then let u_n be the same function in Lemma 2.6 and we have

$$\widetilde{J}_\epsilon(u_n)\to 0 \text{ as } n\to\infty$$

under the condition $(*)$.

Now, repeating the same arguments as in the proof of Theorem 1.1, we can complete the proof of Theorem 1.2. □

3. Energy Estimate

In this section, we study energy estimates. For the sake of simplicity, throughout this section, we assume that $\Omega = (0,1)^3 \subset \mathbb{R}^3$ and $\delta_i < \bar{\delta}$, where $\bar{\delta}$ is a fixed positive constant.

Our main energy theorems in this section are the following. Combining the formulas (2.6) and (2.16) with Theorems 3.1 and 3.2, we can easily prove Theorems 1.4 and 1.5.

THEOREM 3.1. *Let u_ϵ be the minimizer of* (2.7), $\zeta > 0$ *be a small constant and* $m_0 \in (-1+\zeta, 1-\zeta)$. *Then we have the following statements*:

(1) *There exist positive constants $\epsilon_0 = \epsilon_0(\zeta, \bar{\delta})$ and L_0 independent of $\bar{\delta}$, ζ such that*

$$\widetilde{I}_\epsilon(u_\epsilon) \le L_0 \epsilon^{\frac{2}{3}} \left(\delta_1^{\frac{1}{3}} + \delta_2^{\frac{1}{3}} \right)$$

for all $\epsilon \in (0, \epsilon_0)$, $\delta_1, \delta_2 \in (\epsilon, \bar{\delta})$.

(2) *There exist positive constants $\epsilon_1 = \epsilon_1(\zeta, \bar{\delta})$ and $l_0 = l_0(\zeta)$ such that*

$$\widetilde{I}_\epsilon(u_\epsilon) \ge l_0 \epsilon^{\frac{2}{3}} \left(\delta_1^{\frac{1}{3}} + \delta_2^{\frac{1}{3}} \right)$$

for all $\epsilon \in (0, \epsilon_1)$, $\delta_i \in (\epsilon, \bar{\delta})$ $(i = 1, 2)$.

THEOREM 3.2. *Let u_ϵ be the minimizer of* (2.17), $\zeta > 0$ *be a small constant and* $m_0 \in (-1+\zeta, 1-\zeta)$. *Then we have the following statements*:

(1) *There exist positive constants $\epsilon_0{}' = \epsilon_0{}'(\zeta, \bar{\delta})$, L_0 independent of $\bar{\delta}$, ζ such that*

$$\widetilde{J}_\epsilon(u_\epsilon) \le L_0 \epsilon^{\frac{2}{3}} \delta_1^{\frac{1}{3}}$$

for all $\epsilon \in (0, \epsilon_0{}')$, $\delta_1, \delta_2 \in (\epsilon, \bar{\delta})$.

(2) *There exist positive constants $\epsilon_1{}' = \epsilon_1{}'(\zeta, \bar{\delta})$, $l_0 = l_0(\zeta)$ such that*

$$\widetilde{J}_\epsilon(u_\epsilon) \ge l_0 \epsilon^{\frac{2}{3}} \delta_1^{\frac{1}{3}}$$

for all $\epsilon \in (0, \epsilon_1{}')$, $\delta_i \in (\epsilon, \bar{\delta})$ $(i = 1, 2)$.

To prove these theorems, we present several lemmas. The interpolation inequality in Lemma 3.4 is the key for the lower bound estimate. Lemmas 3.1–3.3 are used in the proof of Lemma 3.4. We also prepare Lemma 3.5 for the upper bound estimate.

Also, we define h as follows:

$$h(s) = \int_{-1}^{s} \sqrt{H(t)}\, dt\,.$$

Then there exists a constant $L_2 > 0$ depending only H such that for any $X, Y \in [-1, 1]$,

$$|X - Y|^2 \le L_2 \left| h(X) - h(Y) \right| \,. \tag{3.1}$$

Although (3.1) was stated in [4] without a proof, we present a proof of (3.1) in Appendix B for reader's convenience.

LEMMA 3.1. *Let D be a bounded domain. There exist constants $L_3, L_3' > 0$ such that*

$$\iint_{D^2} |u(x) - u(y)|^2 \, dxdy \le L_3 \, |D| \int_{D'} (|u(x)| - 1)^2 \, dx + L_3' \iint_{D^2} |h(u(x)) - h(u(y))| \, dxdy$$

for any $u \in H^1(D)$, where $D' = \{x \in D;\ |u(x)| > 1\}$.

PROOF. Let $D'' = \{x \in D;\ |u(x)| \le 1\}$. Then we have

$$\begin{aligned}
\iint_{D^2} |u(x) - u(y)|^2 \, dxdy
&= \iint_{D^2} |u(x) - u(y)|^2 \, \chi_{D''}(x)\chi_{D''}(y) dxdy \\
&\quad + 2 \iint_{D^2} |u(x) - u(y)|^2 \, \chi_{D''}(x)\chi_{D'}(y) dxdy \\
&\quad + \iint_{D^2} |u(x) - u(y)|^2 \, \chi_{D'}(x)\chi_{D'}(y) dxdy \\
&=: J_1 + 2J_2 + J_3 \, .
\end{aligned}$$

First, we can readily see

$$J_1 \le L_2 \iint_{D^2} |h(u(x)) - h(u(y))| \, dxdy$$

from (3.1).

Next, we will show

$$J_2 \le 2 \, |D| \int_{D'} (|u(x)| - 1)^2 \, dx + 2L_2 \iint_{D^2} |h(u(x)) - h(u(y))| \, dxdy \, . \tag{3.2}$$

If $|u(x)| \le 1, u(y) > 1$, we have

$$\begin{aligned}
|u(x) - u(y)|^2 &= |u(y) - 1 + 1 - u(x)|^2 \\
&\le 2 \left\{ (|u(y)| - 1)^2 + (1 - u(x))^2 \right\} \\
&\le 2 \, (|u(y)| - 1)^2 + 2L_2 \, |h(1) - h(u(x))| \\
&\le 2 \, (|u(y)| - 1)^2 + 2L_2 \, |h(u(y)) - h(u(x))| \; .
\end{aligned}$$

Similarly, if $|u(x)| \le 1, u(y) < -1$, we also have

$$|u(x) - u(y)|^2 \le 2 \, (|u(y)| - 1)^2 + 2L_2 \, |h(u(y)) - h(u(x))| \; .$$

Thus we have (3.2).

Finally, we will show

$$J_3 \le 6\,|D| \int_{D'} (|u(x)|-1)^2\,dx + 3L_2 \iint_{D^2} |h(u(x)) - h(u(x))|\,dxdy\,. \tag{3.3}$$

If $u(x) > 1$ and $u(y) > 1$, it is clear

$$|u(x) - u(y)|^2 \le |u(y) - 1|^2\,.$$

Then if $u(x) > 1$ and $u(y) < -1$, we can see

$$\begin{aligned} |u(x) - u(y)|^2 &= \{(u(x) - 1) + (1 - (-1)) + (-1 - u(y))\}^2 \\ &\le 3\left\{(|u(x)| - 1)^2 + L_2\,|h(1) - h(-1)| + (|u(y)| - 1)^2\right\} \\ &\le 3\left\{(|u(x)| - 1)^2 + L_2\,|h(u(x)) - h(u(y))| + (|u(y)| - 1)^2\right\}\,. \end{aligned}$$

As a consequence, we have (3.3) and it follows

$$\begin{aligned} &\iint_{D^2} |u(x) - u(y)|^2\,dxdy \\ &\qquad \le 10\,|D| \int_{D'} (|u(x)| - 1)^2\,dx + 8L_2 \iint_{D^2} |h(u(x)) - h(u(x))|\,dxdy \\ &\qquad = L_3\,|D| \int_{D'} (|u(x)| - 1)^2\,dx + L_3' \iint_{D^2} |h(u(x)) - h(u(x))|\,dxdy. \end{aligned}$$

□

We introduce some more notations. First, for fixed $N \in \mathbb{N}$, we define Ω_m ($m \in \{0, 1, 2, \ldots, (N-1)\}^3$) as follows:

$$\Omega_m = \frac{1}{N}(m + \Omega) = \left\{\frac{1}{N}(m + x);\, x \in \Omega\right\}\,.$$

Note that $\{\Omega_m\}$ are disjoint and

$$\Omega = \bigcup_{m \in \{0,1,2,\ldots,(N-1)\}^3} \Omega_m\,.$$

Next, let $\eta \in C_c^\infty(\Omega)$ be a function satisfying $\int_\Omega \eta(x)\,dx = 1$ and $\eta_m \in C_c^\infty(\Omega_m)$ be

$$\eta_m(x) = N^3 \eta(Nx - m)\,, \quad x \in \Omega_m\,.$$

Note $\int_{\Omega_m} \eta_m(x)dx = 1$. Finally, we define the operator $T_N : L^2(\Omega) \to L^2(\Omega)$ defined in [4] as follows:

$$T_N u(x) = \sum_m \left(\int_{\Omega_m} u(y)\eta_m(y)\,dy\right) \chi_{\Omega_m}(x)\,.$$

The next lemma is proved by Lemma 3.1 and the arguments in [4].

LEMMA 3.2. *There exists a constant* $L_4 > 0$ *such that*

$$\int_\Omega |u(x) - m_0 - T_N(u - m_0)|^2\, dx \le L_4 \left\{ \frac{1}{N} \int_\Omega |\nabla(h \circ u)(x)|\, dx + \int_{\Omega'} |u(x) - 1|^2\, dx \right\} \tag{3.4}$$

for any $u \in H^1(\Omega)$, *where* $\Omega' = \{x \in \Omega;\, |u(x)| > 1\}$.

PROOF. By the definition of T_N, we have

$$\begin{aligned}
&\int_\Omega |u - m_0 - T_N(u - m_0)|^2\, dx \\
&= \sum_m \int_{\Omega_m} \left| u(x) - m_0 - \left\{ \sum_k \int_{\Omega_k} (u(y) - m_0)\eta_k(y)\, dy\ \chi_{\Omega_k}(x) \right\} \right|^2 dx \\
&= \sum_m \int_{\Omega_m} \left| u(x) - m_0 - \int_{\Omega_m} (u(y) - m_0)\eta_m(y)\, dy\ \chi_{\Omega_m}(x) \right|^2 dx \\
&= \sum_m \int_{\Omega_m} \left| u(x) - \int_{\Omega_m} u(y)\eta_m(y)\, dy\ \chi_{\Omega_m}(x) \right|^2 dx\,.
\end{aligned}$$

Thus it suffices to show

$$\int_{\Omega_m} \left| u(x) - \int_{\Omega_m} u(y)\eta_m(y)\, dy \right|^2 dx \le L_4 \left\{ \frac{1}{N} \int_{\Omega_m} |\nabla(h \circ u)(x)|\, dx + \int_{\Omega'_m} |u(x) - 1|^2\, dx \right\} \tag{3.5}$$

for each $m \in \{0, 1, \ldots, (N-1)\}^3$, where $\Omega'_m = \{x \in \Omega_m;\, |u(x)| > 1\}$.

Then we can see

$$\begin{aligned}
J_4 &:= \int_{\Omega_m} \left| u(x) - \int_{\Omega_m} u(y)\eta_m(y)\, dy \right|^2 dx \\
&= \int_{\Omega_m} \left| \int_{\Omega_m} (u(x) - u(y))\eta_m(y)\, dy \right|^2 dx \\
&\le |\Omega_m| \iint_{\Omega_m^2} |u(x) - u(y)|^2\, \eta_m^2(y)\, dy\, dx\,.
\end{aligned}$$

From Lemma 3.1 and $|\eta_m(y)|^2 \le \|\eta\|^2_{L^\infty(\Omega)} N^6$, we have

$$J_4 \le \|\eta\|^2_{L^\infty(\Omega)} \left\{ L_3' N^3 \iint_{\Omega_m^2} |h(u(x)) - h(u(y))| \, dydx + L_3 \int_{\Omega_m'} |u(x) - 1|^2 \, dx \right\}.$$

Now we will show

$$J_5 := \iint_{\Omega_m^2} |\phi(x) - \phi(y)| \, dxdy \le \frac{C}{N^4} \int_{\Omega_m} |\nabla \phi(x)| \, dx \,, \tag{3.6}$$

where $\phi = h \circ u$.

First, we can estimate J_5 as follows:

$$\begin{aligned} J_5 &\le \iint_{\Omega_m^2} \int_0^1 |\nabla \phi \left(tx + (1-t)y \right) \cdot (x-y)| \, dtdxdy \\ &\le \frac{\sqrt{3}}{N} \left\{ \int_0^{\frac{1}{2}} \iint_{\Omega_m^2} |\nabla \phi(tx + (1-t)y| \, dydxdt + \int_{\frac{1}{2}}^1 \iint_{\Omega_m^2} |\nabla \phi(tx + (1-t)y| \, dxdydt \right\} \\ &:= \frac{\sqrt{3}}{N} (J_6 + J_7) \,. \end{aligned}$$

We choose $t \in \left(0, \frac{1}{2}\right)$, $x \in \Omega_m$ and define $z = tx + (1-t)y$. Thus from convexity of Ω_m, we have

$$J_6 \le \int_0^{\frac{1}{2}} \iint_{\Omega_m^2} \frac{1}{(1-t)^3} |\nabla \phi(z)| \, dzdxdt \le \frac{8}{N^3} \int_{\Omega_m} |\nabla \phi(z)| \, dz \,.$$

Similarly, we have $J_7 \le \frac{8}{N^3} \int_{\Omega_m} |\nabla \phi(z)| \, dz$. As a result, we can arrive at (3.6).

Finally, we have

$$\begin{aligned} J_4 &\le \|\eta\|^2_{L^\infty(\Omega)} \left\{ \frac{16\sqrt{3} L_3'}{N} \int_{\Omega_m} |\nabla (h \circ u)(x)| \, dx + L_3 \int_{\Omega_m'} |u(x) - 1|^2 \, dx \right\} \\ &\le L_4 \left\{ \frac{1}{N} \int_{\Omega_m} |\nabla (h \circ u)(x)| \, dx + \int_{\Omega_m'} |u(x) - 1|^2 \, dx \right\}. \end{aligned}$$

From these estimates, we can complete the proof of Lemma 3.2. □

We define $e_n : \Omega \to \mathbb{R}$ as

$$\begin{cases} e_n(x) = 1, & n = (0,0,0)\,, \\ e_n(x) = 2^{3/2} \cos n_1 \pi x_1 \cos n_2 \pi x_2 \cos n_3 \pi x_3, & n \ne (0,0,0) \end{cases}$$

for $n = (n_1, n_2, n_3) \in \mathbb{N}^3$. We note that e_n are eigenfunctions of $-\Delta$ under the Neumann boundary condition. Then we can write

$$\frac{\delta_i}{2}\int_\Omega (u-m_0)G_i(u-m_0)\,dx = \frac{\delta_i}{2}\sum_{n\in\mathbb{N}^3}\frac{|\langle u-m_0, e_n\rangle|^2}{\gamma_i+|n|^2\pi^2}\,,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on $L^2(\Omega)$.

Then we will show the following lemma, which is a modification of Lemma 2.3 in [5].

LEMMA 3.3. *There exists a constant* $L_5 > 0$ *such that*

$$\int_\Omega |T_N(u-m_0)|^2\,dx \le L_5 N^2\int_\Omega (u-m_0)G_i(u-m_0)\,dx \quad (i=1,2) \tag{3.7}$$

for any $N\in\mathbb{N}$.

PROOF. For simplicity, let $\gamma=\gamma_i$. We define T_N^* as the adjoint of T_N, which is easily seen to be

$$T_N^* g(x) = \sum_m \left(\int_{\Omega_m} g(y)\,dy\right)\eta_m(x)\,.$$

Then we see that it suffices to show

$$\sum_{n\in\mathbb{N}^3}\max\left\{1, \frac{\gamma+|n|^2\pi^2}{N^2}\right\}\left|\left\langle T_N^*\zeta, e_n\right\rangle\right|^2 \le L_5\int_\Omega |\zeta(x)|^2\,dx \quad (\zeta\in L^2(\Omega)) \tag{3.8}$$

to prove (3.7). From the Cauchy-Schwarz inequality, we have

$$\begin{aligned} N^2 &\int_\Omega (u-m_0)G_i(u-m_0)\,dx \\ &= \sum_n \frac{N^2}{\gamma+|n|^2\pi^2}\,|\langle u-m_0, e_n\rangle|^2 \\ &\ge \sum_n \min\left\{1, \frac{N^2}{\gamma+|n|^2\pi^2}\right\}|\langle u-m_0, e_n\rangle|^2 \\ &= \sup_{a_n\in l^2}\frac{\left(\sum_n \min\left\{1, \sqrt{\frac{N^2}{\gamma+|n|^2\pi^2}}\right\}\langle u-m_0, e_n\rangle\, a_n\right)^2}{\sum_n |a_n|^2}, \end{aligned} \tag{3.9}$$

where $l^2 = \{\{a_n\}_{n\in\mathbb{N}} : a_n\in\mathbb{R}, \sum_{i=1}^\infty |a_n|^2 < \infty\}$. We note that if $f\in L^2$, then $\{\langle f, e_n\rangle\}_{n\in\mathbb{N}}\in l^2$, where $\langle f, e_n\rangle$ is the n-th Fourier coefficient of f. Then (3.8) and (3.9) with $a_n = \max\left(1, \frac{\sqrt{\gamma^2+|n|^2\pi^2}}{N}\right)\left\langle T_N^*\zeta, e_n\right\rangle$ $(n = 1, \dots,)$ yield

$$N^2 \int_\Omega (u - m_0)G_i(u - m_0)\, dx \geq \sup_{\zeta \in L^2(\Omega)} \frac{\left(\sum_n \langle u - m_0, e_n\rangle \left\langle T_N^*\zeta, e_n\right\rangle\right)^2}{\sum_n \max\{1, \frac{\gamma+|n|^2\pi^2}{N^2}\} \left|\left\langle T_N^*\zeta, e_n\right\rangle\right|^2}$$
$$\geq \frac{1}{L_5} \sup_{\zeta \in L^2(\Omega)} \frac{\left(\int_\Omega (T_N(u - m_0))\zeta\, dx\right)^2}{\int_\Omega |\zeta|^2\, dx}$$
$$= \frac{1}{L_5} \int_\Omega |T_N(u - m_0)|^2\, dx\,.$$

For the proof of (3.8), we can see

$$\sum_n \max\left\{1, \frac{\gamma + |n|^2\pi^2}{N^2} \left|\left\langle T_N^*\zeta, e_n\right\rangle\right|^2\right\}$$
$$\leq \sum_n \left\{\frac{\gamma + |n|^2\pi^2}{N^2} \left|\left\langle T_N^*\zeta, e_n\right\rangle\right|^2 + \left|\left\langle T_N^*\zeta, e_n\right\rangle\right|^2\right\}$$
$$= \frac{1}{N^2} \int_\Omega \left|\nabla(T_N^*\zeta)\right|^2 dx + \frac{\gamma + N^2}{N^2} \int_\Omega (T_N^*\zeta)^2 dx$$
$$= \frac{1}{N^2} \int_\Omega \left\{\sum_m \left(\int_{\Omega_m} \zeta(y)\, dy\right) \nabla\eta_m(x)\right\}^2 dx$$
$$+ \frac{\gamma + N^2}{N^2} \int_\Omega \left\{\sum_m \left(\int_{\Omega_m} \zeta(y)\, dy\right) \eta_m(x)\right\}^2 dx$$
$$= \sum_m \left\{\left(\int_{\Omega_m} \zeta(y)\, dy\right)^2 \int_{\Omega_m} \frac{|\nabla\eta_m(x)|^2}{N^2} + \frac{\gamma + N^2}{N^2} |\eta_m(x)|^2\, dx\right\}$$
$$\leq \sum_m \int_{\Omega_m} |\zeta(y)|^2\, dy \int_\Omega |\nabla\eta(X)|^2 + \left(\frac{\gamma}{N^2} + 1\right) |\eta(X)|^2\, dX$$
$$\leq L_5 \int_\Omega |\zeta(y)|^2\, dy\,,$$

where $L_5 = \int_\Omega |\nabla\eta(X)|^2 + (\gamma + 1)\, |\eta(X)|^2\, dX$. Thus we complete the proof of Lemma 3.3. □

We next show an interpolation inequality using Lemmas 3.1-3.3. The following inequality is closely related to the one which is proposed in [15] without a proof.

LEMMA 3.4. *Let $u \in H^1(\Omega)$. Then there exists a positive constant L_6 such that for* *all* $\sigma \in (0,1), i \in \{1,2\}$,

$$\int_\Omega (u-m_0)^2 \le L_6 \left\{ \sigma \int_\Omega |\nabla(h \circ u)| + \int_{\{x\in\Omega : |u(x)|\ge 1\}} (u-1)^2 + \frac{1}{\sigma^2} \int_\Omega (u-m_0) G_i (u-m_0) \right\}.$$

PROOF. First, we prove for the case $\sigma = \frac{1}{N}$ $(N \in \mathbb{N})$. We can easily see

$$\begin{aligned} \int_\Omega (u-m_0)^2 &\le 2\int_\Omega |u(x)-m_0-T_N(u-m_0)|^2 + 2\int_\Omega |T_N(u-m_0)|^2 \\ &\le C_1 \left\{ \frac{1}{N}\int_\Omega |\nabla(h\circ u)| + \int_{\{x\in\Omega:|u(x)|\ge 1\}} (u-1)^2 + N^2 \int_\Omega (u-m_0)G_i(u-m_0) \right\} \end{aligned} \tag{3.10}$$

for any N from Lemmas 3.2 and 3.3, where $C_1 = 2\max\{L_4, L_5\}$.

Next, let $\sigma \in (0,1) \subset \mathbb{R}$. Then there exist $N \in \mathbb{N}$, $t \in (0,1)$ such that $\sigma = \frac{1-t}{N+1} + \frac{t}{N}$. We consider the following two cases:

(i) If $t \in \left[\frac{1}{2}, 1\right]$, since $\sigma \ge \frac{t}{N}$ and $\frac{1}{\sigma^2} \ge N^2$, by (3.10) we have

$$\begin{aligned} \sigma \int_\Omega |\nabla(h\circ u)| &+ \int_{\{x\in\Omega:|u(x)|\ge 1\}} (u-1)^2 + \frac{1}{\sigma^2}\int_\Omega (u-m_0)G_i(u-m_0) \\ &\ge \frac{t}{N}\int_\Omega |\nabla(h\circ u)| + \int_{\{x\in\Omega:|u(x)|\ge 1\}} (u-1)^2 + N^2\int_\Omega (u-m_0)G_i(u-m_0) \\ &\ge \frac{t}{C_1}\int_\Omega (u-m_0)^2 \\ &\ge \frac{1}{2C_1}\int_\Omega (u-m_0)^2. \end{aligned}$$

(ii) If $t \in \left[0, \frac{1}{2}\right]$, since $\sigma \ge \frac{1-t}{N+1} \ge \frac{1-t}{2N} \ge \frac{1}{4N}$ and $\frac{1}{\sigma^2} \ge N^2$, by (3.10), we have

$$\begin{aligned} \sigma \int_\Omega |\nabla(h\circ u)| &+ \int_{\{x\in\Omega:|u(x)|\ge 1\}} (u-1)^2 + \frac{1}{\sigma^2}\int_\Omega (u-m_0)G_i(u-m_0) \\ &\ge \frac{1}{4N}\int_\Omega |\nabla(h\circ u)| + \int_{\{x\in\Omega:|u(x)|\ge 1\}} (u-1)^2 + N^2\int_\Omega (u-m_0)G_i(u-m_0) \\ &\ge \frac{1}{4C_1}\int_\Omega (u-m_0)^2. \end{aligned}$$

Thus we set $L_6 = 4C_1$ and conclude the lemma.

□

The next lemma is used for the upper bound estimate (cf. [4]).

LEMMA 3.5. *Let*

$$u(x) = \begin{cases} 1\,, & 0 < x_1 < \frac{m_0+1}{2}\,, \\ -1\,, & \frac{m_0+1}{2} \le x_1 < 1\,. \end{cases} \tag{3.11}$$

Moreover, we define u_m as follows.

For $x \in [0, 4/m]$,

$$u_m(x_1, x_2, x_3) = \begin{cases} u\left(\frac{m}{2}x_1, x_2, x_3\right)\,, & 0 < x_1 \le \frac{2}{m}\,, \\ u\left(\frac{m}{2}\left(\frac{4}{m} - x_1, x_2, x_3\right)\right)\,, & \frac{2}{m} < x_1 \le \frac{4}{m}\,. \end{cases} \tag{3.12}$$

For $x \in (4/m, 1]$, we define u_m as a periodic function of the period $4/m$, that is,

$$u_m\,(x_1, x_2, x_3) = u_m\left(x_1 - \frac{4}{m}k, x_2, x_3\right)$$

for $x_1 \in [4k/m, 4(k+1)/m]$, $k = 1, 2, \dots, \frac{m}{4} - 1$. Then there exists a positive constant L_7 such that

$$\int_\Omega (u_m - m_0)G_i(u_m - m_0)\,dx \le \frac{L_7}{m^2} \quad (i = 1, 2)$$

for all $m \in \mathbb{N}$.

PROOF. Since $\int_\Omega u(x) - m_0 dx = 0$, there exists a unique solution v to

$$\begin{cases} -\Delta v(x) = u(x) - m_0\,, & x \in \Omega\,, \\ \frac{\partial v}{\partial n}(x) = 0\,, & x \in \partial\Omega\,, \\ \int_\Omega v(x)dx = 0\,. \end{cases} \tag{3.13}$$

Then we note

$$\|v\|_{L^2(\Omega)} = \left\| v - \frac{1}{|\Omega|}\int v\,dx \right\|_{L^2(\Omega)} \le p_0\,\|\nabla v\|_{L^2(\Omega)} \le p_0\,\|u - m_0\|_{L^2(\Omega)}$$

by the Poincaré inequality. Moreover, we define v_m as a periodic function as follows:

$$\begin{cases} v_m(x_1, x_2, x_3) = \frac{4}{m^2}v\left(\frac{m}{2}x_1, x_2, x_3\right)\,, & 0 < x_1 \le \frac{2}{m}\,, \\ v_m(x_1, x_2, x_3) = \frac{4}{m^2}v\left(\frac{m}{2}\left(\frac{4}{m} - x_1, x_2, x_3\right)\right)\,, & \frac{2}{m} < x_1 \le \frac{4}{m}\,, \\ v_m\,(x_1, x_2, x_3) = v_m(x_1 - \frac{4}{m}k, x_2, x_3), & \frac{4k}{m} < x_1 \le \frac{4(k+1)}{m}, \\ & k = 1, 2, \dots, \frac{m}{4} - 1\,. \end{cases} \tag{3.14}$$

Then it is easy to see that v_m is a unique solution to

$$\begin{cases} -\Delta v_m(x) = u_m(x) - m_0 \,, & x \in \Omega \,, \\ \frac{\partial v_m}{\partial n}(x) = 0 \,, & x \in \partial\Omega \,, \\ \int_\Omega v_m(x) dx = 0 \,. \end{cases} \tag{3.15}$$

Therefore, we obtain

$$\int_\Omega (u_m - m_0) v_m \, dx \le \frac{4}{m^2} \int_\Omega (u - m_0) v \, dx \le \frac{4p_0}{m^2} \|u - m_0\|^2_{L^2(\Omega)} \,.$$

From the definition of u,

$$\begin{aligned} \int_\Omega |u - m_0|^2 \, dx &= (1 - m_0)^2 \frac{m_0 + 1}{2} + (1 + m_0)^2 \frac{1 - m_0}{2} \\ &= 1 - m_0^2 \le 1 \,. \end{aligned}$$

Moreover, noting

$$\begin{aligned} \int_\Omega (u_m - m_0) v_m \, dx &= \sum_{n \in \mathbb{N}^3 \setminus (0,0,0)} \frac{|u_n - m_0|^2}{n^2} \ge \sum_{n \in \mathbb{N}^3 \setminus (0,0,0)} \frac{|u_n - m_0|^2}{n^2 + \gamma_i} \\ &= \int_\Omega (u_m - m_0) G_i (u_m - m_0) \, dx \,, \end{aligned}$$

we can see

$$\int_\Omega (u_m - m_0) G_i (u_m - m_0) \, dx \le \frac{4p_0}{m^2} = \frac{L_7}{m^2} \,.$$

□

Finally, we prove Theorems 3.1 and 3.2.

PROOF OF THEOREM 3.1(1). Let $\epsilon < 2\zeta/m$ and $\widetilde{u}_m$ be a periodic function satisfying the following condition:

$$\widetilde{u}_m(x) = \begin{cases} 1 \,, & 0 \le x_1 \le \frac{m_0+1}{2} \frac{2}{m} - \frac{\epsilon}{2} \,, \\ -\frac{2}{\epsilon} \left(x_1 - \frac{m_0+1}{2} \frac{2}{m} \right) \,, & \frac{m_0+1}{2} \frac{2}{m} - \frac{\epsilon}{2} \le x_1 \le \frac{m_0+1}{2} \frac{2}{m} + \frac{\epsilon}{2} \,, \\ -1 \,, & \frac{m_0+1}{2} \frac{2}{m} + \frac{\epsilon}{2} \le x_1 \le \frac{2}{m} \,. \end{cases}$$

$$\widetilde{u}_m(x_1, x_2, x_3) = \widetilde{u}_m \left(\frac{4}{m} - x_1, x_2, x_3 \right) \,, \quad \frac{2}{m} \le x_1 \le \frac{4}{m} \,.$$

$$\widetilde{u}_m \, (x_1, x_2, x_3) = \widetilde{u}_m \left(x_1 - \frac{4}{m} k, , x_2, x_3 \right), \; \frac{4k}{m} \le x_1 \le \frac{4(k+1)}{m}, \; k = 1, 2, \ldots, \frac{m}{4} - 1 \,.$$

Then we readily see

$$\int_\Omega \epsilon^2 |\nabla \widetilde{u}_m|^2 \, dx = \epsilon^2 \cdot \frac{m}{4} \cdot \left|\frac{2}{\epsilon}\right|^2 \cdot \epsilon = m\epsilon ,$$

$$\int_\Omega H(\widetilde{u}_m) \, dx \le \max_{s \in (-1,1)} H(s) \cdot \frac{m}{4} \cdot \epsilon = C_1 m\epsilon .$$

We define e'_j as follows:

$$e'_j(x_1) = \begin{cases} 1 , & j = 0 , \\ 2^{3/2} \cos j\pi x_1 , & j \ne 0 . \end{cases}$$

Then note that

$$\delta_1 \int_\Omega (\widetilde{u}_m - m_0) G_1 (\widetilde{u}_m - m_0) \, dx = \delta_1 \sum_{j=0}^{\infty} \frac{\left|\left\langle \widetilde{u}_m - m_0, e'_j \right\rangle\right|^2}{j^2\pi^2 + \gamma_1} .$$

For any $j \in \mathbb{N}$, we have

$$\begin{aligned} \left|\left\langle \widetilde{u}_m - m_0, e'_j \right\rangle\right| &= \left| \int_\Omega (\widetilde{u}_m - m_0) \cos j\pi x_1 \; dx_1 \right| \\ &\le \left| \int_\Omega (u_m - m_0) \cos j\pi x_1 \; dx_1 \right| + \left| \int_\Omega (\widetilde{u}_m - u_n) \cos j\pi x_1 \; dx_1 \right| \\ &\le \left|\left\langle u_m - m_0, e'_j \right\rangle\right| + m\epsilon , \end{aligned}$$

where u_m is defined in Lemma 3.5. Therefore we obtain

$$\begin{aligned} \delta_1 \int_\Omega (\widetilde{u}_m - m_0) G_1 (\widetilde{u}_m - m_0) \, dx &\le 2\delta_1 \sum_{j=0}^{\infty} \frac{\left|\left\langle u_m - m_0, e'_j \right\rangle\right|^2}{j^2\pi^2 + \gamma_1} + 2\delta_1 \sum_{j=0}^{\infty} \frac{(m\epsilon)^2}{j^2\pi^2 + \gamma_1} \\ &\le 2\delta_1 \left\{ \frac{L_7}{m^2} + \left(\frac{1}{\gamma_1} + 2 \right) (m\epsilon)^2 \right\} \\ &\le C_2 \delta_1 \frac{1}{m^2} + C_3 \delta_1 (m\epsilon)^2 \end{aligned}$$

from Lemma 3.5. Similarly we also have

$$\delta_2 \int_\Omega (\widetilde{u}_m - m_0) G_2 (\widetilde{u}_m - m_0) \, dx \le C_2 \delta_2 \frac{1}{m^2} + C_3 \delta_2 (m\epsilon)^2 .$$

Thus letting $m = \left[\left(\frac{\delta_1 + \delta_2}{\epsilon} \right)^{\frac{1}{3}} \right]$ (where $[x] = \max\{n \in \mathbb{Z};\ n \le x\}$) and $\epsilon_0 > 0$ be small enough so that $(2\bar{\delta})^{\frac{4}{3}}\epsilon_0^{\frac{2}{3}} < 1$, we have

$$\begin{aligned}\widetilde{I}_\epsilon(\widetilde{u}_m) &\le (1+C_1)m\epsilon + C_2(\delta_1+\delta_2)\frac{1}{m^2} + C_3(\delta_1+\delta_2)(m\epsilon)^2 \\ &\le (2+2C_1+2C_2)\epsilon^{\frac{2}{3}}(\delta_1^{\frac{1}{3}}+\delta_2^{\frac{1}{3}}) + 2C_3\cdot(2\bar{\delta})^{\frac{4}{3}}\epsilon_0^{\frac{2}{3}}\cdot\epsilon^{\frac{2}{3}}(\delta_1^{\frac{1}{3}}+\delta_2^{\frac{1}{3}}) \\ &\le L_0\epsilon^{\frac{2}{3}}(\delta_1^{\frac{1}{3}}+\delta_2^{\frac{1}{3}})\end{aligned}$$

for any $\epsilon \in (0,\epsilon_0)$, $\delta_i \in (\epsilon,\bar{\delta})$. □

PROOF OF THEOREM 3.1(2). It suffices to show that

$$I_i(u_\epsilon) = \frac{\epsilon^2}{4}\int_\Omega |\nabla u_\epsilon|^2 + \frac{\delta_i}{2}\int_\Omega (u_\epsilon - m_0)G_i(u_\epsilon - m_0) + \frac{1}{2}\int_\Omega H(u_\epsilon) \ge l_0\epsilon^{\frac{2}{3}}\delta_i^{\frac{1}{3}} \tag{3.16}$$

for $i = 1, 2$. Since

$$\epsilon\,|\nabla u_\epsilon|^2 + \frac{1}{\epsilon}H(u_\epsilon) \ge 2\,|\nabla(h\circ u_\epsilon)|\ ,$$

we have

$$\begin{aligned}\frac{I_i(u_\epsilon)}{\delta_i} &= \frac{\epsilon^{\frac{2}{3}}}{\delta_i^{\frac{2}{3}}}\left\{\int_\Omega \frac{\epsilon^{\frac{1}{3}}}{\delta_i^{\frac{1}{3}}}\left(\frac{\epsilon}{4}|\nabla u_\epsilon|^2 + \frac{1}{2\epsilon}H(u_\epsilon)\right) + \frac{\delta_i^{\frac{2}{3}}}{2\epsilon^{\frac{2}{3}}}(u_\epsilon - m_0)G_i(u_\epsilon - m_0)dx\right\} \\ &\ge \frac{\epsilon^{\frac{2}{3}}}{2\delta_i^{\frac{2}{3}}}\left\{\int_\Omega \frac{\epsilon^{\frac{1}{3}}}{\delta_i^{\frac{1}{3}}}|\nabla(h\circ u_\epsilon)|\,dx + \int_{\{|u_\epsilon(x)|\ge 1\}}(|u_\epsilon|-1)^2\,dx\right. \\ &\quad\left. + \frac{\delta_i^{\frac{2}{3}}}{\epsilon^{\frac{2}{3}}}\int_\Omega (u_\epsilon - m_0)G_i(u_\epsilon - m_0)dx\right\} - \frac{\epsilon^{\frac{2}{3}}}{2\delta_i^{\frac{2}{3}}}\int_{\{|u_\epsilon(x)|\ge 1\}}(|u_\epsilon|-1)^2\,dx\,.\end{aligned}$$

Then from Lemma 3.4 with $\sigma = (\epsilon/\delta_i)^{1/3}$, we have

$$\frac{I_i(u_\epsilon)}{\delta_i} \ge \frac{1}{L_6}\frac{\epsilon^{\frac{2}{3}}}{2\delta_i^{\frac{2}{3}}}\int_\Omega |u_\epsilon - m_0|^2\,dx - \frac{\epsilon^{\frac{2}{3}}}{2\delta_i^{\frac{2}{3}}}\int_{\{|u_\epsilon(x)|\ge 1\}}(|u_\epsilon|-1)^2\,dx$$

for any $0 < \epsilon \le \delta_i$. Letting $\theta > 0$ be a small constant which we will specify later, we have

$$\begin{aligned}\int_\Omega |u_\epsilon - m_0|^2\,dx &\ge \int_{\Omega_{\epsilon,\theta}^c} |u_\epsilon - m_0|^2\,dx \\ &\ge \min\{|1-\theta-m_0|\,,\,|m_0-(-1+\theta)|\}\left(|\Omega| - \left|\Omega_{\epsilon,\theta}\right|\right) \\ &\ge (\zeta-\theta)\left(|\Omega| - \left|\Omega_{\epsilon,\theta}\right|\right)\,,\end{aligned}$$

where $\Omega_{\epsilon,\theta} = \{x \in \Omega; |(|u_\epsilon(x)| - 1)| \geq \theta\}$. On the other hand, by Lemma 2.3 we have

$$\int_{\{|u_\epsilon(x)|\geq 1\}} (|u_\epsilon| - 1)^2 \, dx \leq \int_{\Omega_{\epsilon,\theta}} (|u_\epsilon| - 1)^2 \, dx + \int_{(\Omega_{\epsilon,\theta})^c} (|u_\epsilon| - 1)^2 \, dx$$
$$\leq 2(L_1^2 + 1) \left|\Omega_{\epsilon,\theta}\right| + \theta^2 \left|(\Omega_{\epsilon,\theta})^c\right| .$$

As a consequence, we obtain

$$\frac{I_i(u_\epsilon)}{\delta_i} \geq \frac{\epsilon^{\frac{2}{3}}}{2\delta_i^{\frac{2}{3}}} \left\{ \frac{\zeta}{L_6} |\Omega| - \left(\frac{\zeta - \theta}{L_6} + 2(L_1^2 + 1) \right) \left|\Omega_{\epsilon,\theta}\right| - \left(\frac{\theta}{L_6} + \theta^2 \right) |\Omega| \right\} .$$

We set $\theta > 0$ small so that $\left(\frac{\theta}{L_6} + \theta^2\right) |\Omega| \leq \frac{\zeta}{2L_6} |\Omega|$. From Theorem 3.1(1), we note

$$\begin{aligned} \left|\Omega_{\epsilon,\theta}\right| &\leq \frac{1}{c_\theta} \int_\Omega H(u_\epsilon) \, dx \\ &\leq \frac{L_0}{c_\theta} \epsilon^{\frac{2}{3}} (\delta_1^{\frac{1}{3}} + \delta_2^{\frac{1}{3}}) \\ &\leq \frac{2\bar{\delta}^{\frac{1}{3}} L_0}{c_\theta} \epsilon^{\frac{2}{3}} \end{aligned}$$

for any $\epsilon \in (0, \epsilon_0)$, $\delta_i \in (\epsilon, \bar{\delta})$, where $c_\theta = \min\{H(s); |s| - 1 \geq \theta\} > 0$. Thus letting $\epsilon_1 > 0$ small so that $\epsilon_1 < \epsilon_0$ and

$$\left(\frac{\zeta - \theta}{L_6} + 2(L_1^2 + 1) \right) \left|\Omega_{\epsilon_1,\theta}\right| < \frac{\zeta}{4L_6} |\Omega| ,$$

we can see

$$\frac{I_i(u_\epsilon)}{\delta_i} \geq \frac{\zeta}{4L_6} |\Omega| \cdot \frac{\epsilon^{\frac{2}{3}}}{2\delta_i^{\frac{2}{3}}} = l_0 \frac{\epsilon^{\frac{2}{3}}}{\delta_i^{\frac{2}{3}}}$$

for any $\epsilon \in (0, \epsilon_1)$, $\delta_i \in (\epsilon, \bar{\delta})$. □

PROOF OF THEOREM 3.2. Let $\widetilde{u}_m$ be a function defined in the proof of Theorem 3.1(1). Then we will show

$$\int_\Omega \widetilde{u}_m(x) \, dx = m_0 . \tag{3.17}$$

In fact,

$$\begin{aligned} \int_\Omega \widetilde{u}_m \, dx &= \int_{\Omega'} 2 \cdot \frac{m}{4} \int_0^{\frac{2}{m}} \widetilde{u}_m \, dx_1 dx' \\ &= \frac{m}{2} \left(\int_0^{\frac{2}{m} \frac{1+m_0}{2} - \frac{\epsilon}{2}} 1 \, dx_1 + \int_{\frac{2}{m} \frac{1+m_0}{2} + \frac{\epsilon}{2}}^{\frac{2}{m}} (-1) \, dx_1 + \int_{\frac{2}{m} \frac{1+m_0}{2} - \frac{\epsilon}{2}}^{\frac{2}{m} \frac{1+m_0}{2} + \frac{\epsilon}{2}} \widetilde{u}_m(x_1, x') \, dx_1 \right) , \end{aligned}$$

where $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^2$. We can readily see

$$\frac{m}{2}\left(\int_0^{\frac{2}{m}\frac{1+m_0}{2}-\frac{\epsilon}{2}} 1\,dx_1 + \int_{\frac{2}{m}\frac{1+m_0}{2}+\frac{\epsilon}{2}}^{\frac{2}{m}} (-1)\,dx_1\right) = \left[\frac{2}{m}\left(\frac{1+m_0}{2} - \frac{1-m_0}{2}\right) - \frac{\epsilon}{2} + \frac{\epsilon}{2}\right]\frac{m}{2}$$
$$= m_0\,,$$

$$\begin{aligned}\int_{\frac{2}{m}\frac{1+m_0}{2}-\frac{\epsilon}{2}}^{\frac{2}{m}\frac{1+m_0}{2}+\frac{\epsilon}{2}} \widetilde{u}_m(x_1, x')\,dx_1 &= \int_{\frac{2}{m}\frac{1+m_0}{2}-\frac{\epsilon}{2}}^{\frac{2}{m}\frac{1+m_0}{2}+\frac{\epsilon}{2}} -\frac{2}{\epsilon}\left(x_1 - \frac{m_0+1}{2}\cdot\frac{2}{m}\right)dx_1 \\ &= \int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} -\frac{\epsilon}{2} X\,dX \\ &= 0\,.\end{aligned}$$

Therefore we obtain (3.17). As a consequence, letting $m = (\delta_1/\epsilon)^{\frac{1}{3}}$ and $\epsilon_0{}'$ be small enough so that $\bar{\delta}^{\frac{4}{3}}\epsilon_0{}'^{\frac{2}{3}} < 1$, we can see

$$\begin{aligned}\widetilde{J}_\epsilon(\widetilde{u}_m) &\le (1+C_1)m\epsilon + C_2\delta_1\frac{1}{m^2} + C_3\delta_1(m\epsilon)^2 \\ &\le (1+C_1+C_2)\epsilon^{\frac{2}{3}}\delta_1^{\frac{1}{3}} + C_3\cdot\bar{\delta}^{\frac{4}{3}}\epsilon_0'^{\frac{2}{3}}\cdot\epsilon^{\frac{2}{3}}\delta_1^{\frac{1}{3}} \\ &\le L_0'\epsilon^{\frac{2}{3}}\delta_1^{\frac{1}{3}}\end{aligned}$$

for any $\epsilon \in (0, \epsilon_0')$, $\delta_i \in (\epsilon, \bar{\delta})$ as in the proof of Theorem 3.1(1) and we complete the upper bound estimate.

For the lower bound estimate, we can prove as in the proof of Theorem 3.1(2) since $\left(\int_\Omega (u(x) - m_0)\,dx\right)^2 \ge 0$. □

Appendix A. On the stability of stationary solutions

In this section, we give a proof of Theorem 1.3. We mainly treat the shadow system (1.6) since we can treat the problem (1.1) as in the proof of Theorem 1.2 in [15]. Then we consider the spectrum of the linearized operator $T : \left(L^2(\Omega)\right)^2 \times \mathbb{R} \to \left(L^2(\Omega)\right)^2 \times \mathbb{R}$ defined as follow:

$$\text{(A.1)}\qquad T\Phi = \begin{pmatrix} \Delta\phi + f'(u_0)\phi - \psi - \mu \\ \frac{1}{\tau_1}\left(\Delta\psi - \gamma_1\psi + \delta_1\phi\right) \\ \frac{1}{\tau_2|\Omega|}\int_\Omega(-\gamma_2\mu + \delta_2\phi)\,dx \end{pmatrix}, \quad \Phi = (\phi, \psi, \mu) \in X^2 \times \mathbb{R}\,,$$

where $X = \left\{\phi \in H^2(\Omega);\, \frac{\partial\phi}{\partial n}(x) = 0,\ x \in \partial\Omega\right\}$.

We denote by Σ_p, Σ_c, Σ_r the point spectrum, the continuous spectrum and the residual spectrum of T, respectively. Let $\Sigma = \Sigma_p \cup \Sigma_c \cup \Sigma_r$ be the spectrum of T. Moreover, we define $\|\cdot\| = \|\cdot\|_{(L^2(\Omega)^2\times\mathbb{R})}$, that is, $\|\Phi\| = \|\phi\|_{L^2(\Omega)} + \|\psi\|_{L^2(\Omega)} + |\mu|$ $(\Phi = (\phi, \psi, \mu) \in \{L^2(\Omega)\}^2 \times \mathbb{R})$.

The goal of this section is to show the following stability result:

THEOREM A.1. *Assume that u_0 is a local minimizer of $\widetilde{J}_\epsilon$ and $\frac{\delta_1\tau_1}{\gamma_1^2} + \frac{\delta_2\tau_2}{\gamma_2^2} < 1$. Then the spectrum of T lies in the stable region and (u_0, v_0, ξ_0) is a stable solution to equation* (1.6), *where $v_0 = \delta_1 G_1 u_0$, $\xi_0 = \frac{\delta_2}{|\Omega|}\int_\Omega u_0\, dx$.*

Before the proof of Theorem A.1, we introduce some notations.

Let $N = \left\{\zeta \in \mathbb{C}; \operatorname{Re}\zeta > \max\{-\frac{\gamma_1}{\tau_1}, -\frac{\gamma_2}{\tau_2}\}\right\}$. Noting $\operatorname{Re}(\gamma_i + \lambda\tau_i) > 0$ $(i = 1, 2)$ if $\lambda \in N$, there exists a unique solution $\psi \in L^2(\Omega)$ of the equation

$$\begin{cases} -\Delta\psi + (\gamma_i + \tau_i\lambda)\psi = \phi & \text{in } \Omega\,, \\ \frac{\partial\psi}{\partial n} = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Then we write $\psi = (-\Delta + \gamma_i + \tau_i\lambda)^{-1}\phi$. Moreover we define

$$\mathcal{L}_\lambda\phi = \Delta\phi + f'(u_0)\phi - \delta_1(-\Delta + \gamma_1 + \tau_1\lambda)^{-1}\phi - \frac{\delta_2}{|\Omega|\,(\gamma_2 + \tau_2\lambda)}\int_\Omega \phi\, dx$$

for $\lambda \in N, \phi \in X$. Theorem A.1 follows from Lemmas A.1–A.6. In addition, we mention that throughout this section, the proofs are essentially the same argument as in [15].

LEMMA A.1. (1) *If $\lambda \in \Sigma \cap N$, then there holds either* (a) *or* (b):

(a) *λ is an eigenvalue of $\mathcal{L}_\lambda$.*

(b) *$\overline{\lambda}$ is an eigenvalue of $\mathcal{L}_{\overline{\lambda}}$.*

(2) *Assume $\frac{\delta_1\tau_1}{\gamma_1^2} + \frac{\delta_2\tau_2}{\gamma_2^2} < 1$. Then $\Sigma \cap \left\{\zeta \in \mathbb{C} : \frac{\delta_1\tau_1}{(\gamma_1+\tau_1\operatorname{Re}\zeta)^2} + \frac{\delta_2\tau_2}{(\gamma_2+\tau_2\operatorname{Re}\zeta)^2} < 1\right\}$ consists of real numbers.*

PROOF. (1) First, if $\lambda \in \Sigma_p \cap N$, then there exists a function $\Phi = (\phi, \psi, \mu) \in X^2 \times \mathbb{R} \setminus \{0\}$ such that $T\Phi = \lambda\Phi$, that is,

$$\text{(A.2)}\qquad \begin{cases} \Delta\phi(x) + f'(u_0)\phi(x) - \psi(x) - \mu = \lambda\phi(x)\,, & x \in \Omega\,, \\ \Delta\psi(x) - \gamma_1\psi(x) + \delta_1\phi(x) = \tau_1\lambda\psi(x)\,, & x \in \Omega\,, \\ -\gamma_2\mu + \frac{\delta_2}{|\Omega|}\int_\Omega \phi(x)\, dx = \tau_2\lambda\mu\,, & \\ \frac{\partial\phi}{\partial n}(x) = \frac{\partial\psi}{\partial n}(x) = 0\,, & x \in \partial\Omega\,. \end{cases}$$

Since $\psi = \delta_1(-\Delta + \gamma_1 + \tau_1\lambda)^{-1}\phi$, $\mu = \frac{\delta_2}{(\gamma_2+\tau_2\lambda)|\Omega|}\int_\Omega \phi(x)\, dx$, we can see $\mathcal{L}_\lambda\phi = \lambda\phi$.

Next if $\lambda \in \Sigma_c \cap N$, we have a sequence $F_k = (f_k, g_k, h_k) \in (L^2(\Omega))^2 \times \mathbb{R}$ such that $\|F_k\| \leq \frac{1}{k}$ and $\left\|(\lambda I - T)^{-1}F_k\right\| = 1$ for any $k \in \mathbb{N}$, which means that there exists a sequence $(\phi_k, \psi_k, \mu_k) \in X^2 \times \mathbb{R}$ such that

$$
\text{(A.3)} \qquad \begin{cases} \Delta\phi_k(x) + f'(u_0)\phi_k(x) - \psi_k(x) - \mu_k = \lambda\phi_k(x) + f_k(x)\,, & x \in \Omega\,,\\ \Delta\psi_k(x) - \gamma_1\psi_k(x) + \delta_1\phi_k(x) = \tau_1\lambda\psi_k(x) + g_k(x)\,, & x \in \Omega\,,\\ -\gamma_2\mu_k + \frac{\delta_2}{|\Omega|}\int_\Omega \phi_k\,dx = \tau_2\lambda\mu_k + h_k\,, & \\ \frac{\partial\phi_k}{\partial n}(x) = \frac{\partial\psi_k}{\partial n}(x) = 0\,, & x \in \partial\Omega\,,\\ \|\phi_k\|^2_{L^2(\Omega)} + \|\psi_k\|^2_{L^2(\Omega)} + |\mu_k|^2 = 1\,, & \\ \|f_k\|^2_{L^2(\Omega)} + \|g_k\|^2_{L^2(\Omega)} + |h_k|^2 \le \frac{1}{k}\,. & \end{cases}
$$

From the second and third equations of (A.3), $\|\psi_k\|_{L^2(\Omega)}\,, |\mu_k| \le C_1\,\|\phi_k\|_{L^2(\Omega)} + \frac{C_2}{k}$. Then there exists a positive constant c_3 such that $\liminf_{k\to\infty}\|\phi_k\|_{L^2(\Omega)} \ge c_3 > 0$ from the fourth equation of (A.3). Moreover, since

$$
\|\Delta\phi_k\|_{L^2(\Omega)} = \left\|-f'(u_0)\phi_k + \psi_k + \mu_k + \lambda\phi_k + f_k\right\|_{L^2(\Omega)} \le C_4\,,
$$

we can see $\|\phi_k\|_{H^1(\Omega)} \le C_5$ for any $k \in \mathbb{N}$. This implies there exists $\phi \in H^1(\Omega)$ such that $\phi_k \to \phi$ weakly in $H^1(\Omega)$ and $\phi_k \to \phi$ strongly in $L^2(\Omega)$. Thus we have

$$
\mathcal{L}_\lambda\phi_k - \lambda\phi_k = f_k - (-\Delta + \gamma_1 + \tau_1\lambda)^{-1}g_k - \frac{h_k}{\gamma_2 + \tau_2\lambda} \to 0 \text{ in } L^2(\Omega)\ (k \to \infty)\,.
$$

We define $\mathcal{B}_\lambda$ as follow:

$$
\mathcal{B}_\lambda\phi = \Delta\phi - \frac{\gamma_1}{\tau_1}\phi - \delta_1(-\Delta + \gamma_1 + \tau_1\lambda)^{-1}\phi - \frac{\delta_2}{|\Omega|\,(\gamma_2 + \tau_2\lambda)}\int_\Omega \phi\,dx
$$

for $\phi \in X$. Since $\mathrm{Re}\lambda > -\frac{\gamma_1}{\tau_1}$, $\lambda \notin \sigma(\mathcal{B}_\lambda)$, where $\sigma(\mathcal{B}_\lambda)$ is the spectrum of $\mathcal{B}_\lambda$. Moreover,

$$
\mathcal{B}_\lambda\phi_k - \lambda\phi_k = \mathcal{B}_\lambda\phi_k - \lambda\phi_k + \left(-\frac{\gamma_1}{\tau_1} - f'(u_0)\right)\phi_k \xrightarrow{k\to\infty} \left(-\frac{\gamma_1}{\tau_1} - f'(u_0)\right)\phi\,.
$$

Thus we have

$$
\phi = (\mathcal{B}_\lambda - \lambda)^{-1}\left(-\frac{\gamma_1}{\tau_1} - f'(u_0)\right)\phi\,.
$$

As a consequence, we can see

$$
\mathcal{L}_\lambda\phi - \lambda\phi = (\mathcal{B}_\lambda - \lambda)\,\phi - \left(-\frac{\gamma_1}{\tau_1} - f'(u_0)\right)\phi = 0\,.
$$

Finally, we will show in the case $\lambda \in \Sigma_r \cap N$. We have a function $\widetilde{\Phi} = (\phi, \psi, \mu) \in \mathcal{R}(\lambda I - T)^\perp \setminus \{0\}$, which means that $\widetilde{\Phi} = (\phi, \psi, \mu)$ satisfies that $\left\langle(\lambda I - T)\,\Phi, \widetilde{\Phi}\right\rangle = 0$ for all $\Phi \in \mathcal{D}(T) = X^2 \times \mathbb{R}$, where $\langle\cdot,\cdot\rangle$ denotes the complex inner product on $\left(L^2(\Omega)\right)^3$. This implies that $\widetilde{\Phi} \in \mathcal{D}(T^*)$ and $\overline{\lambda}$ is an eigenvalue of T^* corresponding to $\widetilde{\Phi}$. We can easily see $\overline{\lambda}$ is also an eigenvalue of $\mathcal{L}_{\overline{\lambda}}$ corresponding to $\widetilde{\phi}$.

(2) It suffices to show that if $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathcal{L}_\lambda$ with $\lambda \in \mathbb{C}$ such that $\frac{\delta_1\tau_1}{(\gamma_1+\tau_1\mathrm{Re}\zeta)^2} + \frac{\delta_2\tau_2}{(\gamma_2+\tau_2\mathrm{Re}\zeta)^2} < 1$, then λ is real.

Let $\lambda = x + iy$ be an eigenvalue of $\mathcal{L}_\lambda$ such that $\frac{\delta_1\tau_1}{(\gamma_1+\tau_1 x)^2} + \frac{\delta_2\tau_2}{(\gamma_2+\tau_2 x)^2} < 1$, $y \in \mathbb{R}$, and let ϕ be an eigenfunction corresponding to λ satisfying $\|\phi\|_{L^2(\Omega)} = 1$. Moreover, letting ξ_n be eigenvalues of $-\Delta$ under the Neumann boundary condition and e_n be eigenfunctions corresponding to ξ_n, we can see $\phi = \sum_{n=1}^{\infty} a_n e_n$. Notice $\sum_{n=1}^{\infty} |a_n|^2 = 1$, $e_0 = |\Omega|^{\frac{1}{2}}$ and $\xi_0 = 0$. Since

$$\begin{aligned}
\lambda &= \langle \mathcal{L}_\lambda\phi, \phi\rangle \\
&= \int_\Omega |\nabla\phi|^2\,dx + \int_\Omega f'(u_0)\,|\phi|^2\,dx - \delta_1\int_\Omega \left\{(-\Delta+\gamma_1+\tau_1\lambda)^{-1}\phi\right\}\phi\,dx \\
&\quad - \frac{\delta_2}{(\gamma_2+\tau_2\lambda)\,|\Omega|}\left(\int_\Omega \phi\,dx\right)^2 \\
&= \int_\Omega |\nabla\phi|^2\,dx + \int_\Omega f'(u_0)\,|\phi|^2\,dx - \delta_1\sum_{n=0}^{\infty}\frac{|a_n|^2}{\xi_n+\gamma_1+\tau_1\lambda} - \delta_2\frac{|a_0|^2}{\gamma_2+\tau_2\lambda}\,,
\end{aligned}$$

there holds

$$\begin{aligned}
|y| &= \left|\mathrm{Im}\left\{\sum_{n=0}^{\infty}\frac{\delta_1\,|a_n|^2}{\xi_n+\gamma_1+\tau_1\lambda} + \frac{\delta_2\,|a_0|^2}{\gamma_2+\tau_2\lambda}\right\}\right| \\
&= \sum_{n=0}^{\infty}\frac{\delta_1\,|a_n|^2\,\tau_1\,|y|}{(\xi_n+\gamma_1+\tau_1 x)^2+(\tau_1 y)^2} + \frac{\delta_2\,|a_0|^2\,\tau_2\,|y|}{(\gamma_2+\tau_2 x)^2+(\tau_2 y)^2} \\
&\le |y|\left(\frac{\delta_1\tau_1}{(\gamma_1+\tau_1 x)^2} + \frac{\delta_2\tau_2}{(\gamma_2+\tau_2 x)^2}\right)^2 \sum_{n=0}^{\infty}|a_n|^2\,.
\end{aligned}$$

As $\frac{\delta_1\tau_1}{(\gamma_1+\tau_1 x)^2} + \frac{\delta_2\tau_2}{(\gamma_2+\tau_2 x)^2} < 1$, we have $y = 0$. The proof is complete. □

Before Lemma A.2, we define $h(\lambda)$ as follows:

$$h(\lambda) = \max_{\|\phi\|_{L^2(\Omega)}=1}\langle \mathcal{L}_\lambda\phi, \phi\rangle\,,\quad \lambda \in J = \left(\max\left\{-\frac{\gamma_1}{\tau_1}, -\frac{\gamma_2}{\tau_2}\right\}, \infty\right). \tag{A.4}$$

LEMMA A.2. (1) *If $\lambda \in J$ is an eigenvalue of $\mathcal{L}_\lambda$, then $\lambda \le h(\lambda)$ and λ is an eigenvalue of T.*

(2) *If $\lambda \in J$, $\lambda = h(\lambda)$, then λ is an eigenvalue of T.*

PROOF. (1) If $\lambda \in J$ is an eigenvalue of $\mathcal{L}_\lambda$, then there exists $\widetilde{\phi} \in X$ such that $\mathcal{L}_\lambda\widetilde{\phi} = \lambda\widetilde{\phi}$, $\|\widetilde{\phi}\|_{L^2(\Omega)} = 1$. Letting $\widetilde{\psi} = \delta_1(-\Delta+\gamma_1+\tau_1\lambda_1)^{-1}\widetilde{\phi}$, $\widetilde{\mu} = \frac{\delta_2}{|\Omega|(\gamma_2+\tau_2\lambda)}\int_\Omega \widetilde{\phi}\,dx$, we have $T(\widetilde{\phi}, \widetilde{\psi}, \widetilde{\mu}) = \lambda(\widetilde{\phi}, \widetilde{\psi}, \widetilde{\mu})$. Moreover,

$$\lambda = \langle \mathcal{L}_\lambda \widetilde{\phi}, \widetilde{\phi} \rangle \le \max_{\|\phi\|_{L^2(\Omega)}=1} \langle \mathcal{L}_\lambda \phi, \phi \rangle = h(\lambda)\,.$$

(2) It is clear from the characterization of an eigenvalue of $\mathcal{L}_\lambda$. □

LEMMA A.3. (1) $h(0) \le 0$.

(2) *The function* $h : J \to \mathbb{R}$ *is nondecreasing. Moreover, if* $\lambda_1 < \lambda_2$, *we have* $h(\lambda_2) - h(\lambda_1) \le \left\{ \frac{\delta_1 \tau_1}{(\gamma_1+\tau_1\lambda_1)^2} + \frac{\delta_2 \tau_2}{(\gamma_2+\tau_2\lambda_1)^2} \right\} (\lambda_2 - \lambda_1)\,.$

PROOF. (1) Since u_0 is a local minimizer of J_ϵ,

$$\begin{aligned}\langle \mathcal{L}_0 \phi, \phi \rangle &= -\int_\Omega |\nabla\phi|^2\, dx + \int_\Omega f'(u_0)\phi^2\, dx \\ &\quad -\delta_1 \int_\Omega \left\{ (-\Delta + \gamma_1 + \tau_1\lambda)^{-1} \phi \right\} \phi\, dx - \frac{\delta_2}{\gamma_2\,|\Omega|} \left(\int_\Omega \phi\, dx \right)^2 \\ &= J_\epsilon{}''(u_0)(\phi, \phi) \le 0\end{aligned}$$

for any $\phi \in H^1(\Omega)$. Hence we have $h(0) \le 0$.

(2) For $\lambda_1 < \lambda_2$, there holds

$$\begin{aligned}\langle \mathcal{L}_{\lambda_1} \phi, \phi \rangle &= -\int_\Omega \left\{ |\nabla\phi|^2 + f'(u_0)\phi^2 \right\} dx - \delta_1 \sum_{n=0}^\infty \frac{|a_n|^2}{\xi_n + \gamma_1 + \tau_1\lambda_1} - \delta_2 \frac{|a_0|^2}{\gamma_2 + \tau_2\lambda_1} \\ &\le -\int_\Omega \left\{ |\nabla\phi|^2 + f'(u_0)\phi^2 \right\} dx - \delta_1 \sum_{n=0}^\infty \frac{|a_n|^2}{\xi_n + \gamma_1 + \tau_1\lambda_2} - \delta_2 \frac{|a_0|^2}{\gamma_2 + \tau_2\lambda_2} \\ &\le h(\lambda_2)\end{aligned}$$

for any ϕ with $\|\phi\|_{L^2(\Omega)} = 1$, where $a_n = \langle \phi, e_n \rangle$.

Now if ϕ_2 is an eigenfunction with $\|\phi_2\|_{L^2(\Omega)} = 1$ corresponding to $h(\lambda_2)$, we can see

$$h(\lambda_2) = \int_\Omega |\nabla\phi_2|^2\, dx + \int_\Omega f'(u_0)\,|\phi_2|^2\, dx - \delta_1 \sum_{n=0}^\infty \frac{|a_n|^2}{\xi_n + \gamma_1 + \tau_1\lambda_2} - \frac{\delta_2\,|a_0|^2}{\gamma_2 + \tau_2\lambda_2}\,,$$

$$h(\lambda_1) \ge \int_\Omega |\nabla\phi_2|^2\, dx + \int_\Omega f'(u_0)\,|\phi_2|^2\, dx - \delta_1 \sum_{n=0}^\infty \frac{|a_n|^2}{\xi_n + \gamma_1 + \tau_1\lambda_1} - \frac{\delta_2\,|a_0|^2}{\gamma_2 + \tau_2\lambda_1}\,.$$

Thus we have

$$h(\lambda_2) - h(\lambda_1) \le \delta_1 \sum_{n=0}^\infty \left(\frac{|a_n|^2}{\xi_n + \gamma_1 + \tau_1\lambda_1} - \frac{|a_n|^2}{\xi_n + \gamma_1 + \tau_1\lambda_2} \right)$$

$$
\begin{aligned}
&+ \delta_2 \left(\frac{|a_0|^2}{\gamma_2 + \tau_2\lambda_1} - \frac{|a_0|^2}{\gamma_2 + \tau_2\lambda_2} \right) \\
&= \delta_1 \sum_{n=0}^{\infty} \frac{\tau_1(\lambda_2 - \lambda_1)}{(\xi_n + \gamma_1 + \tau_1\lambda_1)\,(\xi_n + \gamma_1 + \tau_1\lambda_2)} \\
&+ \delta_2 \frac{\tau_2(\lambda_2 - \lambda_1)}{(\gamma_2 + \tau_2\lambda_1)\,(\gamma_2 + \tau_2\lambda_2)} |a_0|^2 \\
&\leq \left\{ \frac{\delta_1\tau_1}{(\gamma_1 + \tau_1\lambda_1)^2} + \frac{\delta_2\tau_2}{(\gamma_2 + \tau_2\lambda_1)^2} \right\} (\lambda_2 - \lambda_1)\,.
\end{aligned}
$$

Therefore, h is locally Lipschitz continuous. □

LEMMA A.4. *Assume* $\frac{\delta_1\tau_1}{\gamma_1^2} + \frac{\delta_2\tau_2}{\gamma_2^2} < 1$. *Then there holds* $h(\lambda) < \lambda$ *for all* $\lambda > 0$.

PROOF. From Lemma A.3, we have

$$
h(\lambda) \leq h(0) + \left(\frac{\delta_1\tau_1}{(\gamma_1 + \tau_1\lambda)^2} + \frac{\delta_2\tau_2}{(\gamma_2 + \tau_2\lambda)^2} \right) \lambda \leq \left(\frac{\delta_1\tau_1}{\gamma_1^2} + \frac{\delta_2\tau_2}{\gamma_2^2} \right) \lambda < \lambda\,.
$$

□

LEMMA A.5. *There holds* $\Sigma \subset \{0\} \cup \{\zeta \in \mathbb{C} : \mathrm{Re}\zeta \leq \alpha\}$ *for some* $\alpha < 0$. *If, in addition,* $h(0) < 0$, *then* $\Sigma \subset \{\zeta \in \mathbb{C} : \mathrm{Re}\zeta \leq \alpha\}$.

PROOF. Let β be a constant such that $\frac{\tau_1\delta_1}{(\gamma_1+\tau_1\beta)^2} + \frac{\tau_2\delta_2}{(\gamma_2+\tau_2\beta)^2} < 1$ and $\beta < 0$. If $\lambda \in \Sigma \cap \{\zeta \in \mathbb{C} : \mathrm{Re}\zeta > \beta\}$, then from Lemma A.1, λ is real and an eigenvalue of $\mathcal{L}_\lambda$. By Lemma A.2(1), we have $\lambda \leq h(\lambda)$. In addition, by Lemma A.4, we have $\lambda \leq 0$.

Next, we will prove the essential spectrum of $\mathcal{L}_\lambda$, denoted by $\sigma_e(\mathcal{L}_\lambda)$ is empty. Now, let $H_0 = \Delta$, $V\phi = f'(u_0)\phi - \delta_1(-\Delta + \gamma_1 + \tau_1\lambda)^{-1}\phi - \frac{\delta_2}{(\gamma_2\tau_2\lambda)|\Omega|}\int_\Omega \phi\,dx$ for $\phi \in L^2(\Omega)$. Then noting that H_0 and V are self adjoint operators from $L^2(\Omega)$ to $L^2(\Omega)$ and V is H_0-*compact*, we have $\sigma_e(\mathcal{L}_\lambda) = \sigma_e(H_0) = \emptyset$. This implies that λ is an eigenvalue with finite multiplicity. Thus, there exist $\lambda_1, \lambda_2, \ldots, \lambda_m \in (\beta, 0]$ such that

$$
\begin{aligned}
\Sigma &= \{\Sigma \cap \{\zeta \in \mathbb{C} : \mathrm{Re}\zeta > \beta\}\} \cup \{\Sigma \cap \{\zeta \in \mathbb{C} : \mathrm{Re}\zeta \leq \beta\}\} \\
&= \{\lambda_1, \lambda_2, \ldots, \lambda_m\} \cup \{\Sigma \cap \{\zeta \in \mathbb{C} : \mathrm{Re}\zeta \leq \beta\}\}\,.
\end{aligned}
$$

The proof is complete. □

We recall the basic center manifold theory (see [9]).

PROPOSITION A.1. *Let* $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$ *be Banach spaces such that* $\mathfrak{Z} \hookrightarrow \mathfrak{Y} \hookrightarrow \mathfrak{X}$ *with continuous embeddings. We consider a differential equation in* $\mathfrak{X}$ *of the form*

$$
\frac{du}{dt} = \mathbf{L}u + \mathbf{R}(u) \tag{A.5}
$$

in which we assume the following hypothesis for the linear part $\mathbf{L}$ *and the nonlinear part* $\mathbf{R}$.

(1) **Hypothesis 1.** *We assume that* $\mathbf{L}$ *and* $\mathbf{R}$ *have the following properties*:

  (i) $\mathbf{L} : \mathfrak{Z} \to \mathfrak{X}$ *is bounded linear operator.*

  (ii) *For some* $k \geq 2$, *there exists a neighborhood* $\mathfrak{V} \subset \mathfrak{Z}$ *of* 0 *such that* $\mathbf{R} \in C^k(\mathfrak{V}, \mathfrak{Y})$ *and* $\mathbf{R}(0), \mathbf{DR}(0) = 0$.

(2) **Hypothesis 2.** *Consider the spectrum* σ *of the linear operator* $\mathbf{L}$ *and write* $\sigma = \sigma_+ \cup \sigma_0 \cup \sigma_-$ *in which* $\sigma_+ = \{\lambda \in \sigma : \mathrm{Re}\lambda > 0\}$, $\sigma_0 = \{\lambda \in \sigma : \mathrm{Re}\lambda = 0\}$, $\sigma_- = \{\lambda \in \sigma : \mathrm{Re}\lambda < 0\}$. *We assume that* (i) *and* (ii):

  (i) *There exists a positive constant* $\gamma > 0$ *such that*

$$\inf_{\lambda \in \sigma_+} \mathrm{Re}\lambda > \gamma, \ \sup_{\lambda \in \sigma_-} \mathrm{Re}\lambda < -\gamma.$$

  (ii) *The set* σ_0 *consists of a finite number of eigenvalues with finite algebraic multiplicities.*

(3) **Hypothesis 3.** *Let* $P_0 : \mathfrak{X} \to \mathfrak{X}$ *be a bounded linear operator and be the the spectral projection corresponding to* σ_0. *Moreover let* $\mathcal{E}_0 = \mathrm{Im}P_0$, $X_h = \mathrm{Im}(I - P_0) = \mathrm{Ker}P_0$, $\mathbf{L}_0 = \mathbf{L}|_{\mathcal{E}_0}$, *and* $\mathbf{L}_h = \mathbf{L}|_{X_h}$. *Then there exist* $\omega_0, C > 0$ *such that for any* $\omega \in \{\omega \in \mathbb{R} : |\omega| \geq \omega_0\}$,

$$i\omega \in \rho(\mathbf{L}) \text{ and } \sup_{\|\phi\|_{\mathfrak{X}}=1} \left\| (i\omega I - \mathbf{L}_h)^{-1} \phi \right\|_{\mathfrak{X}} \leq \frac{C}{|\omega|}.$$

Then there exists a map $\Psi \in C^k(\mathcal{E}_0, \mathfrak{Z}_h)$ *with* $\Psi(0) = 0$, $D\Psi(0) = 0$. *Moreover, a neighborhood* $\mathcal{O}$ *of* 0 *in* $\mathfrak{Z}$ *such that the center manifold*

$$\mathcal{M}_0 = \{u + \Psi(u_0) : u \in \mathcal{E}_0\} \subset \mathfrak{Z}$$

has the following properties:

(1) $\mathcal{M}_0$ *is locally invariant, i.e., if* u *is a solution of* (A.5) *satisfying* $u(0) \in \mathcal{M}_0 \cap \mathcal{O}$, *then* $u(t) \in \mathcal{M}_0 \cap \mathcal{O}$ *for all* $t \in [0, T]$.

(2) $\mathcal{M}_0$ *contains the set of bounded solutions of* (A.5) *staying in* $\mathcal{O}$ *for all* $t \in \mathbb{R}$, *then* $u(0) \in \mathcal{M}_0$.

Now, we show the stability of the local minimizer in the case that 0 is an eigenvalue of T.

LEMMA A.6. *Assume* $h(0) = 0$ *and let* M^c *be the center manifold of the equilibrium point* (u_0, v_0, w_0). *Let* $u = u(t)$, $v = v(t)$, $w = w(t)$ *be a flow on* M^c. *Then* $\frac{d}{dt} I(u(t)) \leq 0$, *in a neighborhood of* (u_0, v_0, w_0) *is a descending flow for the functional* I *restricted to a neighborhood of* (u_0, v_0, w_0) *on* M^c. *Furthermore the equality holds if and only if* $(u, v, w) = (u(t), v(t), w(t))$ *is an equilibrium solution.*

PROOF. First, let $U = u - u_0$, $V = v - v_0$, $W = \xi - \xi_0$ and we rewrite (1.2) as follows:

$$\text{(A.6)}\qquad \begin{cases} U_t = \Delta U + f'(u_0) - V - W + \Delta u_0 \\ \qquad + f(U + u_0) - v_0 - w_0 - f'(u_0)U\,, & x \in \Omega, t > 0\,, \\ \tau_1 V_t = \Delta V - \gamma_1 V + \delta_1 U, & x \in \Omega\,, \quad t > 0\,, \\ \tau_2 W_t = -\gamma_2 W + \frac{\delta_2}{|\Omega|}\int_\Omega U\,dx\,, & t > 0\,, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0\,, & x \in \partial\Omega\,, \quad t > 0\,. \end{cases}$$

Then we define **L**, **R** that

$$\text{(A.7)}\qquad \mathbf{L}\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} \Delta U + f'(u_0)U - V - W \\ \frac{1}{\tau_1}\left(\Delta V - \gamma_1 V + \delta_1 U\right) \\ \frac{1}{\tau_2}\left(-\gamma_2 W + \frac{\delta_2}{|\Omega|}\int_\Omega U\,dx\right) \end{pmatrix},$$

$$\text{(A.8)}\qquad \mathbf{R}\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} \Delta u_0 + f(U + u_0) - v_0 - w_0 - f'(u_0)U \\ 0 \\ 0 \end{pmatrix}$$

for $(U, V, W) \in X^3$. Therefore we have

$$\text{(A.9)}\qquad \frac{d}{dt}(U, V, W) = \mathbf{L}(U, V, W) + \mathbf{R}(U, V, W)\,.$$

Now we check that (A.9) satisfies Hypothesis 1 to 3 to apply Proposition A.1.

(1) **Hypothesis 1.** From definition of **L** and **R**, we can readily check:

- $\mathbf{L} : \mathfrak{Z} \to \mathfrak{X}$ is a bounded linear operator.
- there exists a neighborhood $\mathfrak{V} \subset \mathfrak{Z}$ of u_0 such that $\mathbf{R} \in C^2(\mathfrak{V}, \mathfrak{Y})$ and $\mathbf{R}(0), \mathbf{DR}(0) = 0$,

where $\mathfrak{X} = \mathfrak{Y} = \left(L^2(\Omega)\right)^2 \times \mathbb{R}$, $\mathfrak{Z} = X^2 \times \mathbb{R}$.

(2) **Hypothesis 2.** From Lemma A.5, $\sup_{\lambda\in\sigma_-} \mathrm{Re}\lambda \le \alpha < 0, \sigma_+ = \emptyset$. Moreover, $\sigma_0 = \{0\}$ and the multiplicity of 0 is finite.

(3) **Hypothesis 3.** From Lemma A.5, if $|\omega| \neq 0$, then $i\omega \in \rho(\mathbf{L})$. Let $F = (f, g, h) \in \left(L^2(\Omega)\right)^2 \times \mathbb{R}$ be a function satisfying $\|F\| = \|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)} + |h| = 1$, and let $\Phi = (\phi, \psi, \mu) \in X^2 \times \mathbb{R}$ be a function satisfying $(i\omega I - \mathbf{L})\,\Phi = F$, i.e.

$$\text{(A.10)}\qquad \begin{cases} i\omega\phi - (\Delta\phi + f'(u_0)\phi - \psi - \mu) = f\,, & \text{in } \Omega\,, \\ i\omega\psi - \frac{1}{\tau_1}(\Delta\psi - \gamma_1\psi + \delta_1\phi) = g, & \text{in } \Omega\,, \\ i\omega\mu - \frac{1}{\tau_2}(-\gamma_2\mu + \frac{\delta_2}{|\Omega|}\int_\Omega \phi\,dx) = h\,, \\ \frac{\partial\phi}{\partial n} = \frac{\partial\psi}{\partial n} = 0\,, & \text{on } \partial\Omega\,. \end{cases}$$

Then from the second and third equations of (A.10), we have

$$\psi = (-\Delta + \gamma_1 + i\tau_1\omega)^{-1}(\delta_1\phi + \tau_1 g)\ ,$$
$$\mu = \frac{\delta_2}{(\gamma_2 + i\tau_2\omega)\,|\Omega|}\int_\Omega \phi + \tau_2 h\, dx\ .$$

Noting $i\omega \in \rho(\mathbf{L})$, we have

$$\begin{aligned}\|\psi\|_{L^2(\Omega)} &\le \frac{1}{\tau_1\,|\omega|}\,\|\delta_1\phi + \tau_1 g\|_{L^2(\Omega)}\\ &\le \frac{C'\max\{\delta_1,\tau_1\}}{\tau_1\,|\omega|}\,\|F\|\\ &\le \frac{C}{|\omega|},\\ |\mu| &\le \frac{\delta_2}{\tau_2\,|\omega|\,|\Omega|^{\frac{1}{2}}}\,\|\phi + \tau_2 h\|_{L^2(\Omega)}\\ &\le \frac{C}{|\omega|}.\end{aligned}$$

Moreover, we can see

$$\begin{aligned}\|\nabla\phi\|^2_{L^2(\Omega)} + \big(i\omega - f'(u_0)\big)\,\|\phi\|^2_{L^2(\Omega)} &= \int_\Omega (f - \psi - \mu)\,\phi\, dx\\ \big|f'(u_0)\big|\,\|\phi\|^2_{L^2(\Omega)} + |\omega|\,\|\phi\|^2_{L^2(\Omega)} &\le C\,\|F\|\,\|\phi\|_{L^2(\Omega)}\\ \|\phi\|_{L^2(\Omega)} &\le \frac{C}{|\omega|}\ .\end{aligned}$$

It follows that there is $F = (F_1, F_2, F_3) \in C^2(\mathcal{E}_0, \mathfrak{Z}_h)$ with $F(0) = DF(0) = 0$ and the center manifold M^c can be written

$$\begin{aligned}M^c &= \{\Phi + F(\Phi) : \Phi \in \mathcal{E}\}\\ &= \left\{ \big(\hat{u}(s), \hat{v}(s), \hat{w}(s)\big) : \begin{array}{l} \hat{u}(s) = u_0 + s\phi + U(s)\ ,\\ \hat{v}(s) = v_0 + s\psi + V(s)\ ,\\ \hat{w}(s) = w_0 + s\mu + W(s) \end{array} \right\},\end{aligned}$$

where $\Phi = (\phi, \psi, \mu)$ is a 0-eigenfunction of $\mathbf{L}$ with $\|\phi\|_{L^2(\Omega)} = 1$ and (U, V, W) belongs to the orthogonal complement of span$\{(\phi, \psi, \mu)\}$ in $\big(L^2(\Omega)\big)^2 \times \mathbb{R}$. Then $|U(s)| = o(s)$, $|V(s)| = o(s)$, $|W(s)| = o(s)$, $\big|U'(s)\big| = o(1)$, $\big|V'(s)\big| = o(1)$, $\big|W'(s)\big| = o(1)$ as $|s| \to 0$.

Moreover, we have

$$
\text{(A.11)} \qquad \begin{cases} \Delta\phi + f'(u_0)\phi - \psi - \mu = 0, & \text{in } \Omega\,, \\ \Delta\psi - \gamma_1\psi + \delta_1\phi = 0, & \text{in } \Omega\,, \\ -\gamma_2\mu + \frac{\delta_2}{|\Omega|}\int_\Omega \phi\, dx = 0\,, & \\ \frac{\partial\phi}{\partial n} = \frac{\partial\psi}{\partial n} = 0\,, & \text{on } \Omega\,. \end{cases}
$$

Note that $\psi = \delta_1\left(-\Delta + \gamma_1\right)^{-1}\phi,\ \mu = \frac{\delta_2}{\gamma_2|\Omega|}\int_\Omega \phi\, dx$.

Letting $u = \hat{u}(s(t))$, $v = \hat{v}(s(t))$, $\xi = \hat{\xi}(s(t))$ be a flow on M^c, we have

$$
\text{(A.12)} \qquad \begin{cases} \left(\phi + U'(s)\right)\frac{ds}{dt} = \Delta u + f(u) - v - w\,, \\ \left(\psi + V'(s)\right)\frac{ds}{dt} = \frac{1}{\tau_1}\left(\Delta v - \gamma_1 v + \delta_1 u\right)\,, \\ \left(\mu + W'(s)\right)\frac{ds}{dt} = \frac{1}{\tau_2}\left(-\gamma_2\xi + \frac{\delta_2}{|\Omega|}\int_\Omega u\, dx\right)\,. \end{cases}
$$

In particular, noting $v(t) = v_0 + s\psi + V(s)$, we have

$$
\begin{aligned} \tau_1 v_t &= \Delta V(s) - \gamma_1 V_1(s) + \delta_1 U(s) \\ &= (\Delta - \gamma_1)\left(\delta_1 G_1 U - V\right)\,. \end{aligned}
$$

Similarly, we have $\tau_2\xi_t = -\gamma_2\left(\frac{\delta_2}{\gamma_2|\Omega|}\int_\Omega U\, dx - W\right)$. We can compute

$$
\begin{aligned} \frac{d}{dt}J_\epsilon\left(u(t)\right) &= \int_\Omega \left(-\Delta u + \delta_1 G_1 u + \frac{\delta_2}{\gamma_2\,|\Omega|}\int_\Omega u\, dx - f(u)\right) u_t \\ &= \int_\Omega \left(-\Delta u - f(u) + v + \xi + \delta_1 G_1 u - v + \frac{\delta_2}{\gamma_2\,|\Omega|}\int_\Omega u\, dx - \xi\right) u_t \\ &= \int_\Omega \left(-u_t + \delta_1 G_1 U(s) - V(s) + \frac{\delta_2}{\gamma_2\,|\Omega|}\int_\Omega u\, dx - W(s)\right) u_t \\ &= -\left\|u_t\right\|^2_{L^2(\Omega)} - \langle u_t, V(s) - \delta_1 G_1 U(s)\rangle - \left\langle u_t, W(s) - \frac{\delta_2}{\gamma_2\,|\Omega|}\int_\Omega u\, dx\right\rangle\,. \end{aligned}
$$

Moreover we can see that

$$
\begin{aligned} \left\|u_t\right\|^2_{L^2(\Omega)} &= \left\|\phi + U'(s)\right\|^2_{L^2(\Omega)}\left(\frac{ds}{dt}\right)^2 \\ &= \left\{\left\|\phi\right\|^2_{L^2(\Omega)} + 2\left\langle\phi, U'(s)\right\rangle + \left\|U'(s)\right\|^2_{L^2(\Omega)}\right\}\left(\frac{ds}{dt}\right)^2 \\ &= (1 + o(1))\left(\frac{ds}{dt}\right)^2\,, \end{aligned}
$$

and

$$\begin{aligned}
-\langle u_t, V(s) - \delta_1 G_1 U(s)\rangle &= -\langle u_t, \tau_1 G_1 v_t\rangle \\
&= \tau_1 \langle G_1 u_t, v_t\rangle \\
&= -\tau_1 \left(\frac{ds}{dt}\right)^2 \left\langle G_1\left(\phi + U'(s)\right), \psi + V'(s)\right\rangle \\
&= -\tau_1 \left(\frac{ds}{dt}\right)^2 \left(\delta_1 \|G_1\phi\|^2_{L^2(\Omega)} + o(1)\right) \\
&\le -\tau_1 \frac{\delta_1}{\gamma_1^2}\left(\frac{ds}{dt}\right)^2 + o(1)\,.
\end{aligned}$$

Similarly,

$$\begin{aligned}
-\left\langle u_t, W(s) - \frac{\delta_2}{\gamma_2|\Omega|}\int_\Omega U(s)\,dx\right\rangle &= -\left\langle u_t, \frac{\tau_2}{\gamma_2}\xi_t\right\rangle \left(\frac{ds}{dt}\right)^2 \\
&= -\frac{\tau_2}{\gamma_2}\left(\frac{ds}{dt}\right)^2 \left\langle \phi + U'(s), \mu + W'(s)\right\rangle \\
&\le -\frac{\tau_2}{\gamma_2}\left(\frac{ds}{dt}\right)^2 \frac{\delta_2}{\gamma_2|\Omega|}\left(\int_\Omega \phi\,dx\right)^2 + o(1) \\
&\le -\tau_2\frac{\delta_2}{\gamma_2^2}\left(\frac{ds}{dt}\right)^2 + o(1)\,.
\end{aligned}$$

Thus we have

$$\frac{d}{dt}J_\epsilon(u(t)) \le \left(\frac{ds}{dt}\right)^2\left(1 - \left(\frac{\tau_1\delta_1}{\gamma_1^2} + \frac{\tau_2\delta_2}{\gamma_2^2}\right) + o(s)\right) \le 0\,.$$

If $\frac{d}{dt}J_\epsilon(u(t)) = 0$, then we have $(u, v, \xi) = (u_0, v_0, \xi_0)$. Hence it follows the statement. □

Appendix B. Proof of (3.1)

We present a proof of the inequality (3.1). Before we prove (3.1), we will show Claims 1, 2:

CLAIM 1. Let $\xi_+ > 0$ be a small number such that for any $t \in [1-\xi_+, 1]$, $H'(t) \in \left(\frac{1}{2}H''(1), \frac{3}{2}H''(1)\right)$. Then for any $x, y \in [1-\xi_+, 1]$,

$$|h(x) - h(y)| \ge \frac{1}{4}\sqrt{\frac{H''(1)}{3}}\,|x-y|^2\,.$$

PROOF OF CLAIM 1. We may assume $1-\xi_+ \le y < x \le 1$. Then we have

$$h(x) = h(y) + h'(y)(x-y) + \frac{1}{2}h''(z)(x-y)^2\,,$$

where $z \in (y, x) \subset (1-\xi_+, 1)$. Since $h'(y) = \sqrt{H(y)} > 0$, $h''(z) = \frac{H'(z)}{2\sqrt{H(z)}}$, we have

$$h(x) \ge h(y) + \frac{H'(z)}{4\sqrt{H(z)}}(x-y)^2\,.$$

Moreover,

$$H'(z) = H'(1) + H''(\hat{z})(z-1) = H''(\hat{z})(z-1)\,,$$

$$H(z) = H(1) + H'(1)(z-1) + \frac{1}{2}H''(\tilde{z})(z-1)^2 = \frac{1}{2}H''(\tilde{z})(z-1)^2\,,$$

where $\hat{z}, \tilde{z} \in (z, 1)$. Then it follows

$$\begin{aligned} h(x) &\ge h(y) + \frac{H''(\hat{z})(z-1)}{4\sqrt{\frac{1}{2}H''(\tilde{z})(z-1)^2}}(x-y)^2 \\ &\ge \frac{\sqrt{2}}{4}\frac{\frac{1}{2}H''(1)}{\sqrt{\frac{3}{2}H''(1)}}(x-y)^2 \\ &= \frac{1}{4}\sqrt{\frac{H''(1)}{3}}(x-y)^2\,. \end{aligned}$$

Thus we complete the proof of Claim1.

Similarly, we can show Claim2:

CLAIM 2. Let $\xi_- > 0$ be a small number such that for any $t \in [-1, -1+\xi_-]$, $H'(t) \in \left(\frac{1}{2}H''(-1), \frac{3}{2}H''(-1)\right)$. Then for any $x, y \in [-1, -1+\xi_-]$,

$$|h(x) - h(y)| \ge \frac{1}{4}\sqrt{\frac{H''(-1)}{3}}\,|x-y|^2\,.$$

PROOF OF (3.1). We prove by contradiction. Namely we assume that there exist $x_n, y_n \in [-1, 1]$ such that $x_n > y_n$ and

$$0 < h(x_n) - h(y_n) < \frac{1}{n}(x_n - y_n)^2 \tag{B.1}$$

for all $n \in \mathbb{N}$. Since both $(x_n)_n$ and $(y_n)_n$ are bounded, we may assume that there exist $\bar{x}, \bar{y}$ such that $x_n \to \bar{x}$, $y_n \to \bar{y}$ as $n \to \infty$. Then we can see $h(\bar{x}) = h(\bar{y})$ from (B.1), which implies $\bar{x} = \bar{y}$. Then we consider the following three cases.

(i) *Case*1. $\bar{x} = \bar{y} = 1$.

Since $y_n \leq x_n \leq 1$ and $y_n \to 1$ as $n \to \infty$, x_n, y_n satisfy $x_n, y_n \in [1 - \xi_+, 1]$ and (B.1) for large n. But it clearly contradicts Claim 1.

(ii) *Case*2. $\bar{x} = \bar{y} = -1$.

We can prove in a similar way to Case 1.

(iii) *Case*3. $-1 < \bar{x} = \bar{y} < 1$.

In this case, there exists $\xi_0 > 0$ such that $x_n, y_n \in (-1 + \xi_0, 1 - \xi_0)$. Then we have

$$h(x_n) - h(y_n) = h'(z)(x_n - y_n) \geq \bar{c}(x - y),$$

where $\bar{c} = \min\{\sqrt{H(z)}; z \in (-1 + \xi_0, 1 - \xi_0)\} > 0$. In addition, noting $|x - y| \geq \frac{1}{2}|x - y|^2$ for $x, y \in [-1, 1]$, we obtain

$$h(x_n) - h(y_n) \geq \frac{\bar{c}}{2}(x - y)^2 .$$

But it clearly contradicts (B.1). Thus we complete the proof. □

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Present Addresses:
TAKASHI KAJIWARA
DEPARTMENT OF MATHEMATICS AND INFORMATION SCIENCES,
TOKYO METROPOLITAN UNIVERSITY,
1–1 MINAMI-OHSAWA, HACHIOJI, TOKYO 192–0397, JAPAN.
e-mail: kajiwara-takashi1@ed.tmu.ac.jp

KAZUHIRO KURATA
DEPARTMENT OF MATHEMATICS AND INFORMATION SCIENCES,
TOKYO METROPOLITAN UNIVERSITY,
1–1 MINAMI-OHSAWA, HACHIOJI, TOKYO 192–0397, JAPAN.
e-mail: kurata@tmu.ac.jp