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On the Non-existence of Static Pluriclosed Metrics on Non-Kähler Minimal Complex Surfaces

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Abstract. The pluriclosed flow is an example of Hermitian flows generalizing the Kähler-Ricci flow. We classify static pluriclosed solutions of the pluriclosed flow on non-Kähler minimal compact complex surfaces. We show that there are no static pluriclosed metrics on Kodaira surfaces, non-Kähler minimal properly elliptic surfaces and Inoue surfaces.

1. Introduction

In [10] and [12], Streets and Tian introduced a parabolic evolution equation of pluriclosed metrics with pluriclosed initial metric ω_0 on a Hermitian manifold,

$$\begin{cases} \frac{\partial}{\partial t}\omega(t) = \partial\partial^*\omega(t) + \bar{\partial}\bar{\partial}^*\omega(t) - \mathrm{Ric}(\omega(t))\,, \\ \omega(0) = \omega_0\,, \end{cases}$$

which is called the pluriclosed flow. A solution of the pluriclosed flow with pluriclosed initial condition is equivalent to a solution of the Hermitian curvature flow (HCF) (cf. [10, Proposition 3.3]). If the initial condition ω_0 of the solution to HCF is Kähler, the solution is Kähler and HCF (equivalently the pluriclosed flow) coincides with the Kähler-Ricci flow (cf. [11, Proposition 5.2]). They investigated the relation between the pluriclosed flow and the associated static pluriclosed metrics. We say that a pluriclosed metric ω is static if

$$-\partial\partial^*\omega - \bar{\partial}\bar{\partial}^*\omega + \mathrm{Ric}(\omega) = \lambda\omega$$

for some constant λ. Note that if ω is Kähler and static pluriclosed, then it is Kähler-Einstein. Making the sign of λ agrees with the usual sign for Kähler-Einstein metrics is the reason for the sign of the left hand side. On a complex surface, the existence of static pluriclosed metrics is closely related to the existence of algebraic and topological structures. Streets and Tian classified these static pluriclosed metrics on K3 surfaces and 2-dimensional complex tori, standard Hopf surfaces and Hopf surfaces blown up at finitely many points, complex surfaces of general type, Class VII^+ surfaces (Class VII surfaces with the second Betti number

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$b_2 > 0$). A Class VII surface is a compact complex surface with the first Betti number $b_1 = 1$ and the Kodaira dimension $\kappa = -\infty$. If there exists a static pluriclosed metric on a K3 surface or torus, then it is actually Kähler and Ricci-flat ([10, Proposition 5.6]). In the case of complex surfaces of general type with a static pluriclosed metric, the metric is Kähler-Einstein ([10, Proposition 5.7]).

Thanks to the Kodaira-Enriques classification ([1, p.244]), we know that non-Kähler minimal compact complex surfaces fall into the following four cases:

(1) (Primary and Secondary) Kodaira surfaces,
(2) Minimal properly elliptic surfaces,
(3) Class VII surfaces with $b_2 = 0$ (Inoue surfaces or Hopf surfaces),
(4) Minimal Class VII^+ surfaces.

Streets and Tian have shown that Hopf surfaces admit a static pluriclosed metric with $\lambda = 0$ ([10, Example 6.1]) and Class VII^+ surfaces admit no static pluriclosed metrics ([10, Proposition 6.3]).

In this paper, we will study other remained three cases; Kodaira surfaces, non-Kähler minimal properly elliptic surfaces and Inoue surfaces. Thus the classification problem of static pluriclosed metrics on non-Kähler minimal compact complex surfaces is settled.

Our main result is as follows.

THEOREM 1.1. *Kodaira surfaces, non-Kähler minimal properly elliptic surfaces and Inoue surfaces admit no static pluriclosed metrics.*

We introduce the following definitions as in [10], [12].

DEFINITION 1.1. Let (M, J) be a complex manifold. A metric g is called a pluriclosed metric on M if g is a Hermitian metric whose associated real $(1,1)$-form $\omega = \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$ satisfies

$$\partial\bar{\partial}\omega = 0\,.$$

We also define the Gauduchon metric.

DEFINITION 1.2. Let (M, J) be a compact complex manifold of dimension n. A metric g is called a Gauduchon metric on M if g is a Hermitian metric whose associated real $(1,1)$-form $\omega = \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$ satisfies

$$\partial\bar{\partial}(\omega^{n-1}) = 0\,.$$

We will also refer to the associated real $(1,1)$-form ω as a pluriclosed metric or a Gauduchon metric. The following states that there are lot of Gauduchon metrics on any compact complex manifold M of complex dimension n (cf. [5]).

PROPOSITION 1.1. *Let M be a compact complex manifold of complex dimension n. Any Hermitian metric on M is conformally equivalent to a Gauduchon metric. If $n \geq 2$, then this Gauduchon metric is unique up to a positive factor.*

Since all complex surfaces admit Hermitian metrics, there are always pluriclosed metrics (Gauduchon metrics) on compact complex surfaces.

DEFINITION 1.3. Let (M, J) be a complex manifold with a pluriclosed metric ω on M. We say that ω is a static pluriclosed metric if

$$-\partial\partial^*\omega - \bar{\partial}\bar{\partial}^*\omega + \mathrm{Ric}(\omega) = \lambda\omega$$

for some constant λ.

We put

$$\Phi(\omega) := -\partial\partial^*\omega - \bar{\partial}\bar{\partial}^*\omega + \mathrm{Ric}(\omega)\,.$$

Note that if ω is Kähler and satisfies $\Phi(\omega) = \lambda\omega$ for some constant λ, then it is a Kähler-Einstein metric.

Now let us define the condition of a structure called Hermitian-symplectic.

DEFINITION 1.4. Let (M, J) be a complex manifold of complex dimension n. A Hermitian-symplectic structure on M is a real 2-form $\tilde{\omega}$ such that $d\tilde{\omega} = 0$, and the projection of $\tilde{\omega}$ onto $(1, 1)$-tensors determined by J is positive definite. We say that a complex manifold is Hermitian-symplectic if it admits a Hermitian-symplectic structure.

It is well known that the space of symplectic manifolds is strictly larger than the space of Kähler manifolds. But we do not know whether the space of Hermitian-symplectic manifolds is strictly larger than the space of Kähler manifolds or not. However, for surfaces, we have the following result.

We will crucially use the fact that static pluriclosed metrics with nonzero constant λ automatically imply that there exists a Hermitian-symplectic structure (cf. [10, Proposition 5.10]) and the following proposition in our argument.

PROPOSITION 1.2 ([10, Proposition 1.6]). *A complex surface is Hermitian-symplectic if and only if it is Kähler.*

2. Proof of Theorem 1.1

Let (X, J) be a complex manifold of complex dimension n with a Hermitian metric ω_X on X. $\mathrm{Ric}(\omega_X)$ denotes the Chern-Ricci curvature of ω_X, given locally by

$$\mathrm{Ric}(\omega_X) = -\frac{\sqrt{-1}}{2}\partial\bar{\partial}\log\omega_X^n\,,$$

which determines the Bott-Chern cohomology class denoted by $c_1^{BC}(X) \in H^{1,1}_{BC}(X, \mathbb{R})$, where

$$H^{1,1}_{BC}(X, \mathbb{R}) = \frac{\{d\text{-closed real }(1,1)\text{-forms}\}}{\{\sqrt{-1}\partial\bar{\partial}\psi \,|\, \psi \in C^\infty(M, \mathbb{R})\}}\,.$$

We call it the first Bott-Chern class of X. Note that we omit a factor of π in this paper, and it is independent of the choice of Hermitian metrics. Let

$$A^r(X) = \bigoplus_{p+q=r} A^{p,q}(X)$$

for $0 \le r \le 2n$ denote the decomposition of complex differential r-forms into (p,q)-forms. The exterior differential operator d decomposes into the operators ∂ and $\bar{\partial}$

$$\partial : A^{p,q}(X) \to A^{p+1,q}(X), \quad \bar{\partial} : A^{p,q}(X) \to A^{p,q+1}(X).$$

Let d_g^* denote the L^2-adjoint operator of d. Then d_g^* decomposes into ∂_g^* and $\bar{\partial}_g^*$

$$\partial_g^* : A^{p,q}(X) \to A^{p-1,q}(X), \quad \bar{\partial}_g^* : A^{p,q}(X) \to A^{p,q-1}(X).$$

We here again define a static pluriclosed metric on surfaces.

DEFINITION 2.1. Let (X, J) be a complex surface with a pluriclosed metric ω on X. We say that ω is a static pluriclosed metric if ω satisfies

$$\Phi(\omega) = \lambda\omega$$

for some constant λ, and

$$\mathrm{Vol}(g) = 1.$$

Now we consider a complex surface X with a Hermitian metric ω_X. We rule out the scaling ambiguity by fixing the volume to be 1 as in Definition 2.1. Since then, for a static pluriclosed metric ω satisfying $\Phi(\omega) = \lambda\omega$ on X, we have $\int_X \omega \wedge \omega = 2$, we can easily have that (cf. [10, Proposition 5.2])

$$(\dagger)_\omega \qquad 2\int_X |\partial_g^*\omega|^2 = d - 2\lambda,$$

where

$$d := \deg(X) = \int_X c_1^{BC}(X) \wedge \omega = \int_X \mathrm{Ric}(\omega_X) \wedge \omega$$

denotes the degree of the pluriclosed metric ω (cf. [10, Definition 3.7], [12, Definition 7.2]). Note that the value of degree does not depend on the representative of $c_1^{BC}(X)$ since the definition is made with respect to a fixed pluriclosed metric.

First, let M be a Kodaira surface and ω be a pluriclosed metric on M. A Kodaira surface is a non-Kähler minimal compact complex surface with the Kodaira dimension $\kappa(M) = 0$, which can be classified into the following two cases: A primary Kodaira surface is a surface with the first Betti number $b_1(M) = 3$, admitting a holomorphic locally trivial fibration over an elliptic curve with an elliptic curve as typical fibre. A secondary Kodaira surface is a surface with $b_1(M) = 1$, admitting a primary Kodaira surface as unramified covering. These surfaces are elliptic fibre spaces over rational curves. In either case, since some power of the canonical bundle lK_M, $l \geq 1$ is holomorphically trivial (i.e., M has torsion canonical bundle), then we have $c_1^{BC}(M) = 0$. Hence there exists a Gauduchon metric ω_0 on M such that $\mathrm{Ric}(\omega_0) = 0$ (cf. [13], [14]). Then we have

$$d = \int_M c_1^{BC}(M)\wedge\omega = \int_M \mathrm{Ric}(\omega)\wedge\omega = \int_M (\mathrm{Ric}(\omega)-\mathrm{Ric}(\omega_0))\wedge\omega = \int_M \sqrt{-1}\partial\bar{\partial}\varphi\wedge\omega = 0\,,$$

where φ is a real-valued smooth function on M and we used that ω is pluriclosed at the last equality.

Here, we claim the following statement.

PROPOSITION 2.1. *Let M be a non-Kähler compact complex surface. If we have that $d \leq 0$, then there is no static pluriclosed metrics on M.*

REMARK 2.1. Note that the standard Hopf surface H admits a static pluriclosed metric with $\lambda = 0$ (cf. [10, Example 6.1]). In this case, a pluriclosed metric $\omega_H = \frac{\sqrt{-1}}{\rho^2}\partial\bar{\partial}\rho^2$, where $\rho^2 = \sum_{i=1}^2 |z_i|^2$ for $(z_1, z_2) \in \mathbb{C}^2 \setminus \{0\}$, satisfies that $\mathrm{Ric}(\omega_H) \geq 0$ on the surface H. Hence we have $c_1^{BC}(H) \geq 0$ and then $d \geq 0$ (cf. [15, Section 4], [16, Section 8]). Then the metric ω_H is a static pluriclosed metric with $\lambda = 0$ satisfying $\Phi(\omega_H) = 0$. On the other hand, a Hopf surface blown up at $p > 0$ points admits no static pluriclosed metrics ([10, Proposition 5.9]).

We need the following proposition to prove our statement above:

PROPOSITION 2.2 ([10, Proposition 5.10]). *Let (M, J) be a compact complex manifold with a static metric ω. If $\lambda \neq 0$, then M is a Hermitian-symplectic manifold.*

PROOF (PROPOSITION 2.1). We assume that there exists a static pluriclosed metric ω on M, then we have from the equality $(\dagger)_\omega$,

$$2\int_M |\partial_g^*\omega|^2 = d - 2\lambda \leq -2\lambda$$

since we have assumed that $d \leq 0$.

If $\lambda \geq 0$, then we have $\int_M |\partial_g^*\omega|^2 = 0$ and $\partial_g^*\omega = 0$, equivalently we have $\partial\omega = 0$. This means that ω is a Kähler metric on M, which contradicts with that M is non-Kähler. Hence we must have $\lambda < 0$. Since we have $\lambda < 0$, we may apply Proposition 2.2, and hence M must be Hermitian-symplectic. On the other hand, from Proposition 1.2, then M must be Kähler, which is again a contradiction. Therefore, M admits no static pluriclosed metrics. □

Since we have $d = 0$ on Kodaira surfaces M, we conclude that there is no static pluriclosed metric on M by Proposition 2.1.

Second, let M be a non-Kähler properly elliptic surface. Here, non-Kähler means that M admits no Kähler metrics. A non-Kähler properly elliptic surface is a compact complex surface with its first Betti number $b_1(M) =$ odd and the Kodaira dimension $\kappa(M) = 1$ which admits an elliptic fibration $\pi : M \to S$ to a smooth compact curve S. The Kodaira-Enriques classification tells us that minimal non-Kähler properly elliptic surfaces are the only one case for minimal non-Kähler compact complex surfaces with the Kodaira dimension $\kappa = 1$.

We assume that M is minimal, that is, there is no (-1)-curve on M. It has been shown that the universal cover of M is $\mathbb{C} \times H$ [8, Theorem 28], where H is the upper half plane in $\mathbb{C}$. Also, it is known that there is a finite unramified covering $p : M' \to M$ which is also a minimal properly elliptic surface $\pi' : M' \to S'$ and π' is an elliptic fiber bundle over a compact Riemann surface S' of genus at least 2, with fiber an elliptic curve E (cf. [3, Lemma 1, 2], [7], [15] and [17]). By considering the following holomorphic covering map

$$h : \mathbb{C} \times H \to \mathbb{C}^* \times H, \quad h(z, z') = (e^{-\frac{z}{2}}, z'),$$

we may work with $\mathbb{C}^* \times H$ instead of $\mathbb{C} \times H$, where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. We will write (z_1, z_2) for the coordinates on $\mathbb{C}^* \times H$ and $z_i = x_i + \sqrt{-1}y_i$, $x_i, y_i \in \mathbb{R}$ for $i = 1, 2$, which means that we have $y_2 > 0$.

It has been shown by Maehara in [9] that there exists a discrete subgroup $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$ with $H/\Gamma = S$, together with $\mu \in \mathbb{C}^*$ with $|\mu| \neq 1$ and $\mathbb{C}^*/\langle \mu \rangle = E$, and with a character $\chi : \Gamma \to \mathbb{C}^*$ such that M' is biholomorphic to the quotient of $\mathbb{C}^* \times H$ by the $\Gamma \times \mathbb{Z}$ -action defined by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, n\right) \cdot (z_1, z_2) = \left((cz_2 + d) \cdot z_1 \cdot \mu^n \cdot \chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right), \frac{az_2 + b}{cz_2 + d}\right),$$

and the map $\pi' : M' \to S'$ is induced by the projection $\mathbb{C}^* \times H \to H$ (cf. [2, Proposition 2], [20, Theorem 7.4]). We define two forms on $\mathbb{C}^* \times H$ below:

$$\alpha := \frac{\sqrt{-1}}{2y_2^2} dz_2 \wedge d\bar{z}_2, \quad \gamma := \sqrt{-1}\left(-\frac{2}{z_1}dz_1 + \frac{\sqrt{-1}}{y_2}dz_2\right) \wedge \left(-\frac{2}{\bar{z}_1}d\bar{z}_1 - \frac{\sqrt{-1}}{y_2}d\bar{z}_2\right).$$

Note that we may work in a single compact fundamental domain for M' in $\mathbb{C}^* \times H$ using z_1, z_2 as local coordinates and we may assume that z_1, z_2 are uniformly bounded and that y_2 is uniformly bounded below away from zero.

LEMMA 2.1. *The forms α and γ are invariant under the $\Gamma \times \mathbb{Z}$ -action.*

PROOF. It suffices to show that the forms on $\mathbb{C}^* \times H$;

$$\frac{\sqrt{-1}}{y_2^2} dz_2 \wedge d\bar{z}_2, \quad -\frac{2}{z_1}dz_1 + \frac{\sqrt{-1}}{y_2}dz_2$$

are $\Gamma \times \mathbb{Z}$ -invariant. For any $(A, n) \in \Gamma \times \mathbb{Z}$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc = 1$, we compute

$$\frac{\sqrt{-1}}{\mathrm{Im}\left(\frac{az_2+b}{cz_2+d}\right)} d\left(\frac{az_2 + b}{cz_2 + d}\right) \wedge d\left(\frac{a\bar{z}_2 + b}{c\bar{z}_2 + d}\right) = \frac{\sqrt{-1}}{y_2^2} \frac{|cz_2 + d|^4}{(cz_2 + d)^2(c\bar{z}_2 + d)^2} dz_2 \wedge d\bar{z}_2$$

$$= \frac{\sqrt{-1}}{y_2^2} dz_2 \wedge d\bar{z}_2$$

and

$$\begin{aligned} &-\frac{2}{(cz_2+d)\cdot z_1\cdot \mu^n\cdot \chi(A)} d((cz_2+d)\cdot z_1\cdot \mu^n\cdot \chi(A)) + \frac{\sqrt{-1}}{\mathrm{Im}\left(\frac{az_2+b}{cz_2+d}\right)} d\left(\frac{az_2+b}{cz_2+d}\right) \\ &= -\frac{2c}{cz_2+d} dz_2 - \frac{2}{z_1} dz_1 + \frac{\sqrt{-1}}{y_2} \frac{|cz_2+d|^2}{(cz_2+d)^2} dz_2 \\ &= -\frac{2}{z_1} dz_1 + \frac{\sqrt{-1}}{y_2} \left(\frac{|cz_2+d|^2 + \sqrt{-1} 2cy_2(cz_2+d)}{(cz_2+d)^2} \right) dz_2 \\ &= -\frac{2}{z_1} dz_1 + \frac{\sqrt{-1}}{y_2} dz_2 \end{aligned}$$

Therefore, the forms α and γ are invariant under the $\Gamma \times \mathbb{Z}$ -action. □

It follows that they descend to M' and we may define a smooth strictly positive volume form Ω on M' by

$$\Omega = 2\alpha \wedge \gamma$$

since the volume form Ω is invariant under the $\Gamma \times \mathbb{Z}$ -action as we see in the following lemma.

LEMMA 2.2. *The volume form Ω is $\Gamma \times \mathbb{Z}$ -invariant and satisfies*

$$\mathrm{Ric}(\Omega) = -\alpha \in c_1^{BC}(M') = -c_1^{BC}(K_{M'}),$$

where $K_{M'}$ is the canonical bundle over M'.

PROOF. Since the forms α and γ are $\Gamma \times \mathbb{Z}$ -invariant from Lemma 2.1, so is Ω. We compute

$$\begin{aligned} \mathrm{Ric}(\Omega) &= -\sqrt{-1}\partial\bar{\partial}\log\Omega \\ &= -\sqrt{-1}\partial\bar{\partial}\log\left(\frac{4}{y_2^2|z_1|^2}\right) \\ &= \sqrt{-1}\partial\bar{\partial}\log y_2^2 \\ &= \sqrt{-1}\partial\left(\frac{\sqrt{-1}}{y_2}\right) d\bar{z}_2 \\ &= -\frac{\sqrt{-1}}{2y_2^2} dz_2 \wedge d\bar{z}_2 = -\alpha\,. \end{aligned}$$

□

The form α induces a unique Kähler-Einstein metric $\omega_{S'}$ with $\mathrm{Ric}(\omega_{S'}) = -\omega_{S'}$ on S'. From Lemma 2.2, we obtain that

$$\mathrm{Ric}(\Omega) = -\pi'^{*}\omega_{S'} = \pi'^{*}\,\mathrm{Ric}(\omega_{S'})\,.$$

It follows that we have that

$$c_1^{BC}(M') = \pi'^{*}c_1(S')\,.$$

Since the genus of S' is at least 2, we have $c_1(S') < 0$. Therefore we have $c_1^{BC}(M') \leq 0$, which implies that the canonical bundle $K_{M'}$ is nef. We say that a holomorphic line bundle L over a compact complex surface N is nef if we have $\int_C c_1^{BC}(L) \geq 0$ for all curves C in N. If C is not smooth, then we integrate over C_{reg}, the set of points $p \in C$ for which C is a submanifold of N near p, since Stokes' Theorem still holds for C_{reg} (cf. [6, p. 33]).

LEMMA 2.3. *If the canonical bundle $K_{M'}$ is nef, then the canonical bundle K_M is nef.*

PROOF. Since $p : M' \to M$ is as unramified finite covering, i.e., for a sufficiently small open set $U \subset M$ we have that $p^{-1}(U)$ is a disjoint union of finitely many copies U_j of U and then $p : U_j \to U$ is biholomorphic for each j, we may compute that for any Hermitian metric ω and any curve C on M,

$$\int_C (-\,\mathrm{Ric}(\omega)) = \int_{p^*C} (-p^*\,\mathrm{Ric}(\omega)) = \int_{p^*C} (-\,\mathrm{Ric}(p^*\omega)) = \int_{p^*C} c_1^{BC}(K_{M'}) \geq 0$$

since K'_M is assumed to be nef, where $p^*\omega$ is Hermitian and p^*C is a curve on M'. Hence K_M is also nef. □

If we have $c_1^{BC}(K_M) = -c_1^{BC}(M) < 0$, for any Hermitian metric ω_M on M, there exists a real smooth function F on M such that $-\,\mathrm{Ric}(\omega_M) + \frac{\sqrt{-1}}{2}\partial\bar{\partial}F < 0$ and then for any curve C in M, using Stokes' Theorem,

$$\int_C c_1^{BC}(K_M) = \int_C (-\,\mathrm{Ric}(\omega_M)) = \int_C \left(-\,\mathrm{Ric}(\omega_M) + \frac{\sqrt{-1}}{2}\partial\bar{\partial}F\right) < 0\,,$$

which contradicts to the result that K_M is nef in Lemma 2.3. Hence, that $c_1^{BC}(M') \leq 0$ gives us that $c_1^{BC}(M) \leq 0$. With using this result, we obtain that

$$d = \int_M c_1^{BC}(M) \wedge \omega \leq 0$$

for any pluriclosed metric ω on M. By applying Proposition 2.1, we conclude that there is no static pluriclosed metric on M.

Finally, we study a static pluriclosed metric on Inoue surfaces. Inoue surfaces form three families, S_M, $S^+_{N,p,q,r;\mathbf{t}}$ and $S^-_{N,p,q,r}$. First of all, we construct the Inoue surface of type S_M. Let $M \in \mathrm{SL}(3,\mathbb{Z})$ be a matrix with one real eigenvalue $\tau > 1$ and two complex conjugate eigenvalues $\eta \neq \bar{\eta}$. Let (l_1, l_2, l_3) be a real eigenvector for M with eigenvalue τ and (m_1, m_2, m_3) be an eigenvector with eigenvalue η. We write $z_1 = x_1 + \sqrt{-1}y_1$ for the coordinate on $\mathbb{C}$ and $z_2 = x_2 + \sqrt{-1}y_2$ for the coordinate on $H = \{y_2 > 0\}$, the upper half plane in $\mathbb{C}$. Let G_M be the group of automorphisms of $\mathbb{C} \times H$, which is generated by

$$f_0(z_1, z_2) = (\eta z_1, \tau z_2), \quad f_i(z_1, z_2) = (z_1 + m_i, z_2 + l_i)$$

for $i = 1, 2, 3$, $(z_1, z_2) \in \mathbb{C} \times H$, $1 \leq i \leq 3$. We define S_M to be the quotient surface $(\mathbb{C} \times H)/G_M$, which is a T^3-torus bundle over a circle.

We may assume that z_1 and z_2 are uniformly bounded and that y_2 is uniformly bounded below away from zero since we may work in a single compact fundamental domain for S_M in $\mathbb{C} \times H$.

On $\mathbb{C} \times H$, we define nonnegative $(1,1)$-forms α' and β by

$$\alpha' := \frac{\sqrt{-1}}{4y_2^2} dz_2 \wedge d\bar{z}_2, \quad \beta := \sqrt{-1}y_2 dz_1 \wedge d\bar{z}_1 .$$

Since these forms are invariant under the G_M -action (note that $\tau|\eta|^2 = 1$), they descend to S_M. We then can define a Hermitian metric on S_M, so called the Tricerri metric ω_T (cf. [4, Section 2], [15, Section 5] and [18]) by

$$\omega_T := 4\alpha' + \beta ,$$

which is pluriclosed on S_M. Then we have

$$\mathrm{Ric}(\omega_T) = -\sqrt{-1}\partial\bar{\partial} \log \omega_T^2 = \sqrt{-1}\partial\bar{\partial} \log y_2 = -\alpha' \leq 0$$

and $c_1^{BC}(S_M) \leq 0$, which implies that we have $d \leq 0$.

We next construct the Inoue surface of type $S^+_{N,p,q,r;\mathbf{t}}$. Let $N = (n_{ij}) \in \mathrm{SL}(2, \mathbb{Z})$ with two real eigenvalues $\tau > 1$ and $\frac{1}{\tau}$. Let (a_1, a_2) and (b_1, b_2) be two real eigenvectors for N with eigenvalues τ and $\frac{1}{\tau}$, respectively. Fix integers $p, q, r \in \mathbb{Z}$ with $r \neq 0$ and a complex number $\mathbf{t} \in \mathbb{C}$. Define

$$e_i := \frac{1}{2} n_{i1}(n_{i1} - 1)a_1 b_1 + \frac{1}{2} n_{i2}(n_{i2} - 1)a_2 b_2 + n_{i1} n_{i2} b_1 a_2$$

for $i = 1, 2$. Using N, a_i, b_i, p, q, r, one gets two real numbers (c_1, c_2) as solutions of the linear equation

$$(c_1, c_2) = (c_1, c_2) \cdot N^t + (e_1, e_2) + \frac{b_1 a_2 - b_2 a_1}{r}(p, q) .$$

Let G^+_N be the group of automorphism of $\mathbb{C} \times H$ generated by

$$f_0(z_1, z_2) = (z_1 + \mathbf{t}, \tau z_2) ,$$

$$f_i(z_1, z_2) = (z_1 + b_i z_2 + c_i, z_2 + a_i) ,$$

$$f_3(z_1, z_2) = \Big(z_1 + \frac{b_1a_2 - b_2a_1}{r}, z_2\Big)$$

for $i = 1, 2$, $(z_1, z_2) \in \mathbb{C} \times H$. We define $S^+_{N,p,q,r;\mathbf{t}}$ to be the quotient surface $(\mathbb{C} \times H)/G^+_N$, which is diffeomorphic to a bundle over a circle with fiber a compact 3-manifold X.

We finally construct the Inoue surface of type $S^-_{N,p,q,r}$. Let $N = (n_{ij}) \in \mathrm{GL}(2, \mathbb{Z})$ with $\det N = -1$ and with two real eigenvalues $\tau > 1$ and $-\frac{1}{\tau}$. Let (a_1, a_2) and (b_1, b_2) be two real eigenvectors for N with eigenvalues τ and $-\frac{1}{\tau}$, respectively. Fix integers $p, q, r \in \mathbb{Z}$ with $r \neq 0$. One gets two real numbers (c_1, c_2) as solutions of the following linear equation

$$-(c_1, c_2) = (c_1, c_2) \cdot N^t + (e_1, e_2) + \frac{b_1a_2 - b_2a_1}{r}(p, q)\,,$$

where e_i for each $i = 1, 2$ is defined as in the case $S^+_{N,p,q,r;\mathbf{t}}$. Let G^-_N be the group of automorphism of $\mathbb{C} \times H$ generated by

$$f_0(z_1, z_2) = (-z_1, \tau z_2)\,,$$
$$f_i(z_1, z_2) = (z_1 + b_i z_2 + c_i, z_2 + a_i)\,,$$
$$f_3(z_1, z_2) = \Big(z_1 + \frac{b_1a_2 - b_2a_1}{r}, z_2\Big)$$

for $i = 1, 2$ and for $(z_1, z_2) \in \mathbb{C} \times H$. We define $S^-_{N,p,q,r}$ to be the quotient surface $(\mathbb{C} \times H)/G^-_N$. Note that every surface $S^-_{N,p,q,r}$ has as an unramified double cover an Inoue surface $S^+_{N^2,p',q',r;0}$ for suitable integers p', q'. In fact, we have the involution of $S^+_{N^2,p',q',r;0}$: $\iota(z_1, z_2) = (-z_1, \tau z_2)$ satisfies $\iota^2 = \mathrm{Id}$ and $S^-_{N,p,q,r} = S^+_{N^2,p',q',r;0}/\iota$.

As in the case of the surface S_M, we may assume that local holomorphic coordinates z_1 and z_2 are uniformly bounded and that y_2 is uniformly bounded below away from zero.

Since $\tau > 1$, we may write $\mathrm{Im}\mathbf{t} = m \log \tau$ for some $m \in \mathbb{R}$. Note that $\mathbf{t}$ is real if and only if $m = 0$. We define $(1, 1)$-forms α'' and γ' on $\mathbb{C} \times H$ by

$$\alpha'' := \frac{\sqrt{-1}}{2y_2^2} dz_2 \wedge d\bar{z}_2, \quad \gamma' := \sqrt{-1}\Big(dz_1 - \frac{y_1 - m \log y_2}{y_2} dz_2\Big) \wedge \Big(d\bar{z}_1 - \frac{y_1 - m \log y_2}{y_2} d\bar{z}_2\Big)\,.$$

Since these forms are invariant under the G^+_N -action, they descend to $S^+_{N,p,q,r;\mathbf{t}}$.

Then we can define the Vaisman metric on $S^+_{N,p,q,r;\mathbf{t}}$

$$\omega_V := 2\alpha'' + \gamma'\,,$$

which is a pluriclosed metric on $S^+_{N,p,q,r;\mathbf{t}}$ (cf. [4, Section 3], [15, Section 6] and [19]). This metric satisfies on $S^+_{N,p,q,r;\mathbf{t}}$,

$$\mathrm{Ric}(\omega_V) = -\sqrt{-1}\partial\bar{\partial} \log \omega_V^2 = \sqrt{-1}\partial\bar{\partial} \log y_2^2 = -\alpha'' \leq 0\,.$$

In the case of $S^-_{N,p,q,r}$ for a matrix $N \in \mathrm{GL}(2,\mathbb{Z})$ with $\det N = -1$ and arbitrary fixed $p, q, r \in \mathbb{Z}$ with $r \neq 0$, we consider forms on $\mathbb{C} \times H$

$$\alpha'' = \frac{\sqrt{-1}}{2y_2^2} dz_2 \wedge d\bar{z}_2 \,, \quad \gamma'^- := \sqrt{-1}\Big(dz_1 - \frac{y_1}{y_2} dz_2\Big) \wedge \Big(d\bar{z}_1 - \frac{y_1}{y_2} d\bar{z}_2\Big)\,,$$

which are invariant under the G^-_N -action and so they descend to $S^-_{N,p,q,r}$. Hence we can define a Hermitian metric on $S^-_{N,p,q,r}$,

$$\omega^-_V := 2\alpha'' + \gamma'^- \,,$$

which is pluriclosed.

Denote

$$\sigma : S^+_{N^2,p',q',r;0} \to S^-_{N,p,q,r}$$

the quotient map for suitable $p', q' \in \mathbb{Z}$, which is an unramified double covering. We pull back the metric ω^-_V via σ to $S^+_{N^2,p',q',r;0}$. The metric $\sigma^*\omega^-_V$ coincides with the metric $(\omega_V)_0$ which is the metric ω_V with $m = 0$. Then we have

$$\sigma^* \operatorname{Ric}(\omega^-_V) = \operatorname{Ric}(\sigma^*\omega^-_V) = \operatorname{Ric}((\omega_V)_0) = -\alpha'' \leq 0\,.$$

Since α'' is invariant under the $G^+_{N^2}$ and G^-_N -actions, we obtain on $S^-_{N,p,q,r}$,

$$\operatorname{Ric}(\omega^-_V) = -\alpha'' \leq 0\,.$$

Hence we obtain that $c_1^{BC}(S^\pm) \leq 0$ and this implies that we have $d \leq 0$ in both cases $S^\pm$.

By applying Proposition 2.1, we conclude that Inoue surfaces of all types S_M, S^+ and S^- admit no static pluriclosed metrics.

For these reasons, we conclude that there are no static pluriclosed metrics on Kodaira surfaces, non-Kähler minimal properly elliptic surfaces and Inoue surfaces.

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