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Precise Asymptotic Formulae for the First Hitting Times of Bessel Processes

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Abstract. We study the first hitting time to b of a Bessel process with index ν starting from a, which is denoted by $\tau_{a,b}^{(\nu)}$, in the case when $0 < b < a$. When $\nu > 1$ and $\nu - 1/2$ is not an integer, we obtain that $\mathbf{P}(t < \tau_{a,b}^{(\nu)} < \infty)$ is asymptotically equal to $\kappa_1^{(\nu)} t^{-\nu} + \kappa_2^{(\nu)} t^{-\nu-1}$ as $t \to \infty$ for some explicit constants $\kappa_1^{(\nu)}$ and $\kappa_2^{(\nu)}$. The constant $\kappa_1^{(\nu)}$ is known and the aim is to get $\kappa_2^{(\nu)}$. Combining our result with the known facts, we obtain the precise asymptotic formula for every index ν.

1. Introduction and main result

For $\nu \in \mathbf{R}$ let $R^{(\nu)} = \{R_t^{(\nu)}\}_{t \geqq 0}$ be a Bessel process with index ν defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ starting from a fixed point $a > 0$ and denote by $\tau_{a,b}^{(\nu)}$ the first hitting time to b of $R^{(\nu)}$,

$$\tau_{a,b}^{(\nu)} = \inf\{t > 0\,;\ R_t^{(\nu)} = b\} \quad (\inf \emptyset = \infty)\,.$$

The Laplace transform of the distribution of $\tau_{a,b}^{(\nu)}$ is obtained by solving an eigenvalue problem and it is expressed in the form of a ratio of the modified Bessel functions ([3, 11, 12]).

In this paper we are concerned with the asymptotic behavior of the tail probability $\mathbf{P}(\tau_{a,b}^{(\nu)} > t)$ or $\mathbf{P}(t < \tau_{a,b}^{(\nu)} < \infty)$ as $t \to \infty$ when $0 < b < a$. This case is interesting because we need to consider the natural boundary ∞. Since

$$\mathbf{P}(t < \tau_{a,b}^{(\nu)} < \infty) = \left(\frac{b}{a}\right)^{2\nu} \mathbf{P}(\tau_{a,b}^{(-\nu)} > t)$$

for $\nu > 0$ ([10, 14]), we only consider the case where $\nu \geqq 0$.

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While the first hitting times of diffusion processes are quite classical objects, some explicit expressions for the distributions for the Bessel processes have recently been found and the asymptotic behavior has been also studied. See [1, 7, 8, 15, 16, 17]. In the next section we recall the results in [7], on which the arguments in this article are based. It should be mentioned here that this study is applied to a study on heat conductance ([17]) and the Wiener sausage ([5, 6, 9]).

For the asymptotic behavior of $\mathbf{P}\big(t < \tau_{a,b}^{(\nu)} < \infty\big)$ it has been shown in [8] that

$$\mathbf{P}\big(t < \tau_{a,b}^{(\nu)} < \infty\big) = \kappa_1^{(\nu)} t^{-\nu} + o(t^{-\nu-\varepsilon})$$

for any $\varepsilon \in (0, \nu/(1+\nu))$, where the constant $\kappa_1^{(\nu)}$ is given by

$$\kappa_1^{(\nu)} = \Big(\frac{b^3}{2a}\Big)^{\nu} \Big\{\Big(\frac{a}{b}\Big)^{\nu} - \Big(\frac{b}{a}\Big)^{\nu}\Big\} \frac{1}{\Gamma(\nu+1)} .$$

Recently Hariya [10] has shown a precise estimate for $\nu \leqq 1$ and have that when $\nu < 1$,

$$\mathbf{P}\big(t < \tau_{a,b}^{(\nu)} < \infty\big) = \kappa_1^{(\nu)} t^{-\nu} + \xi_\nu t^{-2\nu} + o(t^{-2\nu}) ,$$

where

$$\xi_\nu = \frac{b^{2\nu}\kappa_1^{(\nu)}}{2^{\nu}\Gamma(\nu+1)} \Big\{1 - \nu \int_1^{\infty} \frac{(x+1)^{2\nu} - x^{2\nu}}{x^{\nu+1}} dx\Big\} ,$$

and when $\nu = 1$,

$$\mathbf{P}\big(t < \tau_{a,b}^{(1)} < \infty\big) = \kappa_1^{(1)} t^{-1} - \frac{b^4}{a^2}\frac{a^2 - b^2}{2} t^{-2}\log t + o(t^{-2}\log t) . \tag{1}$$

In the case where $\nu > 1$, the following estimate is also given in [10]:

$$\begin{aligned} -\infty &< \liminf_{t\to\infty} t^{\nu+1}\big\{\mathbf{P}\big(t < \tau_{a,b}^{(\nu)} < \infty\big) - \kappa_1^{(\nu)} t^{-\nu}\big\} \\ &\leqq \limsup_{t\to\infty} t^{\nu+1}\big\{\mathbf{P}\big(t < \tau_{a,b}^{(\nu)} < \infty\big) - \kappa_1^{(\nu)} t^{-\nu}\big\} < 0 \end{aligned} \tag{2}$$

and this estimate is used in the following argument.

Moreover, Chiba [2] has shown a similar precise estimate when $\nu - 1/2$ is an integer and $R^{(\nu)}$ is identical in law with the radial part of $(2\nu+2)$-dimensional standard Brownian motion.

The purpose of this paper is to give the precise estimate for the tail probability $\mathbf{P}(t < \tau_{a,b}^{(\nu)} < \infty)$ when $\nu > 1$ and $\nu - 1/2$ is not an integer. Our argument is applicable for other cases, but we concentrate on this case. Combining all the results, we obtain the precise estimates for all the Bessel processes.

From now on, we use

$$c = \frac{a}{b}$$

for simplicity. The following is the main theorem of this paper.

THEOREM 1. *If $\nu > 1$ and $\nu - 1/2$ is not an integer, then*

$$\mathbf{P}\big(t < \tau_{a,b}^{(\nu)} < \infty\big) = \kappa_1^{(\nu)} t^{-\nu} - \kappa_2^{(\nu)} t^{-\nu-1} + o(t^{-\nu-1})$$

holds as $t \to \infty$, where the constant $\kappa_2^{(\nu)}$ is given by

$$\kappa_2^{(\nu)} = \frac{1}{\Gamma(\nu)}\Big(\frac{b^2}{2}\Big)^{\nu+1}\Big(\frac{c^2 - c^{-2\nu}}{\nu+1} + \frac{c^{-2\nu+2} - 2c^{-2\nu} + 1}{\nu - 1}\Big).$$

The proof is based on the representation for the distribution function of $\tau_{a,b}^{(\nu)}$ given in [7] and the estimates for it shown in [1] and [10].

By a similar method, we can prove the identity (1) when $\nu = 1$ and

$$\begin{aligned}\mathbf{P}(t < \tau_{a,b}^{(\nu)} < \infty) =& \Big(\frac{b^3}{2a}\Big)^{\nu} \frac{c^{\nu} - c^{-\nu}}{\Gamma(\nu+1)} t^{-\nu} \\ &+ \Big(\frac{b^5}{a}\Big)^{\nu} \frac{B(\nu, 1-\nu)}{B(\nu, 1/2)} \frac{(c^{\nu} - c^{-\nu})\cos(\pi\nu)}{\Gamma(1+\nu)^2} t^{-2\nu} + o(t^{-2\nu})\end{aligned}$$

when $0 < \nu < 1$ and $\nu \neq 1/4, 1/2, 3/4$. Since Chiba [2] and Hariya [10] have studied these cases, we omit the details.

2. Preliminaries

In this section we recall some known facts concerning $\tau_{a,b}^{(\nu)}$ and modified Bessel functions. We assume that $\nu > 1$ and $\nu - 1/2$ is not an integer.

The Laplace transform of $\tau_{a,b}^{(\nu)}$ is given by

$$\mathbf{E}[e^{-\lambda\tau_{a,b}^{(\nu)}}] = \frac{a^{-\nu} K_{\nu}(a\sqrt{2\lambda})}{b^{-\nu} K_{\nu}(b\sqrt{2\lambda})}$$

for $\lambda > 0$ when $a > b > 0$, where $\mathbf{E}$ denotes the expectation with respect to $\mathbf{P}$ and K_{ν} is the modified Bessel function of the second kind or the Macdonald function. We need the explicit expression for the distribution of $\tau_{a,b}^{(\nu)}$ given in [7], obtained by inverting the Laplace transform.

It is known that, if $\nu > 3/2$, the function K_{ν} has finite number of simple complex zeros, which we denote by $z_{\nu,j}$ ($j = 1, 2, \dots, N(\nu)$), and that $\mathrm{Re}\, z_{\nu,j} < 0$ for each j. The complex conjugate of a zero is also a zero and the number $N(\nu)$ is $\nu - 1/2$ if $\nu - 1/2$ is an integer and is the even number closest to $\nu - 1/2$ otherwise. Thus we have $N(\nu) = 2$ when $3/2 < \nu < 7/2$, $N(\nu) = 4$ when $7/2 < \nu < 11/2$ and so on. If $\nu - 1/2$ is an odd integer, K_{ν} has a negative zero and the situation is different. See [9, 13, 18] about the modified Bessel functions.

It is shown in [7] by some residue calculus that, if $1 < \nu < 3/2$,

$$\mathbf{P}(\tau_{a,b}^{(\nu)} > t) = 1 - c^{-2\nu}(1 - \Psi_1(t)) + c^{-\nu}\cos(\pi\nu)\Psi_3^{(\nu)}(t)$$

and, if $\nu > 3/2$,

$$\mathbf{P}\big(\tau_{a,b}^{(\nu)} > t\big) = 1 - c^{-2\nu}(1 - \Psi_1(t)) + c^{-\nu} \sum_{j=1}^{N(\nu)} \frac{K_\nu(cz_{\nu,j})}{z_{\nu,j} K_{\nu+1}(z_{\nu,j})} \Psi_2(t; z_{\nu,j}) + c^{-\nu} \cos(\pi\nu) \Psi_3^{(\nu)}(t)\,,$$

where the functions Ψ_1, Ψ_2 and $\Psi_3^{(\nu)}$ are given by

$$\Psi_1(t) = \sqrt{\frac{2}{\pi}} \int_0^{\frac{a-b}{\sqrt{t}}} e^{-\frac{1}{2}u^2} du\,,$$

$$\Psi_2(t; z) = \sqrt{\frac{2}{\pi}} \int_{\frac{a-b}{\sqrt{t}}}^{\infty} e^{-\frac{1}{2}u^2 + \frac{z\sqrt{t}u}{b}} du, \quad \operatorname{Re} z < 0\,,$$

$$\Psi_3^{(\nu)}(t) = \sqrt{\frac{2}{\pi}} \int_{\frac{a-b}{\sqrt{t}}}^{\infty} e^{-\frac{1}{2}u^2} du \int_0^\infty \frac{L_c^{(\nu)}(x)}{x} e^{-\frac{x\sqrt{t}u}{b}} dx\,,$$

$$L_c^{(\nu)}(x) = \frac{I_\nu(cx)K_\nu(x) - I_\nu(x)K_\nu(cx)}{K_\nu(x)^2 + \pi^2 I_\nu(x)^2 + 2\pi \sin(\pi\nu) K_\nu(x) I_\nu(x)}, \quad x > 0\,.$$

Here I_ν has denoted the modified Bessel function of the first kind. We mention that $L_c^{(\nu)}(x) > 0$ for any $x > 0$, which can be derived by the monotonicity of I_ν and K_ν. Note that $\mathbf{P}\big(\tau_{a,b}^{(\nu)} = \infty\big) = 1 - c^{-2\nu}$. Hence, what we should consider is that

$$\mathbf{P}\big(t < \tau_{a,b}^{(\nu)} < \infty\big) = c^{-2\nu}\Psi_1(t) + c^{-\nu} \cos(\pi\nu) \Psi_3^{(\nu)}(t)$$

for $\nu \in (1, 3/2)$ and that

$$\mathbf{P}\big(t < \tau_{a,b}^{(\nu)} < \infty\big) = c^{-2\nu}\Psi_1(t) + c^{-\nu} \sum_{j=1}^{N(\nu)} \frac{K_\nu(cz_{\nu,j})}{z_{\nu,j} K_{\nu+1}(z_{\nu,j})} \Psi_2(t; z_{\nu,j}) + c^{-\nu} \cos(\pi\nu) \Psi_3^{(\nu)}(t)$$

for $\nu > 3/2$.

Next we recall some estimates for the tail probability in [1] and [10]. It is shown in [1] that, for any $\nu > 0$, there exists a constant $C_1^{(\nu)}$ such that

$$\mathbf{P}\big(t < \tau_{a,b}^{(\nu)} < \infty\big) \leqq C_1^{(\nu)} t^{-\nu}\,. \tag{3}$$

On the other hand, (2) yields that, if $\nu > 1$, there exists a constant $C_2^{(\nu)}$ such that

$$\Big|\mathbf{P}\big(t < \tau_{a,b}^{(\nu)} < \infty\big) - \kappa_1^{(\nu)} t^{-\nu}\Big| \leqq C_2^{(\nu)} t^{-\nu-1} \tag{4}$$

for $t \geqq 1$.

3. Proof of the theorem

The proof of the theorem follows from the asymptotic formulae for the functions Ψ_1, Ψ_2 and $\Psi_3^{(\nu)}$. For $\mu \in \mathbf{R}$ we set $m(\mu) = [\mu - 1/2]$, the largest integer less than or equal to $\mu - 1/2$. Note that

$$m(\mu) + \frac{1}{2} \leqq \mu < m(\mu) + \frac{3}{2}, \quad m(\mu+1) = m(\mu) + 1 .$$

We write

$$\alpha_1(m) = \frac{\phi_m (a-b)^{2m+1}}{2m+1}$$

and

$$\alpha_2(m; z) = -(2m)!\, \phi_m \Big(\frac{b}{z}\Big)^{2m+1} e^{z(c-1)} \sum_{k=0}^{2m} \frac{\{-z(c-1)\}^k}{k!}$$

for $m \geqq 0$ and $\operatorname{Re} z < 0$, where

$$\phi_m = \sqrt{\frac{2}{\pi}} \Big(-\frac{1}{2}\Big)^m \frac{1}{m!} .$$

The following lemma gives limiting behavior of Ψ_1 and Ψ_2, which is shown in [7].

LEMMA 1. *We have that*

$$\Psi_1(t) = \sum_{m=0}^{m(\nu+1)} \alpha_1(m) t^{-m-1/2} + O(t^{-m(\nu)-5/2}) ,$$

$$\Psi_2(t; z) = \sum_{m=0}^{m(\nu+1)} \alpha_2(m; z) t^{-m-1/2} + O(t^{-m(\nu)-5/2})$$

as $t \to \infty$.

For simplicity we let

$$\rho_\nu = \frac{c^\nu - c^{-\nu}}{2^{2\nu-1} \nu \Gamma(\nu)^2}, \quad \beta_1^{(\nu)} = -\frac{\phi_{m(\nu+1)} \rho_\nu b^{2\nu} \Gamma(2\nu)(a-b)^{2m(\nu)-2\nu+3}}{2m(\nu) - 2\nu + 3} ,$$

$$\beta_2^{(\nu)} = b^{2m(\nu)+3} \phi_{m(\nu+1)} \int_0^\infty \frac{L_c^{(\nu)}(x) - \rho_\nu x^{2\nu}}{x^{2m(\nu)+4}} dx \int_{(c-1)x}^\infty v^{2m(\nu)+2} e^{-v} dv .$$

We need to give a large time asymptotics of $\Psi_3^{(\nu)}$ and the following lemma is important to derive the main terms of $\mathbf{P}\big(t < \tau_{a,b}^{(\nu)} < \infty\big)$.

LEMMA 2. *Set*

$$\alpha_3^{(\nu)}(m) = (2m)!\,\phi_m b^{2m+1} \sum_{k=0}^{2m} \frac{(c-1)^k}{k!} \int_0^\infty \frac{L_c^{(\nu)}(x)}{x^{2m-k+2}} e^{-x(c-1)} dx$$

for $m = 0, 1, \dots, m(\nu)$ *and* $\alpha_3^{(\nu)}(m(\nu+1)) = \beta_1^{(\nu)} + \beta_2^{(\nu)}$. *Then we have that*

$$\Psi_3^{(\nu)}(t) = \sum_{m=0}^{m(\nu+1)} \alpha_3^{(\nu)}(m) t^{-m-1/2} + \frac{c^\nu \kappa_1^{(\nu)}}{\cos(\pi\nu)} t^{-\nu} - \frac{c^\nu \kappa_2^{(\nu)}}{\cos(\pi\nu)} t^{-\nu-1} + o(t^{-\nu-1})$$

as $t \to \infty$.

We postpone the proof of Lemma 2 to the next section, and we here give a proof of Theorem 1 admitting the lemma as proved. For $m = 0, 1, \dots, m(\nu+1)$ we set

$$\gamma^{(\nu)}(m) = c^{-2\nu}\alpha_1(m) + c^{-\nu}\cos(\pi\nu)\alpha_3^{(\nu)}(m)$$

if $1 < \nu < 3/2$ and

$$\gamma^{(\nu)}(m) = c^{-2\nu}\alpha_1(m) + c^{-\nu} \sum_{j=1}^{N(\nu)} \frac{K_\nu(cz_{\nu,j})}{z_{\nu,j} K_{\nu+1}(z_{\nu,j})} \alpha_2(m; z_{\nu,j}) + c^{-\nu}\cos(\pi\nu)\alpha_3^{(\nu)}(m)$$

if $\nu > 3/2$. By Lemmas 1 and 2 we have

$$\mathbf{P}\big(t < \tau_{a,b}^{(\nu)} < \infty\big) = \sum_{m=0}^{m(\nu+1)} \gamma^{(\nu)}(m) t^{-m-1/2} + \kappa_1^{(\nu)} t^{-\nu} - \kappa_2^{(\nu)} t^{-\nu-1} + o(t^{-\nu-1}) . \tag{5}$$

From (3) we see that each coefficient of $t^{-m-1/2}$ is zero for $m = 0, 1, ..., m(\nu)$. Then, by (4), the coefficient of $t^{-m(\nu)-3/2}$ is zero. Therefore, the first term of the right hand side of (5) is zero and we obtain the assertion of the theorem.

4. Proof of Lemma 2

In this section we give a proof of Lemma 2 to complete the proof of the theorem. We set

$$P^{(\mu)}(x) = \sqrt{\frac{2}{\pi}} e^{-\frac{1}{2}x^2} - \sum_{m=0}^{m(\mu)} \phi_m x^{2m}$$

for $\mu > 1/2$. Note that there exists a suitable constant $C_3^{(\mu)}$ such that

$$|P^{(\mu)}(x)| \leqq C_3^{(\mu)} x^{2m(\mu)} \min\{x^2, 1\} . \tag{6}$$

We write

$$\Psi_3^{(\nu)}(t) = \eta_1^{(\nu)}(t) + \eta_2^{(\nu)}(t) + \eta_3^{(\nu)}(t) ,$$

where

$$\eta_1^{(\nu)}(t) = \sum_{m=0}^{m(\nu)} \phi_m \int_0^\infty \frac{L_c^{(\nu)}(x)}{x} dx \int_{\frac{a-b}{\sqrt{t}}}^\infty u^{2m} e^{-\frac{x\sqrt{t}u}{b}} du\,,$$

$$\eta_2^{(\nu)}(t) = \int_{\frac{a-b}{\sqrt{t}}}^\infty P^{(\nu)}(u) du \int_0^\infty \rho_\nu x^{2\nu-1} e^{-\frac{x\sqrt{t}u}{b}} dx\,,$$

$$\eta_3^{(\nu)}(t) = \int_{\frac{a-b}{\sqrt{t}}}^\infty P^{(\nu)}(u) du \int_0^\infty \left\{\frac{L_c^{(\nu)}(x)}{x} - \rho_\nu x^{2\nu-1}\right\} e^{-\frac{x\sqrt{t}u}{b}} dx\,.$$

For $\eta_1^{(\nu)}(t)$ elementary computations give

$$\int_{\frac{a-b}{\sqrt{t}}}^\infty u^{2m} e^{-\frac{x\sqrt{t}u}{b}} du = \int_0^\infty \left(u + \frac{a-b}{\sqrt{t}}\right)^{2m} e^{-(c-1)x} e^{-\frac{x\sqrt{t}u}{b}} du$$

$$= e^{-(c-1)x}(2m)!b^{2m+1} \sum_{k=0}^{2m} \frac{(c-1)^k}{k!} \frac{1}{x^{2m-k+1}} t^{-m-1/2}\,,$$

which yields

$$\eta_1^{(\nu)}(t) = \sum_{m=0}^{m(\nu)} \alpha_3^{(\nu)}(m) t^{-m-1/2}\,.$$

To estimate the second term $\eta_2^{(\nu)}(t)$, we prepare the following lemma.

LEMMA 3. *Let $\mu > 1/2$ and $\mu - 1/2 \notin \mathbf{Z}$. We have that*

$$\int_0^\infty u^{-2p} P^{(\mu)}(u) du = \frac{1}{2^p\sqrt{\pi}} \Gamma\left(\frac{1}{2} - p\right), \tag{7}$$

$$\Gamma(2p) \int_0^\infty u^{-2p} P^{(\mu)}(u) du = \frac{2^{p-1}\Gamma(p)}{\cos(\pi p)} \tag{8}$$

for $m(\mu) + 1/2 < p < m(\mu) + 3/2$.

PROOF. We can deduce (7) directly from

$$\int_0^\infty x^{q-1}\left\{e^{-x} + \sum_{m=1}^{n} \frac{(-1)^m}{(m-1)!} x^{m-1}\right\} dx = \Gamma(q)$$

for $-n < q < -n+1$, $n = 0, 1, 2, \dots$ (cf. [4, p. 361]).

Recall the formulae

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z)\Gamma\left(z + \frac{1}{2}\right), \quad \Gamma\left(z + \frac{1}{2}\right)\Gamma\left(\frac{1}{2} - z\right) = \frac{\pi}{\cos(\pi z)}$$

(cf. [13, p. 3]). Then we can easily see that (7) gives (8). □

It follows from (8) that $\eta_2^{(\nu)}(t)$ is equal to

$$\frac{\rho_\nu b^{2\nu} 2^{\nu-1}\Gamma(\nu)}{\cos(\pi\nu)} t^{-\nu} - \rho_\nu b^{2\nu}\Gamma(2\nu) t^{-\nu} \int_0^{\frac{a-b}{\sqrt{t}}} u^{-2\nu} P^{(\nu)}(u) du \,. \tag{9}$$

A simple calculation shows that the first term of (9) is

$$\frac{1}{\cos\pi\nu}\Big(\frac{b^2}{2}\Big)^\nu \frac{c^\nu - c^{-\nu}}{\Gamma(\nu+1)} t^{-\nu} = \frac{c^\nu \kappa_1^{(\nu)}}{\cos(\pi\nu)} t^{-\nu} \,.$$

Since $m(\nu+1) = m(\nu) + 1$ and

$$P^{(\nu+1)}(u) = P^{(\nu)}(u) - \phi_{m(\nu+1)} u^{2m(\nu+1)} \,, \tag{10}$$

the second term of (9) is the sum of the following two integrals:

$$\rho_\nu b^{2\nu}\Gamma(2\nu) t^{-\nu} \int_0^{\frac{a-b}{\sqrt{t}}} u^{-2\nu} P^{(\nu+1)}(u) du \,, \tag{11}$$

$$\rho_\nu b^{2\nu} \phi_{m(\nu+1)} \Gamma(2\nu) t^{-\nu} \int_0^{\frac{a-b}{\sqrt{t}}} u^{2m(\nu)-2\nu+2} du \,. \tag{12}$$

The estimate (6) yields that (11) is dominated by

$$\rho_\nu b^{2\nu}\Gamma(2\nu) t^{-\nu} C_3^{(\nu+1)} \int_0^{\frac{a-b}{\sqrt{t}}} u^{-2\nu} u^{2(m(\nu+1)+1)} du = O(t^{-m(\nu)-5/2}) \,.$$

Moreover, since (12) is

$$\frac{\phi_{m(\nu+1)}\rho_\nu b^{2\nu}\Gamma(2\nu)(a-b)^{2m(\nu)-2\nu+3}}{2m(\nu)-2\nu+3} t^{-m(\nu)-3/2} \,,$$

we obtain

$$\eta_2^{(\nu)}(t) = \frac{c^\nu \kappa_1^{(\nu)}}{\cos(\pi\nu)} t^{-\nu} - \beta_1^{(\nu)} t^{-m(\nu)-3/2} + O(t^{-m(\nu)-5/2}) \,.$$

To show the asymptotic behavior of $\eta_3^{(\nu)}(t)$, we need an estimate for the function $L_c^{(\nu)}(x)$. For this purpose we prepare a precise estimate for the Macdonald function $K_\nu(x)$ as $x \downarrow 0$.

LEMMA 4. *If* $\nu > 3/2$ *and* $\nu - 1/2$ *is not an integer*,

$$K_\nu(x) = \frac{\Gamma(\nu)}{2}\Big(\frac{2}{x}\Big)^\nu \Big\{1 - \frac{x^2}{4(\nu-1)} + O(x^3)\Big\}$$

and, *if* $1 < \nu < 3/2$,

$$K_\nu(x) = \frac{\Gamma(\nu)}{2}\Big(\frac{2}{x}\Big)^\nu \Big\{1 - \frac{x^2}{4(\nu-1)} + O(x^{2\nu})\Big\} \,.$$

PROOF. The first assertion is shown in Lemma 4.3 in [9] and we here give a proof in the case where $1 < \nu < 3/2$.

We start from the formula ([13, 18]):

$$K_\nu(x) = \frac{\sqrt{\pi}}{(2x)^\nu} \frac{e^{-x}}{\Gamma(\nu+1/2)} \int_0^\infty e^{-y} y^{2\nu-1} \Big(1 + \frac{2x}{y}\Big)^{\nu-1/2} dy, \quad \nu > -\frac{1}{2}.$$

A simple calculation shows that

$$\begin{aligned}\int_0^\infty e^{-y} y^{2\nu-1} \Big(1 + \frac{2x}{y}\Big)^{\nu-1/2} dy =& \Gamma(2\nu) + \Gamma(2\nu)x + \frac{2\nu-3}{4(\nu-1)} \Gamma(2\nu)x^2 \\ &+ \int_0^\infty e^{-y} y^{2\nu-1} f_\nu\Big(\frac{2x}{y}\Big) dy,\end{aligned}$$

where

$$f_\nu(z) = (1+z)^{\nu-1/2} - 1 - \Big(\nu - \frac{1}{2}\Big)z - \frac{1}{2}\Big(\nu - \frac{1}{2}\Big)\Big(\nu - \frac{3}{2}\Big)z^2$$

for $z > 0$.

Since $1/2 < \nu - 1/2 < 1$, we have

$$\lim_{z\to\infty} \frac{f_\nu(z)}{z^2} = -\frac{1}{2}\Big(\nu - \frac{1}{2}\Big)\Big(\nu - \frac{3}{2}\Big)$$

and hence there exists a positive constant $C_4^{(\nu)}$ such that $|f_\nu(z)| \leqq C_4^{(\nu)} z^2$ for $z \geqq 1$. On the other hand, it is easy to see $f_\nu(z) = O(z^3)$ as $z \downarrow 0$. From this observation we can conclude

$$\int_0^\infty e^{-y} y^{2\nu-1} f_\nu\Big(\frac{2x}{y}\Big) dy = O(x^{2\nu})$$

as $x \downarrow 0$. □

REMARK 1. It may be worthwhile to note that

$$K_1(x) = \frac{1}{x}\Big\{1 - \frac{1}{2}x^2 \log\frac{1}{x} + O(x^2)\Big\}$$

and, if $0 < \nu < 1$,

$$K_\nu(x) = \frac{\Gamma(\nu)}{2}\Big(\frac{2}{x}\Big)^\nu \Big\{1 - \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\Big(\frac{x}{2}\Big)^{2\nu} + O(x^2)\Big\}$$

as $x \downarrow 0$.

From Lemma 4 we obtain the asymptotic behavior of $L_c^{(\nu)}(x)$ as $x \downarrow 0$. Let

$$\zeta_\nu = \frac{1}{4(c^\nu - c^{-\nu})}\Big(\frac{c^{\nu+2} - c^{-\nu}}{\nu+1} + \frac{c^{-\nu+2} - c^\nu}{\nu-1}\Big) + \frac{1}{2(\nu-1)}.$$

LEMMA 5. *We have that, if* $\nu > 3/2$ *and* $\nu - 1/2$ *is not an integer,*

$$L_c^{(\nu)}(x) = \rho_\nu x^{2\nu}\{1 + \zeta_\nu x^2 + O(x^3)\}$$

and that, if $1 < \nu < 3/2$,

$$L_c^{(\nu)}(x) = \rho_\nu x^{2\nu}\{1 + \zeta_\nu x^2 + O(x^{2\nu})\}\,.$$

PROOF. We only give a proof for the first assertion. The second assertion can be proved in the same manner.

By the series representation for $I_\nu(x)$ we have

$$\begin{aligned} I_\nu(x) &= \Big(\frac{x}{2}\Big)^\nu \Big\{\frac{1}{\Gamma(\nu+1)} + \frac{1}{\Gamma(\nu+2)}\Big(\frac{x}{2}\Big)^2 + O(x^4)\Big\} \\ &= \frac{1}{\Gamma(\nu+1)}\Big(\frac{x}{2}\Big)^\nu \Big\{1 + \frac{1}{4(\nu+1)}x^2 + O(x^4)\Big\} \end{aligned}$$

as $x \downarrow 0$. Combining this formula with Lemma 4, we see that the asymptotic behavior of the denominator of $L_c^{(\nu)}(x)$ is

$$\frac{\Gamma(\nu)^2}{4}\Big(\frac{2}{x}\Big)^{2\nu}\Big\{1 - \frac{1}{2(\nu-1)}x^2 + O(x^3)\Big\}$$

and that of the numerator is given by

$$\frac{c^\nu - c^{-\nu}}{2\nu} + \frac{1}{8\nu}\Big(\frac{c^{\nu+2} - c^{-\nu}}{\nu+1} + \frac{c^{-\nu+2} - c^{\nu}}{\nu-1}\Big)x^2 + O(x^3)\,.$$

From these asymptotic results we obtain the assertion of the lemma. □

We go back to the proof of Lemma 2. It remains to calculating $\eta_3^{(\nu)}$. We deduce from (10) that

$$\eta_3^{(\nu)}(t) = \xi_1^{(\nu)}(t) + \xi_2^{(\nu)}(t)\,,$$

where

$$\xi_1^{(\nu)}(t) = \int_{\frac{a-b}{\sqrt{t}}}^\infty P^{(\nu+1)}(u)du \int_0^\infty \frac{L_c^{(\nu)}(x) - \rho_\nu x^{2\nu}}{x} e^{-\frac{x\sqrt{tu}}{b}}dx\,,$$

$$\xi_2^{(\nu)}(t) = \phi_{m(\nu+1)} \int_{\frac{a-b}{\sqrt{t}}}^\infty u^{2m(\nu)+2}du \int_0^\infty \frac{L_c^{(\nu)}(x) - \rho_\nu x^{2\nu}}{x} e^{-\frac{x\sqrt{tu}}{b}}dx\,.$$

Changing the variables by $y = (\sqrt{tu}/b)x$, we get

$$\xi_1^{(\nu)}(t) = \frac{b^{2\nu+2}}{t^{\nu+1}} \int_0^\infty 1_{[\frac{a-b}{\sqrt{t}},\infty)}(u)\frac{P^{(\nu+1)}(u)}{u^{2\nu+2}}du \int_0^\infty Q_c^{(\nu)}\Big(\frac{by}{\sqrt{tu}}\Big)y^{2\nu+1}e^{-y}dy\,,$$

where

$$Q_c^{(\nu)}(x) = \frac{L_c^{(\nu)}(x) - \rho_\nu x^{2\nu}}{x^{2\nu+2}}.$$

Lemma 5 yields that $Q_c^{(\nu)}(x)$ converges to $\rho_\nu \zeta_\nu$ as $x \downarrow 0$, which implies that $Q_c^{(\nu)}$ is bounded on (0, 1]. In [7] it is shown that

$$L_c^{(\nu)}(x) = \frac{\pi}{\sqrt{c}} e^{(c-3)x}(1 + o(1))$$

as $x \to \infty$. This yields that there is a constant $C_5^{(\nu)}$ such that

$$|Q_c^{(\nu)}(x)| \leqq C_5^{(\nu)} e^{(c-3)x} + \rho_\nu$$

for $x \geqq 1$. Hence it follows that

$$|Q_c^{(\nu)}(x)| \leqq C_6^{(\nu)} \max\{e^{(c-3)x}, 1\}$$

for a suitable constant $C_6^{(\nu)}$. By the argument used in [7, p. 5253] we obtain that

$$1_{[\frac{a-b}{\sqrt{t}},\infty)}(u) u^{-2(\nu+1)} |P^{(\nu+1)}(u)| \cdot \left| Q_c^{(\nu)}\Big(\frac{by}{\sqrt{tu}}\Big)\right| y^{2\nu-1} e^{-y}$$

is dominated by a constant multiple of

$$u^{-2(\nu+1)} |P^{(\nu+1)}(u)| y^{2\nu-1} e^{-\delta y}. \tag{13}$$

Here a constant δ has been used for $\min\{1, 2b/(a-b)\}$. It follows from (6) that (13) is integrable on $(0,\infty) \times (0,\infty)$. The dominated convergence theorem yields

$$\xi_1^{(\nu)}(t) = \rho_\nu \zeta_\nu b^{2\nu+2} \Gamma(2\nu+2) t^{-\nu-1} \int_0^\infty u^{-2(\nu+1)} P^{(\nu+1)}(u) du + o(t^{-\nu-1}).$$

We easily see by (8) that the coefficient of the term of $t^{-\nu-1}$ is equal to

$$-\Big(\frac{b^2}{2}\Big)^\nu \frac{2b^2(c^\nu - c^{-\nu})\zeta_\nu}{\Gamma(\nu)\cos(\pi\nu)} = -\frac{c^\nu \kappa_2^{(\nu)}}{\cos(\pi\nu)}.$$

To show that the Fubini theorem can be applied to $\xi_2^{(\nu)}(t)$, we need to check that

$$\int_0^\infty du \int_0^\infty 1_{\big[\frac{a-b}{\sqrt{t}},\infty\big)}(u) u^{2m(\nu)+2} |Q_c^{(\nu)}(x)| x^{2\nu+1} e^{-\frac{x\sqrt{tu}}{b}} dx \tag{14}$$

converges. Changing the variable x to y given by $y = (\sqrt{t}u/b)x$, we have that (14) is equal to

$$\frac{b^{2\nu+2}}{t^{\nu+1}} \int_0^\infty du \int_0^\infty 1_{\big[\frac{a-b}{\sqrt{t}},\infty\big)}(u) u^{2m(\nu)-2\nu} \left| Q_c^{(\nu)}\Big(\frac{by}{\sqrt{tu}}\Big)\right| y^{2\nu+1} e^{-y} dy. \tag{15}$$

Similarly to $\xi_1^{(\nu)}(t)$, we deduce from $2m(\nu)-2\nu < -1$ that the integrand of (15) is integrable on $(0,\infty)\times(0,\infty)$. This yields that

$$\begin{aligned}\xi_2^{(\nu)}(t) &= \phi_{m(\nu+1)} \int_0^\infty \frac{L_c^{(\nu)}(x) - \rho_\nu x^{2\nu}}{x} dx \int_{\frac{a-b}{\sqrt{t}}}^\infty u^{2m(\nu)+2} e^{-\frac{x\sqrt{t}u}{b}} du \\ &= \beta_2^{(\nu)} t^{-m(\nu)-3/2}.\end{aligned}$$

We now obtain an estimate for $\eta_3^{(\nu)}(t)$ and, combining it to the estimates for $\eta_1^{(\nu)}(t)$ and $\eta_2^{(\nu)}(t)$ above, we complete the proof of Lemma 2.

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