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A Diffusion Process with a Random Potential Consisting of Two Contracted Self-Similar Processes

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Abstract. We study a limiting behavior of a one-dimensional diffusion process with a random potential. The potential consists of two independent contracted self-similar processes with different indices for the right and the left hand sides of the origin. Brox (1986) and Schumacher (1985) studied a diffusion process with a Brownian potential, and showed, roughly speaking, after a long time with high probability the process is at the bottom of a valley. Their result was extended to a diffusion process in an asymptotically self-similar random environment by Kawazu, Tamura and Tanaka (1989). Our model is a variant of their models. But we show, roughly speaking, after a long time it is possible that our process is not at the bottom of a valley. We also study asymptotic behaviors of the minimum process and the maximum process of our process.

1. Model and results

Denote by $\mathbb{W}$ the set of $\mathbb{R}$-valued functions w on $\mathbb{R}$ satisfying:

(i) $w(0) = 0$,

(ii) w is right-continuous and has left limits on $[0, \infty)$,

(iii) w is left-continuous and has right limits on $(-\infty, 0]$.

Let $\alpha_1, \alpha_2 > 0$, and P_{α_1,α_2} be the probability measure on $\mathbb{W}$ which satisfies:

(i) $\{w(-x), x \geq 0, P_{\alpha_1,\alpha_2}\}$ is an ${\alpha_1}^{-1}$-self-similar process with time parameter x,

(ii) $\{w(x), x \geq 0, P_{\alpha_1,\alpha_2}\}$ is an ${\alpha_2}^{-1}$-self-similar process with time parameter x,

(iii) two processes above are independent.

In words, we have

$$\{T_\xi w, P_{\alpha_1,\alpha_2}\} \stackrel{\mathrm{d}}{=} \{w, P_{\alpha_1,\alpha_2}\} \text{ for all } \xi > 0,$$

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where

$$\left(T_\xi w\right)(x) = \begin{cases} \xi^{-1} w(\xi^{\alpha_1} x) & \text{for } x \le 0\,, \\ \xi^{-1} w(\xi^{\alpha_2} x) & \text{for } x > 0\,, \end{cases}$$

and $\stackrel{\mathrm{d}}{=}$ means the equality in distribution. Let $c \in (0, \min\{1/(2\alpha_1), 1/(2\alpha_2)\})$ be fixed. For $w \in \mathbb{W}$ and $\lambda > 0$, define $w_\lambda \in \mathbb{W}$ by

$$w_\lambda(x) = \lambda e^{-c\lambda} w(x)\,, \quad x \in \mathbb{R}\,.$$

Then for each $\lambda > 0$ we also have

$$\{T_\xi w_\lambda,\, P_{\alpha_1,\alpha_2}\} \stackrel{\mathrm{d}}{=} \{w_\lambda,\, P_{\alpha_1,\alpha_2}\} \quad \text{for all } \xi > 0\,.$$

Let Ω denote the set of $\mathbb{R}$-valued continuous functions on $[0, \infty)$, and for $\omega \in \Omega$ and $t \ge 0$ set $X(t) = X(t, \omega) = \omega(t)$. For $w \in \mathbb{W}$ and $x_0 \in \mathbb{R}$, let $P_w^{x_0}$ be the probability measure on Ω such that $\{X(t), t \ge 0, P_w^{x_0}\}$ is a diffusion process with generator

$$\mathcal{L}_w = \frac{1}{2} e^{w(x)} \frac{d}{dx}\left(e^{-w(x)} \frac{d}{dx}\right)$$

whose starting point is x_0. Define $\mathcal{P}_\lambda^{x_0}$, a probability measure on $\mathbb{W} \times \Omega$, by

$$\mathcal{P}_\lambda^{x_0}(dw d\omega) = P_{\alpha_1,\alpha_2}(dw) P_{w_\lambda}^{x_0}(d\omega)\,.$$

For each $\lambda > 0$, we regard $\{X(t), t \ge 0, \mathcal{P}_\lambda^{x_0}\}$ as a process defined on the probability space $(\mathbb{W} \times \Omega, \mathcal{P}_\lambda^{x_0})$, which we call a diffusion process with a random potential consisting of two contracted self-similar processes. We investigate the behavior of the process $\{X(t), t \ge 0, \mathcal{P}_\lambda^0\}$ at $t = e^\lambda$ $(\lambda \to \infty)$.

A diffusion process with a Brownian potential was studied by Brox ([1]) and Schumacher ([8]). Their result was extended to a diffusion process in an asymptotically self-similar random environment by Kawazu, Tamura and Tanaka ([5], [6]). On the other hand, in [3] and [4] a diffusion process with a one-sided Brownian potential was investigated. Moreover, in [9] the results in [5] and [6] were extended to a diffusion process with a random potential consisting of two self-similar processes with different indices. Namely, in [9] the limiting behavior of the process $\{X(t), t \ge 0, \mathcal{P}_{\alpha_1,\alpha_2}^0\}$ as time goes to infinity is studied, where $\mathcal{P}_{\alpha_1,\alpha_2}^0$ is the probability measure on $\mathbb{W} \times \Omega$ defined by

$$\mathcal{P}_{\alpha_1,\alpha_2}^0(dw d\omega) = P_{\alpha_1,\alpha_2}(dw) P_w^0(d\omega)\,.$$

She shows that, roughly speaking, for large t with high probability $X(t)$ is at the bottom of a valley. This result corresponds to the results in [5] and [6], where the case $\alpha_1 = \alpha_2$ is studied. Our present model is a variant of the model in [9], but our result concerning the behavior of $X(e^\lambda)$ is much different from the results in [1], [5], [6] and [9]. Roughly speaking, for large λ it is possible that $X(e^\lambda)$ is not at the bottom of a valley in the case $\alpha_1 \ne \alpha_2$. For the precise meaning of this, see Theorems 1.1 and 1.2 combined with Theorems 1.6, 1.10 and Remark 1.2. We remark, in [10] she gets a similar result to ours for a diffusion process with a Brownian potential including a zero potential part.

To state our results, we prepare some notation. Set $\widetilde{\alpha}_i = c\alpha_i (< 1/2), i = 1, 2,$ and for $w \in \mathbb{W}$ and $\lambda > 0$ define $\tau_\lambda w \in \mathbb{W}$ by

$$(\tau_\lambda w)\,(x) = \begin{cases} \lambda^{-1} w(e^{\widetilde{\alpha}_1 \lambda} x) & \text{for } x \le 0\,, \\ \lambda^{-1} w(e^{\widetilde{\alpha}_2 \lambda} x) & \text{for } x > 0\,. \end{cases}$$

Then it is easily seen that

$$\{\tau_\lambda w_\lambda,\, P_{\alpha_1,\alpha_2}\} \overset{\mathrm{d}}{=} \{w,\, P_{\alpha_1,\alpha_2}\} \quad \text{for all } \lambda > 0\,. \tag{1.1}$$

Denote by $\mathbb{W}^\#$ the set of functions $w \in \mathbb{W}$ satisfying

$$\limsup_{x\to\infty} \left\{ w(x) - \inf_{0\le y\le x} w(y) \right\} = \limsup_{x\to-\infty} \left\{ w(x) - \inf_{x\le y\le 0} w(y) \right\} = \infty\,,$$

and for $w \in \mathbb{W}^\#$ set

$$\zeta_1 = \zeta_1(w) = \sup\{x < 0 : w^*(x) - \inf_{x<y\le 0} w(y) \ge 1 - 2\widetilde{\alpha}_1\}\,,$$
$$\zeta_2 = \zeta_2(w) = \inf\{x > 0 : w^*(x) - \inf_{0\le y<x} w(y) \ge 1 - 2\widetilde{\alpha}_2\}\,,$$

where

$$w^*(x) = w(x-) \vee w(x+)\,,\ \ w_*(x) = w(x-) \wedge w(x+)\,,\ \ x \in \mathbb{R}\,.$$

We notice $-\infty < \zeta_1(w) < 0$ and $0 < \zeta_2(w) < \infty$ for any $w \in \mathbb{W}^\#$. We also define some functions of $w \in \mathbb{W}^\#$ as follows (cf. [6], [9]):

$$V_1 = V_1(w) = \inf\{w_*(x) : \zeta_1 < x \le 0\}\,,$$
$$V_2 = V_2(w) = \inf\{w_*(x) : 0 \le x < \zeta_2\}\,,$$
$$\mathbf{b}_1 = \mathbf{b}_1(w) = \begin{cases} \{\zeta_1 < x \le 0 : w_*(x) = V_1\} & \text{if } w(\zeta_1) = V_1\,, \\ \{\zeta_1 \le x \le 0 : w_*(x) = V_1\} & \text{if } w(\zeta_1) \ne V_1\,, \end{cases}$$
$$\mathbf{b}_2 = \mathbf{b}_2(w) = \begin{cases} \{0 \le x < \zeta_2 : w_*(x) = V_2\} & \text{if } w(\zeta_2) = V_2\,, \\ \{0 \le x \le \zeta_2 : w_*(x) = V_2\} & \text{if } w(\zeta_2) \ne V_2\,, \end{cases}$$
$$b_1 = b_1(w) = \min \mathbf{b}_1(w)\,,\ \ b_2 = b_2(w) = \max \mathbf{b}_2(w)\,,$$
$$M_1 = M_1(w) = \begin{cases} \sup\{w^*(x) : b_1 < x \le 0\} & \text{if } b_1 < 0\,, \\ 0 & \text{if } b_1 = 0\,, \end{cases}$$
$$M_2 = M_2(w) = \begin{cases} \sup\{w^*(x) : 0 \le x < b_2\} & \text{if } b_2 > 0\,, \\ 0 & \text{if } b_2 = 0\,, \end{cases}$$

$$\mathbf{a}_1 = \mathbf{a}_1(w) = \begin{cases} \{b_1 \le x \le 0 : w^*(x) = M_1\} & \text{if } w(b_1) \le w(b_1+)\,, \\ \{b_1 < x \le 0 : w^*(x) = M_1\} & \text{if } w(b_1) > w(b_1+)\,, \end{cases}$$

$$\mathbf{a}_2 = \mathbf{a}_2(w) = \begin{cases} \{0 \le x \le b_2 : w^*(x) = M_2\} & \text{if } w(b_2) \le w(b_2-)\,, \\ \{0 \le x < b_2 : w^*(x) = M_2\} & \text{if } w(b_2) > w(b_2-)\,, \end{cases}$$

$$a_i^+ = a_i^+(w) = \max \mathbf{a}_i(w)\,,\ a_i^- = a_i^-(w) = \min \mathbf{a}_i(w)\,, \quad i = 1, 2\,,$$

$$K_i = K_i(w) = M_i(w) \vee (V_i(w) + 1 - 2\widetilde{\alpha}_i)\,, \quad i = 1, 2\,.$$

Moreover, for $w \in \mathbb{W}^\#$ and $\varepsilon > 0$ we set

$$b_{1,\varepsilon} = b_{1,\varepsilon}(w) = \begin{cases} b_1(w) & \text{if } w(b_1) > w(b_1+)\,, \\ b_1(w) - \varepsilon & \text{if } w(b_1) \le w(b_1+)\,, \end{cases}$$

$$b_{2,\varepsilon} = b_{2,\varepsilon}(w) = \begin{cases} b_2(w) & \text{if } w(b_2) > w(b_2-)\,, \\ b_2(w) + \varepsilon & \text{if } w(b_2) \le w(b_2-)\,. \end{cases}$$

We divide $\mathbb{W}^\#$ into two subsets (cf. [7], [9]):

$$\mathbb{A} = \{w \in \mathbb{W}^\# : K_1 + \widetilde{\alpha}_1 < K_2 + \widetilde{\alpha}_2\}\,,$$
$$\mathbb{B} = \{w \in \mathbb{W}^\# : K_1 + \widetilde{\alpha}_1 > K_2 + \widetilde{\alpha}_2\}\,,$$

and for $\lambda > 0$ we set

$$\mathbb{A}_\lambda = \{w \in \mathbb{W}^\# : \tau_\lambda w_\lambda \in \mathbb{A}\}\,,$$
$$\mathbb{B}_\lambda = \{w \in \mathbb{W}^\# : \tau_\lambda w_\lambda \in \mathbb{B}\}\,.$$

By (1.1), we note $P_{\alpha_1,\alpha_2}\{\mathbb{A}_\lambda\} = P_{\alpha_1,\alpha_2}\{\mathbb{A}\}$ and $P_{\alpha_1,\alpha_2}\{\mathbb{B}_\lambda\} = P_{\alpha_1,\alpha_2}\{\mathbb{B}\}$ for all $\lambda > 0$.

Now we state our results on the behaviors of $\{e^{-\widetilde{\alpha}_1\lambda}X(e^\lambda), \mathcal{P}_\lambda^0\}$ and $\{e^{-\widetilde{\alpha}_2\lambda}X(e^\lambda), \mathcal{P}_\lambda^0\}$. In the following theorems, $P_{\alpha_1,\alpha_2}\{\cdot|\cdot\}$ denotes the conditional probability.

THEOREM 1.1. *Suppose $P_{\alpha_1,\alpha_2}\{\mathbb{W}^\#\} = 1$. Then for any $\varepsilon > 0$ and $M > 0$ the following* (i)–(iii) *hold.*

(i) $\lim_{\lambda\to\infty} P_{\alpha_1,\alpha_2}\{\mathbb{E}^-_{1,\lambda,\varepsilon}|\mathbb{A}_\lambda\} = 1\,.$

(ii) $\lim_{\lambda\to\infty} P_{\alpha_1,\alpha_2}\{\mathbb{E}^+_{1,\lambda,\varepsilon}|\mathbb{B}_\lambda\} = 1\,,$ *if* $\alpha_1 > \alpha_2\,.$

(iii) $\lim_{\lambda\to\infty} P_{\alpha_1,\alpha_2}\{\mathbb{E}^+_{1,\lambda,M,\varepsilon}|\mathbb{B}_\lambda\} = 1\,,$ *if* $\alpha_1 < \alpha_2\,.$

Here

$$\mathbb{E}^\pm_{1,\lambda,\varepsilon} = \{w \in \mathbb{W}^\# : p^\pm_{1,\lambda,\varepsilon}(w) > 1 - \varepsilon\}\,,$$
$$\mathbb{E}^+_{1,\lambda,M,\varepsilon} = \{w \in \mathbb{W}^\# : p^+_{1,\lambda,M}(w) > 1 - \varepsilon\}\,,$$
$$p^-_{1,\lambda,\varepsilon}(w) = P^0_{w_\lambda}\{e^{-\widetilde{\alpha}_1\lambda}X(e^\lambda) \in U_\varepsilon(\mathbf{b}_1(\tau_\lambda w_\lambda)) \cap (b_{1,\varepsilon}(\tau_\lambda w_\lambda), 0)\}\,,$$
$$p^+_{1,\lambda,\varepsilon}(w) = P^0_{w_\lambda}\{0 < e^{-\widetilde{\alpha}_1\lambda}X(e^\lambda) < \varepsilon\}\,,$$
$$p^+_{1,\lambda,M}(w) = P^0_{w_\lambda}\{e^{-\widetilde{\alpha}_1\lambda}X(e^\lambda) > M\}\,,$$

and $U_\varepsilon(\mathbf{a})$ *denotes the open* ε*-neighborhood of the (generic) set* $\mathbf{a}$ *in* $\mathbb{R}$.

THEOREM 1.2. *Suppose* $P_{\alpha_1,\alpha_2}\{\mathbb{W}^\#\} = 1$. *Then for any* $\varepsilon > 0$ *and* $M > 0$ *the following* (i)–(iii) *hold.*

(i) $\lim_{\lambda\to\infty} P_{\alpha_1,\alpha_2}\{\mathbb{E}^-_{2,\lambda,M,\varepsilon}|\mathbb{A}_\lambda\} = 1\,,\quad$ *if* $\alpha_1 > \alpha_2$.

(ii) $\lim_{\lambda\to\infty} P_{\alpha_1,\alpha_2}\{\mathbb{E}^-_{2,\lambda,\varepsilon}|\mathbb{A}_\lambda\} = 1\,,\quad$ *if* $\alpha_1 < \alpha_2$.

(iii) $\lim_{\lambda\to\infty} P_{\alpha_1,\alpha_2}\{\mathbb{E}^+_{2,\lambda,\varepsilon}|\mathbb{B}_\lambda\} = 1$.

Here

$$\mathbb{E}^-_{2,\lambda,M,\varepsilon} = \{w \in \mathbb{W}^\# : p^-_{2,\lambda,M}(w) > 1-\varepsilon\}\,,$$
$$\mathbb{E}^\pm_{2,\lambda,\varepsilon} = \{w \in \mathbb{W}^\# : p^\pm_{2,\lambda,\varepsilon}(w) > 1-\varepsilon\}\,,$$
$$p^-_{2,\lambda,M}(w) = P^0_{w_\lambda}\{e^{-\widetilde{\alpha}_2\lambda}X(e^\lambda) < -M\}\,,$$
$$p^-_{2,\lambda,\varepsilon}(w) = P^0_{w_\lambda}\{-\varepsilon < e^{-\widetilde{\alpha}_2\lambda}X(e^\lambda) < 0\}\,,$$
$$p^+_{2,\lambda,\varepsilon}(w) = P^0_{w_\lambda}\{e^{-\widetilde{\alpha}_2\lambda}X(e^\lambda) \in U_\varepsilon(\mathbf{b}_2(\tau_\lambda w_\lambda)) \cap (0, b_{2,\varepsilon}(\tau_\lambda w_\lambda))\}\,.$$

EXAMPLE 1.1 (cf. [9]). Let $\alpha_1, \alpha_2 \in (0,2)$, and P_{α_1,α_2} be the probability measure on $\mathbb{W}$ such that $\{w(-x), x \geq 0, P_{\alpha_1,\alpha_2}\}$ and $\{w(x), x \geq 0, P_{\alpha_1,\alpha_2}\}$ are, respectively, symmetric α_1-stable and symmetric α_2-stable Lévy motions with time parameter x, and these two processes are independent. Then P_{α_1,α_2} satisfies our conditions and $P_{\alpha_1,\alpha_2}\{\mathbb{W}^\#\} = P_{\alpha_1,\alpha_2}\{\mathbb{A}\cup\mathbb{B}\} = 1$.

REMARK 1.1 (cf. [9]). Let $\alpha_1 = \alpha_2 = \alpha \in (0,1)$, and P_{α_1,α_2} be the probability measure on $\mathbb{W}$ such that $\{w(-x), x \geq 0, P_{\alpha_1,\alpha_2}\}$ and $\{w(x), x \geq 0, P_{\alpha_1,\alpha_2}\}$ are independent α-stable subordinators with time parameter x. Then P_{α_1,α_2} satisfies our conditions and $P_{\alpha_1,\alpha_2}\{\mathbb{W}^\#\} = P_{\alpha_1,\alpha_2}\{K_1 + \widetilde{\alpha}_1 = K_2 + \widetilde{\alpha}_2\} = 1$. In this case $\mathbf{b}_1(w) = \mathbf{b}_2(w) = \{0\}$ for any $w \in \mathbb{W}^\#$ and the following holds for any $\varepsilon > 0$:

$$\lim_{\lambda\to\infty} \mathcal{P}^0_\lambda\big\{e^{-\widetilde{\alpha}_1\lambda}X(e^\lambda) \in (-\varepsilon,\varepsilon)\big\} = 1\,.$$

The following corollaries are obtained from the proofs of Theorems 1.1 and 1.2.

COROLLARY 1.3. *Suppose* $P_{\alpha_1,\alpha_2}\{\mathbb{W}^\#\} = 1$. *Then for any* $\varepsilon > 0$ *and* $M > 0$ *the following* (i)–(iii) *hold.*

(i) $\lim_{\lambda\to\infty} P_{\alpha_1,\alpha_2}\{\mathbb{E}^-_{3,\lambda,\varepsilon}|\mathbb{A}_\lambda\} = 1$.

(ii) $\lim_{\lambda\to\infty} P_{\alpha_1,\alpha_2}\{\mathbb{E}^+_{3,\lambda,\varepsilon}|\mathbb{B}_\lambda\} = 1\,,\quad$ *if* $\alpha_1 > \alpha_2$.

(iii) $\lim_{\lambda\to\infty} P_{\alpha_1,\alpha_2}\{\mathbb{E}^+_{3,\lambda,M,\varepsilon}|\mathbb{B}_\lambda\} = 1\,,\quad$ *if* $\alpha_1 < \alpha_2$.

Here

$$\mathbb{E}^\pm_{3,\lambda,\varepsilon} = \{w \in \mathbb{W}^\# : p^\pm_{3,\lambda,\varepsilon}(w) > 1-\varepsilon\}\,,$$

$$\mathbb{E}^{+}_{3,\lambda,M,\varepsilon} = \{w \in \mathbb{W}^{\#} : p^{+}_{3,\lambda,M,\varepsilon}(w) > 1-\varepsilon\},$$

$$p^{-}_{3,\lambda,\varepsilon}(w) = P^{0}_{w_\lambda}\left\{e^{-\lambda}\int_0^{e^\lambda} \mathbf{1}_{U_\varepsilon(\mathbf{b}_1(\tau_\lambda w_\lambda))\cap(b_{1,\varepsilon}(\tau_\lambda w_\lambda),0)}(e^{-\widetilde{\alpha}_1\lambda}X(s))ds > 1-\varepsilon\right\},$$

$$p^{+}_{3,\lambda,\varepsilon}(w) = P^{0}_{w_\lambda}\left\{e^{-\lambda}\int_0^{e^\lambda} \mathbf{1}_{(0,\varepsilon)}(e^{-\widetilde{\alpha}_1\lambda}X(s))ds > 1-\varepsilon\right\},$$

$$p^{+}_{3,\lambda,M,\varepsilon}(w) = P^{0}_{w_\lambda}\left\{e^{-\lambda}\int_0^{e^\lambda} \mathbf{1}_{(M,\infty)}(e^{-\widetilde{\alpha}_1\lambda}X(s))ds > 1-\varepsilon\right\},$$

and $\mathbf{1}_{\mathbf{a}}$ *denotes the indicator function of the* (*generic*) *set* $\mathbf{a}$.

COROLLARY 1.4. *Suppose* $P_{\alpha_1,\alpha_2}\{\mathbb{W}^{\#}\} = 1$. *Then for any* $\varepsilon > 0$ *and* $M > 0$ *the following* (i)–(iii) *hold.*

(i) $\lim_{\lambda\to\infty} P_{\alpha_1,\alpha_2}\{\mathbb{E}^{-}_{4,\lambda,M,\varepsilon}|\mathbb{A}_\lambda\} = 1$, *if* $\alpha_1 > \alpha_2$.

(ii) $\lim_{\lambda\to\infty} P_{\alpha_1,\alpha_2}\{\mathbb{E}^{-}_{4,\lambda,\varepsilon}|\mathbb{A}_\lambda\} = 1$, *if* $\alpha_1 < \alpha_2$.

(iii) $\lim_{\lambda\to\infty} P_{\alpha_1,\alpha_2}\{\mathbb{E}^{+}_{4,\lambda,\varepsilon}|\mathbb{B}_\lambda\} = 1$.

Here

$$\mathbb{E}^{-}_{4,\lambda,M,\varepsilon} = \{w \in \mathbb{W}^{\#} : p^{-}_{4,\lambda,M,\varepsilon}(w) > 1-\varepsilon\},$$

$$\mathbb{E}^{\pm}_{4,\lambda,\varepsilon} = \{w \in \mathbb{W}^{\#} : p^{\pm}_{4,\lambda,\varepsilon}(w) > 1-\varepsilon\},$$

$$p^{-}_{4,\lambda,M,\varepsilon}(w) = P^{0}_{w_\lambda}\left\{e^{-\lambda}\int_0^{e^\lambda} \mathbf{1}_{(-\infty,-M)}(e^{-\widetilde{\alpha}_2\lambda}X(s))ds > 1-\varepsilon\right\},$$

$$p^{-}_{4,\lambda,\varepsilon}(w) = P^{0}_{w_\lambda}\left\{e^{-\lambda}\int_0^{e^\lambda} \mathbf{1}_{(-\varepsilon,0)}(e^{-\widetilde{\alpha}_2\lambda}X(s))ds > 1-\varepsilon\right\},$$

$$p^{+}_{4,\lambda,\varepsilon}(w) = P^{0}_{w_\lambda}\left\{e^{-\lambda}\int_0^{e^\lambda} \mathbf{1}_{U_\varepsilon(\mathbf{b}_2(\tau_\lambda w_\lambda))\cap(0,b_{2,\varepsilon}(\tau_\lambda w_\lambda))}(e^{-\widetilde{\alpha}_2\lambda}X(s))ds > 1-\varepsilon\right\}.$$

Next we study the minimum process and the maximum process of $\{X(t), t \geq 0, \mathcal{P}^0_\lambda\}$. For $\omega \in \Omega$ and $t \geq 0$, we set $\underline{X}(t) = \underline{X}(t,\omega) = \min_{0\leq s\leq t} X(s,\omega)$ and $\overline{X}(t) = \overline{X}(t,\omega) = \max_{0\leq s\leq t} X(s,\omega)$. We investigate the behaviors of $\{\underline{X}(t), t \geq 0, \mathcal{P}^0_\lambda\}$ and $\{\overline{X}(t), t \geq 0, \mathcal{P}^0_\lambda\}$ at $t = e^\lambda$ ($\lambda \to \infty$). In the following, we define some functions of $w \in \mathbb{W}^{\#}$. For $\gamma \in \mathbb{R}$, we set

$$\zeta_1(\gamma) = \zeta_1(\gamma, w) = \sup\{x < 0 : w^*(x) - \inf_{x<y\leq 0} w(y) \geq 1 - 2\widetilde{\alpha}_1 + \gamma\},$$

$$\zeta_2(\gamma) = \zeta_2(\gamma, w) = \inf\{x > 0 : w^*(x) - \inf_{0\leq y<x} w(y) \geq 1 - 2\widetilde{\alpha}_2 + \gamma\}.$$

We are interested in $\zeta_i(\gamma)$, $i = 1, 2$, in the case $|\gamma|$ is sufficiently small. Note that $\zeta_i(0) = \zeta_i$,

$i = 1, 2$. Moreover, we set for $\varepsilon > 0$

$$\zeta_{1,\varepsilon} = \zeta_{1,\varepsilon}(w) = \begin{cases} \zeta_1(\varepsilon) & \text{if } w(\zeta_1(\varepsilon)) \le w(\zeta_1(\varepsilon)+)\,, \\ \zeta_1(\varepsilon) - \varepsilon & \text{if } w(\zeta_1(\varepsilon)) > w(\zeta_1(\varepsilon)+)\,, \end{cases}$$

$$\zeta_{2,\varepsilon} = \zeta_{2,\varepsilon}(w) = \begin{cases} \zeta_2(\varepsilon) & \text{if } w(\zeta_2(\varepsilon)) \le w(\zeta_2(\varepsilon)-)\,, \\ \zeta_2(\varepsilon) + \varepsilon & \text{if } w(\zeta_2(\varepsilon)) > w(\zeta_2(\varepsilon)-)\,. \end{cases}$$

We also set, for $\gamma \in \mathbb{R}$

$$\rho_1(\gamma) = \rho_1(\gamma, w) = \sup\{x < 0 : w(x) > K_2 - \widetilde{\alpha}_1 + \widetilde{\alpha}_2 + \gamma\}\,,$$
$$\rho_2(\gamma) = \rho_2(\gamma, w) = \inf\{x > 0 : w(x) > K_1 + \widetilde{\alpha}_1 - \widetilde{\alpha}_2 + \gamma\}\,.$$

In the above (and also in the below) we understand $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$. In particular, we set

$$\rho_1 = \rho_1(0) = \rho_1(0, w)\,,$$
$$\rho_2 = \rho_2(0) = \rho_2(0, w)\,,$$
$$v_1 = v_1(w) = \inf\{w_*(x) : \rho_1 \le x \le 0\}\,,$$
$$v_2 = v_2(w) = \inf\{w_*(x) : 0 \le x \le \rho_2\}\,.$$

Moreover, we set for $\varepsilon > 0$

$$\rho_{1,\varepsilon} = \rho_{1,\varepsilon}(w) = \begin{cases} -\infty & \text{if } \rho_1(\varepsilon) = -\infty\,, \\ \rho_1(\varepsilon) & \text{if } -\infty < \rho_1(\varepsilon) \ne 0 \text{ and } w(\rho_1(\varepsilon)) = w(\rho_1(\varepsilon)+)\,, \\ \rho_1(\varepsilon) - \varepsilon & \text{if } -\infty < \rho_1(\varepsilon) \text{ and } w(\rho_1(\varepsilon)) > w(\rho_1(\varepsilon)+)\,, \\ -\varepsilon & \text{if } \rho_1(\varepsilon) = 0\,, \end{cases}$$

$$\rho_{2,\varepsilon} = \rho_{2,\varepsilon}(w) = \begin{cases} \infty & \text{if } \rho_2(\varepsilon) = \infty\,, \\ \rho_2(\varepsilon) & \text{if } 0 \ne \rho_2(\varepsilon) < \infty \text{ and } w(\rho_2(\varepsilon)) = w(\rho_2(\varepsilon)-)\,, \\ \rho_2(\varepsilon) + \varepsilon & \text{if } \rho_2(\varepsilon) < \infty \text{ and } w(\rho_2(\varepsilon)) > w(\rho_2(\varepsilon)-)\,, \\ \varepsilon & \text{if } \rho_2(\varepsilon) = 0\,. \end{cases}$$

PROPOSITION 1.5. (i) *If $w \in \mathbb{A}$, then the following* (1)–(3) *hold.*

(1) $\rho_1 \le \zeta_1 < 0 \le \rho_2 \le \zeta_2 < \infty$.

(2) $K_1 - v_2 < 1 - \widetilde{\alpha}_1 - \widetilde{\alpha}_2$.

(3) $V_1 - \widetilde{\alpha}_1 < v_2 - \widetilde{\alpha}_2$.

(ii) *If $w \in \mathbb{B}$, then the following* (1)–(3) *hold.*

(1) $-\infty < \zeta_1 \le \rho_1 \le 0 < \zeta_2 \le \rho_2$.

(2) $K_2 - v_1 < 1 - \widetilde{\alpha}_1 - \widetilde{\alpha}_2$.

(3) $V_2 - \widetilde{\alpha}_2 < v_1 - \widetilde{\alpha}_1$.

PROOF. We let $w \in \mathbb{A}$ and prove (i)(2) and (i)(3). In this case, we have either

$$0 \le \rho_2 < \zeta_2 < \infty \tag{1.2}$$

or

$$\begin{cases} 0 < \rho_2 = \zeta_2 < \infty\,, \\ K_2 = V_2 + 1 - 2\widetilde{\alpha}_2\,. \end{cases} \tag{1.3}$$

In the case (1.2), we get (i)(2) by combining $w(\rho_2) - v_2 < 1 - 2\widetilde{\alpha}_2$ and $w(\rho_2) \geq K_1 + \widetilde{\alpha}_1 - \widetilde{\alpha}_2$. In the case (1.3), we get (i)(2) because of $K_1 + \widetilde{\alpha}_1 - \widetilde{\alpha}_2 < K_2 = v_2 + 1 - 2\widetilde{\alpha}_2$. Noticing $V_1 + 1 - 2\widetilde{\alpha}_1 \leq K_1$, we obtain (i)(3) from (i)(2). □

REMARK 1.2. For $w \in \mathbb{W}^{\#}$, set

$$\mathbf{b}_1' = \mathbf{b}_1'(w) = \{\rho_1 \leq x \leq 0 : w_*(x) = v_1\}\,,$$
$$\mathbf{b}_2' = \mathbf{b}_2'(w) = \{0 \leq x \leq \rho_2 : w_*(x) = v_2\}\,.$$

In the case $\alpha_1 > \alpha_2$ and $w \in \mathbb{A}$ it is possible that $V_1 > v_2$ and $(\{\zeta_1\}, \mathbf{b}_2', \{\rho_2\})$ is a valley of w, and in the case $\alpha_1 < \alpha_2$ and $w \in \mathbb{B}$ it is possible that $V_2 > v_1$ and $(\{\rho_1\}, \mathbf{b}_1', \{\zeta_2\})$ is a valley of w. (See [6] for the definition of a valley of w.)

To state our results on the minimum process and the maximum process, we divide each of $\mathbb{A}$ and $\mathbb{B}$ into two subsets (cf. [10]):

$$\mathbb{A}^1 = \{w \in \mathbb{A} : V_1 + 1 - 2\widetilde{\alpha}_1 > M_1\}\,,$$
$$\mathbb{A}^2 = \{w \in \mathbb{A} : V_1 + 1 - 2\widetilde{\alpha}_1 < M_1\}\,,$$
$$\mathbb{B}^1 = \{w \in \mathbb{B} : V_2 + 1 - 2\widetilde{\alpha}_2 > M_2\}\,,$$
$$\mathbb{B}^2 = \{w \in \mathbb{B} : V_2 + 1 - 2\widetilde{\alpha}_2 < M_2\}\,.$$

Moreover, for $\lambda > 0$ and $i = 1, 2$, we set

$$\mathbb{A}^i_\lambda = \{w \in \mathbb{A}_\lambda : \tau_\lambda w_\lambda \in \mathbb{A}^i\}\,,$$
$$\mathbb{B}^i_\lambda = \{w \in \mathbb{B}_\lambda : \tau_\lambda w_\lambda \in \mathbb{B}^i\}\,.$$

By (1.1), we have $P_{\alpha_1,\alpha_2}\{\mathbb{A}^i_\lambda\} = P_{\alpha_1,\alpha_2}\{\mathbb{A}^i\}$ and $P_{\alpha_1,\alpha_2}\{\mathbb{B}^i_\lambda\} = P_{\alpha_1,\alpha_2}\{\mathbb{B}^i\}$ for all $\lambda > 0$ and $i = 1, 2$.

First we state our result on the behavior of $\{e^{-\widetilde{\alpha}_1\lambda}\underline{X}(e^\lambda), \mathcal{P}^0_\lambda\}$.

THEOREM 1.6. *Suppose* $P_{\alpha_1,\alpha_2}\{\mathbb{W}^{\#}\} = 1$. *Then for any* $\varepsilon > 0$ *and* $\varepsilon(\lambda) > 0, \lambda > 0$, *satisfying* $\lim_{\lambda\to\infty} \varepsilon(\lambda) = 0$ *and* $\lim_{\lambda\to\infty} \lambda\varepsilon(\lambda) = \infty$, *the following* (i)–(iii) *hold.*

(i) $\displaystyle\lim_{\lambda\to\infty} P_{\alpha_1,\alpha_2}\{\mathbb{E}_{5,\lambda,\varepsilon}|\mathbb{A}_\lambda\} = 1\,.$

(ii) $\displaystyle\lim_{\lambda\to\infty} P_{\alpha_1,\alpha_2}\{\mathbb{E}_{6,\lambda,\varepsilon}|\mathbb{B}^1_\lambda\} = 1\,.$

(iii) $\displaystyle\lim_{\lambda\to\infty} P_{\alpha_1,\alpha_2}\{\mathbb{E}_{7,\lambda,\varepsilon}|\mathbb{B}^2_\lambda\} = 1\,.$

Here

$$\mathbb{E}_{i,\lambda,\varepsilon} = \{w \in \mathbb{W}^{\#} : p_{i,\lambda,\varepsilon}(w) > 1 - \varepsilon\}\,, \quad 5 \leq i \leq 7\,,$$

$$p_{5,\lambda,\varepsilon}(w) = P^0_{w_\lambda}\{\zeta_{1,\varepsilon}(\tau_\lambda w_\lambda) < e^{-\widetilde{\alpha}_1\lambda}\underline{X}(e^\lambda) < \zeta_1(-\varepsilon(\lambda), \tau_\lambda w_\lambda)\},$$
$$p_{6,\lambda,\varepsilon}(w) = P^0_{w_\lambda}\{\rho_{1,\varepsilon}(\tau_\lambda w_\lambda) < e^{-\widetilde{\alpha}_1\lambda}\underline{X}(e^\lambda) < \rho_1(-\varepsilon(\lambda), \tau_\lambda w_\lambda)\},$$
$$p_{7,\lambda,\varepsilon}(w) = P^0_{w_\lambda}\{\rho_{1,\varepsilon}(\tau_\lambda w_\lambda) < e^{-\widetilde{\alpha}_1\lambda}\underline{X}(e^\lambda) < \rho_1(-\varepsilon, \tau_\lambda w_\lambda)\}.$$

By Proposition 1.5 and the proof of Theorem 1.6, we obtain the following corollary.

COROLLARY 1.7. *Suppose* $P_{\alpha_1,\alpha_2}\{\mathbb{W}^\#\} = 1$. *Then for any* $\varepsilon > 0$

$$\lim_{\lambda\to\infty} \mathcal{P}^0_\lambda \left\{\ell_{1,\varepsilon}(\tau_\lambda w_\lambda) < e^{-\widetilde{\alpha}_1\lambda}\underline{X}(e^\lambda) < \ell_1(-\varepsilon, \tau_\lambda w_\lambda)\right\} = 1.$$

Here

$$\ell_1(-\varepsilon, w) = \rho_1(-\varepsilon, w) \vee \zeta_1(-\varepsilon, w), \ell_{1,\varepsilon}(w) = \rho_{1,\varepsilon}(w) \vee \zeta_{1,\varepsilon}(w).$$

Next we state our result concerning the behavior of $\left\{e^{-\widetilde{\alpha}_1\lambda}\overline{X}(e^\lambda), \mathcal{P}^0_\lambda\right\}$.

THEOREM 1.8. *Suppose* $P_{\alpha_1,\alpha_2}\{\mathbb{W}^\#\} = 1$. *Then for any* $\varepsilon > 0$ *and* $M > 0$ *the following* (i)–(ii) *hold.*

(i) $\lim_{\lambda\to\infty} \mathcal{P}^0_\lambda \left\{0 < e^{-\widetilde{\alpha}_1\lambda}\overline{X}(e^\lambda) < \varepsilon\right\} = 1$, *if* $\alpha_1 > \alpha_2$.

(ii) $\lim_{\lambda\to\infty} \mathcal{P}^0_\lambda \left\{e^{-\widetilde{\alpha}_1\lambda}\overline{X}(e^\lambda) > M\right\} = 1$, *if* $\alpha_1 < \alpha_2$.

The following theorems are concerning the behaviors of $\left\{e^{-\widetilde{\alpha}_2\lambda}\underline{X}(e^\lambda), \mathcal{P}^0_\lambda\right\}$ and $\left\{e^{-\widetilde{\alpha}_2\lambda}\overline{X}(e^\lambda), \mathcal{P}^0_\lambda\right\}$.

THEOREM 1.9. *Suppose* $P_{\alpha_1,\alpha_2}\{\mathbb{W}^\#\} = 1$. *Then for any* $M > 0$ *and* $\varepsilon > 0$ *the following* (i)–(ii) *hold.*

(i) $\lim_{\lambda\to\infty} \mathcal{P}^0_\lambda\{e^{-\widetilde{\alpha}_2\lambda}\underline{X}(e^\lambda) < -M\} = 1$, *if* $\alpha_1 > \alpha_2$.

(ii) $\lim_{\lambda\to\infty} \mathcal{P}^0_\lambda\{-\varepsilon < e^{-\widetilde{\alpha}_2\lambda}\underline{X}(e^\lambda) < 0\} = 1$, *if* $\alpha_1 < \alpha_2$.

THEOREM 1.10. *Suppose* $P_{\alpha_1,\alpha_2}\{\mathbb{W}^\#\} = 1$. *Then for any* $\varepsilon > 0$ *and* $\varepsilon(\lambda) > 0, \lambda > 0$, *satisfying* $\lim_{\lambda\to\infty} \varepsilon(\lambda) = 0$ *and* $\lim_{\lambda\to\infty} \lambda\varepsilon(\lambda) = \infty$, *the following* (i)–(iii) *hold.*

(i) $\lim_{\lambda\to\infty} P_{\alpha_1,\alpha_2}\{\mathbb{E}_{8,\lambda,\varepsilon}|\mathbb{A}^1_\lambda\} = 1$.

(ii) $\lim_{\lambda\to\infty} P_{\alpha_1,\alpha_2}\{\mathbb{E}_{9,\lambda,\varepsilon}|\mathbb{A}^2_\lambda\} = 1$.

(iii) $\lim_{\lambda\to\infty} P_{\alpha_1,\alpha_2}\{\mathbb{E}_{10,\lambda,\varepsilon}|\mathbb{B}_\lambda\} = 1$.

Here

$$\mathbb{E}_{i,\lambda,\varepsilon} = \{w \in \mathbb{W}^\# : p_{i,\lambda,\varepsilon}(w) > 1 - \varepsilon\}, \quad 8 \le i \le 10,$$
$$p_{8,\lambda,\varepsilon}(w) = P^0_{w_\lambda}\{\rho_2(-\varepsilon(\lambda), \tau_\lambda w_\lambda) < e^{-\widetilde{\alpha}_2\lambda}\overline{X}(e^\lambda) < \rho_{2,\varepsilon}(\tau_\lambda w_\lambda)\},$$
$$p_{9,\lambda,\varepsilon}(w) = P^0_{w_\lambda}\{\rho_2(-\varepsilon, \tau_\lambda w_\lambda) < e^{-\widetilde{\alpha}_2\lambda}\overline{X}(e^\lambda) < \rho_{2,\varepsilon}(\tau_\lambda w_\lambda)\},$$

$$p_{10,\lambda,\varepsilon}(w) = P^0_{w_\lambda}\left\{\zeta_2(-\varepsilon(\lambda), \tau_\lambda w_\lambda) < e^{-\widetilde{\alpha}_2\lambda}\overline{X}(e^\lambda) < \zeta_{2,\varepsilon}(\tau_\lambda w_\lambda)\right\}.$$

By Proposition 1.5 and the proof of Theorem 1.10, we obtain the following corollary.

COROLLARY 1.11. *Suppose $P_{\alpha_1,\alpha_2}\{\mathbb{W}^\#\} = 1$. Then for any $\varepsilon > 0$*

$$\lim_{\lambda\to\infty} \mathcal{P}^0_\lambda\left\{\ell_2(-\varepsilon, \tau_\lambda w_\lambda) < e^{-\widetilde{\alpha}_2\lambda}\overline{X}(e^\lambda) < \ell_{2,\varepsilon}(\tau_\lambda w_\lambda)\right\} = 1.$$

Here

$$\ell_2(-\varepsilon, w) = \rho_2(-\varepsilon, w) \wedge \zeta_2(-\varepsilon, w),\ \ell_{2,\varepsilon}(w) = \rho_{2,\varepsilon}(w) \wedge \zeta_{2,\varepsilon}(w).$$

To state more precise results than Theorem 1.6 (ii)–(iii) and Theorem 1.10 (i)–(ii) in particular cases, we introduce two subsets of $\mathbb{W}$:

$$\mathbb{A}^- = \{w \in \mathbb{A} : K_1 + \widetilde{\alpha}_1 - \widetilde{\alpha}_2 < 0\},$$
$$\mathbb{B}^- = \{w \in \mathbb{B} : K_2 - \widetilde{\alpha}_1 + \widetilde{\alpha}_2 < 0\}.$$

We also define, for $\lambda > 0$

$$\mathbb{A}^-_\lambda = \{w \in \mathbb{A}_\lambda : \tau_\lambda w_\lambda \in \mathbb{A}^-\},$$
$$\mathbb{B}^-_\lambda = \{w \in \mathbb{B}_\lambda : \tau_\lambda w_\lambda \in \mathbb{B}^-\}.$$

In the case $\alpha_1 > \alpha_2$, we note $\rho_1(-\varepsilon) = 0$ for any $w \in \mathbb{B}^-$ and $\varepsilon > 0$. Therefore, in this case, we have, by Theorem 1.6 (ii)–(iii),

$$\lim_{\lambda\to\infty} P_{\alpha_1,\alpha_2}\left\{P^0_{w_\lambda}\{\rho_{1,\varepsilon}(\tau_\lambda w_\lambda) < e^{-\widetilde{\alpha}_1\lambda}\underline{X}(e^\lambda) < 0\} > 1-\varepsilon\middle|\mathbb{B}^-_\lambda\right\} = 1. \tag{1.4}$$

In the case $\alpha_1 < \alpha_2$, we note $\rho_2(-\varepsilon) = 0$ for any $w \in \mathbb{A}^-$ and $\varepsilon > 0$. Therefore, in this case, we have, by Theorem 1.10 (i)–(ii),

$$\lim_{\lambda\to\infty} P_{\alpha_1,\alpha_2}\left\{P^0_{w_\lambda}\{0 < e^{-\widetilde{\alpha}_2\lambda}\overline{X}(e^\lambda) < \rho_{2,\varepsilon}(\tau_\lambda w_\lambda)\} > 1-\varepsilon\middle|\mathbb{A}^-_\lambda\right\} = 1. \tag{1.5}$$

To state more precise estimates than (1.4) and (1.5), we prepare some notation. For $w \in \mathbb{W}^\#$ and $\varepsilon > 0$, we set

$$H_1 = H_1(w) = \sup\{w^*(x) : \zeta_1 < x \le 0\},$$
$$H_2 = H_2(w) = \sup\{w^*(x) : 0 \le x < \zeta_2\},$$
$$H_{1,\varepsilon} = H_{1,\varepsilon}(w) = \sup\{w^*(x) : \zeta_{1,\varepsilon} < x \le 0\},$$
$$H_{2,\varepsilon} = H_{2,\varepsilon}(w) = \sup\{w^*(x) : 0 \le x < \zeta_{2,\varepsilon}\}.$$

We note $H_1 \le K_1 \le H_{1,\varepsilon}$ and $H_2 \le K_2 \le H_{2,\varepsilon}$, and therefore $H_1 < \widetilde{\alpha}_2 - \widetilde{\alpha}_1$ for $w \in \mathbb{A}^-$ and $H_2 < \widetilde{\alpha}_1 - \widetilde{\alpha}_2$ for $w \in \mathbb{B}^-$. The following Theorems 1.12 and 1.13 are more precise results than (1.4) and (1.5), respectively.

THEOREM 1.12. *Suppose* $\alpha_1 > \alpha_2$ *and* $P_{\alpha_1,\alpha_2}\{\mathbb{W}^\#\} = 1$. *Then for any* $\varepsilon > 0$ *and* $M > 0$

$$\lim_{\lambda\to\infty} P_{\alpha_1,\alpha_2}\{\mathbb{E}_{11,\lambda,M,\varepsilon}|\mathbb{B}_\lambda^-\} = 1\,.$$

Here

$$\mathbb{E}_{11,\lambda,M,\varepsilon} = \{w \in \mathbb{W}^\# : p_{11,\lambda,M,\varepsilon}(w) > 1-\varepsilon\}\,,$$
$$p_{11,\lambda,M,\varepsilon}(w) = P^0_{w_\lambda}\left\{g_{1,\lambda}(\varepsilon, \tau_\lambda w_\lambda) < e^{-\widetilde{\alpha}_1\lambda}\underline{X}(e^\lambda) < g_{2,\lambda}(M, \varepsilon, \tau_\lambda w_\lambda)\right\},$$
$$g_{1,\lambda}(\varepsilon, w) = \rho_{1,\varepsilon}(w) \vee (-e^{\lambda(H_{2,\varepsilon}(w)+\widetilde{\alpha}_2-\widetilde{\alpha}_1+\varepsilon)})\,,$$
$$g_{2,\lambda}(M, \varepsilon, w) = (-Me^{\lambda(\widetilde{\alpha}_2-\widetilde{\alpha}_1)}) \wedge (-e^{\lambda(H_2(w)+\widetilde{\alpha}_2-\widetilde{\alpha}_1-\varepsilon)})\,.$$

THEOREM 1.13. *Suppose* $\alpha_1 < \alpha_2$ *and* $P_{\alpha_1,\alpha_2}\{\mathbb{W}^\#\} = 1$. *Then for any* $\varepsilon > 0$ *and* $M > 0$

$$\lim_{\lambda\to\infty} P_{\alpha_1,\alpha_2}\{\mathbb{E}_{12,\lambda,M,\varepsilon}|\mathbb{A}_\lambda^-\} = 1\,.$$

Here

$$\mathbb{E}_{12,\lambda,M,\varepsilon} = \{w \in \mathbb{W}^\# : p_{12,\lambda,M,\varepsilon}(w) > 1-\varepsilon\}\,,$$
$$p_{12,\lambda,M,\varepsilon}(w) = P^0_{w_\lambda}\left\{g_{3,\lambda}(M, \varepsilon, \tau_\lambda w_\lambda) < e^{-\widetilde{\alpha}_2\lambda}\overline{X}(e^\lambda) < g_{4,\lambda}(\varepsilon, \tau_\lambda w_\lambda)\right\},$$
$$g_{3,\lambda}(M, \varepsilon, w) = Me^{\lambda(\widetilde{\alpha}_1-\widetilde{\alpha}_2)} \vee e^{\lambda(H_1(w)+\widetilde{\alpha}_1-\widetilde{\alpha}_2-\varepsilon)}\,,$$
$$g_{4,\lambda}(\varepsilon, w) = \rho_{2,\varepsilon}(w) \wedge e^{\lambda(H_{1,\varepsilon}(w)+\widetilde{\alpha}_1-\widetilde{\alpha}_2+\varepsilon)}\,.$$

2. Preliminaries

For $w \in \mathbb{W}$ and $\lambda > 0$, we define $G_\lambda w \in \mathbb{W}$ by

$$(G_\lambda w)(x) = \begin{cases} \lambda w(x) & \text{for } x \le 0\,, \\ \lambda w(e^{(\widetilde{\alpha}_1-\widetilde{\alpha}_2)\lambda}x) & \text{for } x > 0\,, \end{cases}$$

and consider the diffusion process $\{X(t), t \ge 0, P^{x_0}_{G_\lambda w}\}$ for $x_0 \in \mathbb{R}$. This process can be constructed on a probability space $(\widetilde{\Omega}, \widetilde{P})$ in the following way ([2], see also [4]). Let $\{B(t), t \ge 0\}$ be a one-dimensional Brownian motion starting from 0 defined on $(\widetilde{\Omega}, \widetilde{P})$. We set

$$S_{G_\lambda w}(x) = \int_0^x e^{(G_\lambda w)(y)}dy, \quad x \in \mathbb{R}\,,$$
$$\begin{aligned} A_{G_\lambda w}(t) &= \int_0^t e^{-2(G_\lambda w)({S_{G_\lambda w}}^{-1}(B(s)))}ds \\ &= \int_{-\infty}^{\infty} e^{-2(G_\lambda w)({S_{G_\lambda w}}^{-1}(x))}L(t, x)dx\,, \quad t \ge 0\,, \end{aligned} \tag{2.1}$$

$$L(t,x)=\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t \mathbf{1}_{[x,x+\varepsilon)}(B(s))ds\,,\quad t\ge 0,\ x\in\mathbb{R}\,,\quad \text{(local time)}$$

$$X(t;0,G_\lambda w)=S_{G_\lambda w}{}^{-1}(B(A_{G_\lambda w}{}^{-1}(t)))\,,\quad t\ge 0\,, \tag{2.2}$$

$$X(t;x_0,G_\lambda w)=x_0+X(t;0,(G_\lambda w)^{x_0})\,,\quad t\ge 0\,,$$

$$(G_\lambda w)^{x_0}(x)=(G_\lambda w)(x+x_0)\,,\ x\in\mathbb{R}\,.$$

The process $\{X(t;x_0,G_\lambda w), t\ge 0\}$ is the one which we desired.

LEMMA 2.1. *For any* $w\in\mathbb{W}$ *and* $\lambda>0$

$$\left\{X(t),t\ge 0,P^0_{G_\lambda(\tau_\lambda w_\lambda)}\right\}\overset{\mathrm{d}}{=}\left\{e^{-\widetilde{\alpha}_1\lambda}X(e^{2\widetilde{\alpha}_1\lambda}t),t\ge 0,P^0_{w_\lambda}\right\}\,. \tag{2.3}$$

PROOF. Using the equality

$$(G_\lambda(\tau_\lambda w_\lambda))(x)=w_\lambda(e^{\widetilde{\alpha}_1\lambda}x)\,,\quad x\in\mathbb{R}\,,$$

we get (2.3) in the same way as [6] (see also [1]). □

The following proposition is used to prove Theorems 1.1 and 1.2.

PROPOSITION 2.2. *Suppose* $P_{\alpha_1,\alpha_2}\{\mathbb{W}^\#\}=1$. *Then there exists a subset* $\mathbb{W}^\#_0$ *of* $\mathbb{W}^\#$ *with* $P_{\alpha_1,\alpha_2}\{\mathbb{W}^\#_0\}=1$ *such that the following* (i)–(ii) *hold.*

(i) *If* $w\in\mathbb{A}\cap\mathbb{W}^\#_0$, *then for any* $\varepsilon>0$

$$\lim_{\lambda\to\infty}P^0_{G_\lambda w}\left\{X(e^{\lambda(1-2\widetilde{\alpha}_1)})\in U_\varepsilon(\mathbf{b}_1)\cap(b_{1,\varepsilon},0)\right\}=1\,.$$

(ii) *If* $w\in\mathbb{B}\cap\mathbb{W}^\#_0$, *then for any* $\varepsilon>0$

$$\lim_{\lambda\to\infty}P^0_{G_\lambda w}\left\{e^{(\widetilde{\alpha}_1-\widetilde{\alpha}_2)\lambda}X(e^{\lambda(1-2\widetilde{\alpha}_1)})\in U_\varepsilon(\mathbf{b}_2)\cap(0,b_{2,\varepsilon})\right\}=1\,.$$

We use the following proposition to prove Theorems 1.6 and 1.9.

PROPOSITION 2.3. *Suppose* $P_{\alpha_1,\alpha_2}\{\mathbb{W}^\#\}=1$. *Then there exists a subset* $\mathbb{W}^\#_0$ *of* $\mathbb{W}^\#$ *with* $P_{\alpha_1,\alpha_2}\{\mathbb{W}^\#_0\}=1$ *such that the following* (i)–(iii) *hold.*

(i) *If* $w\in\mathbb{A}\cap\mathbb{W}^\#_0$, *then for any* $\varepsilon>0$ *and* $\varepsilon(\lambda)>0$, $\lambda>0$, *satisfying* $\lim_{\lambda\to\infty}\varepsilon(\lambda)=0$ *and* $\lim_{\lambda\to\infty}\lambda\varepsilon(\lambda)=\infty$

$$\lim_{\lambda\to\infty}P^0_{G_\lambda w}\left\{\zeta_{1,\varepsilon}<\underline{X}(e^{\lambda(1-2\widetilde{\alpha}_1)})<\zeta_1(-\varepsilon(\lambda))\right\}=1\,.$$

(ii) *If* $w\in\mathbb{B}^1\cap\mathbb{W}^\#_0$, *then for any* $\varepsilon>0$ *and* $\varepsilon(\lambda)>0$, $\lambda>0$, *satisfying* $\lim_{\lambda\to\infty}\varepsilon(\lambda)=0$ *and* $\lim_{\lambda\to\infty}\lambda\varepsilon(\lambda)=\infty$

$$\lim_{\lambda\to\infty}P^0_{G_\lambda w}\{\rho_{1,\varepsilon}<\underline{X}(e^{\lambda(1-2\widetilde{\alpha}_1)})<\rho_1(-\varepsilon(\lambda))\}=1\,.$$

(iii) *If* $w \in \mathbb{B}^2 \cap \mathbb{W}_0^\#$, *then for any* $\varepsilon > 0$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\rho_{1,\varepsilon} < \underline{X}(e^{\lambda(1-2\widetilde{\alpha}_1)}) < \rho_1(-\varepsilon)\} = 1\,.$$

We use the following proposition to show Theorems 1.8 and 1.10.

PROPOSITION 2.4. *Suppose* $P_{\alpha_1,\alpha_2}\{\mathbb{W}^\#\} = 1$. *Then there exists a subset* $\mathbb{W}_0^\#$ *of* $\mathbb{W}^\#$ *with* $P_{\alpha_1,\alpha_2}\{\mathbb{W}_0^\#\} = 1$ *such that the following* (i)–(iii) *hold.*

(i) *If* $w \in \mathbb{A}^1 \cap \mathbb{W}_0^\#$, *then for any* $\varepsilon > 0$ *and* $\varepsilon(\lambda) > 0,\ \lambda > 0$, *satisfying* $\lim_{\lambda\to\infty}\varepsilon(\lambda) = 0$ *and* $\lim_{\lambda\to\infty}\lambda\varepsilon(\lambda) = \infty$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\rho_2(-\varepsilon(\lambda)) < e^{(\widetilde{\alpha}_1-\widetilde{\alpha}_2)\lambda}\overline{X}(e^{\lambda(1-2\widetilde{\alpha}_1)}) < \rho_{2,\varepsilon}\} = 1\,.$$

(ii) *If* $w \in \mathbb{A}^2 \cap \mathbb{W}_0^\#$, *then for any* $\varepsilon > 0$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\rho_2(-\varepsilon) < e^{(\widetilde{\alpha}_1-\widetilde{\alpha}_2)\lambda}\overline{X}(e^{\lambda(1-2\widetilde{\alpha}_1)}) < \rho_{2,\varepsilon}\} = 1\,.$$

(iii) *If* $w \in \mathbb{B} \cap \mathbb{W}_0^\#$, *then for any* $\varepsilon > 0$ *and* $\varepsilon(\lambda) > 0,\ \lambda > 0$, *satisfying* $\lim_{\lambda\to\infty}\varepsilon(\lambda) = 0$ *and* $\lim_{\lambda\to\infty}\lambda\varepsilon(\lambda) = \infty$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\zeta_2(-\varepsilon(\lambda)) < e^{(\widetilde{\alpha}_1-\widetilde{\alpha}_2)\lambda}\overline{X}(e^{\lambda(1-2\widetilde{\alpha}_1)}) < \zeta_{2,\varepsilon}\} = 1\,.$$

The following proposition, which is a more precise result than Proposition 2.3 (ii)–(iii) for $w \in \mathbb{B}^-$, is used to prove Theorems 1.9 and 1.12.

PROPOSITION 2.5. *Suppose* $\alpha_1 > \alpha_2$ *and* $P_{\alpha_1,\alpha_2}\{\mathbb{W}^\#\} = 1$. *Then there exists a subset* $\mathbb{W}_0^\#$ *of* $\mathbb{W}^\#$ *with* $P_{\alpha_1,\alpha_2}\{\mathbb{W}_0^\#\} = 1$ *such that the following holds*: *for any* $w \in \mathbb{B}^- \cap \mathbb{W}_0^\#$, $\varepsilon > 0$ *and* $M > 0$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{g_{1,\lambda}(\varepsilon, w) < \underline{X}(e^{\lambda(1-2\widetilde{\alpha}_1)}) < g_{2,\lambda}(M, \varepsilon, w)\} = 1\,.$$

The following proposition, which is a more precise result than Proposition 2.4 (i)–(ii) for $w \in \mathbb{A}^-$, is used to prove Theorems 1.8 and 1.13.

PROPOSITION 2.6. *Suppose* $\alpha_1 < \alpha_2$ *and* $P_{\alpha_1,\alpha_2}\{\mathbb{W}^\#\} = 1$. *Then there exists a subset* $\mathbb{W}_0^\#$ *of* $\mathbb{W}^\#$ *with* $P_{\alpha_1,\alpha_2}\{\mathbb{W}_0^\#\} = 1$ *such that the following holds*: *for any* $w \in \mathbb{A}^- \cap \mathbb{W}_0^\#$, $\varepsilon > 0$ *and* $M > 0$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{g_{3,\lambda}(M, \varepsilon, w) < e^{(\widetilde{\alpha}_1-\widetilde{\alpha}_2)\lambda}\overline{X}(e^{\lambda(1-2\widetilde{\alpha}_1)}) < g_{4,\lambda}(\varepsilon, w)\} = 1\,.$$

In Section 3 we prepare lemmas on hitting times of the diffusion process introduced in Section 2. In Section 4 we prove Theorem 1.1 and in Section 5 we prove Theorems 1.6–1.13.

3. Lemmas on hitting times

In this section we show lemmas on hitting times of the process $\{X(t), t \geq 0, P^{x_0}_{G_\lambda w}\}$ by using the methods in [1], [4], [9] and [10]. We set, for $\omega \in \Omega$ and $x \in \mathbb{R}$

$$\tau(x) = \tau(x, \omega) = \inf\{t > 0 : X(t, \omega) = x\}.$$

LEMMA 3.1. (i) *Let* $w \in \mathbb{W}$ *and* $p \leq p_\lambda < 0$ *for all sufficiently large* $\lambda > 0$. *Suppose* $w(p+) \geq w^*(x)$ *for all* $x \in (p, 0)$ *and* $q \equiv \inf\{x > 0 : w(x) > w(p+) + \widetilde{\alpha}_1 - \widetilde{\alpha}_2\} < \infty$. *Then for any* $q' > q$ *and* $f(\lambda) > 0, \lambda > 0$, *satisfying* $\lim_{\lambda\to\infty} \lambda f(\lambda) = \infty$,

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\tau(p_\lambda) < e^{\lambda J_1(\lambda)}\} = 1, \tag{3.1}$$

where

$$J_1(\lambda) = \max\{J_{1,1}(\lambda), J_{1,2}(\lambda)\} + f(\lambda),$$
$$J_{1,1}(\lambda) = \sup_{p_\lambda<x<0} w(x) - \inf_{p_\lambda<x<0} w(x),$$
$$J_{1,2}(\lambda) = \sup_{p_\lambda<x<0} w(x) - \inf_{0<x<q'} w(x) - (\widetilde{\alpha}_1 - \widetilde{\alpha}_2).$$

(ii) *Let* $w \in \mathbb{W}$ *and* $p < 0$. *Suppose* $w(p+) \geq w^*(x)$ *for all* $x \in (p, 0)$ *and* $q \equiv \inf\{x > 0 : w(x) > w(p+) + \widetilde{\alpha}_1 - \widetilde{\alpha}_2\} < \infty$. *Then for any* $\varepsilon > 0$ *and* $q' > q$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\tau(p) < e^{\lambda(J_1+\varepsilon)}\} = 1, \tag{3.2}$$

where

$$J_1 = \max\{J_{1,1}, J_{1,2}\},$$
$$J_{1,1} = w(p+) - \inf_{p<x<0} w(x), \quad J_{1,2} = w(p+) - \inf_{0<x<q'} w(x) - (\widetilde{\alpha}_1 - \widetilde{\alpha}_2).$$

PROOF. First we prove (i). On $(\widetilde{\Omega}, \widetilde{P})$ we define hitting times:

$$\tau(x; x_0, G_\lambda w) = \inf\{t > 0 : X(t; x_0, G_\lambda w) = x\}, \quad x \in \mathbb{R},$$
$$T(x) = \inf\{t > 0 : B(t) = x\}, \quad x \in \mathbb{R}.$$

We let $q' > q$, and $f(\lambda) > 0,\ \lambda > 0$, satisfy $\lim_{\lambda\to\infty} \lambda f(\lambda) = \infty$, and show

$$\lim_{\lambda\to\infty} \widetilde{P}\{\tau(p_\lambda; 0, G_\lambda w) < e^{\lambda J_1(\lambda)}\} = 1, \tag{3.3}$$

which is equivalent to (3.1). For $\lambda > 0$, we set

$$E_\lambda = \{\tau(p_\lambda; 0, G_\lambda w) < \tau(e^{(\widetilde{\alpha}_2-\widetilde{\alpha}_1)\lambda} q'; 0, G_\lambda w)\}.$$

Noting $\sup_{0<x<q'} w(x) > w(p+) + \widetilde{\alpha}_1 - \widetilde{\alpha}_2$, we have

$$\lim_{\lambda\to\infty} \widetilde{P}\{\tau(p; 0, G_\lambda w) < \tau(e^{(\widetilde{\alpha}_2-\widetilde{\alpha}_1)\lambda} q'; 0, G_\lambda w)\} = 1$$

and therefore

$$\lim_{\lambda\to\infty} \widetilde{P}\{E_\lambda\} = 1\,. \tag{3.4}$$

Owing to (2.1) and (2.2), we have

$$\tau(p_\lambda; 0, G_\lambda w) = \int_{p_\lambda}^{\infty} e^{-(G_\lambda w)(x)} L(T(S_{G_\lambda w}(p_\lambda)), S_{G_\lambda w}(x))dx\,. \tag{3.5}$$

On E_λ, $\tau(p_\lambda; 0, G_\lambda w)$ is equal to

$$\begin{aligned}
&\int_{p_\lambda}^{e^{(\widetilde{\alpha}_2-\widetilde{\alpha}_1)\lambda}q'} e^{-(G_\lambda w)(x)} L(T(S_{G_\lambda w}(p_\lambda))\,, S_{G_\lambda w}(x))dx \\
&\stackrel{\mathrm{d}}{=} |S_{G_\lambda w}(p_\lambda)| \int_{p_\lambda}^{e^{(\widetilde{\alpha}_2-\widetilde{\alpha}_1)\lambda}q'} e^{-(G_\lambda w)(x)} L\left(T(-1)\,, \frac{S_{G_\lambda w}(x)}{|S_{G_\lambda w}(p_\lambda)|}\right)dx \\
&= \int_{p_\lambda}^{0} e^{\lambda w(y)}dy \int_{p_\lambda}^{0} e^{-\lambda w(z)} L\left(T(-1)\,, \frac{S_{G_\lambda w}(z)}{|S_{G_\lambda w}(p_\lambda)|}\right)dz \\
&\quad + \int_{p_\lambda}^{0} e^{\lambda w(y)}dy \int_{0}^{q'} e^{-\lambda w(z)} L\left(T(-1)\,, \frac{S_{G_\lambda w}(e^{(\widetilde{\alpha}_2-\widetilde{\alpha}_1)\lambda}z)}{|S_{G_\lambda w}(p_\lambda)|}\right)e^{(\widetilde{\alpha}_2-\widetilde{\alpha}_1)\lambda}dz \\
&\equiv I_\lambda + II_\lambda\,.
\end{aligned} \tag{3.6}$$

We can estimate each of I_λ and II_λ for all sufficiently large $\lambda > 0$ as follows:

$$\begin{cases} I_\lambda \le |p|^2 K(T(-1))e^{\lambda J_{1,1}(\lambda)}\,, & \widetilde{P}\text{-a.s.,} \\ II_\lambda \le |p|q' K(T(-1))e^{\lambda J_{1,2}(\lambda)}\,, & \widetilde{P}\text{-a.s.,} \end{cases} \tag{3.7}$$

where $K(t) = \sup_{x\in\mathbb{R}} L(t, x)$, $t > 0$. Notice $0 < K(T(-1)) < \infty$ ($\widetilde{P}$-a.s.) and set $E'_\lambda = \{K(T(-1)) < e^{\lambda f(\lambda)/2}\}$, $\lambda > 0$. By the assumptions for $f(\lambda)$, $\lambda > 0$, we have

$$\lim_{\lambda\to\infty} \widetilde{P}\{E'_\lambda\} = 1\,. \tag{3.8}$$

Using (3.8), we get

$$\lim_{\lambda\to\infty} \widetilde{P}\{I_\lambda + II_\lambda < e^{\lambda J_1(\lambda)}\} = 1\,. \tag{3.9}$$

By (3.4), (3.6) and (3.9), we obtain (3.3).

As to (ii), besides the assumptions in (i), let $p_\lambda = p$ for all $\lambda > 0$, and $f(\lambda)$, $\lambda > 0$, satisfy $\lim_{\lambda\to\infty} f(\lambda) = 0$. Then we get (ii) from (i). □

The following three lemmas can be proved in the same way as Lemma 3.1.

LEMMA 3.2. (i) *Let $w \in \mathbb{W}$ and $p \le p_\lambda < x_0 < 0$ for all sufficiently large $\lambda > 0$. Suppose $w(p+) \ge w^*(x)$ for all $x \in (p, x_0)$ and $q \equiv \inf\{x > x_0 : w(x) > w(p+)\} < 0$.*

Then for any $q' \in (q, 0)$ *and* $f(\lambda) > 0$, $\lambda > 0$, *satisfying* $\lim_{\lambda\to\infty} \lambda f(\lambda) = \infty$,

$$\lim_{\lambda\to\infty} P^{x_0}_{G_\lambda w}\{\tau(p_\lambda) < e^{\lambda J_2(\lambda)}\} = 1\,, \tag{3.10}$$

where

$$J_2(\lambda) = \sup_{p_\lambda < x < x_0} w(x) - \inf_{p_\lambda < x < q'} w(x) + f(\lambda)\,.$$

(ii) *Let* $w \in \mathbb{W}$ *and* $p < x_0 < 0$. *Suppose* $w(p+) \geq w^*(x)$ *for all* $x \in (p, x_0)$ *and* $q \equiv \inf\{x > x_0 : w(x) > w(p+)\} < 0$. *Then for any* $\varepsilon > 0$

$$\lim_{\lambda\to\infty} P^{x_0}_{G_\lambda w}\{\tau(p) < e^{\lambda(J_2+\varepsilon)}\} = 1\,, \tag{3.11}$$

where

$$J_2 = w(p+) - \inf_{p<x<q} w(x)\,.$$

LEMMA 3.3. (i) *Let* $w \in \mathbb{W}$ *and* $0 < q_\lambda \leq q$ *for all sufficiently large* $\lambda > 0$. *Suppose* $w(q-) \geq w^*(x)$ *for all* $x \in (0, q)$ *and* $p \equiv \sup\{x < 0 : w(x) > w(q-) - (\widetilde{\alpha}_1 - \widetilde{\alpha}_2)\} > -\infty$. *Then for any* $p' < p$ *and* $f(\lambda) > 0$, $\lambda > 0$, *satisfying* $\lim_{\lambda\to\infty} \lambda f(\lambda) = \infty$,

$$\lim_{\lambda\to\infty} P^{0}_{G_\lambda w}\{\tau(e^{(\widetilde{\alpha}_2-\widetilde{\alpha}_1)\lambda} q_\lambda) < e^{\lambda J_3(\lambda)}\} = 1\,, \tag{3.12}$$

where

$$\begin{aligned}
J_3(\lambda) &= \max\{J_{3,1}(\lambda), J_{3,2}(\lambda)\} + f(\lambda)\,,\\
J_{3,1}(\lambda) &= \sup_{0<x<q_\lambda} w(x) - \inf_{p'<x<0} w(x) - (\widetilde{\alpha}_1 - \widetilde{\alpha}_2)\,,\\
J_{3,2}(\lambda) &= \sup_{0<x<q_\lambda} w(x) - \inf_{0<x<q_\lambda} w(x) - 2(\widetilde{\alpha}_1 - \widetilde{\alpha}_2)\,.
\end{aligned}$$

(ii) *Let* $w \in \mathbb{W}$ *and* $0 < q$. *Suppose* $w(q-) \geq w^*(x)$ *for all* $x \in (0, q)$ *and* $p \equiv \sup\{x < 0 : w(x) > w(q-) - (\widetilde{\alpha}_1 - \widetilde{\alpha}_2)\} > -\infty$. *Then for any* $\varepsilon > 0$ *and* $p' < p$

$$\lim_{\lambda\to\infty} P^{0}_{G_\lambda w}\{\tau(e^{(\widetilde{\alpha}_2-\widetilde{\alpha}_1)\lambda} q) < e^{\lambda(J_3+\varepsilon)}\} = 1\,, \tag{3.13}$$

where

$$\begin{aligned}
J_3 &= \max\{J_{3,1}, J_{3,2}\}\,,\\
J_{3,1} &= w(q-) - \inf_{p'<x<0} w(x) - (\widetilde{\alpha}_1 - \widetilde{\alpha}_2)\,,\\
J_{3,2} &= w(q-) - \inf_{0<x<q} w(x) - 2(\widetilde{\alpha}_1 - \widetilde{\alpha}_2)\,.
\end{aligned}$$

LEMMA 3.4. (i) *Let* $w \in \mathbb{W}$ *and* $0 < x_0 < q_\lambda \leq q$ *for all sufficiently large* $\lambda > 0$. *Suppose* $w(q-) \geq w^*(x)$ *for all* $x \in (x_0, q)$ *and* $p \equiv \sup\{x < x_0 : w(x) > w(q-)\} > 0$.

Then for any $p' \in (0, p)$ *and* $f(\lambda) > 0, \lambda > 0$, *satisfying* $\lim_{\lambda\to\infty} \lambda f(\lambda) = \infty$,

$$\lim_{\lambda\to\infty} P_{G_\lambda w}^{e^{(\widetilde{\alpha}_2-\widetilde{\alpha}_1)\lambda}x_0}\{\tau(e^{(\widetilde{\alpha}_2-\widetilde{\alpha}_1)\lambda}q_\lambda) < e^{\lambda J_4(\lambda)}\} = 1\,, \tag{3.14}$$

where

$$J_4(\lambda) = \sup_{x_0<x<q_\lambda} w(x) - \inf_{p'<x<q_\lambda} w(x) - 2(\widetilde{\alpha}_1 - \widetilde{\alpha}_2) + f(\lambda)\,.$$

(ii) *Let* $w \in \mathbb{W}$ *and* $0 < x_0 < q$. *Suppose* $w(q-) \geq w^*(x)$ *for all* $x \in (x_0, q)$ *and* $p \equiv \sup\{x < x_0 : w(x) > w(q-)\} > 0$. *Then for any* $\varepsilon > 0$

$$\lim_{\lambda\to\infty} P_{G_\lambda w}^{e^{(\widetilde{\alpha}_2-\widetilde{\alpha}_1)\lambda}x_0}\{\tau(e^{(\widetilde{\alpha}_2-\widetilde{\alpha}_1)\lambda}q) < e^{\lambda(J_4+\varepsilon)}\} = 1\,, \tag{3.15}$$

where

$$J_4 = w(q-) - \inf_{p<x<q} w(x) - 2(\widetilde{\alpha}_1 - \widetilde{\alpha}_2)\,.$$

LEMMA 3.5. (i) *Let* $w \in \mathbb{W}$ *and* $p < x_0 \leq 0$. *Suppose* $w(p+) > w^*(x)$ *for all* $x \in (p, x_0)$ *and* $w(p+) > w(x_0)$. *Then for any* $\varepsilon > 0$

$$\lim_{\lambda\to\infty} P_{G_\lambda w}^{x_0}\{\tau(p) > e^{\lambda(J_5-\varepsilon)}\} = 1\,, \tag{3.16}$$

where

$$J_5 = w(p+) - \inf_{p<x<x_0} w(x)\,.$$

(ii) *Let* $w \in \mathbb{W}$ *and* $p < x_0 \leq 0$. *Suppose* $w(p) > w^*(x)$ *for all* $x \in (p, x_0)$ *and* $w(p) > w(x_0)$. *Then for any* $\varepsilon_1 > 0$ *and* $\varepsilon > 0$

$$\lim_{\lambda\to\infty} P_{G_\lambda w}^{x_0}\{\tau(p-\varepsilon_1) > e^{\lambda(J_6-\varepsilon)}\} = 1\,, \tag{3.17}$$

where

$$J_6 = \sup_{p-\varepsilon_1<x<p} w(x) - \inf_{p<x<x_0} w(x)\,.$$

(iii) *Let* $w \in \mathbb{W}$ *and* $p < 0 < q$. *Suppose* $\sup_{p<x<0} w(x) > w^*(x) - (\widetilde{\alpha}_1 - \widetilde{\alpha}_2)$ *for all* $x \in [0, q)$. *Then for any* $\varepsilon > 0$

$$\lim_{\lambda\to\infty} P_{G_\lambda w}^{0}\{\tau(p) > e^{\lambda(J_7-\varepsilon)}\} = 1\,, \tag{3.18}$$

where

$$J_7 = \sup_{p<x<0} w(x) - \inf_{0<x<q} w(x) - (\widetilde{\alpha}_1 - \widetilde{\alpha}_2)\,.$$

PROOF. We prove (i) for $x_0 = 0$ and (iii). As in the proof of Lemma 3.1, we observe

$$\tau(p; 0, G_\lambda w) \stackrel{\mathrm{d}}{=} |S_{G_\lambda w}(p)| \int_p^\infty e^{-(G_\lambda w)(x)} L\left(T(-1), \frac{S_{G_\lambda w}(x)}{|S_{G_\lambda w}(p)|}\right) dx \equiv III_\lambda . \quad (3.19)$$

To prove (i) for $x_0 = 0$, we estimate III_λ in (3.19) as follows:

$$III_\lambda \geq \int_p^0 e^{\lambda w(y)} dy \int_p^0 e^{-\lambda w(x)} L\left(T(-1), \frac{S_{G_\lambda w}(x)}{|S_{G_\lambda w}(p)|}\right) dx \equiv IV_\lambda . \quad (3.20)$$

By the assumptions in (i) for $x_0 = 0$, we note that $S_{G_\lambda w}(x)/|S_{G_\lambda w}(p)|$ in IV_λ tends to 0 as $\lambda \to \infty$ uniformly on any closed interval contained in $(p, 0]$. This yields

$$L\left(T(-1), \frac{S_{G_\lambda w}(x)}{|S_{G_\lambda w}(p)|}\right) \to L(T(-1)\,, \;\; 0) > 0\,, \quad \widetilde{P}\text{-a.s.}, \quad (3.21)$$

as $\lambda \to \infty$ uniformly on any closed interval contained in $(p, 0]$. By (3.21) and the classical Laplace method, we get

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \log IV_\lambda = w(p+) - \inf_{p<x<0} w(x) = J_5\,, \quad \widetilde{P}\text{-a.s.},$$

and therefore, for any $\varepsilon > 0$

$$\lim_{\lambda\to\infty} \widetilde{P}\{IV_\lambda > e^{\lambda(J_5-\varepsilon)}\} = 1\,. \quad (3.22)$$

By (3.19), (3.20) and (3.22), we get (3.16) for $x_0 = 0$.

To prove (iii), we estimate III_λ in (3.19) as follows:

$$\begin{aligned} III_\lambda &\geq \int_p^0 e^{\lambda w(y)} dy \int_0^q e^{-\lambda w(z)} L\left(T(-1), \frac{S_{G_\lambda w}(e^{(\widetilde{\alpha}_2-\widetilde{\alpha}_1)\lambda} z)}{|S_{G_\lambda w}(p)|}\right) e^{(\widetilde{\alpha}_2-\widetilde{\alpha}_1)\lambda} dz \\ &\equiv V_\lambda. \end{aligned} \quad (3.23)$$

By the assumptions in (iii), we note that $S_{G_\lambda w}(e^{(\widetilde{\alpha}_2-\widetilde{\alpha}_1)\lambda} z)/|S_{G_\lambda w}(p)|$ in V_λ tends to 0 as $\lambda \to \infty$ uniformly on any closed interval contained in $[0, q)$. From this, we get

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \log V_\lambda = J_7\,, \quad \widetilde{P}\text{-a.s.}$$

in the same way as above. Therefore, for any $\varepsilon > 0$ we have

$$\lim_{\lambda\to\infty} \widetilde{P}\{V_\lambda > e^{\lambda(J_7-\varepsilon)}\} = 1\,. \quad (3.24)$$

By (3.19), (3.23) and (3.24), we obtain (3.18). □

The following lemma can be shown in the same way as Lemma 3.5.

LEMMA 3.6. (i) *Let* $w \in \mathbb{W}$ *and* $0 \le x_0 < q$. *Suppose* $w(q-) > w^*(x)$ *for all* $x \in (x_0, q)$ *and* $w(q-) > w(x_0)$. *Then for any* $\varepsilon > 0$

$$\lim_{\lambda\to\infty} P_{G_\lambda w}^{e^{(\widetilde{\alpha}_2-\widetilde{\alpha}_1)\lambda}x_0}\{\tau(e^{(\widetilde{\alpha}_2-\widetilde{\alpha}_1)\lambda}q) > e^{\lambda(J_8-\varepsilon)}\} = 1\,, \tag{3.25}$$

where

$$J_8 = w(q-) - \inf_{x_0<x<q} w(x) - 2(\widetilde{\alpha}_1 - \widetilde{\alpha}_2).$$

(ii) *Let* $w \in \mathbb{W}$ *and* $0 \le x_0 < q$. *Suppose* $w(q) > w^*(x)$ *for all* $x \in (x_0, q)$ *and* $w(q) > w(x_0)$. *Then for any* $\varepsilon_1 > 0$ *and* $\varepsilon > 0$

$$\lim_{\lambda\to\infty} P_{G_\lambda w}^{e^{(\widetilde{\alpha}_2-\widetilde{\alpha}_1)\lambda}x_0}\{\tau(e^{(\widetilde{\alpha}_2-\widetilde{\alpha}_1)\lambda}(q+\varepsilon_1)) > e^{\lambda(J_9-\varepsilon)}\} = 1\,, \tag{3.26}$$

where

$$J_9 = \sup_{q<x<q+\varepsilon_1} w(x) - \inf_{x_0<x<q} w(x) - 2(\widetilde{\alpha}_1 - \widetilde{\alpha}_2)\,.$$

(iii) *Let* $w \in \mathbb{W}$ *and* $p < 0 < q$. *Suppose* $\sup_{0<x<q} w(x) > w^*(x) + (\widetilde{\alpha}_1 - \widetilde{\alpha}_2)$ *for all* $x \in (p, 0]$. *Then for any* $\varepsilon > 0$

$$\lim_{\lambda\to\infty} P_{G_\lambda w}^{0}\{\tau(e^{(\widetilde{\alpha}_2-\widetilde{\alpha}_1)\lambda}q) > e^{\lambda(J_{10}-\varepsilon)}\} = 1\,, \tag{3.27}$$

where

$$J_{10} = \sup_{0<x<q} w(x) - \inf_{p<x<0} w(x) - (\widetilde{\alpha}_1 - \widetilde{\alpha}_2)\,.$$

4. Proof of Theorem 1.1

In this section we prove Theorem 1.1 by using Proposition 2.2. (We can prove Theorem 1.2 in a similar fashion.) First we prove Proposition 2.2 (i). Proposition 2.2 (ii) can be shown in the same way as (i). To prove Proposition 2.2 (i), we prepare three lemmas. We show them by using the methods in [3], [4] and [9] (see also [10]).

LEMMA 4.1. *Suppose* $P_{\alpha_1,\alpha_2}\{\mathbb{W}^\#\} = 1$. *Then there exists a subset* $\mathbb{W}_0^\#$ *of* $\mathbb{W}^\#$ *with* $P_{\alpha_1,\alpha_2}\{\mathbb{W}_0^\#\} = 1$ *such that the following holds*: *for any* $w \in \mathbb{A} \cap \mathbb{W}_0^\#$ *and sufficiently small* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\lim_{\lambda\to\infty} P_{G_\lambda w}^{0}\{\tau(b_{1,\varepsilon}) < e^{\lambda(1-2\widetilde{\alpha}_1-\delta)}\} = 1\,. \tag{4.1}$$

PROOF. In the case $w(b_1) \le w(b_1+)$, we show (4.1) for sufficiently small $\varepsilon > 0$ satisfying $b_1 - \varepsilon > \zeta_1$. First we let $w \in \mathbb{A}^1$. We define $w_0 \in \mathbb{W}$ by

$$w_0(x) = \begin{cases} w(x) & \text{for } x > b_{1,\varepsilon} , \\ w(b_{1,\varepsilon}+) & \text{for } x = b_{1,\varepsilon} , \\ -x + w(b_{1,\varepsilon}+) + b_{1,\varepsilon} & \text{for } x < b_{1,\varepsilon} , \end{cases}$$

and take $p_0 < b_{1,\varepsilon}$ satisfying

$$M_1 < w_0(p_0) < V_1 + 1 - 2\widetilde{\alpha}_1 = K_1 .$$

We can apply Lemma 3.1 (ii) to $w = w_0$ and $p = p_0$, since $q_0 \equiv \inf\{x > 0 : w_0(x) > w_0(p_0) + \widetilde{\alpha}_1 - \widetilde{\alpha}_2\} \le \rho_2 < \infty$ by Proposition 1.5 (i)(1). By the lemma and

$$K_1 - v_2 - (\widetilde{\alpha}_1 - \widetilde{\alpha}_2) < 1 - 2\widetilde{\alpha}_1 \tag{4.2}$$

which is derived from Proposition 1.5 (i)(2), we have for some $\delta > 0$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w_0}\{\tau(p_0) < e^{\lambda(1-2\widetilde{\alpha}_1-\delta)}\} = 1 .$$

Therefore we get (4.1) in this case.

Next we let $w \in \mathbb{A}^2$. We take $p_k \le u_k, k = 1, 2, \dots, n$, for some integer $n \ge 2$ satisfying $0 > u_1 > \cdots > u_{n-1} > u_n = b_{1,\varepsilon}$,

$$\begin{cases} u_1 \le a_1^- , \\ w_1(p_1) = M_1 , \\ \widehat{J_1} \equiv w_1(p_1) - \inf_{u_1<x<0} w_1(x) < 1 - 2\widetilde{\alpha}_1 , \end{cases} \tag{4.3}$$

and for $k \ge 2$

$$\begin{cases} M_1 > w_k(p_k) \ge w_k^*(x) \quad \text{for all } x \in (p_k, u_{k-1}) , \\ \widehat{J_k} \equiv w_k(p_k) - \inf_{u_k<x<q_k} w_k(x) < 1 - 2\widetilde{\alpha}_1 , \\ q_k \equiv \inf\{x > u_{k-1} : w_k(x) > w_k(p_k)\} (< 0) . \end{cases} \tag{4.4}$$

Here $w_k \in \mathbb{W}$, $1 \le k \le n$, is defined by

$$w_k(x) = \begin{cases} w(x) & \text{for } x > u_k , \\ w(u_k+) & \text{for } x = u_k , \\ -x + w(u_k+) + u_k & \text{for } x < u_k . \end{cases}$$

We can apply Lemma 3.1 (ii) to $w = w_1$ and $p = p_1$ because $q_1 \equiv \inf\{x > 0 : w_1(x) > w_1(p_1) + \widetilde{\alpha}_1 - \widetilde{\alpha}_2\} = \rho_2 < \infty$. From the lemma, we have, for any $\varepsilon_1 > 0$ and $q_1' > q_1$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w_1}\{\tau(p_1) < e^{\lambda(J_1+\varepsilon_1)}\} = 1 ,$$

where $J_1 = \max\{\widehat{J_1}, K_1 - \inf_{0<x<q_1'} w(x) - (\widetilde{\alpha}_1 - \widetilde{\alpha}_2)\}$. Therefore we have

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\tau(u_1) < e^{\lambda(J_1+\varepsilon_1)}\} = 1 . \tag{4.5}$$

Taking $q_1'(> q_1)$ sufficiently close to q_1, we have $J_1 < 1 - 2\widetilde{\alpha}_1$ because of (4.2) and (4.3). Moreover, for any $k \geq 2$ and $\varepsilon_k > 0$ we have, by (4.4) and Lemma 3.2 (ii),

$$\lim_{\lambda\to\infty} P^{u_{k-1}}_{G_\lambda w_k}\{\tau(p_k) < e^{\lambda(\widehat{J}_k+\varepsilon_k)}\} = 1 .$$

Therefore we have

$$\lim_{\lambda\to\infty} P^{u_{k-1}}_{G_\lambda w}\{\tau(u_k) < e^{\lambda(\widehat{J}_k+\varepsilon_k)}\} = 1 . \tag{4.6}$$

By (4.5), (4.6) and the strong Markov property, we obtain (4.1) in this case, too. □

LEMMA 4.2. *Suppose* $P_{\alpha_1,\alpha_2}\{\mathbb{W}^\#\} = 1$. *Then there exists a subset* $\mathbb{W}_0^\#$ *of* $\mathbb{W}^\#$ *with* $P_{\alpha_1,\alpha_2}\{\mathbb{W}_0^\#\} = 1$ *such that the following holds*: *for any* $w \in \mathbb{A} \cap \mathbb{W}_0^\#$ *and* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\tau(\zeta_{1,\varepsilon}) > e^{\lambda(1-2\widetilde{\alpha}_1+\delta)}\} = 1 . \tag{4.7}$$

PROOF. It is enough to show (4.7) for sufficiently small $\varepsilon > 0$ satisfying $\inf\{w_*(x) : \zeta_{1,\varepsilon} < x < \zeta_1\} > V_1$. (By [6], we note that the set of $w \in \mathbb{A}$ for which there is no $\varepsilon > 0$ satisfying this is P_{α_1,α_2}-negligible.) First we let $w \in \mathbb{A}^1$. In the case $w(\zeta_1(\varepsilon)) \leq w(\zeta_1(\varepsilon)+)$, we can apply Lemma 3.5 (i) to $p = \zeta_1(\varepsilon)$ and $x_0 = 0$, and have $J_5 > 1 - 2\widetilde{\alpha}_1$ in the lemma. Therefore we get (4.7) in this case. In the case $w(\zeta_1(\varepsilon)) > w(\zeta_1(\varepsilon)+)$, we can apply Lemma 3.5 (ii) to $p = \zeta_1(\varepsilon)$ and $x_0 = 0$, and have $J_6 > 1 - 2\widetilde{\alpha}_1$ for any $\varepsilon_1 > 0$ in the lemma. Therefore we obtain (4.7) in this case, too.

Next we let $w \in \mathbb{A}^2$. In this case we have

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\tau(u_{n-1}) < e^{\lambda(1-2\widetilde{\alpha}_1-\delta_0)}\} = 1 \tag{4.8}$$

for some $\delta_0 > 0$ by the proof of Lemma 4.1. In the case $w(\zeta_1(\varepsilon)) \leq w(\zeta_1(\varepsilon)+)$, we can apply Lemma 3.5 (i) to $p = \zeta_1(\varepsilon)$ and $x_0 = u_{n-1}$, and have $J_5 > 1 - 2\widetilde{\alpha}_1$ in the lemma. Therefore we get, for some $\delta_1 > 0$

$$\lim_{\lambda\to\infty} P^{u_{n-1}}_{G_\lambda w}\{\tau(\zeta_1(\varepsilon)) > e^{\lambda(1-2\widetilde{\alpha}_1+\delta_1)}\} = 1 . \tag{4.9}$$

Using (4.8), (4.9) and the strong Markov property, we obtain (4.7) in this case. In the case $w(\zeta_1(\varepsilon)) > w(\zeta_1(\varepsilon)+)$, we can apply Lemma 3.5 (ii) to $p = \zeta_1(\varepsilon)$ and $x_0 = u_{n-1}$, and have $J_6 > 1 - 2\widetilde{\alpha}_1$ for any $\varepsilon_1 > 0$ in the lemma. Therefore we obtain, for some $\delta_2 > 0$

$$\lim_{\lambda\to\infty} P^{u_{n-1}}_{G_\lambda w}\{\tau(\zeta_1(\varepsilon) - \varepsilon) > e^{\lambda(1-2\widetilde{\alpha}_1+\delta_2)}\} = 1 . \tag{4.10}$$

By (4.8) and (4.10), we get (4.7) in this case, too. □

LEMMA 4.3. *Suppose* $P_{\alpha_1,\alpha_2}\{\mathbb{W}^\#\} = 1$. *Then there exists a subset* $\mathbb{W}_0^\#$ *of* $\mathbb{W}^\#$ *with* $P_{\alpha_1,\alpha_2}\{\mathbb{W}_0^\#\} = 1$ *such that the following holds*: *for any* $w \in \mathbb{W}_0^\#$ *and* $\varepsilon > 0$ *satisfying*

$\rho_2(\varepsilon) < \infty$ *there exists* $\delta > 0$ *such that*

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\tau(e^{(\widetilde{\alpha}_2-\widetilde{\alpha}_1)\lambda}\rho_{2,\varepsilon}) > e^{\lambda(1-2\widetilde{\alpha}_1+\delta)}\} = 1\,. \tag{4.11}$$

PROOF. Since

$$\sup_{0<x<\rho_{2,\varepsilon}} w(x) > K_1 + \widetilde{\alpha}_1 - \widetilde{\alpha}_2 \geq w^*(x) + \widetilde{\alpha}_1 - \widetilde{\alpha}_2 \quad \text{for all } x \in (\zeta_1, 0]\,,$$

we can apply Lemma 3.6 (iii) to $p = \zeta_1$ and $q = \rho_{2,\varepsilon}$, and have $J_{10} > K_1 - V_1 \geq 1 - 2\widetilde{\alpha}_1$ in the lemma. Therefore we get (4.11). □

Let us prove Proposition 2.2 (i) by making use of the coupling method in [6] (see also [9]).

PROOF OF PROPOSITION 2.2 (i). Let $w \in \mathbb{A}$. In the case $w(b_1) \leq w(b_1+)$, it is enough to show this proposition for sufficiently small $\varepsilon > 0$ satisfying $b_1 - \varepsilon > \zeta_1$. We take $\varepsilon_1 > 0$ satisfying $\inf\{w_*(x) : \zeta_{1,\varepsilon_1} < x < \zeta_1\} > V_1$. (By [6], the set of $w \in \mathbb{A}$ for which there is no $\varepsilon_1 > 0$ satisfying this is P_{α_1,α_2}-negligible.) Moreover, in the case $K_1 + \widetilde{\alpha}_1 - \widetilde{\alpha}_2 > 0$ we take $\varepsilon_2 > 0$ satisfying $\rho_2(\varepsilon_2) < \infty$ and $\inf\{w_*(x) : \rho_2 < x < \rho_{2,\varepsilon_2}\} > v_2$. In the case $K_1 + \widetilde{\alpha}_1 - \widetilde{\alpha}_2 < 0$ we note $\rho_2 = v_2 = 0$ and therefore $V_1 - \widetilde{\alpha}_1 + \widetilde{\alpha}_2 < 0$ by Proposition 1.5 (i)(3). In this case we take $\varepsilon_2 > 0$ satisfying

$$\begin{cases} K_1 + \widetilde{\alpha}_1 - \widetilde{\alpha}_2 + \varepsilon_2 < 0\,, \\ (0 \geq) \inf\{w_*(x) : 0 \leq x < \varepsilon_2\} > V_1 - \widetilde{\alpha}_1 + \widetilde{\alpha}_2\,. \end{cases} \tag{4.12}$$

For $\lambda > 0$, we set $D_\lambda = [\zeta_{1,\varepsilon_1}, e^{(\widetilde{\alpha}_2-\widetilde{\alpha}_1)\lambda}\rho_{2,\varepsilon_2}]$ and define m_λ, a probability measure on D_λ, by

$$m_\lambda(dx) = \frac{e^{-(G_\lambda w)(x)}dx}{\int_{D_\lambda} e^{-(G_\lambda w)(y)}dy}\,.$$

This is the invariant probability measure for the reflecting $\mathcal{L}_{G_\lambda w}$-diffusion process on D_λ. We have

$$m_\lambda\{U_\varepsilon(\mathbf{b}_1) \cap (b_{1,\varepsilon}, 0)\} = \frac{h_1(\lambda)}{h_1(\lambda) + h_2(\lambda) + h_3(\lambda)}\,, \tag{4.13}$$

where

$$h_1(\lambda) = \int_{U_\varepsilon(\mathbf{b}_1)\cap(b_{1,\varepsilon},0)} e^{-\lambda w(x)}dx\,,$$

$$h_2(\lambda) = \int_{(\zeta_{1,\varepsilon_1},0)\setminus(U_\varepsilon(\mathbf{b}_1)\cap(b_{1,\varepsilon},0))} e^{-\lambda w(x)}dx\,,$$

$$h_3(\lambda) = e^{(\widetilde{\alpha}_2-\widetilde{\alpha}_1)\lambda}\int_0^{\rho_{2,\varepsilon_2}} e^{-\lambda w(x)}dx\,.$$

We can estimate the limiting behaviors of $h_1(\lambda)$ and $h_2(\lambda)$ as $\lambda \to \infty$ as follows:

$$\lim_{\lambda\to\infty} \lambda^{-1} \log h_1(\lambda) = -V_1\,, \tag{4.14}$$

$$\lim_{\lambda\to\infty} \lambda^{-1} \log h_2(\lambda) < -V_1. \tag{4.15}$$

As to the limiting behavior of $h_3(\lambda)$, in the case $K_1 + \widetilde{\alpha}_1 - \widetilde{\alpha}_2 > 0$ we have

$$\lim_{\lambda\to\infty} \lambda^{-1} \log h_3(\lambda) = \widetilde{\alpha}_2 - \widetilde{\alpha}_1 - v_2 < -V_1 \tag{4.16}$$

by Proposition 1.5 (i)(3). In the case $K_1 + \widetilde{\alpha}_1 - \widetilde{\alpha}_2 < 0$ we note $\rho_{2,\varepsilon_2} = \varepsilon_2$ and

$$\lim_{\lambda\to\infty} \lambda^{-1} \log h_3(\lambda) = \widetilde{\alpha}_2 - \widetilde{\alpha}_1 - \inf\{w_*(x) : 0 \le x < \varepsilon_2\} < -V_1 \tag{4.17}$$

by (4.12). Therefore, by (4.13)–(4.17), we get

$$\lim_{\lambda\to\infty} m_\lambda\{U_\varepsilon(\mathbf{b}_1) \cap (b_{1,\varepsilon}, 0)\} = 1\,. \tag{4.18}$$

Let $\{X_\lambda^{(R)}(t), t \ge 0\}$ denote the reflecting $\mathcal{L}_{G_\lambda w}$-diffusion process on D_λ with initial distribution m_λ defined on $(\widetilde{\Omega}, \widetilde{P})$. Since this is a stationary process, we have, for any $t \ge 0$

$$\lim_{\lambda\to\infty} \widetilde{P}\{X_\lambda^{(R)}(t) \in U_\varepsilon(\mathbf{b}_1) \cap (b_{1,\varepsilon}, 0)\} = 1 \tag{4.19}$$

by (4.18). We couple $\{X(t; 0, G_\lambda w), t \ge 0\}$ and $\{X_\lambda^{(R)}(t), t \ge 0\}$ as follows ([6]): these processes move independently until they first meet each other; after the first meeting time they move together until they go out from $(\zeta_{1,\varepsilon_1}, e^{(\widetilde{\alpha}_2-\widetilde{\alpha}_1)\lambda}\rho_{2,\varepsilon_2})$; after going out from the interval they move independently again. We set

$$\begin{aligned} \sigma_\lambda &= \inf\{t > 0 : X(t; 0, G_\lambda w) = X_\lambda^{(R)}(t)\}\,, \\ \sigma'_\lambda &= \inf\{t > \sigma_\lambda : X(t; 0, G_\lambda w) \notin (\zeta_{1,\varepsilon_1}, e^{(\widetilde{\alpha}_2-\widetilde{\alpha}_1)\lambda}\rho_{2,\varepsilon_2})\}\,. \end{aligned}$$

By virtue of (4.19), we have

$$\lim_{\lambda\to\infty} \widetilde{P}\{\sigma_\lambda < \tau(b_{1,\varepsilon}; 0, G_\lambda w)\} = 1\,. \tag{4.20}$$

Combining (4.20) with Lemma 4.1, we get, for some $\delta_0 > 0$

$$\lim_{\lambda\to\infty} \widetilde{P}\{\sigma_\lambda < e^{\lambda(1-2\widetilde{\alpha}_1-\delta_0)}\} = 1\,. \tag{4.21}$$

Moreover, by Lemmas 4.2 and 4.3, we have for some $\delta_1 > 0$

$$\lim_{\lambda\to\infty} \widetilde{P}\{\sigma'_\lambda > e^{\lambda(1-2\widetilde{\alpha}_1+\delta_1)}\} = 1\,. \tag{4.22}$$

By (4.19), (4.21) and (4.22), we arrive at

$$\lim_{\lambda\to\infty} \widetilde{P}\{X(e^{\lambda(1-2\widetilde{\alpha}_1)}; 0, G_\lambda w) \in U_\varepsilon(\mathbf{b}_1) \cap (b_{1,\varepsilon}, 0)\} = 1$$

in the same way as [6] (see also [9]), which is the desired result. □

Now we prove Theorem 1.1 by using Proposition 2.2 (cf. [1], [3], [9]).

PROOF OF THEOREM 1.1. First we prove (i). By Lemma 2.1, we observe, for any $w \in \mathbb{W}$, $\varepsilon > 0$ and $\lambda > 0$

$$p^-_{1,\lambda,\varepsilon}(w) = \widehat{p}^-_{1,\lambda,\varepsilon}(\tau_\lambda w_\lambda)\,, \tag{4.23}$$

where

$$\widehat{p}^-_{1,\lambda,\varepsilon}(w) = P^0_{G_\lambda w}\{X(e^{\lambda(1-2\widetilde{\alpha}_1)}) \in U_\varepsilon(\mathbf{b}_1(w)) \cap (b_{1,\varepsilon}(w), 0)\}\,.$$

By (4.23) and (1.1), we have

$$P_{\alpha_1,\alpha_2}\{\mathbb{E}^-_{1,\lambda,\varepsilon} \cap \mathbb{A}_\lambda\} = P_{\alpha_1,\alpha_2}\{\widehat{\mathbb{E}}^-_{1,\lambda,\varepsilon} \cap \mathbb{A}\}\,, \tag{4.24}$$

where

$$\widehat{\mathbb{E}}^-_{1,\lambda,\varepsilon} = \{w \in \mathbb{W}^\# : \widehat{p}^-_{1,\lambda,\varepsilon}(w) > 1-\varepsilon\}\,.$$

By Proposition 2.2 (i), $\lim_{\lambda\to\infty} \widehat{p}^-_{1,\lambda,\varepsilon}(w) = 1$ for any $w \in \mathbb{A} \cap \mathbb{W}^\#_0$, where $\mathbb{W}^\#_0$ is a subset of $\mathbb{W}^\#$ satisfying $P_{\alpha_1,\alpha_2}\{\mathbb{W}^\#_0\} = 1$. Therefore the right-hand side of (4.24) converges to $P_{\alpha_1,\alpha_2}\{\mathbb{A}\}$ as $\lambda \to \infty$ and we obtain (i).

Next we prove (ii). In the same way as above, we have

$$P_{\alpha_1,\alpha_2}\{\mathbb{E}^+_{1,\lambda,\varepsilon} \cap \mathbb{B}_\lambda\} = P_{\alpha_1,\alpha_2}\{\widehat{\mathbb{E}}^+_{1,\lambda,\varepsilon} \cap \mathbb{B}\}\,, \tag{4.25}$$

where

$$\widehat{\mathbb{E}}^+_{1,\lambda,\varepsilon} = \{w \in \mathbb{W}^\# : \widehat{p}^+_{1,\lambda,\varepsilon}(w) > 1-\varepsilon\}\,,$$
$$\widehat{p}^+_{1,\lambda,\varepsilon}(w) = P^0_{G_\lambda w}\{0 < X(e^{\lambda(1-2\widetilde{\alpha}_1)}) < \varepsilon\}\,.$$

By Proposition 2.2 (ii), we have, in the case $w \in \mathbb{B} \cap \mathbb{W}^\#_0$ for some subset $\mathbb{W}^\#_0$ of $\mathbb{W}^\#$ satisfying $P_{\alpha_1,\alpha_2}\{\mathbb{W}^\#_0\} = 1$,

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{0 < X(e^{\lambda(1-2\widetilde{\alpha}_1)}) < e^{\lambda(\widetilde{\alpha}_2-\widetilde{\alpha}_1)} b_{2,\varepsilon}\} = 1\,,$$

and therefore $\lim_{\lambda\to\infty} \widehat{p}^+_{1,\lambda,\varepsilon}(w) = 1$. From this, we observe that the right-hand side of (4.25) converges to $P_{\alpha_1,\alpha_2}\{\mathbb{B}\}$ as $\lambda \to \infty$ and get (ii).

As to (iii), we have

$$P_{\alpha_1,\alpha_2}\{\mathbb{E}^+_{1,\lambda,M,\varepsilon} \cap \mathbb{B}_\lambda\} = P_{\alpha_1,\alpha_2}\{\widehat{\mathbb{E}}^+_{1,\lambda,M,\varepsilon} \cap \mathbb{B}\}\,, \tag{4.26}$$

where

$$\widehat{\mathbb{E}}^+_{1,\lambda,M,\varepsilon} = \{w \in \mathbb{W}^\# : \widehat{p}^+_{1,\lambda,M}(w) > 1-\varepsilon\}\,,$$

$$\widehat{p}^{+}_{1,\lambda,M}(w) = P^0_{G_\lambda w}\{X(e^{\lambda(1-2\widetilde{\alpha}_1)}) > M\}\,.$$

By Proposition 2.2 (ii), we have, in the case $w \in \mathbb{B} \cap \mathbb{W}^{\#}_0$ for some subset $\mathbb{W}^{\#}_0$ of $\mathbb{W}^{\#}$ satisfying $P_{\alpha_1,\alpha_2}\{\mathbb{W}^{\#}_0\} = 1$,

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{X(e^{\lambda(1-2\widetilde{\alpha}_1)}) > e^{\lambda(\widetilde{\alpha}_2-\widetilde{\alpha}_1)}(b_2^- - \varepsilon)\} = 1\,,$$

where $b_2^- \equiv \min \mathbf{b}_2$ and $\varepsilon \in (0, b_2^-)$. Therefore, in this case, we get $\lim_{\lambda\to\infty} \widehat{p}^{+}_{1,\lambda,M}(w) = 1$. Hence the right-hand side of (4.26) converges to $P_{\alpha_1,\alpha_2}\{\mathbb{B}\}$ as $\lambda \to \infty$ and we obtain (iii). □

5. Proofs of Theorems 1.6–1.13

We can show Theorems 1.6 and 1.10 by using, respectively, Propositions 2.3 and 2.4 in the same way as proving Theorem 1.1 (i) by using Proposition 2.2 (i). We can also show Theorems 1.12 and 1.13 by using, respectively, Propositions 2.5 and 2.6 in a similar fashion. (For these, see also the proof of Theorem 1.8 in the end of this section.) Proposition 2.3 (i) is obtained from Lemma 4.2 and the following lemma. Note that, in Proposition 2.3 (i), $\zeta_1(-\varepsilon(\lambda)) = \zeta_1$ for all sufficiently large $\lambda > 0$ in the case $w(\zeta_1+) < V_1+1-2\widetilde{\alpha}_1$. Proposition 2.4 (iii) can be shown in the same way as Proposition 2.3 (i).

LEMMA 5.1. *Suppose $P_{\alpha_1,\alpha_2}\{\mathbb{W}^{\#}\} = 1$. Then there exists a subset $\mathbb{W}^{\#}_0$ of $\mathbb{W}^{\#}$ with $P_{\alpha_1,\alpha_2}\{\mathbb{W}^{\#}_0\} = 1$ such that the following* (i)–(ii) *hold.*

(i) *For any $w \in \mathbb{A} \cap \mathbb{W}^{\#}_0$ satisfying $w(\zeta_1+) < V_1 + 1 - 2\widetilde{\alpha}_1$ there exists $\delta > 0$ such that*

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\tau(\zeta_1) < e^{\lambda(1-2\widetilde{\alpha}_1-\delta)}\} = 1\,.$$

(ii) *For any $w \in \mathbb{A} \cap \mathbb{W}^{\#}_0$ satisfying $w(\zeta_1+) = V_1 + 1 - 2\widetilde{\alpha}_1$ and any $\varepsilon(\lambda) > 0, \lambda > 0$, satisfying $\lim_{\lambda\to\infty} \varepsilon(\lambda) = 0$ and $\lim_{\lambda\to\infty} \lambda\varepsilon(\lambda) = \infty$*

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\tau(\zeta_1(-\varepsilon(\lambda))) < e^{\lambda(1-2\widetilde{\alpha}_1-\varepsilon(\lambda)/2)}\} = 1\,. \tag{5.1}$$

PROOF. We can prove (i) in the same way as proving Lemma 4.1. To prove (ii), first we let $w \in \mathbb{A}^1$ and $w(\zeta_1+) = V_1 + 1 - 2\widetilde{\alpha}_1$. In this case we can apply Lemma 3.1 (i) to $p = \zeta_1$, $p_\lambda = \zeta_1(-\varepsilon(\lambda))$ and $f(\lambda) = \varepsilon(\lambda)/2$ because $q \equiv \inf\{x > 0 : w(x) > w(\zeta_1+) + \widetilde{\alpha}_1 - \widetilde{\alpha}_2\} = \rho_2 < \infty$. We note $\sup_{\zeta_1(-\varepsilon(\lambda))<x<0} w(x) \le V_1 + 1 - 2\widetilde{\alpha}_1 - \varepsilon(\lambda) = K_1 - \varepsilon(\lambda)$ for all sufficiently large $\lambda > 0$. Moreover, in the case $K_1 + \widetilde{\alpha}_1 - \widetilde{\alpha}_2 > 0$, by taking $q'(> \rho_2 > 0)$ sufficiently close to ρ_2 we have $\inf_{0<x<q'} w(x) = v_2$. As to the case $K_1 + \widetilde{\alpha}_1 - \widetilde{\alpha}_2 < 0$, we notice $\rho_2 = v_2 = 0$ and $V_1 - \widetilde{\alpha}_1 + \widetilde{\alpha}_2 < 0$. In this case, for sufficiently small $q' > 0$ we have $(0 \ge) \inf_{0<x<q'} w(x) > V_1 - \widetilde{\alpha}_1 + \widetilde{\alpha}_2$. Therefore in both cases we have (3.1) for $J_1(\lambda) \le 1 - 2\widetilde{\alpha}_1 - \varepsilon(\lambda)/2$, which yields (5.1).

Next we let $w \in \mathbb{A}^2$ and $w(\zeta_1+) = V_1+1-2\widetilde{\alpha}_1$. In this case we have (4.8) for some $\delta_0 > 0$, and can apply Lemma 3.2 (i) to $p = \zeta_1$, $p_\lambda = \zeta_1(-\varepsilon(\lambda))$, $x_0 = u_{n-1}$ and $f(\lambda) = \varepsilon(\lambda)/4$

because $q \equiv \inf\{x > u_{n-1} : w(x) > w(\zeta_1+)\} < 0$. Noting $\sup_{\zeta_1(-\varepsilon(\lambda))<x<u_{n-1}} w(x) \le V_1 + 1-2\widetilde{\alpha}_1-\varepsilon(\lambda)$ for all sufficiently large $\lambda > 0$, we have (3.10) for $J_2(\lambda) \le 1-2\widetilde{\alpha}_1-(3/4)\varepsilon(\lambda)$. Combining this with (4.8), we get (5.1) in this case, too. □

We obtain Proposition 2.4 (i)–(ii) from Lemma 4.3 and the following lemma. Note that $\rho_2(\varepsilon) < \infty$ for $w \in \mathbb{A}$ and sufficiently small $\varepsilon > 0$. We can show Proposition 2.3 (ii)–(iii) in the same way as Proposition 2.4 (i)–(ii).

LEMMA 5.2. *Suppose* $P_{\alpha_1,\alpha_2}\{\mathbb{W}^\#\} = 1$. *Then there exists a subset* $\mathbb{W}_0^\#$ *of* $\mathbb{W}^\#$ *with* $P_{\alpha_1,\alpha_2}\{\mathbb{W}_0^\#\} = 1$ *such that the following* (i)–(ii) *hold.*

(i) *For any* $w \in \mathbb{A}^1 \cap \mathbb{W}_0^\#$ *and any* $\varepsilon(\lambda) > 0, \lambda > 0$, *satisfying* $\lim_{\lambda\to\infty} \varepsilon(\lambda) = 0$ *and* $\lim_{\lambda\to\infty} \lambda\varepsilon(\lambda) = \infty$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\tau(e^{(\widetilde{\alpha}_2-\widetilde{\alpha}_1)\lambda}\rho_2(-\varepsilon(\lambda))) < e^{\lambda(1-2\widetilde{\alpha}_1-\varepsilon(\lambda)/2)}\} = 1 . \tag{5.2}$$

(ii) *For any* $w \in \mathbb{A}^2 \cap \mathbb{W}_0^\#$ *and* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\tau(e^{(\widetilde{\alpha}_2-\widetilde{\alpha}_1)\lambda}\rho_2(-\varepsilon)) < e^{\lambda(1-2\widetilde{\alpha}_1-\delta)}\} = 1 . \tag{5.3}$$

PROOF. It is enough to show this lemma just in the case $K_1 + \widetilde{\alpha}_1 - \widetilde{\alpha}_2 > 0$. To prove (i), we first let $w \in \mathbb{A}^1$ and $w(\rho_2-) = K_1 + \widetilde{\alpha}_1 - \widetilde{\alpha}_2$. In this case we can apply Lemma 3.3 (i) to $q_\lambda = \rho_2(-\varepsilon(\lambda)), q = \rho_2$ and $f(\lambda) = \varepsilon(\lambda)/2$, since $p \equiv \sup\{x < 0 : w(x) > w(\rho_2-) - (\widetilde{\alpha}_1 - \widetilde{\alpha}_2)\} = \sup\{x < 0 : w(x) > V_1 + 1 - 2\widetilde{\alpha}_1\} > -\infty$. By taking $p'(< p)$ sufficiently close to p, we have $\inf_{p'<x<0} w(x) = V_1$. (By [6], we notice that the set of $w \in \mathbb{A}^1$ for which $p = -\infty$ or there is no p' satisfying this property is P_{α_1,α_2}-negligible.) Moreover, we note $\sup_{0<x<\rho_2(-\varepsilon(\lambda))} w(x) \le K_1 + \widetilde{\alpha}_1 - \widetilde{\alpha}_2 - \varepsilon(\lambda)$ and $\inf_{0<x<\rho_2(-\varepsilon(\lambda))} w(x) \ge v_2$. Therefore we have (3.12) for $J_3(\lambda) \le 1 - 2\widetilde{\alpha}_1 - \varepsilon(\lambda)/2$, which yields (5.2). In the case $w \in \mathbb{A}^1$ and $w(\rho_2-) < K_1 + \widetilde{\alpha}_1 - \widetilde{\alpha}_2$, we can prove (5.2) by using the method of proving Lemma 4.1 for $w \in \mathbb{A}^1$.

To prove (ii), we let $w \in \mathbb{A}^2$. It is enough to show (ii) for sufficiently small $\varepsilon > 0$ satisfying $K_1 + \widetilde{\alpha}_1 - \widetilde{\alpha}_2 - \varepsilon > 0$. In the case $w(\rho_2(-\varepsilon)-) = K_1 + \widetilde{\alpha}_1 - \widetilde{\alpha}_2 - \varepsilon$, we can apply Lemma 3.3 (ii) to $q = \rho_2(-\varepsilon)$, since $\overline{p} \equiv \sup\{x < 0 : w(x) > w(\rho_2(-\varepsilon)-) - (\widetilde{\alpha}_1 - \widetilde{\alpha}_2)\} = \sup\{x < 0 : w(x) > M_1 - \varepsilon\} \ge a_1^+ > -\infty$. Noting $M_1 - \inf_{p'<x<0} w(x) < 1 - 2\widetilde{\alpha}_1$ for $p'(< \overline{p})$ sufficiently close to $\overline{p}$, we obtain (5.3) in this case. In the case $w(\rho_2(-\varepsilon)-) < K_1 + \widetilde{\alpha}_1 - \widetilde{\alpha}_2 - \varepsilon$, we can show (5.3) by using the method of proving Lemma 4.1 for $w \in \mathbb{A}^1$. □

Next we show Proposition 2.6 and then prove Theorem 1.8 by using Propositions 2.4 and 2.6. We can prove Proposition 2.5 and Theorem 1.9 in a similar fashion. To prove Proposition 2.6, we prepare a lemma.

LEMMA 5.3. *Let* $w \in \mathbb{W}$, $\alpha_1 < \alpha_2$ *and* $p < 0$.

(i) *Suppose* $a < \sup_{p<x\le 0} w(x) \wedge (\widetilde{\alpha}_2 - \widetilde{\alpha}_1)$. *Then*

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\tau(p) > \tau(e^{a\lambda})\} = 1\,.$$

(ii) *Suppose* $\sup_{p<x\le 0} w(x) < a < \widetilde{\alpha}_2 - \widetilde{\alpha}_1$. *Then*

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\tau(p) < \tau(e^{a\lambda})\} = 1\,.$$

PROOF. First we prove (i). We observe

$$P^0_{G_\lambda w}\{\tau(p) < \tau(e^{a\lambda})\} = \frac{g(\lambda)}{\int_p^0 e^{\lambda w(x)}dx + g(\lambda)}\,, \tag{5.4}$$

where

$$g(\lambda) = g(\lambda, w) = e^{\lambda(\widetilde{\alpha}_2-\widetilde{\alpha}_1)}\int_0^{e^{\lambda(a+\widetilde{\alpha}_1-\widetilde{\alpha}_2)}} e^{\lambda w(y)}dy\,.$$

Since, for any $\varepsilon > 0$, we notice $e^{\lambda(a+\widetilde{\alpha}_1-\widetilde{\alpha}_2)} < \varepsilon$ for all sufficiently large $\lambda > 0$, we have

$$g(\lambda) \le \exp\{\lambda(a + \sup_{0<y<\varepsilon} w(y))\} \tag{5.5}$$

for all sufficiently large $\lambda > 0$. By the classical Laplace method and (5.5), we have, for any $\varepsilon' > 0$

$$\frac{\int_p^0 e^{\lambda w(x)}dx}{g(\lambda)} \ge \exp\left\{\lambda\left(\sup_{p<x<0} w(x) - \varepsilon' - a - \sup_{0<y<\varepsilon} w(y)\right)\right\} \tag{5.6}$$

for all sufficiently large $\lambda > 0$. The right-hand side of (5.6) tends to ∞ as $\lambda \to \infty$ for sufficiently small $\varepsilon > 0$ and $\varepsilon' > 0$. Therefore we obtain (i).

Next we show (ii). In the same way as above, we have, for any $\varepsilon > 0$ and $\varepsilon' > 0$

$$\frac{\int_p^0 e^{\lambda w(x)}dx}{g(\lambda)} \le \exp\left\{\lambda\left(\sup_{p<x<0} w(x) + \varepsilon' - a - \inf_{0<y<\varepsilon} w(y)\right)\right\} \tag{5.7}$$

for all sufficiently large $\lambda > 0$. Choosing $\varepsilon > 0$ and $\varepsilon' > 0$ sufficiently small, we observe that the right-hand side of (5.7) converges to 0 as $\lambda \to \infty$ and obtain (ii). □

PROOF OF PROPOSITION 2.6. We prove this proposition by using the method of proving [3, Lemma 6.1]. First we let $w \in \mathbb{A}^-$. We note $H_1 \le K_1 < \widetilde{\alpha}_2 - \widetilde{\alpha}_1$ and, for sufficiently small $\varepsilon > 0$

$$\sup_{\zeta_1(-\varepsilon/2)<x\le 0} w(x) \ge H_1 - \varepsilon/2 > H_1 - \varepsilon\,.$$

Therefore, by Lemma 5.3 (i), we have

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\tau(\zeta_1(-\varepsilon/2)) > \tau(e^{\lambda(H_1-\varepsilon)})\} = 1\,. \tag{5.8}$$

Combining (5.8) with Lemma 5.1, we have

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\tau(e^{\lambda(H_1-\varepsilon)}) < e^{\lambda(1-2\widetilde{\alpha}_1-\varepsilon(\lambda)/2)}\} = 1\,,$$

where $\varepsilon(\lambda) > 0$, $\lambda > 0$, satisfies the assumptions in Lemma 5.1 (ii). Therefore we get

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\overline{X}(e^{\lambda(1-2\widetilde{\alpha}_1)}) > e^{\lambda(H_1-\varepsilon)}\} = 1\,. \tag{5.9}$$

From this, we notice, for any $M > 0$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\overline{X}(e^{\lambda(1-2\widetilde{\alpha}_1)}) > M\} = 1\,. \tag{5.10}$$

By (5.9) and (5.10), we obtain, for any $\varepsilon > 0$ and $M > 0$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{e^{(\widetilde{\alpha}_1-\widetilde{\alpha}_2)\lambda}\overline{X}(e^{\lambda(1-2\widetilde{\alpha}_1)}) > g_{3,\lambda}(M,\varepsilon,w)\} = 1\,. \tag{5.11}$$

Next we let $w \in \mathbb{A}^-$ satisfy $H_{1,\varepsilon} + \varepsilon < \widetilde{\alpha}_2 - \widetilde{\alpha}_1$ for all sufficiently small $\varepsilon > 0$. Then, by Lemma 5.3 (ii), we have

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\tau(\zeta_{1,\varepsilon}) < \tau(e^{\lambda(H_{1,\varepsilon}+\varepsilon)})\} = 1\,. \tag{5.12}$$

Combining (5.12) with Lemma 4.2, we have

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\tau(e^{\lambda(H_{1,\varepsilon}+\varepsilon)}) > e^{\lambda(1-2\widetilde{\alpha}_1+\delta)}\} = 1$$

for some $\delta > 0$. Therefore we get

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\overline{X}(e^{\lambda(1-2\widetilde{\alpha}_1)}) < e^{\lambda(H_{1,\varepsilon}+\varepsilon)}\} = 1\,. \tag{5.13}$$

On the other hand, by Proposition 2.4 (i)–(ii), we have, for any $w \in \mathbb{A}^-$ and $\varepsilon > 0$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\overline{X}(e^{\lambda(1-2\widetilde{\alpha}_1)}) < \rho_{2,\varepsilon}e^{\lambda(\widetilde{\alpha}_2-\widetilde{\alpha}_1)}\} = 1\,. \tag{5.14}$$

Therefore, by (5.13) and (5.14), we get, for any $w \in \mathbb{A}^-$ and $\varepsilon > 0$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{e^{(\widetilde{\alpha}_1-\widetilde{\alpha}_2)\lambda}\overline{X}(e^{\lambda(1-2\widetilde{\alpha}_1)}) < g_{4,\lambda}(\varepsilon,w)\} = 1\,. \tag{5.15}$$

Combining (5.11) and (5.15), we obtain the desired result. □

Now we prove Theorem 1.8 (cf. [1], [3]).

PROOF OF THEOREM 1.8. First we prove (i). For any $\varepsilon > 0$ we observe, by Lemma 2.1 and (1.1),

$$\begin{aligned}\mathcal{P}^0_\lambda\{0 < e^{-\widetilde{\alpha}_1\lambda}\overline{X}(e^\lambda) < \varepsilon\} &= \int P^0_{w_\lambda}\{0 < e^{-\widetilde{\alpha}_1\lambda}\overline{X}(e^\lambda) < \varepsilon\}P_{\alpha_1,\alpha_2}(dw)\\ &= \int P^0_{G_\lambda(\tau_\lambda w_\lambda)}\{0 < \overline{X}(e^{\lambda(1-2\widetilde{\alpha}_1)}) < \varepsilon\}P_{\alpha_1,\alpha_2}(dw)\\ &= \int P^0_{G_\lambda w}\{0 < \overline{X}(e^{\lambda(1-2\widetilde{\alpha}_1)}) < \varepsilon\}P_{\alpha_1,\alpha_2}(dw)\,.\end{aligned}$$

To prove (i), it is enough to show, for almost all $w \in \mathbb{W}^\#$ (with respect to P_{α_1,α_2})

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{0 < \overline{X}(e^{\lambda(1-2\widetilde{\alpha}_1)}) < \varepsilon\} = 1\,. \tag{5.16}$$

By Proposition 2.4 and noting, for sufficiently small $\varepsilon' > 0$, $\lim_{\lambda\to\infty} e^{(\widetilde{\alpha}_2-\widetilde{\alpha}_1)\lambda}\rho_{2,\varepsilon'} = 0$ in the case $w \in \mathbb{A}$ and $\lim_{\lambda\to\infty} e^{(\widetilde{\alpha}_2-\widetilde{\alpha}_1)\lambda}\zeta_{2,\varepsilon'} = 0$ in the case $w \in \mathbb{B}$, we get (5.16). Therefore we obtain (i).

Next we show (ii). In the same way as above we have, for any $M > 0$

$$\mathcal{P}^0_\lambda\{e^{-\widetilde{\alpha}_1\lambda}\overline{X}(e^\lambda) > M\} = \int P^0_{G_\lambda w}\{\overline{X}(e^{\lambda(1-2\widetilde{\alpha}_1)}) > M\}P_{\alpha_1,\alpha_2}(dw)\,.$$

To prove (ii), it is enough to show, for almost all $w \in \mathbb{W}^\#$ (with respect to P_{α_1,α_2})

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\overline{X}(e^{\lambda(1-2\widetilde{\alpha}_1)}) > M\} = 1\,. \tag{5.17}$$

In the case $w \in \mathbb{A}\backslash\mathbb{A}^-$, we notice $\rho_2(-\varepsilon) > 0$ for sufficiently small $\varepsilon > 0$. (The set of $w \in \mathbb{A}$ for which there is no $\varepsilon > 0$ satisfying this is P_{α_1,α_2}-negligible.) Therefore, in this case, by Proposition 2.4 (i)–(ii) and noting $\lim_{\lambda\to\infty} e^{(\widetilde{\alpha}_2-\widetilde{\alpha}_1)\lambda}\rho_2(-\varepsilon) = \infty$, we get (5.17). In the case $w \in \mathbb{A}^-$, we have (5.17) by Proposition 2.6. In the case $w \in \mathbb{B}$, we get (5.17) by Proposition 2.4 (iii) and noting $\lim_{\lambda\to\infty} e^{(\widetilde{\alpha}_2-\widetilde{\alpha}_1)\lambda}\zeta_2(-\varepsilon) = \infty$ for sufficiently small $\varepsilon > 0$. Hence we obtain (ii). □

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