TOKYO J. MATH.
VOL. 40, NO. 2, 2017
DOI: 10.3836/tjm/1502179241

# On Concentration Phenomena of Least Energy Solutions to Nonlinear Schrödinger Equations with Totally Degenerate Potentials

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(Communicated by H. Nawa)

**Abstract.** We study concentration phenomena of the least energy solutions of the following nonlinear Schrödinger equation:

$$h^2 \Delta u - V(x)u + f(u) = 0 \quad \text{in } \mathbb{R}^N,\ u > 0,\ u \in H^1(\mathbb{R}^N),$$

for a totally degenerate potential $V$. Here $h > 0$ is a small parameter, and $f$ is an appropriate, superlinear and Sobolev subcritical nonlinearity.

In [16], Lu and Wei proved that when the parameter $h$ approaches zero, the least energy solutions concentrate at the most centered point of the totally degenerate set $\Omega = \{x \in \mathbb{R}^N \mid V(x) = \min_{y\in\mathbb{R}^N} V(y)\}$ when $f(u) = u^p$.

In this paper, we show that this kind of result holds for more general $f$. In particular, our proof does not need a so-called uniqueness-nondegeneracy assumption (see, the next-to-last paragraph in Section 1) on the limiting equation (2.6) in Section 2. Furthermore, in [16] Lu and Wei made a technical assumption for $V$, that is,

$$V(x) - \min_{y\in\mathbb{R}^N} V(y) \geq Cd(x, \partial\Omega)^2 \quad \text{for } x \in \Omega^c,$$

where $C$ is a positive constant, but our proof does not need this assumption.

In our proof, we employ a modification of the argument which has been developed by del Pino and Felmer in [9] using Schwarz's symmetrization.

## 1. Introduction

We consider the following nonlinear Schrödinger equation with a potential $V$:

$$\begin{cases} -h^2 \Delta u + V(x)u = f(u) & \text{in } \mathbb{R}^N, \\ u > 0,\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $h > 0$, $\Delta$ is the Laplace operator, $N \geq 1$, $u$ is real-valued, $f$ is a nonlinear term, and $V$ satisfies the following conditions:

(V0) $V \in C^1(\mathbb{R}^N; \mathbb{R})$.
(V1) $V(x) \to +\infty$ as $|x| \to +\infty$.

Received March 17, 2016; revised August 31, 2016
*Mathematics Subject Classification*: 35B40, 35J20, 35Q55
*Key words and phrases*: Nonlinear Schrödinger equation, least energy solution, concentration phenomenon

(V2) $\inf_{x\in\mathbb{R}^N} V(x) = 1$.

The simplest model for the nonlinearity $f$ is given by $f(t) = t^p$ with $1 < p < +\infty$ if $N = 1, 2$, and $1 < p < (N+2)/(N-2)$ if $N \geq 3$.

The study of concentrating solutions for (1.1) began with the result of Floer and Weinstein. In [11], Floer and Weinstein proved that there exists a single-peaked solution of (1.1) concentrating at each given nondegenerate critical point of the potential $V$ when $f(t) = t^3$, $N = 1$, and $V$ is bounded. In [19], Oh generalized this result to $N \geq 2$ and $f(t) = t^p$. Moreover, in [20], Oh proved that there exists a multi-peaked solution of (1.1) concentrating at a given finite collection of nondegenerate critical points of $V$ for $N \geq 2$, $f(t) = t^p$ and $V$ is in the class $(V)_a$ in the sense of Kato for some $a$. Later, the existence of a concentrating solution of (1.1) has long been studied extensively. For example, in [1], Ambrosetti, Badiale and Cingolani proved that there exists a solution of (1.1) concentrating at local minima or maxima of $V$ with nondegenerate $m$-th derivative for some integer $m$, for $f(t) = t^p$. In [7, 8], del Pino and Felmer constructed the solutions of (1.1) concentrating at the degenerate critical point of $V$ for more general $f$. In [10], del Pino, Felmer and Wei constructed a solution of (1.1) concentrating on curves for $f(t) = t^p$. Also, in [5], Byeon and Jeanjean studied the optimal condition of $f$ for the existence of solutions of (1.1). Recently, in [6], Cingolani, Jeanjean and Tanaka studied the multiplicity of solutions concentrating at local minimum points.

On the other hand, the study of a "least energy solution" of (1.1) has been studied in several papers, where the energy functional associated to (1.1) is defined by

$$I_h[u] := \frac{1}{2}\int_{\mathbb{R}^N} h^2|\nabla u(x)|^2 + V(x)u(x)^2\,dx - \int_{\mathbb{R}^N} F(u(x))\,dx, \quad u \in \mathcal{H}, \tag{1.2}$$

where $\mathcal{H} := \{u \in H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V(x)u(x)^2\,dx < +\infty\}$. Moreover, the least energy associated to (1.1) is defined by

$$e_h := \inf_{u\in\mathcal{H}\setminus\{0\}} \sup_{t>0} I_h[tu]\,. \tag{1.3}$$

In [21], Rabinowitz proved that there exists a positive least energy solution of (1.1) for any $h > 0$ if $\limsup_{|x|\to+\infty} V(x) = \sup_{x\in\mathbb{R}^N} V(x)$ or $\liminf_{|x|\to+\infty} V(x) > \inf_{x\in\mathbb{R}^N} V(x)$. In [22], Wang showed that when $f(t) = t^p$, for small $h > 0$, a least energy solution has only one global maximum point $x_h$ and $\lim_{h\to 0} V(x_h) = \inf_{x\in\mathbb{R}^N} V(x)$. In [13], Grossi and Pistoia obtained that for $f(t) = t^p$ if $V$ attains its global minimum at $k$ different points $x_1, \dots, x_k$, which are nondegenerate critical points of $V$, then $\lim_{h\to 0} \Delta V(x_h) = \min\{\Delta V(x_1), \dots, \Delta V(x_k)\}$. In [16], Lu and Wei proved that for $\Omega = \{x \in \mathbb{R}^N \mid V(x) = \inf_{y\in\mathbb{R}^N} V(y)\}$, $\lim_{h\to 0} d(x_h, \partial\Omega) = \max_{x\in\Omega} d(x, \partial\Omega)$. In particular, since one of the results of Lu and Wei is closely related to our result, we state their result precisely. Here, for a set $A \subset \mathbb{R}^N$, $d(x, A)$ denotes the distance from $x$ to $A$, $A^\circ$ denotes the interior of $A$, and $A^{\mathrm{c}}$ denotes the complement of $A$.

THEOREM 1.1 ([16, Theorem 2.2]). *Assume that* $\Omega^\circ$ *is connected,* $f(t) = t^p$ *with* $1 < p < +\infty$ *if* $N = 2$ *and* $1 < p < (N+2)/(N-2)$ *if* $N \geq 3$, *and*

$$V(x) - \inf_{y \in \mathbb{R}^N} V(y) \geq c d(x, \Omega)^2 \quad \textit{for } x \in \Omega^c\,. \tag{1.4}$$

*Let* $u_h$ *be a positive least energy solution of* (1.1). *For* $h$ *sufficiently small, let* $x_h$ *be a unique global maximum point of* $u_h$. *Then, for* $h$ *sufficiently small, we have* $x_h \in \Omega^\circ$ *and*

$$d(x_h, \partial\Omega) \to \max_{x \in \Omega} d(x, \partial\Omega) \quad \textit{as } h \to 0\,. \tag{1.5}$$

In this paper, we study the precise asymptotic location of the concentration point of the least energy solutions for more general nonlinearities $f(u)$. More precisely, the function $f$ satisfies the following conditions:

- (f0) $f \in C^1(\mathbb{R}; \mathbb{R})$.
- (f1) $f(t) \equiv 0$ for $t \leq 0$ and $f(t) = o(t)$ as $t \to 0$.
- (f2) $f'(t) = O(t^{p-1})$ as $t \to +\infty$ for some $1 < p < (N+2)/(N-2)$ if $N \geq 3$ and $1 < p < +\infty$ if $N = 1, 2$.
- (f3) There exists a constant $\theta > 2$ such that $\theta F(t) \leq t f(t)$ for $t \geq 0$, where

$$F(t) = \int_0^t f(s)\,ds\,.$$

- (f4) $f(t) < f'(t)t$ for $t > 0$.

REMARK 1.1. (f0)–(f4) yield the following basic properties:

1. $f(t) > 0$ for $t > 0$.
2. $f(t) = O(t^p)$ as $t \to +\infty$. Actually, by (f2), there exists $M \geq 1$ such that

$$|f'(s)| \leq C s^{p-1} \text{ for } s \geq M\,.$$

Hence we obtain that for $t \geq M$,

$$\begin{aligned} f(t) = \int_0^t f'(s)\,ds &= \int_0^M f'(s)\,ds + \int_M^t f'(s)\,ds \\ &\leq C + \int_M^t C s^{p-1}\,ds \\ &\leq C(1 + t^p) \leq C t^p\,, \end{aligned}$$

where $C$ is a positive constant. Hence, we obtain that $f(t) = O(t^p)$ as $t \to +\infty$.

3. $f(t)/t$ is strictly increasing for $t > 0$.

Finally, the problem (1.1) with a totally degenerate potential is closely related to the Dirichlet problem considered by Ni and Wei in [18] as Lu and Wei pointed out in [16]. Although Ni and Wei in [18] made a so-called uniqueness-nondegeneracy assumption on (2.6) below in Sect.2, del Pino and Felmer in [9] developed a new method without using the assumption, where we say that (2.6) satisfies the uniqueness-nondegeneracy condition if the problem (2.6) has a unique solution and its linearized problem around the solution $w$ to (2.6):

$$\Delta\varphi(x) - \varphi(x) + f^{'}(w(x))\varphi(x) = 0, \quad x \in \mathbb{R}^N, \quad \varphi \in H^1(\mathbb{R}^N)\,, \tag{1.6}$$

does not have nontrivial solutions other than linear combinations of the functions $\partial w/\partial x_i$, $i = 1, \ldots, N$.

In what follows, $B(x; r)$ denotes the open ball of radius $r > 0$ centered at $x$ in $\mathbb{R}^N$ and the following abbreviations, $B_r(x) = B(x; r)$ and $B_r = B_r(0)$ are used. We note that $B_r(x)$ denotes the empty set $\emptyset$ if $r = 0$.

## 2. The main result

Instead of studying the problem (1.1), we study the following problem which is obtained by putting $v(y) := u(hy)$ on (1.1):

$$\begin{cases} -\Delta v + V(hy)v = f(v) \quad \text{in } \mathbb{R}^N\,, \\ v > 0,\ v \in H^1(\mathbb{R}^N)\,. \end{cases} \tag{2.1}$$

To study (2.1), we define $E_h(V)$ by

$$E_h(V) := \left\{ v \in H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V(hy)v(y)^2\, dy < +\infty \right\}, \tag{2.2}$$

define the norm $\|\cdot\|_{E_h(V)}$ on $E_h(V)$ by

$$\|v\|_{E_h(V)} := \left( \int_{\mathbb{R}^N} |\nabla v(y)|^2 + V(hy)v(y)^2\, dy \right)^{\frac{1}{2}} \quad \text{for } v \in E_h(V)\,, \tag{2.3}$$

and define the energy functional $J_h[\cdot; V]$ associated to (2.1) by

$$J_h[v; V] := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v(y)|^2 + V(hy)v(y)^2\, dy - \int_{\mathbb{R}^N} F(v(y))\, dy, \quad v \in E_h(V)\,. \tag{2.4}$$

Moreover, we define the least energy $c_h(V)$ associated to (2.1) by

$$c_h(V) := \inf_{v \in E_h(V)\setminus\{0\}} \sup_{t>0} J_h[tv; V]\,. \tag{2.5}$$

Note that $e_h = h^N c_h(V)$ holds.

First, we prepare basic facts for solutions to the limiting problem:

$$\begin{cases} -\Delta w + w = f(w) \quad \text{in } \mathbb{R}^N\,, \\ w > 0,\ \max_{x\in\mathbb{R}^N} w(x) = w(0),\ w \in H^1(\mathbb{R}^N)\,. \end{cases} \tag{2.6}$$

We define the energy functional $I[\cdot]$ associated to (2.6) by

$$I[w] := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla w|^2 + w^2\,dx - \int_{\mathbb{R}^N} F(w)\,dx, \quad w \in H^1(\mathbb{R}^N), \tag{2.7}$$

and define the least energy $c_*$ associated to (2.6) by

$$c_* := \inf_{w\in H^1(\mathbb{R}^N)\setminus\{0\}} \sup_{t>0} I[tw]. \tag{2.8}$$

REMARK 2.1. Under the assumption (f0)–(f4), it is known that there exists a least energy solution $w \in H^1(\mathbb{R}^N)$ of (2.6) such that $w > 0$, $w \in C^2(\mathbb{R}^N)$ and $w^* = w$, where $w^*$ denotes the standard radially decreasing rearrangement of $w$. Moreover, it is also known that $w(r) \le C\exp(-r)$, where $C$ is a positive constant which independent of $r$. For the proof, see, e.g. [3, 4, 12].

Next, we state the existence and the basic properties of the least energy solutions to (1.1).

PROPOSITION 2.1. *We assume* (f0)–(f4) *and* (V0)–(V2) *hold. Then, for all* $h > 0$, *there exists a least energy solution* $u_h \in E_h(V)\cap C^2(\mathbb{R}^N)$ *of* (1.1) *such that* $u_h > 0$. *Moreover, the following statements hold*:

(i) *For* $h$ *sufficiently small,* $u_h$ *has a unique local maximum point* $x_h \in \mathbb{R}^N$. *Moreover,* $\{x_h\}_{h>0}$ *is bounded in* $\mathbb{R}^N$.

(ii) *Passing to a subsequence, we may assume that* $x_{h_j} \to x_0$ *as* $j \to +\infty$. *Then,* $V(x_0) = \inf_{x\in\mathbb{R}^N} V(x) = 1$.

(iii) *For all* $M > 0$, *there exists* $a(j, M) \in \mathbb{R}$ *such that for* $j$ *sufficiently large,*

$$u_{h_j}(x) \le \exp\left[-\frac{|x - x_{h_j}| + a(j, M)}{h_j}\right], \quad |x| \le M, \tag{2.9}$$

*where* $a(j, M) \to 0$ *as* $j \to +\infty$.

(iv) *There exists a least energy solution* $w \in H^1(\mathbb{R}^N)$ *of* (2.6) *such that*

$$v_{h_j} \to w \text{ in } H^1(\mathbb{R}^N) \text{ as } j \to +\infty, \tag{2.10}$$

*where* $v_{h_j}(y) := u_{h_j}(h_j y + x_{h_j})$.

When $f(u) = u^p$, this result has been proved by Wang ([22]). Since we cannot find the exact reference for the general case $f(u)$, we shall give a proof in Appendix for the sake of completeness to the proof of Proposition 2.1. For the precise asymptotic location of the maximum point $x_h$ and the precise asymptotic expansion of the least energy $c_h(V)$, we show the following theorem, which is the main result of this paper.

THEOREM 2.1. *We assume* (f0)–(f4), (V0)–(V2) *and* $\Omega^\circ \neq \emptyset$ *hold, where* $\Omega = \{x \in \mathbb{R}^N \mid V(x) = \inf_{y\in\mathbb{R}^N} V(y)\}$. *Let* $u_h$ *be a least energy solution of* (1.1) *and* $x_h$ *a point where* $u_h$ *reaches its maximum value. Then,*

(i) *For* $h$ *sufficiently small,* $x_h \in \Omega^\circ$.
(ii) $d(x_h, \partial\Omega) \to \max_{x\in\Omega} d(x, \partial\Omega)$ *as* $h \to 0$.
(iii) *Passing to a subsequence, we have*

$$e_{h_j} = h_j^N c_{h_j}(V) = h_j^N \left[ c_* + \exp\left( -\frac{2}{h_j}[d(x_{h_j}, \partial\Omega) + o(1)] \right) \right] \quad \text{as } j \to +\infty\,,$$

*where we recall the symbols* $e_{h_j}$, $c_{h_j}(V)$ *and* $c_*$ *denote the least energies which are defined by* (2.6), (1.3) *and* (2.9) *respectively.*

This result is closely related to Theorem 1.1. In Theorem 1.1, they proved the same concentration phenomena for the case $f(u) = u^p$. They also imposed a additional assumption on the connectivity of $\Omega^\circ$ and the condition (1.4) for $V(x)$. Our proof of Theorem 2.1 is based on a modification of the idea in [9] employing the rearrangement technique. Especially, we do not need to assume the uniqueness-nondegeneracy condition for (2.6).

This paper is organized as follows. In section 3, we give the proof of the precise asymptotic expansion of the least energy in the case the potential $V(x)$ is radially symmetric, increasing and the totally degenerate set $\Omega$ is the unit ball. In particular, Lemma 3.2 plays a very important role in the proof of the lower bound. In section 4, we give the proof of our main result. In Appendix, we recall the definition and basic properties of the increasing and decreasing rearrangement, and we give the proof of Propositions 2.1 and 3.2.

## 3. In the case $V = V_*$

In this section, we prove the asymptotic expansion of $c_h(V)$ in the case that $V = V_*$ and the totally degenerate set $\Omega$ is the unit ball. More precisely, we assume the following:

(Va) $\Omega = \{x \in \mathbb{R}^N \mid \inf_{y\in\mathbb{R}^N} V(y) = V(x)\} = \bar{B}(0; 1)$.
(Vb) $V$ is a Borel measurable function such that $V_* = V$.

THEOREM 3.1. *Assume that* (f0)–(f4), (V0)–(V2) *and* (Va)–(Vb) *hold. Then, passing to a subsequence, we have*

$$c_{h_j}(V) = c_* + \exp\left( -\frac{2}{h_j}(1 + o(1)) \right) \quad \text{as } j \to \infty\,. \tag{3.1}$$

**3.1. Upper bound.** We can prove the upper bound essentially in the same way as in the proof of the upper bound in [9, Lemma 2.1].

First, we shall show the following important Lemma (cf. [9, Lemma 2.3]).

LEMMA 3.1. *Let $z_\rho$ be a solution of the equation*

$$\begin{cases} -z_\rho''(r) - \frac{N-1}{r} z_\rho'(r) + z_\rho(r) = 0\,, & \rho - 1 \le r \le \rho\,, \\ z_\rho(\rho - 1) = 1,\ z_\rho(\rho) = 0\,. \end{cases}$$

*Then, it holds that*

$$\limsup_{\rho \to +\infty} \{-z_\rho'(\rho - 1)\} \le \frac{1 + e^{-2}}{1 - e^{-2}}\,.$$

PROOF. Let $\widetilde{z_\rho}$ be a solution of the equation

$$\begin{cases} -\widetilde{z_\rho}''(r) - \frac{N-1}{\rho-1} \widetilde{z_\rho}'(r) + \widetilde{z_\rho}(r) = 0\,, & 0 \le r \le 1\,, \\ \widetilde{z_\rho}(0) = 1,\ \widetilde{z_\rho}(1) = 0\,. \end{cases}$$

We define $u_\rho$ as $u_\rho(r) := \widetilde{z_\rho}(r - \rho + 1)$. Note that $u_\rho'(r) < 0$ holds. Then, we have

$$\begin{aligned} &-\Delta(u_\rho - z_\rho)(x) + (u_\rho(x) - z_\rho(x)) \\ &= -u_\rho''(|x|) - \frac{N-1}{|x|} u_\rho'(|x|) + u_\rho(|x|) \\ &\le -u_\rho''(|x|) - \frac{N-1}{\rho - 1} u_\rho'(|x|) + u_\rho(|x|) = 0 \quad \text{for } x \in B_\rho \setminus \bar{B}_{\rho-1}\,. \end{aligned}$$

By the weak maximum principle, we see that $u_\rho(r) \le z_\rho(r)$. We then also have $\widetilde{z_\rho}'(0) = u_\rho'(\rho - 1) \le z_\rho'(\rho - 1)$. We study the behavior of $\widetilde{z_\rho}'(0)$. Let $\lambda_1(\rho) < 0 < \lambda_2(\rho)$ be solutions of the equation

$$\lambda^2 + \frac{N-1}{\rho - 1}\lambda - 1 = 0\,,$$

then we see $\widetilde{z_\rho}(r) = \alpha(\rho) e^{\lambda_1(\rho) r} + \beta(\rho) e^{\lambda_2(\rho) r}$, where $\alpha(\rho)$ and $\beta(\rho)$ satisfy

$$\begin{cases} 1 = \alpha(\rho) + \beta(\rho)\,, \\ 0 = \alpha(\rho) e^{\lambda_1(\rho)} + \beta(\rho) e^{\lambda_2(\rho)}\,. \end{cases}$$

By the elementary calculation, we have

$$\alpha(\rho) = \frac{1}{1 - \exp[\lambda_1(\rho) - \lambda_2(\rho)]}\,.$$

$$\begin{aligned} \widetilde{z_\rho}'(0) &= \alpha(\rho)\lambda_1(\rho) + \beta(\rho)\lambda_2(\rho) \\ &= \alpha(\rho) \left[ \lambda_1(\rho) - \lambda_2(\rho) e^{\lambda_1(\rho) - \lambda_2(\rho)} \right] \\ &\to \frac{-1 - e^{-2}}{1 - e^{-2}} \quad \text{as } \rho \to \infty\,. \end{aligned}$$

Hence, we see that

$$\liminf_{\rho\to\infty} z_\rho'(\rho-1) \geq -\frac{1+e^{-2}}{1-e^{-2}}\,.$$

□

Now, we can prove the upper bound of $c_h(V)$.

PROPOSITION 3.1. *Assume that* (f0)–(f4), (V0)–(V2) *and* (Va) *hold. Then*

$$c_h(V) \leq c_* + \exp\Big(-\frac{2}{h}[1+o(1)]\Big) \quad \text{as } h \to 0\,.$$

REMARK 3.1. Assume (f0)–(f4), (V0)–(V2) and $\Omega = B(0;r)$ for some $r > 0$. Put $\tilde{V}(x) := V(rx)$. Then, note that $\tilde{V}$ satisfies (V0)–(V2) and (Va). By Proposition 3.1, we obtain that

$$c_h(V) = c_{h/r}(\tilde{V}) \leq c_* + \exp\Big(-\frac{2}{h}[r+o(1)]\Big) \quad \text{as } h \to 0\,.$$

PROOF OF PROPOSITION 3.1. We put $\rho_h := 1/h$. Let $w \in H^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$ be a least energy solution of the problem

$$\begin{cases} -\Delta w + w = f(w) & \text{in } \mathbb{R}^N\,, \\ w^* = w,\ w > 0\,. \end{cases} \tag{3.2}$$

Let $w_h \in H^1(B_{\rho_h} \setminus \bar{B}_{\rho_h-1})$ be a unique radially symmetric solution of the equation

$$\begin{cases} -\Delta w_h + w_h = 0 & \text{in } B_{\rho_h} \setminus \bar{B}_{\rho_h-1}\,, \\ w_h(\rho_h-1) = w(\rho_h-1),\ w_h(\rho_h) = 0\,. \end{cases}$$

Note that $w_h(r) = w(\rho_h-1)z_{\rho_h}(r)$ using the notation of Lemma 3.1. We define $\overline{w_h}$ as

$$\overline{w_h}(r) := \begin{cases} w(r) & 0 \leq r \leq \rho_h - 1\,, \\ w_h(r) & \rho_h - 1 \leq r \leq \rho_h\,. \end{cases}$$

Then, we see that $\overline{w_h} \in E_h \setminus \{0\}$ by the zero extension, and hence

$$c_h(V) \leq \sup_{t>0} J_h[t\overline{w_h}; V] = J_h[t_h\overline{w_h}; V]\,,$$

where, $t_h$ is a unique positive constant such that the last equality holds. Remark that the uniqueness of $t_h$ follows from (f4).

CLAIM 1. *For $h$ sufficiently small, $t_h \leq 2$.*

Assume that there exists a subsequence $\{h_k\}_{k=1}^\infty \subset \{h\}$ such that

$$t_{h_k} > 2,\ h_k \to 0 \text{ as } k \to \infty\,.$$

Using the definition of $t_{h_k}$ and $V(h_k y) = 1$ on $|y| \le 1/h_k$ by the assumption (Va), we obtain that

$$\begin{aligned}
&\int_{B_{\rho_{h_k}-1}} |\nabla w|^2 + w^2\, dy + \int_{B_{\rho_{h_k}} \setminus B_{\rho_{h_k}-1}} |\nabla w_{h_k}|^2 + {w_{h_k}}^2\, dy \\
&= \int_{\mathbb{R}^N} |\nabla \overline{w_{k_j}}|^2 + V(h_{j_k} y)\overline{w_{k_j}}^2\, dy \\
&= \frac{1}{t_{h_k}} \int_{\mathbb{R}^N} f(t_{h_k}\overline{w_{h_k}})\overline{w_{h_k}}\, dy \\
&> \frac{1}{2} \int_{\mathbb{R}^N} f(2\overline{w_{h_k}})\overline{w_{h_k}}\, dy \\
&\ge \frac{1}{2} \int_{B_{\rho_{h_k}-1}} f(2w)w\, dy.
\end{aligned}$$

By integration by parts, we may estimate

$$\begin{aligned}
&\int_{B_{\rho_{h_k}} \setminus B_{\rho_{h_k}-1}} |\nabla w_{h_k}|^2 + {w_{h_k}}^2\, dy \\
&= \int_{\partial B_{\rho_{h_k}-1}} \nabla w_{h_k}(y) \cdot \frac{-y}{|y|} w_{h_k}(y)\, dS \\
&= -w_{h_k}'(\rho_{h_k}-1)w(\rho_{h_k}-1)|\partial B_1|(\rho_{h_k}-1)^{N-1} \\
&\le C w(\rho_{h_k}-1)^2(\rho_{h_k}-1)^{N-1} \\
&\le C \exp[-2\rho_{h_k}(1+o(1))]\,, \qquad (3.3)
\end{aligned}$$

where the first inequality follows from Lemma 3.1 and the second inequality follows from $w(r) \le C\exp(-r)$. Hence, we have

$$\int_{B_{\rho_{h_k}-1}} |\nabla w|^2 + w^2\, dy + C\exp[-2\rho_{h_k}(1+o(1))] \ge \frac{1}{2} \int_{B_{\rho_{h_k}-1}} f(2w)w\, dy\,.$$

As $k \to \infty$, we see that

$$\int_{\mathbb{R}^N} |\nabla w|^2 + w^2\, dy \ge \frac{1}{2} \int_{\mathbb{R}^N} f(2w)w\, dy > \int_{\mathbb{R}^N} f(w)w\, dy\,.$$

This is impossible because $w$ is a solution of (3.2).

Now, we have

$$\begin{aligned}
c_h(V) \le J_h[t_h\overline{w_h}; V] \le \frac{t_h^2}{2} \int_{B_{\rho_h-1}} |\nabla w|^2 + w^2\, dy - \int_{B_{\rho_h}-1} F(t_h w)\, dy \\
+ \frac{t_h^2}{2} \int_{B_{\rho_h} \setminus \bar{B}_{\rho_h-1}} |\nabla w_h|^2 + w_h^2\, dy
\end{aligned}$$

$$= \frac{t_h^2}{2} \int_{B_{\rho_h - 1}} |\nabla w|^2 + w^2 \left[ 1 - \frac{2F(t_h w)}{t_h^2 w^2} \right] dy$$

$$+ \frac{t_h^2}{2} \int_{B_{\rho_h} \setminus \bar{B}_{\rho_h - 1}} |\nabla w_h|^2 + w_h^2 \, dy$$

$$\leq I[t_h w] + \frac{t_h^2}{2} \int_{B_{\rho_h} \setminus \bar{B}_{\rho_h - 1}} |\nabla w_h|^2 + w_h^2 \, dy$$

$$\leq \sup_{t>0} I[tw] + \frac{t_h^2}{2} \int_{B_{\rho_h} \setminus \bar{B}_{\rho_h - 1}} |\nabla w_h|^2 + w_h^2 \, dy \,, \tag{3.4}$$

where the second inequality holds by the following inequality which follows by (f1), Claim 1, and $w(r) \leq C \exp(-r)$:

$$1 - \frac{2F(t_h w(r))}{t_h^2 w(r)^2} \geq 0 \quad \text{for } \rho_h - 1 \leq r < +\infty \,.$$

Since $w$ is a least energy solution of (3.2), we see that

$$\sup_{t>0} I[tw] = I[w] = c_* \,. \tag{3.5}$$

Also, by Claim 1 and (3.3), we obtain that

$$\frac{t_h^2}{2} \int_{B_{\rho_h} \setminus \bar{B}_{\rho_h - 1}} |\nabla w_h|^2 + w_h^2 \, dy \leq C \exp[-2\rho_h(1 + o(1))] \,. \tag{3.6}$$

(3.4), (3.5) and (3.6) yield that

$$c_h(V) \leq c_* + \exp[-2\rho_h(1 + o(1))] \,.$$

□

**3.2. Lower Bound.** Next we shall prove the lower bound of $c_h(V)$. We need to modify the method of the proof of lower bound in [9, Lemma 2.1]. We will prepare some results for the proof of lower bound of $c_h(V)$.

First, we note the following result which gives the information of least energy solutions under the assumption (Va) and (Vb).

PROPOSITION 3.2. *Assume that* (f0)–(f4), (V1), (V2), (Va), (Vb) *hold. Then, the following statements hold*:

(i) *There exists a least energy solution* $v_h \in E_h(V) \cap C^1(\mathbb{R}^N)$ *of* (2.1) *such that* $v_h > 0$ *and* $v_h^* = v_h$.

(ii) *For all* $\delta > 0$, *there exist a subsequence* $\{h_j\}_{j=1}^{\infty} \subset \{h\}$ *and a constant* $C(\delta) > 0$ *such that*

$$v_{h_j}(y) \leq C(\delta) \exp(-(1 - \delta)|y|) \quad \textit{for } y \in \mathbb{R}^N \,.$$

(iii) *Passing to a subsequence, we have a least energy solution* $w \in H^1(\mathbb{R}^N)$ *of* (2.6) *such that* $v_{h_j} \to w$ *in* $H^1(\mathbb{R}^N) \cap C^1_{\rm loc}(\mathbb{R}^N)$.

(iv) *For all* $\beta > 0$*, there exists* $C > 0$ *such that*

$$|v'_{h_j}(y)| \le C \quad \text{for } |y| \le \beta/h_j\,.$$

We can show Proposition 3.2 by the well-known argument, but it is very long. So, it will be included in the Appendix. The next Lemma plays a very important role in the proof of the lower bound of $c_h$.

LEMMA 3.2. *Assume* (f0)–(f4), (V1), (V2), (Va) *and* (Vb). *Let* $v_h$ *be a least energy solution of* (2.1) *such that* $v_h^* = v_h$. *Then for all* $\varepsilon > 0$*, there exists a subsequence* $\{h_k\}_{k=1}^{\infty} \subset \{h\}$ *such that*

$$v_{h_k}((1+\varepsilon)\rho_k) \ge \exp\bigl(-\rho_k(\sqrt{V(1+2\varepsilon)}(1+\varepsilon)+o(1)\bigr) \quad \text{as } k \to \infty\,.$$

PROOF. Take any $R > 0$. Put $A_\varepsilon := V(1+2\varepsilon)$. Let $z(r)$ be a solution of the equation

$$\begin{cases} z''(r) + \frac{N-1}{R} z'(r) - A_\varepsilon z(r) = 0 \quad R \le r \le (1+2\varepsilon)\rho_k\,, \\ z(R) = v_{h_k}(R),\ z((1+2\varepsilon)\rho_k) = 0\,. \end{cases} \tag{3.7}$$

Let $\tilde{z}(r)$ be a solution of the equation

$$\begin{cases} \tilde{z}''(r) + \frac{N-1}{R} \tilde{z}'(r) - A_\varepsilon \tilde{z}(r) = 0 \quad 0 \le r \le (1+2\varepsilon)\rho_k - R\,, \\ \tilde{z}(0) = 1,\ \tilde{z}((1+2\varepsilon)\rho_k - R) = 0\,. \end{cases} \tag{3.8}$$

Note that $z(r) = \tilde{z}(r-R)v_{h_k}(R)$. Remarking $z'(r) < 0$, we may estimate

$$\begin{aligned} &-\Delta(z - v_{h_k})(|y|) + A_\varepsilon(z - v_{h_k})(|y|) \\ &= (V(h_k y) - A_\varepsilon)v_{h_k}(|y|) - z''(|y|) - \frac{N-1}{r} z'(|y|) + A_\varepsilon z(|y|) \\ &\le -z''(|y|) - \frac{N-1}{R} z'(|y|) + A_\varepsilon z(|y|) = 0 \quad \text{for } |y| \in (R, (1+2\varepsilon)\rho_k)\,. \end{aligned}$$

By the weak maximum principle, we have $z(r) \le v_{h_k}(r)$ $(R \le r \le (1+2\varepsilon)\rho_k)$. In particular, $z((1+\varepsilon)\rho_k) \le v_{h_k}((1+\varepsilon)\rho_k)$.

**Step 1.** *We show the estimate of* $\tilde{z}((1+\varepsilon)\rho_k - R)$.

Let $\lambda_1(R) < 0 < \lambda_2(R)$ be a solution of the equation

$$\lambda^2 + \frac{N-1}{R}\lambda - A_\varepsilon = 0\,. \tag{3.9}$$

Then, we see that $\lambda_1(R) \to -\sqrt{A_\varepsilon},\ \lambda_2 \to \sqrt{A_\varepsilon}$ as $R \to \infty$. Moreover, it follows that,

$$\tilde{z}(r) = \alpha(k)\exp(\lambda_1(R)r) + \beta(k)\exp(\lambda_2(R)r)\,, \tag{3.10}$$

where $\alpha(k)$ and $\beta(k)$ satisfy

$$\begin{cases} 1 & = \alpha(k) + \beta(k)\,, \\ 0 & = \alpha(k)\exp[\lambda_1(R)\{(1+2\varepsilon)\rho_k - R\}] + \beta(k)\exp[\lambda_2(R)\{(1+2\varepsilon)\rho_k - R\}]\,. \end{cases}$$

Hence, we have

$$\tilde{z}((1+\varepsilon)\rho_k - R) = e^{\lambda_1(R)\{(1+\varepsilon)\rho_k - R\}}\left[\alpha(k) + \beta(k)e^{(\lambda_2(R)-\lambda_1(R))\{(1+\varepsilon)\rho_k - R\}}\right]. \tag{3.11}$$

We will prove that $\alpha(k) \to 1$ as $k \to \infty$ and $\beta(k)e^{(\lambda_2-\lambda_1)\{(1+\varepsilon)\rho_k - R\}} \to 0$ as $k \to \infty$. By the elementary calculation, we have

$$\beta(k)e^{(\lambda_2(R)-\lambda_1(R))\{(1+\varepsilon)\rho_k - R\}} = \frac{1}{e^{(\lambda_1(R)-\lambda_2(R))\{(1+\varepsilon)\rho_k - R\}} - e^{(\lambda_2(R)-\lambda_1(R))\varepsilon\rho_k}}\,.$$

Hence we see that $\beta(k)e^{(\lambda_2-\lambda_1)\{(1+\varepsilon)\rho_k - R\}} \to 0$ as $k \to \infty$. Furthermore, by $\beta(k) \to 0$ as $k \to \infty$, we have $\alpha(k) \to 1$ as $k \to \infty$. By (3.11), it follows that

$$\tilde{z}((1+\varepsilon)\rho_k - R) \geq \frac{1}{2}e^{-\rho_k\left\{-\lambda_1(R)(1+\varepsilon) - \frac{R}{\rho_k}\right\}}\,.$$

Passing to a subsequence, we have

$$\tilde{z}((1+\varepsilon)\rho_{k_j} - R) \geq e^{-\rho_{k_j}\{\sqrt{A_\varepsilon}(1+\varepsilon)+o(1)\}} \quad \text{as } j \to \infty\,. \tag{3.12}$$

**Step 2.** *We prove the estimate of $v_{h_k}(R)$.*

By $v_{h_k}(R) \to w(R)$ as $k \to \infty$, it follows that $v_{h_k}(R) \geq \frac{1}{2}w(R) > 0$ as $k \to \infty$. This inequality and (3.11) yield

$$v_{h_k}((1+\varepsilon)\rho_k) \geq \exp\left(-\rho_k(\sqrt{V(1+2\varepsilon)}(1+\varepsilon) + o(1)\right) \quad \text{as } k \to \infty\,.$$

□

The next Lemma also plays an important role (cf. [9, Lemma 2.3]).

LEMMA 3.3. *We assume $u_\rho \in H^1(\mathbb{R}^N \setminus B_\rho)$ is a solution of equation*

$$\begin{cases} -u_\rho''(r) - \frac{N-1}{r}u_\rho'(r) + a_\rho u_\rho(r) = 0\,,\ \rho \leq r < +\infty\,, \\ u_\rho(\rho) = 1,\ u_\rho(+\infty) = 0\,, \end{cases} \tag{3.13}$$

*where $a_\rho \in \mathbb{R}$ and $\lim_{\rho\to+\infty} a_\rho =: \alpha > 0$. Then, the following statements follow*:

(i)

$$\lim_{\rho\to+\infty} u_\rho'(\rho) = -\sqrt{\alpha}\,.$$

(ii)

$$u_\rho(r) \le C_\rho e^{-\frac{\sqrt{\alpha}}{2}r} \text{ for } r \ge \rho\,,$$

*where $C_\rho$ is a positive constant which depends on $\rho$.*

(iii)

$$|u_\rho^{'}(r)| \le C_\rho \text{ for } r \ge 2\rho\,,$$

*where $C_\rho$ is a positive constant which depends on $\rho$.*

PROOF. Let $\overline{u_\rho}$ and $\widetilde{u_\rho}$ be solutions of the equations

$$\begin{cases} -\overline{u_\rho}''(r) - \frac{N-1}{\rho}\overline{u_\rho}'(r) + a_\rho \overline{u_\rho}(r) = 0\,, & \rho \le r < +\infty\,, \\ \overline{u_\rho}(\rho) = 1\,,\ \overline{u_\rho}(+\infty) = 0\,, \end{cases}$$

$$\begin{cases} -\widetilde{u_\rho}''(r) + a_\rho \widetilde{u_\rho}(r) = 0\,, & \rho \le r < +\infty\,, \\ \widetilde{u_\rho}(\rho) = 1,\ \widetilde{u_\rho}(+\infty) = 0\,. \end{cases}$$

Then, by the elementary calculation, we have

$$\lim_{\rho\to+\infty} \overline{u_\rho}'(\rho) = -\sqrt{\alpha},\ \lim_{\rho\to+\infty} \widetilde{u_\rho}'(\rho) = -\sqrt{\alpha}\,. \tag{3.14}$$

On the other hand, by the weak maximum principle, we see that $\overline{u_\rho}$ is a subsolution of (3.13), and $\widetilde{u_\rho}$ is a supersolution. Then, we have $\widetilde{u_\rho}'(\rho) \ge u_\rho^{'}(\rho) \ge \overline{u_\rho}'(\rho)$. By (3.14), we obtain

$$\lim_{\rho\to+\infty} u_\rho^{'}(\rho) = -\sqrt{\alpha}\,.$$

Also, by the elementary calculation, we see that

$$\widetilde{u_\rho}(r) = e^{-\sqrt{a_\rho}(r-\rho)}\,.$$

By $u_\rho(r) \le \widetilde{u_\rho}(r)$ and $\lim_{\rho\to+\infty} a_\rho = \alpha$, (ii) follows. Finally, we will prove (iii). By Lemma B.3, for any $|z| \ge 2\rho$, we have

$$\sup_{B_{3/4}(z)} |\nabla u_\rho|^2 \le C(\|u_\rho\|^2_{L^{N+1}(B_1(z))} + \|\nabla u_\rho\|^2_{L^2(B_1(z))})\,.$$

By $B_1(z) \subset \mathbb{R}^N \setminus B_\rho$, (ii) and $u_\rho \in H^1(\mathbb{R}^N \setminus B_\rho)$, (iii) follows. □

Now, we can prove the lower bound of $c_h(V)$.

PROPOSITION 3.3. *Assume that* (f0)–(f4), (V1), (V2), (Va) *and* (Vb). *Then passing to a subsequence, we have*

$$c_{h_k}(V) \ge c_* + \exp\Big(-\frac{2}{h_k}[1 + o(1)]\Big) \quad \textit{as } k \to \infty\,.$$

REMARK 3.2. If we assume (f0)–(f4), (V1), (V2), (Vb) and $\Omega = B(0; r)$ for some $r > 0$, then we have

$$c_{h_k}(V) \geq c_* + \exp\Big(-\frac{2}{h_k}[r + o(1)]\Big) \quad \textit{as } k \to \infty\,,$$

by the similar argument to Remark 3.1.

PROOF OF PROPOSITION 3.3. Take any $\varepsilon > 0$. Let $v_{h_k} \in E_{h_k}(V) \cap C^1(\mathbb{R}^N)$ be a least energy solution of (2.1) such that $v_h^* = v_h$.

$$c_{h_k}(V) \geq J_{h_k}[tv_{h_k}; V]\,.$$

Let $w_k \in H^1(\mathbb{R}^N \setminus B_{(1+\varepsilon)\rho_k})$ be a solution of the equation

$$\begin{cases} -\Delta w_k + w_k = 0 & \text{in } \mathbb{R}^N \setminus \bar{B}_{(1+\varepsilon)\rho_k}\,, \\ w_k((1+\varepsilon)\rho_k) = v_{h_k}((1+\varepsilon)\rho_k),\ w_k(x) \to 0 & \text{as } |x| \to \infty\,. \end{cases}$$

Put

$$\overline{v_k}(x) := \begin{cases} v_{h_k}(x) \text{ for } |x| \leq (1+\varepsilon)\rho_k\,, \\ w_k(x) \text{ for } |x| \geq (1+\varepsilon)\rho_k\,. \end{cases}$$

We decompose $J_{h_k}[tv_{h_k}; V]$ into the following two parts.

$$\begin{aligned} J_{h_k}[tv_{h_k}; V] &= \left\{\frac{t^2}{2}\int_{B_{(1+\varepsilon)\rho_k}} |\nabla v_{h_k}|^2 + V(h_k y)v_{h_k}^2\,dy - \int_{B_{(1+\varepsilon)\rho_k}} F(tv_{h_k})\,dy\right\} \\ &\quad + \left\{\frac{t^2}{2}\int_{\mathbb{R}^N\setminus B_{(1+\varepsilon)\rho_k}} |\nabla v_{h_k}|^2 + V(h_k y)v_{h_k}^2\,dy - \int_{\mathbb{R}^N\setminus B_{(1+\varepsilon)\rho_k}} F(tv_{h_k})\,dy\right\} \\ &=: I_1(t,k,\varepsilon) + I_2(t,k,\varepsilon)\,. \end{aligned}$$

First, we shall estimate $I_1(t,k,\varepsilon)$. By $\min_{x\in\mathbb{R}^N} V(x) = 1$, we obtain that

$$\begin{aligned} I_1(t,k,\varepsilon) &= \frac{t^2}{2}\int_{B_{(1+\varepsilon)\rho_k}} |\nabla v_{h_k}|^2 + v_{h_k}^2\,dy - \int_{B_{(1+\varepsilon)\rho_k}} F(tv_{h_k})\,dy \\ &\quad + \frac{t^2}{2}\int_{B_{(1+\varepsilon)\rho_k}} (V(h_k y) - 1)v_{h_k}^2\,dy \\ &\geq \frac{t^2}{2}\int_{B_{(1+\varepsilon)\rho_k}} |\nabla v_{h_k}|^2 + v_{h_k}^2\,dy - \int_{B_{(1+\varepsilon)\rho_k}} F(tv_{h_k})\,dy \\ &= I[t\overline{v_{h_k}}] - \left\{\frac{t^2}{2}\int_{\mathbb{R}^N\setminus B_{(1+\varepsilon)\rho_k}} |\nabla w_k|^2 + w_k^2\,dy - \int_{\mathbb{R}^N\setminus B_{(1+\varepsilon)\rho_k}} F(tw_k)\,dy\right\} \\ &=: I[t\overline{v_{h_k}}] - K(t,k,\varepsilon)\,. \end{aligned}$$

By integration by parts, we have

$$\begin{aligned}
-K(t,k,\varepsilon) &\geq -\frac{t^2}{2}\int_{\mathbb{R}^N\setminus B_{(1+\varepsilon)\rho_k}} |\nabla w_k|^2 + w_k^2\,dy \\
&= -\frac{t^2}{2}\lim_{R\to+\infty}\int_{B_R\setminus B_{(1+\varepsilon)\rho_k}} |\nabla w_k|^2 + w_k^2\,dy \\
&= \frac{t^2}{2}\lim_{R\to+\infty}\Bigg[-\int_{\partial B_R}\nabla w_k(y)\cdot\frac{y}{|y|}w_k(y)\,dS \\
&\quad + \int_{\partial B_{(1+\varepsilon)\rho_k}}\nabla w_k(y)\cdot\frac{y}{|y|}w_k(y)\,dS\Bigg] \\
&\geq \frac{t^2}{2}\int_{\partial B_{(1+\varepsilon)\rho_k}}\nabla w_k(y)\cdot\frac{y}{|y|}w_k(y)\,dS \\
&= \frac{t^2}{2}|\partial B_1|\rho_k^{N-1}(1+\varepsilon)^{N-1}v_{h_k}((1+\varepsilon)\rho_k)w_k^{'}((1+\varepsilon)\rho_k)\,,
\end{aligned}$$

where the second inequality follows from Lemma 3.3 (ii) and (iii). Hence, we have

$$I_1(t,k,\varepsilon) \geq I[t\overline{v_{h_k}}] + \frac{t^2}{2}|\partial B_1|\rho_k^{N-1}(1+\varepsilon)^{N-1}v_{h_k}((1+\varepsilon)\rho_k)w_k^{'}((1+\varepsilon)\rho_k)\,. \tag{3.15}$$

Next, we shall estimate $I_2$.

$$\begin{aligned}
&I_2(t,k,\varepsilon) \\
&= \frac{t^2}{2}\int_{\mathbb{R}^N\setminus B_{(1+\varepsilon)\rho_k}} |\nabla v_{h_k}|^2 + V(h_k y)v_{h_k}^2\,dy - \int_{\mathbb{R}^N\setminus B_{(1+\varepsilon)\rho_k}} F(tv_{h_k})\,dy \\
&= \frac{t^2}{2}\int_{\mathbb{R}^N\setminus B_{(1+\varepsilon)\rho_k}} |\nabla v_{h_k}|^2 + V(h_k y)v_{h_k}^2\left[1 - \frac{2F(tv_{h_k})}{t^2v_{h_k}^2}\right]dy \\
&\geq \frac{t^2}{2}\int_{\mathbb{R}^N\setminus B_{(1+\varepsilon)\rho_k}} |\nabla v_{h_k}|^2 + V(h_k y)v_{h_k}^2\left[1 - \frac{f((t+1)v_{h_k})}{(t+1)v_{h_k}}\right]dy \\
&\geq \frac{t^2}{2}\min_{|y|\geq(1+\varepsilon)\rho_k}\left[1 - \frac{f((t+1)v_{h_k})}{(t+1)v_{h_k}}\right]\int_{\mathbb{R}^N\setminus B_{(1+\varepsilon)\rho_k}} |\nabla v_{h_k}|^2 + V(h_k y)v_{h_k}^2\,dy \\
&\geq \frac{t^2}{2}\min_{|y|\geq(1+\varepsilon)\rho_k}\left[1 - \frac{f((t+1)v_{h_k})}{(t+1)v_{h_k}}\right]\int_{B_{3\rho_k}\setminus B_{(1+\varepsilon)\rho_k}} |\nabla v_{h_k}|^2 + V(h_k y)v_{h_k}^2\,dy\,.
\end{aligned}$$

By integration by parts, we may estimate

$$\int_{B_{3\rho_k}\setminus B_{(1+\varepsilon)\rho_k}} |\nabla v_{h_k}|^2 + V(h_k y)v_{h_k}^2\,dy$$

$$
\begin{aligned}
&= \int_{B_{3\rho_k}\setminus B_{(1+\varepsilon)\rho_k}} f(v_{h_k})v_{h_k}\,dy - \int_{\partial B_{(1+\varepsilon)\rho_k}} \nabla v_{h_k}(y)\cdot\frac{y}{|y|}v_{h_k}(y)\,dS \\
&\quad + \int_{\partial B_{3\rho_k}} \nabla v_{h_k}(y)\cdot\frac{y}{|y|}v_{h_k}(y)\,dS \\
&\geq -\int_{\partial B_{(1+\varepsilon)\rho_k}} \nabla v_{h_k}(y)\cdot\frac{y}{|y|}v_{h_k}(y)\,dS + \int_{\partial B_{3\rho_k}} \nabla v_{h_k}(y)\cdot\frac{y}{|y|}v_{h_k}(y)\,dS \\
&= |\partial B_1|\rho_k^{N-1} \\
&\quad \times \Big[(1+\varepsilon)^{N-1}v_{h_k}((1+\varepsilon)\rho_k)(-v_{h_k}'((1+\varepsilon)\rho_k)) + 3^{N-1}v_{h_k}(3\rho_k)v_{h_k}'(3\rho_k)\Big] \\
&\geq |\partial B_1|\rho_k^{N-1}(1+\varepsilon)^{N-1}v_{h_k}((1+\varepsilon)\rho_k)(-v_{h_k}'((1+\varepsilon)\rho_k)) - \exp\Big(-\frac{5}{2}\rho_k\Big),
\end{aligned}
$$

where the last inequality yields by Proposition 3.2 (ii), (iv). Putting

$$
e(t,k,\varepsilon) := \min_{|y|\geq(1+\varepsilon)\rho_k}\left[1 - \frac{f((t+1)v_{h_k})}{(t+1)v_{h_k}}\right],
$$

then we have

$$
\begin{aligned}
&I_2(t,k,\varepsilon) \qquad (3.16)\\
&\geq \frac{t^2}{2}e(t,k,\varepsilon)\Big[|\partial B_1|\rho_k^{N-1}(1+\varepsilon)^{N-1}v_{h_k}((1+\varepsilon)\rho_k)(-v_{h_k}'((1+\varepsilon)\rho_k)) \\
&\quad - \exp\Big(-\frac{5}{2}\rho_k\Big)\Big].
\end{aligned}
$$

Take $t_k > 0$ which satisfies $\sup_{t>0} I[t\overline{v_k}] = I[t_k\overline{v_k}]$. Then

$$
I[t_k\overline{v_{h_k}}] \geq \inf_{\substack{w\in H^1(\mathbb{R}^N)\\ w\not\equiv 0}} \sup_{t>0} I[tw] =: c_*.
$$

Hence, by (3.15) and (3.16), we have

$$
\begin{aligned}
c_{h_k}(V) \geq{}& c_* - \frac{t_k^2}{2}\exp\Big(-\frac{5}{2}\rho_k\Big) \\
&+ \frac{t_k^2}{2}|\partial B_1|(1+\varepsilon)^{N-1}\rho_k^{N-1}v_{h_k}((1+\varepsilon)\rho_k) \\
&\times\big[-e(t_k,k,\varepsilon)v_{h_k}'((1+\varepsilon)\rho_k) + w_k'((1+\varepsilon)\rho_k)\big]. \qquad (3.17)
\end{aligned}
$$

CLAIM 1. *$t_k \to 1$ as $k \to \infty$.*

First, we will prove $\bar{v}_k \to w$ in $H^1(\mathbb{R}^N)$.

$$
\|w_k\|^2_{H^1(\mathbb{R}^N\setminus B_{(1+\varepsilon)\rho_k})}
$$

$$
\begin{aligned}
&= \lim_{R\to+\infty} \|w_k\|^2_{H^1(B_R\setminus B_{(1+\varepsilon)\rho_k})} \\
&= \lim_{R\to+\infty} \left[ \int_{\partial B_R} \nabla w_k(y)\cdot \frac{y}{|y|} w_k(y)dS - \int_{\partial B_{(1+\varepsilon)\rho_k}} \nabla w_k(y)\cdot \frac{y}{|y|} w_k(y)dS \right] \\
&\le -|\partial B_1|(1+\varepsilon)^{N-1}\rho_k^{N-1} w_k'((1+\varepsilon)\rho_k) v_{h_k}((1+\varepsilon)\rho_k)\,,
\end{aligned}
$$

where the inequality follows by Lemma 3.3 (ii) and (iii). By Lemma 3.3 (i), we have

$$
-w_k'((1+\varepsilon)\rho_k)/v_{h_k}((1+\varepsilon)\rho_k) \le 3/2\,,
$$

and by Proposition 3.2, it holds that

$$
v_{h_k}((1+\varepsilon)\rho_k) \le C\exp(-\alpha\rho_k)\,,
$$

for some positive constants $C$ and $\alpha$ which are independent of $k$. Hence, we have

$$
-\rho_k^{N-1} w_k'((1+\varepsilon)\rho_k) v_{h_k}((1+\varepsilon)\rho_k) \le C\rho_k^{N-1}\exp(-\alpha\rho_k) \to 0\,,
$$

as $k\to\infty$. Therefore, we obtain that

$$
\|w_k\|^2_{H^1(\mathbb{R}^N\setminus B_{(1+\varepsilon)\rho_k})} \to 0 \quad \text{as } k\to\infty\,. \tag{3.18}
$$

By $\{v_{h_k}\}$ is bounded in $H^1(\mathbb{R}^N)$, we see that $\{\overline{v_k}\}$ is bounded in $H^1(\mathbb{R}^N)$. Hence,

$$
\overline{v_k} \rightharpoonup \tilde{w} \;\text{ in } H^1(\mathbb{R}^N)\,.
$$

For any $\phi\in C_c^\infty(\mathbb{R}^N)$,

$$
\int_{\mathbb{R}^N} w\phi\, dy = \lim_{k\to\infty}\int_{\mathbb{R}^N} v_{h_k}\phi\, dy = \lim_{k\to\infty}\int_{\mathbb{R}^N} \overline{v_k}\phi\, dy = \int_{\mathbb{R}^N} \tilde{w}\phi\, dy\,.
$$

Hence, $w=\tilde{w}$.

$$
\begin{aligned}
&\left|\|\overline{v_k}\|^2_{H^1(\mathbb{R}^N)} - \|w\|^2_{H^1(\mathbb{R}^N)}\right| \\
&\le \left|\|v_{h_k}\|^2_{H^1(B_{(1+\varepsilon)\rho_k})} - \|w\|^2_{H^1(B_{(1+\varepsilon)\rho_k})}\right| \\
&\quad + \left|\|w_k\|^2_{H^1(\mathbb{R}^N\setminus B_{(1+\varepsilon)\rho_k})} - \|w\|^2_{H^1(\mathbb{R}^N\setminus B_{(1+\varepsilon)\rho_k})}\right| \\
&\to 0 \text{ as } k\to\infty\,,
\end{aligned}
$$

where the last limit is observed by $v_{h_k}\to w$ in $H^1(\mathbb{R}^N)$, $\|w_k\|_{H^1(\mathbb{R}^N\setminus B_{(1+\varepsilon)\rho_k})}\to 0$ and $\|w\|_{H^1(\mathbb{R}^N\setminus B_{(1+\varepsilon)\rho_k})}\to 0$. Hence $\|\overline{v_k}\|_{H^1(\mathbb{R}^N)}\to\|w\|_{H^1(\mathbb{R}^N)}$, so we have $\overline{v_k}\to w$ in $H^1(\mathbb{R}^N)$.

We will prove Claim 1. Assume that there exists a subsequence $\{k_j\}\subset\{k\}$ and a constant $\delta_0>0$ such that

$$
t_{k_j} \le 1-\delta_0 \tag{3.19}
$$

or

$$t_{k_j} \geq 1 + \delta_0 \,. \tag{3.20}$$

If (3.19) holds,

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla \overline{v_k}|^2 + \overline{v_k}^2 dy &= \frac{1}{t_k} \int_{\mathbb{R}^N} f(t_k \overline{v_k}) \overline{v_k} dy \\ &\leq \frac{1}{1-\delta_0} \int_{\mathbb{R}^N} f((1-\delta_0)\overline{v_k}) \overline{v_k} dy \,.\end{aligned}$$

By using $\overline{v_k} \to w$ in $H^1(\mathbb{R}^N)$, it follows that

$$\int_{\mathbb{R}^N} |\nabla w|^2 + w^2 dy \leq \frac{1}{1-\delta_0} \int_{\mathbb{R}^N} f((1-\delta_0)w) w dy < \int_{\mathbb{R}^N} f(w) w dy \,.$$

It is impossible since $w$ is a least energy solution of (2.6). We can show that the case of (3.20) is also impossible in a similar way.

By Claim 1 and $v_{h_k}(y) \leq C \exp(-\alpha |y|)$, we have

$$e(t_k, k, \varepsilon) \to 1 \quad \text{as } k \to \infty \,. \tag{3.21}$$

We will estimate the square bracket part in (3.17). Let $z_k \in H^1(\mathbb{R}^N \setminus B_{(1+\varepsilon)\rho_k})$ be a solution of the equation

$$\begin{cases} -\Delta z_k + e(t_k, k, \varepsilon) M_\varepsilon z_k = 0 & \text{in } \mathbb{R}^N \setminus \bar{B}_{(1+\varepsilon)\rho_k} \,, \\ z_k((1+\varepsilon)\rho_k) = v_{h_k}((1+\varepsilon)\rho_k), \; z_k(x) \to 0 \; (|x| \to \infty) \,, \end{cases}$$

where $1 < M_\varepsilon := V(1+\varepsilon)$.

CLAIM 2. $\left( \frac{\partial z_k}{\partial \nu} - \frac{\partial v_{h_k}}{\partial \nu} \right) \leq 0$ *on* $\partial B_{(1+\varepsilon)\rho_k}$.

Take any $k$ and any $R > 2\rho_k$.

$$\begin{aligned} &- \Delta(v_{h_k} - z_k)(y) + M_\varepsilon e(t_k, k, \varepsilon)(v_{h_k} - z_k)(y) \\ &= f(v_{h_k}(y)) - V(h_k y) v_{h_k}(y) + v_{h_k}(y) e(t_k, k, \varepsilon) M_\varepsilon \\ &\leq f(v_{h_k}(y)) - V(h_k y) v_{h_k}(y) + \left[ 1 - \frac{f((t_k+1) v_{h_k}(y))}{(t_k+1) v_{h_k}(y)} \right] v_{h_k}(y) M_\varepsilon \\ &\leq f(v_{h_k}(y))[1 - M_\varepsilon] + v_{h_k}(y)[M_\varepsilon - V(h_k y)] \\ &\leq 0 \quad \text{for } y \in B_R \setminus \bar{B}_{(1+\varepsilon)\rho_k} \,. \end{aligned}$$

Hence by the weak maximum principle,

$$v_{h_k}(y) - z_k(y) \leq \max_{\substack{|y|=R \\ or \\ |y|=(1+\varepsilon)\rho_k}} \left[ v_{h_k}(y) - z_k(y) \right]_+ \quad \text{for } (1+\varepsilon)\rho_k \leq |y| \leq 2\rho_k \,.$$

As $R \to \infty$, we have $v_{h_k} \le z_k$ in $\bar{B}_{2\rho_k} \setminus B_{(1+\varepsilon)\rho_k}$. Hence Claim 2 follows.

By Claim 2, we have

$$-v_{h_k}^{'}((1+\varepsilon)\rho_k) \ge -z_k^{'}((1+\varepsilon)\rho_k)\,. \tag{3.22}$$

Lemma 3.3 yields that

$$\lim_{k\to\infty} \frac{-z_k^{'}((1+\varepsilon)\rho_k)}{v_{h_k}((1+\varepsilon)\rho_k)} = \sqrt{M_\varepsilon}\,, \tag{3.23}$$

and

$$\lim_{k\to\infty} \frac{w_k^{'}((1+\varepsilon)\rho_k)}{v_{h_k}((1+\varepsilon)\rho_k)} = -1\,. \tag{3.24}$$

Therefore, by (3.21)–(3.26), we obtain that

$$-e(t_k,k,\varepsilon)v_{h_k}^{'}((1+\varepsilon)\rho_k)+w_k^{'}((1+\varepsilon)\rho_k) \ge \frac{\sqrt{M_\varepsilon}-1}{2}v_{h_k}((1+\varepsilon)\rho_k) \quad \text{as } k \to \infty\,. \tag{3.25}$$

Hence, we have

$$\begin{aligned} c_{h_k}(V) &\ge c_* + Cv_{h_k}((1+\varepsilon)\rho_k)^2 - \exp\Big(-\frac{5}{2}\rho_k\Big) \\ &\ge c_* + C\exp\left(-2\rho_k\left[\sqrt{V(1+2\varepsilon)}(1+\varepsilon) - \frac{\log C(\varepsilon)}{\rho_k} + o(1)\right]\right) \\ &\quad - \exp\Big(-\frac{5}{2}\rho_k\Big)\,. \end{aligned}$$

Since $V$ is lower semicontinuous and nondecreasing, and $V(1) = 1$, passing to a subsequence, we obtain that

$$c_{h_k}(V) \ge c_* + \exp(-2\rho_k(1+o(1))) \quad k \to \infty\,.$$

□

The following result holds from Proposition 3.3. We need this result to show Theorem 2.1.

COROLLARY 3.1. *Assume* (f0)–(f4), (V1), (V2) *and* (Vb). *Moreover, assume*

$$\Omega = \Big\{x \in \mathbb{R}^N \mid V(x) = \inf_{y\in\mathbb{R}^N} V(y) = 1\Big\} = \{0\}\,. \tag{3.26}$$

*Then for any $\varepsilon > 0$ there exists a subsequence $\{h_k\}_{k=1}^{\infty} \subset \{h\}$ such that*

$$c_{h_k}(V) \ge c_* + \exp\Big(-\frac{2}{h_k}[\varepsilon + o(1)]\Big) \quad \textit{as } k \to \infty\,.$$

PROOF. Fix any $\varepsilon > 0$. Put

$$\bar{V}(z) := \chi_{\mathbb{R}^N \setminus \bar{B}_\varepsilon}(z) V(z) + \chi_{\bar{B}_\varepsilon}(z) . \tag{3.27}$$

Then, we note that $\bar{V}$ is Borel measurable, and satisfies (V1), (V2). Moreover, note that

$$\{y \in \mathbb{R}^N \mid \bar{V}(y) = 1\} = \bar{B}_\varepsilon , \tag{3.28}$$

and $\bar{V}(z) \le V(z)$. Then we obtain that

$$c_h(V) \ge c_h(\bar{V}) . \tag{3.29}$$

For any $v \in E_h(\bar{V}) \setminus \{0\}$, we obtain that

$$J_h[v; \bar{V}] \ge J_h[v^*; \bar{V}_*] , \tag{3.30}$$

by Propositions A.1–A.3, where $\bar{V}_*$ denotes the increasing rearrangement of $\bar{V}$. Remarking that $v^* \in E_h(\bar{V}_*)$ holds for any $v \in E_h(\bar{V})$, we have

$$\sup_{t>0} J_h[tv; \bar{V}] \ge \sup_{t>0} J_h[tv^*; \bar{V}_*] \ge c_h(\bar{V}_*) , \tag{3.31}$$

for any $v \in E_h(\bar{V})$ and $t > 0$. Hence, we obtain that

$$c_h(\bar{V}) \ge c_h(\bar{V}_*) . \tag{3.32}$$

Note that

$$\{y \in \mathbb{R}^N \mid \bar{V}_*(y) = 1\} = \bar{B}_\varepsilon ,$$

and $\bar{V}_*$ satisfies (V1), (V2). By Remark 3.2, passing to a subsequence, we obtain

$$c_{h_k}(\bar{V}_*) \ge c_* + \exp\Big(-\frac{2}{h_k}[\varepsilon + o(1)]\Big) \quad \text{as } k \to \infty . \tag{3.33}$$

(3.29), (3.32) and (3.33) yield the estimate

$$c_{h_k}(V) \ge c_* + \exp\Big(-\frac{2}{h_k}[\varepsilon + o(1)]\Big) \quad \text{as } k \to \infty .$$

□

## 4. Proof of Theorem 2.1

First, we shall prove (iii) of Theorem 2.1.

**4.1. Proof of the upper bound.** Take $x_0 \in \Omega$ such that $\operatorname{dist}(x_0, \partial\Omega) = \max_{x\in\Omega} \operatorname{dist}(x, \partial\Omega)$, and put $r := \operatorname{dist}(x_0, \partial\Omega)$. Put

$$\tilde{V}(y) := V(y) + (|y - x_0|^2 - r^2)^3 \chi_{\mathbb{R}^N \setminus \bar{B}(x_0,r)}(y) .$$

Then, $\tilde{V} \in C^1(\mathbb{R}^N)$, $\inf_{x\in\mathbb{R}^N} \tilde{V}(x) = 1$, $\lim_{|x|\to\infty} \tilde{V}(x) = +\infty$ and $\{\tilde{V} \equiv 1\} = \bar{B}(x_0, r)$. By $V(y) \le \tilde{V}(y)$, we obtain that

$$J_h[tv; V] \le J_h[tv; \tilde{V}] \quad \text{for any } t > 0 \text{ and } v \in E_h(\tilde{V}) \setminus \{0\}\,.$$

Hence we have $c_h(V) \le c_h(\tilde{V})$ by $E_h(\tilde{V}) \subset E_h(V)$. By Remark 3.1, it holds that

$$c_h(V) \le c_* + \exp\Big(-\frac{2}{h}[r + o(1)]\Big) \quad \text{as } h \to 0\,.$$

**4.2. Proof of the lower bound.** By Proposition 2.1 (ii), $x_{h_k} \to x_0 \in \Omega$ as $k \to \infty$.

$$d_k := \operatorname{dist}(x_{h_k}, \partial\Omega) \to \operatorname{dist}(x_0, \partial\Omega) =: d_0 \quad \text{as } k \to \infty\,.$$

Choose any $\delta > 0$. Put $\delta' := \delta/2$. Let us choose a number $d_0' \ge 0$ so that

$$|B(x_0; d_0')| = |\Omega \cap B(x_0; d_0 + \delta')|\,.$$

Note that $d_0 + \delta' > d_0'$.
Take $R > 0$ so that $\Omega \subset B(x_0; R)$. Take $\eta \in C^\infty(\mathbb{R}^N)$ such that

$$\eta \equiv 0 \quad \text{on } \bar{B}(x_0; d_0 + \delta') \cup (\mathbb{R}^N \setminus B(x_0; R))\,,$$
$$1 \ge \eta > 0 \quad \text{on } B(x_0; R) \setminus \bar{B}(x_0; d_0 + \delta')\,.$$

Put

$$\tilde{V}(x) := V(x) + \eta(x)\,.$$

Then $\{x \in \mathbb{R}^N \mid \tilde{V}(x) = 1\} = \Omega \cap \bar{B}(x_0; d_0 + \delta')$. For all $0 < t \le 2$,

$$\begin{aligned}
c_{h_k}(V) &= J_{h_k}[v_{h_k}; V] \\
&\ge J_{h_k}[tv_{h_k}; V] \\
&= \frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla v_{h_k}|^2 + \tilde{V}(h_k y)v_{h_k}^2\,dy - \int_{\mathbb{R}^N} F(tv_{h_k})\,dy - \frac{t^2}{2}\int_{\mathbb{R}^N} \eta(h_k y)v_{h_k}^2\,dy \\
&=: I_1 - I_2\,.
\end{aligned}$$

First, we shall estimate $I_2$. By Proposition 2.1, we may estimate

$$\begin{aligned}
I_2 &= \frac{1}{h_k^N}\frac{t^2}{2}\int_{\mathbb{R}^N} \eta(x)u_{h_k}(x)^2\,dx \\
&\le \frac{C}{h_k^N}\frac{2^2}{2}\int_{B(x_0;R)\setminus B(x_0;d_0+\delta')} \exp\Big(-\frac{2|x - x_{h_k}|}{h_k}\Big)\,dx \\
&\le \frac{C}{h_k^N}\int_{B(x_0;R)\setminus B(x_0;d_0+\delta')} \exp\Big(-\frac{2|x - x_0| - 2|x_0 - x_{h_k}|}{h_k}\Big)\,dx
\end{aligned}$$

$$\le \frac{C}{h_k^N}\int_{B(x_0;R)\setminus B(x_0;d_0+\delta')} \exp\Big(-\frac{2(d_0+\delta')-2|x_0-x_{h_k}|}{h_k}\Big)\,dx$$
$$\le \exp\Big(-\frac{2}{h_k}\Big[d_0+\delta'-|x_0-x_{h_k}|-\frac{h_k}{2}\log\Big(\frac{C}{h_k^N}\Big)\Big]\Big)$$
$$= \exp\Big(-\frac{2}{h_k}[d_0+\delta'+o(1)]\Big)\,. \tag{4.1}$$

Next, we shall estimate $I_1$. By Propositions A.1–A.3, we obtain that

$$I_1 \ge \frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla v_{h_k}^*|^2 + \tilde{V}_*(h_k y)(v_{h_k}^*)^2\,dy - \int_{\mathbb{R}^N} F(t v_{h_k}^*)\,dy\,. \tag{4.2}$$

By (4.1) and (4.2), we obtain

$$c_{h_k}(V) \ge J_{h_k}[t v_{h_k}^*;\tilde{V}_*] - \exp\Big(-\frac{2}{h_k}[d_0+\delta'+o(1)]\Big)\,. \tag{4.3}$$

Let us choose a number $t_k > 0$ so that

$$J_{h_k}[t_k v_{h_k}^*;\tilde{V}_*] = \sup_{t>0} J_{h_k}[t v_{h_k}^*;\tilde{V}_*]\,.$$

CLAIM 1. *For sufficiently large $k$, $t_k \le 2$.*

Assume that there exists a subsequence such that $t_k > 2$. By (f4), we obtain that

$$\begin{aligned}
A_k &:= \int_{\mathbb{R}^N} |\nabla v_{h_k}|^2 + \tilde{V}(h_k y)v_{h_k}^2\,dy \\
&\ge \int_{\mathbb{R}^N} |\nabla v_{h_k}^*|^2 + \tilde{V}_*(h_k y)(v_{h_k}^*)^2\,dy \\
&= \frac{1}{t_k}\int_{\mathbb{R}^N} f(t_k v_{h_k}^*)v_{h_k}^*\,dy \\
&> \frac{1}{2}\int_{\mathbb{R}^N} f(2v_{h_k}^*)v_{h_k}^*\,dy \\
&=: B_k\,.
\end{aligned}$$

$$\begin{aligned}
A_k &= \int_{\mathbb{R}^N} |\nabla v_{h_k}|^2 + V(h_k y)v_{h_k}^2\,dy + \int_{\mathbb{R}^N} \eta(h_k y)v_{h_k}^2\,dy \\
&= \int_{\mathbb{R}^N} f(v_{h_k})v_{h_k}\,dy + \frac{1}{{h_k}^N}\int_{B(x_0;R)\setminus B(x_0;d_0+\delta')} \eta(x)u_{h_k}(x)^2\,dx \\
&\le \int_{\mathbb{R}^N} f(u_{h_k}(h_k y+x_{h_k}))u_{h_k}(h_k y+x_{h_k})\,dy + \exp\Big(-\frac{2}{h_k}[d_0+\delta'+o(1)]\Big)\,.
\end{aligned}$$

On the other hand,

$$B_k = \frac{1}{2}\int_{\mathbb{R}^N} f(2u_{h_k}(h_k y + x_{h_k}))u_{h_k}(h_k y + x_{h_k})\,dy\,.$$

Hence, we have

$$\int_{\mathbb{R}^N} f(u_{h_k}(h_k y + x_{h_k}))u_{h_k}(h_k y + x_{h_k})\,dy + \exp\Big(-\frac{2}{h_k}[d_0 + \delta' + o(1)]\Big) \geq \frac{1}{2}\int_{\mathbb{R}^N} f(2u_{h_k}(h_k y + x_{h_k}))u_{h_k}(h_k y + x_{h_k})\,dy\,. \quad (4.4)$$

By $u_{h_k}(h_k \cdot + x_{h_k}) \to w$ in $H^1(\mathbb{R}^N)$, taking $k \to \infty$ on (4.4), we have

$$\int_{\mathbb{R}^N} f(w)w\,dy \geq \frac{1}{2}\int_{\mathbb{R}^N} f(2w)w\,dy > \int_{\mathbb{R}^N} f(w)w\,dy\,.$$

This is impossible.

Hence,

$$c_{h_k}(V) \geq c_{h_k}(\tilde{V}_*) - \exp\Big(-\frac{2}{h_k}[d_0 + \delta' + o(1)]\Big) \quad \text{as } k \to \infty\,.$$

We will prove the next claim to use Proposition 3.2.

CLAIM 2. $\{\tilde{V}_* = 1\} = \bar{B}(0; d_0')$.

We will prove $\tilde{V}_*(d_0') = 1$. Assume $\tilde{V}_*(d_0') > 1$. Then we see that there exists $t_0 > 1$ such that $d_0' \in \mathbb{R}^N \setminus \{\tilde{V} < t_0\}^*$. Put $B(0; r_0) := \{\tilde{V} < t_0\}^*$. Note that $r_0 \leq d_0'$. Since $\{\tilde{V} = 1\} \subset \{\tilde{V} < t_0\}^*$, we have $B(0; d_0') \subset B(0; r_0)$. Thus, $d_0' \leq r_0$. Hence, $d_0' = r_0$. By

$$|\{\tilde{V} = 1\}| + |\{1 < \tilde{V} < t_0\}| = |\{\tilde{V} < t_0\}| = |B(0; r_0)| = |B(0; d_0')| = |\{\tilde{V} = 1\}|\,,$$

it follows that $|\{1 < \tilde{V} < t_0\}| = 0$. On the other hand, by the continuity of $\tilde{V}$, $\{\tilde{V} = 1\} \neq$, and $\tilde{V}(x) \to +\infty$ as $|x| \to \infty$, we have $|\{1 < \tilde{V} < t_0\}| > 0$. This is impossible.

**Case 1.** $d_0' = 0$ *for some* $\delta > 0$.

By Claim 2, we can use Corollary 3.1, hence we have

$$c_{h_k}(\tilde{V}_*) \geq c_* + \exp\Big(-\frac{2}{h_k}[\varepsilon + o(1)]\Big) \quad \text{as } k \to \infty\,, \quad (4.5)$$

for any $\varepsilon > 0$. On the other hand, by subsection 5.1, we have

$$c_{h_k}(V) \leq c_* + \exp\Big(-\frac{2}{h_k}\Big[\max_{x\in\Omega} d(x, \partial\Omega) + o(1)\Big]\Big) \quad \text{as } k \to \infty\,. \quad (4.6)$$

By (4.5) and (4.6), we have $\varepsilon \geq \max_{x\in\Omega} d(x, \partial\Omega)$ for any $\varepsilon > 0$. This is impossible.

**Case 2.** $d_o^{'} > 0$ *for any* $\delta > 0$.

By Claim 2, we can use Proposition 3.3, hence we have

$$c_{h_k}(\tilde{V}_*) \geq c_* + \exp\Big(-\frac{2}{h_k}[d_0^{'} + o(1)]\Big) \quad \text{as } k \to \infty\,.$$

Hence,

$$\begin{aligned} c_{h_k}(V) &\geq c_* + \exp\Big(-\frac{2}{h_k}[d_0^{'} + o(1)]\Big) - \exp\Big(-\frac{2}{h_k}[d_0 + \delta^{'} + o(1)]\Big) \\ &= c_* + \exp\Big(-\frac{2}{h_k}[d_0 + \delta]\Big) \\ &\quad \times \Big\{\exp\Big(-\frac{2}{h_k}[d_0^{'} - d_0 - \delta + o(1)]\Big) - \exp\Big(-\frac{2}{h_k}[\delta^{'} - \delta + o(1)]\Big)\Big\}\,. \end{aligned}$$

CLAIM 3.

$$\Big\{\exp\Big(-\frac{2}{h_k}[d_0^{'} - d_0 - \delta + o(1)]\Big) - \exp\Big(-\frac{2}{h_k}[\delta^{'} - \delta + o(1)]\Big)\Big\} \geq 1 \quad \textit{as } k \to \infty\,.$$

Since $\delta^{'} - \delta < 0$ and $d_0^{'} - d_0 - \delta^{'} < 0$, we have

$$\begin{aligned} &\Big\{\exp\Big(-\frac{2}{h_k}[d_0^{'} - d_0 - \delta + o(1)]\Big) - \exp\Big(-\frac{2}{h_k}[\delta^{'} - \delta + o(1)]\Big)\Big\} \\ &= \exp\Big(-\frac{2}{h_k}[\delta^{'} - \delta + o(1)]\Big)\Big\{\exp\Big(-\frac{2}{h_k}[d_0^{'} - d_0 - \delta^{'} + o(1)]\Big) - 1\Big\} \\ &\geq 1\,. \end{aligned}$$

Hence,

$$c_{h_k}(V) \geq c_* + \exp\Big(-\frac{2}{h_k}[d(x_0, \partial\Omega) + o(1)]\Big) \quad \text{as } k \to \infty\,.$$

**4.3. Proof of (i) and (ii) of Theorem 2.1.** First, we shall prove (i). Assume that there exists a subsequence such that $x_{h_k} \in \Omega^{\mathrm{c}}$. Then passing to a subsequence, we learn $x_{h_k} \to x_0 \in \Omega$. By the argument in the proof of (iii), we have

$$c_{h_k}(V) \geq c_* + \exp\Big(-\frac{2}{h_k}[d(x_0, \partial\Omega) + o(1)]\Big) \quad \text{as } k \to \infty\,, \tag{4.7}$$

$$c_{h_k}(V) \leq c_* + \exp\Big(-\frac{2}{h_k}[d_0 + 0(1)]\Big) \quad \text{as } k \to \infty\,, \tag{4.8}$$

where $d_0 := \max_{x\in\Omega} d(x, \partial\Omega)$. By (4.7) and (4.8), we see that $0 = d(x_0, \partial\Omega) \geq d_0 > 0$. This is impossible. Hence, we have proved (i).

Next, we shall prove (ii). Assume that there exist a constant $\delta > 0$ and a subsequence such that for $k$ sufficiently large,

$$d(x_{h_k}, \partial\Omega) \geq d_0 + \delta \tag{4.9}$$

or

$$d(x_{h_k}, \partial\Omega) \leq d_0 - \delta\,. \tag{4.10}$$

Passing to a subsequence, we have $x_{h_k} \to x_0 \in \Omega$. By (iii), it follows that $d(x_0, \partial\Omega) \geq d_0$. On the other hand, from (4.9) or (4.10), we see that $d(x_0, \partial\Omega) \geq d_0+\delta$ or $d(x_0, \partial\Omega) \leq d_0-\delta$. This is impossible. Hence, we have proved (ii).

## A. Rearrangements

The rearrangement is a key in the proof of our main result. So, we prepare properties of the decreasing rearrangement and increasing rearrangement.

First, we recall the decreasing rearrangement.

DEFINITION A.1. Assume that $f : \mathbb{R}^N \to \mathbb{R}$ is Borel measurable and for any $t > 0$, $|\{|f| > t\}| < \infty$. We define the decreasing rearrangement $f^*$ of $f$ as

$$f^*(x) := \int_0^\infty \chi_{\{|f|>t\}^*}(x)\, dt, \quad x \in \mathbb{R}^N\,,$$

where for a Borel measurable set $A \subset \mathbb{R}^N$, there exists $r \geq 0$ such that $|A| = |B(0; r)|$, and put $A^* := B(0; r)$.

The following results are well known (see, [15]).

PROPOSITION A.1. *Assume that $f : \mathbb{R}^N \to \mathbb{R}$ is Borel measurable and for any $t > 0$, $|\{|f| > t\}| < \infty$. Then the following statements hold*:

(i) *$f^*$ is radially symmetric and decreasing, i.e.,*

$$f^*(x) = f^*(y) \quad \text{if } |x| = |y|\,,$$

*and*

$$f^*(x) \geq f^*(y) \quad \text{if } |x| \leq |y|\,.$$

(ii) *If $F : \mathbb{R} \to \mathbb{R}$ is Borel measurable and $F \geq 0$, then*

$$\int_{\mathbb{R}^N} F(f^*(x))\, dx = \int_{\mathbb{R}^N} F(f(x))\, dx\,.$$

(iii) *If $\psi : \mathbb{R} \to \mathbb{R}$ is non-decreasing, then*

$$(\psi(f))^* = \psi(f^*)\,.$$

(iv) *If $f \in H^1(\mathbb{R}^N)$, then*

$$\int_{\mathbb{R}^N} |\nabla f^*(x)|^2\, dx \leq \int_{\mathbb{R}^N} |\nabla f(x)|^2\, dx\,.$$

Next, we define the increasing rearrangement.

DEFINITION A.2. Assume that $f : \mathbb{R}^N \to \mathbb{R}$ is Borel measurable and for any $t > 0$, $|\{|f| < t\}| < \infty$. We define the increasing rearrangement $f_*$ of $f$ as

$$f_*(x) := \int_0^\infty \chi_{\mathbb{R}^N \setminus \{|f|<t\}^*}(x)\, dt, \quad x \in \mathbb{R}^N .$$

PROPOSITION A.2. *Assume that $f : \mathbb{R}^N \to \mathbb{R}$ is Borel measurable and for any $t > 0$, $|\{|f| < t\}| < \infty$. Then the following statements hold*:

(i) *$f_*$ is radially symmetric and increasing, i.e.,*

$$f_*(x) = f_*(y) \quad \text{if } |x| = |y|$$

*and*

$$f_*(x) \le f_*(y) \quad \text{if } |x| \le |y| .$$

(ii) *For all $t > 0$,*

$$\{x \in \mathbb{R}^N \mid f_*(x) < t\} = \{x \in \mathbb{R}^N \mid |f(x)| < t\}^* .$$

(iii) *$f_*$ is right continuous, i.e., for $r \ge 0$,*

$$\lim_{h \to +0} f_*(r + h) = f_*(r) .$$

Since we cannot find the exact reference for the increasing rearrangement in $\mathbb{R}^N$, we shall give a proof here.

PROOF OF PROPOSITION A.2. It is easy to see that (i) holds. So, we will prove (ii) and (iii). Assume that

$$f_*(x) < t . \tag{A.1}$$

If $x \notin \{t > |f|\}^*$, then

$$\begin{aligned} t &> \int_0^\infty \chi_{\mathbb{R}^N \setminus \{s>|f|\}^*}(x)\, ds \\ &\ge \int_0^t \chi_{\mathbb{R}^N \setminus \{s>|f|\}^*}(x)\, ds \\ &\ge \int_0^t \chi_{\mathbb{R}^N \setminus \{t>|f|\}^*}(x)\, ds = t , \end{aligned}$$

where the last equality holds by $\chi_{\mathbb{R}^N \setminus \{t>|f|\}^*}(x) = 1$. This is impossible. Therefore, we obtain $x \in \{t > |f|\}^*$. Hence

$$\{x \in \mathbb{R}^N \mid f_*(x) < t\} \subset \{t > |f|\}^* . \tag{A.2}$$

Next, we will prove

$$\{x \in \mathbb{R}^N \mid f_*(x) < t\} \supset \{t > |f|\}^* . \tag{A.3}$$

Assume that $x \in \{t > |f|\}^*$. Here, we will show the next claim.

CLAIM 1. *There exists* $0 < t_0 < t$ *such that* $x \in \{|f| < t_0\}^*$.

If not, then $x \notin \{|f| < s\}^*$ for all $0 < s < t$. Take $r > 0$ such that $B(0; r) = \{|f| < t\}^*$ and $r_s \geq 0$ such that $B(0; r_s) = \{|f| < s\}^*$. Note that $|x| < r$ and $|x| \geq r_s$. Then, we may estimate

$$\begin{aligned} |B(0;r)| &= \left|\left\{x \in \mathbb{R}^N \mid |f(x)| < t\right\}\right| \\ &= \left|\bigcup_{0<s<t} \left\{x \in \mathbb{R}^N \mid |f(x)| < s\right\}\right| \\ &= \lim_{s\uparrow t}\left|\left\{x \in \mathbb{R}^N \mid |f(x)| < s\right\}\right| \\ &= \lim_{s\uparrow t} |B(0;r_s)| \\ &\leq |B(0;|x|)| \\ &< |B(0;r)|\,. \end{aligned}$$

This is impossible. Hence, Claim 1 holds.

By $x \in \{|f| < t_0\}^*$, we obtain that

$$\chi_{\mathbb{R}^N\setminus\{s>|f|\}^*}(x) \leq \chi_{\mathbb{R}^N\setminus\{t_0>|f|\}^*}(x) = 0 \quad \text{for any } t_0 \leq s < +\infty\,. \tag{A.4}$$

Hence, we have

$$\begin{aligned} f_*(x) &= \int_0^{t_0} \chi_{\mathbb{R}^N\setminus\{s>|f|\}^*}(x)\,ds + \int_{t_0}^{+\infty} \chi_{\mathbb{R}^N\setminus\{s>|f|\}^*}(x)\,ds \\ &= \int_0^{t_0} \chi_{\mathbb{R}^N\setminus\{s>|f|\}^*}(x)\,ds \\ &\leq t_0 < t\,. \end{aligned}$$

Therefore, we have proved (ii).

Finally, we will prove (iii). Let $x \in \mathbb{R}^N$ be $|x| = r$. Note that for all $t > 0$, $\{y \in \mathbb{R}^N \mid f_*(y) < t\}$ is an open set since the right hand of (ii) is an open ball. Therefore, since for any $r \geq 0$ and any $\varepsilon > 0$, $\{y \in \mathbb{R}^N \mid f_*(y) < f_*(x) + \varepsilon\}$ is an open set, there exists $\delta > 0$ such that

$$B(x;\delta) \subset \left\{y \in \mathbb{R}^N \mid f_*(y) < f_*(x) + \varepsilon\right\}. \tag{A.5}$$

(A.5) and (i) yield that

$$0 \leq f_*(r+h) - f_*(r) < \varepsilon \quad \text{for } 0 < h < \delta\,.$$

Hence, (iii) holds. □

In particular, the following result is very important.

PROPOSITION A.3. *Assume that $f : \mathbb{R}^N \to \mathbb{R}$ and $g : \mathbb{R}^N \to \mathbb{R}$ are Borel measurable and for any $t > 0$, $|\{|f| > t\}| < \infty$ and $|\{|g| < t\}| < \infty$. Then,*

$$\int_{\mathbb{R}^N} f^*(x) g_*(x)\, dx \le \int_{\mathbb{R}^N} |f(x)||g(x)|\, dx\,. \tag{A.6}$$

PROOF OF PROPOSITION A.3. First, we remark the following inequality.

CLAIM 1. *Let $A, B \subset \mathbb{R}^N$ be measurable sets such that $|A| < \infty$ and $|B| < \infty$. Then, it follows that*

$$|A \setminus B| \ge |A^* \setminus B^*|\,. \tag{A.7}$$

If $|B| \ge |A|$, we have $B^* \supset A^*$, hence we obtain that $|A^* \setminus B^*| = 0$. Therefore, (A.7) holds in this case. Assume that $|B| < |A|$. Note that $B^* \subset A^*$. Then, we may estimate

$$|A \setminus B| \ge |A| - |B| = |A^*| - |B^*| = |A^* \setminus B^*|\,.$$

Hence, (A.7) holds in this case too.

We will prove (A.6). By Fubini's Theorem and Claim 1, we may estimate

$$\begin{aligned}
&\int_{\mathbb{R}^N} |f(x)||g(x)|\, dx \\
&= \int_{\mathbb{R}^N} \int_0^{+\infty} \int_0^{+\infty} \chi_{\{s<|f(x)|\}}(s) \chi_{\{t\le|g(x)|\}}(t)\, dsdtdx \\
&= \int_0^{+\infty} \int_0^{+\infty} \int_{\mathbb{R}^N} \chi_{\{s<|f(x)|\}}(x) \chi_{\mathbb{R}^N\setminus\{t>|g(x)|\}}(x)\, dxdsdt \\
&= \int_0^{+\infty} \int_0^{+\infty} |\{x \in \mathbb{R}^N \mid s < |f(x)|\} \setminus \{x \in \mathbb{R}^N \mid t > |g(x)|\}|\, dsdt \\
&\ge \int_0^{+\infty} \int_0^{+\infty} |\{s < |f|\}^* \setminus \{t > |g|\}^*|\, dsdt \\
&= \int_0^{+\infty} \int_0^{+\infty} \int_{\mathbb{R}^N} \chi_{\{s<|f|\}^*}(x) \chi_{\mathbb{R}^N\setminus\{t>|g|\}^*}(x)\, dxdsdt \\
&= \int_{\mathbb{R}^N} f^*(x) g_*(x)\, dx\,.
\end{aligned}$$

□

## B. Proof of Propositions 2.1 and 3.2

We will prove Propositions 2.1 and 3.2 in this section. For simplicity, we use symbols $E_h$, $J_h[v]$ and $c_h$ instead of $E_h(V)$, $J_h[v; V]$ and $c_h(V)$ respectively.

First, We will use the next Lemma to show Proposition 2.1 and Proposition 3.2. This Lemma gives another representation of the least energy. This result is well known. For the proof, see e.g. [21, Proposition 3.11].

LEMMA B.1. *Assume that* (f0)–(f4) *hold and* $V$ *is Borel measurable and satisfies* (V2)*. Then, it holds that*

$$\inf_{v\in N_h} J_h[v] = \inf_{\substack{v\in E_h \\ v\not\equiv 0}} \sup_{t>0} J_h[tv]\,,$$

*where* $N_h := \{v \in E_h \setminus \{0\} \mid \langle J_h'[v], v\rangle = 0\}$.

Now, we shall prove Proposition 2.1.

PROOF OF PROPOSITION 2.1. Our proof is carried out in eleven steps. In Steps 1–3, we show that the existence of a least energy solution $u_h$. In Step 4, we show that the existence of a point $x_h$ where $u_h$ reaches its maximum value, and in Steps 5–7, we show the properties for the maximum point $x_h$. In Steps 8–11, we show the properties for a least energy solution.

**Step 1.** *There exists* $v_h \in N_h$ *such that* $m_h := \inf_{v\in N_h} J_h[v] = J_h[v_h]$.

By the definition of $m_h$, there exists $\{v_j\}_{j=1}^{\infty} \subset N_h$ such that $J_h[v_j] \to m_h$ as $j \to \infty$. Then, for $j$ sufficiently large,

$$\begin{aligned} m_h + 1 &\geq J_h[v_j] \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_j|^2 + V(hy)v_j^2\,dy - \int_{\mathbb{R}^N} F(v_j)\,dy \\ &\geq \frac{1}{2}\|v_j\|_{E_h}^2 - \frac{1}{\theta}\int_{\mathbb{R}^N} f(v_j)v_j\,dy \\ &= \left(\frac{1}{2} - \frac{1}{\theta}\right)\|v_j\|_{E_h}^2\,. \end{aligned}$$

Hence, $\|v_j\|_{E_h}$ is bounded uniformly for $j$. Moreover,

$$\begin{aligned} \|v_j\|_{E_h}^2 &= \int_{\mathbb{R}^N} f(v_j)v_j\,dy \\ &\leq \int_{\mathbb{R}^N} \frac{1}{2}v_j^2 + Cv_j^{p+1}\,dy \\ &\leq \frac{1}{2}\|v_j\|_{E_h}^2 + C\|v_j\|_{E_h}^{p+1}\,, \end{aligned}$$

hence, there exists $\delta > 0$ such that $\delta \leq \|v_j\|_{E_h} = \int_{\mathbb{R}^N} f(v_j)v_j\,dy$ holds uniformly for $j$. By using the compact embedding $E_h \hookrightarrow L^q(\mathbb{R}^N)$ for $2 \leq q < (2N)/(N-2)$ if $N \geq 3$ and $2 \leq q < \infty$ if $N = 1, 2$ (see e.g. [2]), we have

$$v_j \rightharpoonup w_h \quad \text{weakly in } E_h\,, \tag{B.1}$$

$$v_j \to w_h \quad \text{in } L^q(\mathbb{R}^N)\,, \tag{B.2}$$

$$v_j(y) \to w_h(y) \quad \text{a.e. } y \in \mathbb{R}^N\,, \tag{B.3}$$

where $w_h \in E_h$. We will prove $w_h \in N_h$ and $m_h = J_h[w_h]$. By $\delta \le \int_{\mathbb{R}^N} f(v_j)v_j\,dy$, taking $j \to \infty$, we have $\delta \le \int_{\mathbb{R}^N} f(w_h)w_h\,dy$. Hence, $w_h \not\equiv 0$. Moreover, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N} |\nabla w_h|^2 + V(hy)w_h^2\,dy - \int_{\mathbb{R}^N} f(w_h)w_h\,dy\\&\le \liminf_{j\to\infty}\left[\int_{\mathbb{R}^N} |\nabla v_j|^2 + V(hy)(v_j)^2\,dy - \int_{\mathbb{R}^N} f(v_j)v_j\,dy\right] = 0\,.\end{aligned}$$

Thus there exists a constant $0 < t_0 \le 1$ such that $t_0 w_h \in N_h$.

$$\begin{aligned}m_h \le J_h[t_0 w_h] &= \int_{\mathbb{R}^N} \frac{1}{2} f(t_0 w_h)t_0 w_h - F(t_0 w_h)\,dy\\&\le \int_{\mathbb{R}^N} \frac{1}{2} f(w_h)w_h - F(w_h)\,dy\\&= \lim_{j\to\infty}\int_{\mathbb{R}^N} \frac{1}{2} f(v_j)v_j - F(v_j)\,dy\\&= \lim_{j\to\infty} J_h[v_j] = m_h\,.\end{aligned}$$

Hence, by (f4), we have $t_0 = 1$. Therefore, we can conclude $w_h \in N_h$ and $m_h = J_h[w_h]$.

**Step 2.** *We claim that $J_h^{'}[v_h] = 0$ holds.*

Let $\varphi \in E_h \setminus \{0\}$. Then, there exists $\varepsilon > 0$ such that $v_h + s\varphi \not\equiv 0$ $(-\varepsilon < s < \varepsilon)$. We put $\gamma(s,t) := \langle J_h^{'}[t(v_h + s\varphi)], t(v_h + s\varphi)\rangle$. Note that $\gamma \in C^1(\mathbb{R}^2)$ by (f2), and $\gamma(0,1) = 0$ by $v_h \in N_h$. We calculate

$$\begin{aligned}\gamma_t(s,t) &= 2t\int_{\mathbb{R}^N} |\nabla(v_h + s\varphi)|^2 + V(hy)(v_h + s\varphi)^2\,dy\\&\quad - \int_{\mathbb{R}^N} f^{'}(t(v_h + s\varphi))t(v_h + s\varphi)^2\,dy - \int_{\mathbb{R}^N} f(t(v_h + s\varphi))(v_h + s\varphi)^2\,dy\,.\end{aligned}$$

By using (f4),

$$\gamma_t(0,1) = \int_{\mathbb{R}^N} f(v_h)v_h - f^{'}(v_h)v_h^2\,dy < 0\,.$$

Hence, by Implicit Function Theorem, there exists $t \in C^1((-\varepsilon_0, \varepsilon_0))$ such that $t(0) = 1$ and $\gamma(s, t(s)) = 0$. We put $\rho(s) := J_h[t(s)(v_h + s\varphi)]$. By $\gamma(s, t(s)) = 0$ and $t(s)(v_h + s\varphi) \in N_h$, we have $\rho(0) = J_h[v_h] = \inf_{v\in N_h} J_h[v] \le \rho(s)$ $(-\varepsilon_0 < s < \varepsilon_0)$. Hence, $\rho^{'}(0) = 0$, and

$$0 = \rho^{'}(0) = \langle J_h^{'}[v_h], t^{'}(0)v_h\rangle + \langle J_h^{'}[v_h], \varphi\rangle = \langle J_h^{'}[v_h], \varphi\rangle\,.$$

**Step 3.** *We claim that a least energy solution $v_h$ of (2.1) satisfies $v_h > 0$.*

By Step 1, Step 2 and Lemma B.1, we see that there exists a least energy solution $v_h \in E_h$. In particular, the maximum principle yields $v_h > 0$.

**Step 4.** *The existence of a point where $u_h$ reaches its maximum value.*

Note that $u_h(0) > 0$. Put $v_h(y) := u_h(hy)$. Then

$$-\Delta v_h(y) + d_h(y)v_h(y) \le 0, \ \ y \in \mathbb{R}^N ,$$

where $d_h(y) := -Cv_h(y)^{p-1}$. Since $u_h$ is a least energy solution, we have $\|v_h\|_{H^1(\mathbb{R}^N)} \le \|v_h\|_{E_h} \le C$ for all $h > 0$, hence $\|d_h\|_{L^{2^*/(p-1)}(\mathbb{R}^N)} \le C$, where $2^*/(p-1) > N/2$. By Theorem 4.1 in [14], for all $z \in \mathbb{R}^N$,

$$\sup_{y \in B_{\frac{1}{4}}(z)} v_h(y) \le C\|v_h\|_{L^2(B_1(z))} , \tag{B.4}$$

where $C$ is independent of $z$. Take $R = R(h) > 0$ such that $C\|v_h\|_{L^2(\mathbb{R}^N \setminus B_R)} \le u_h(0)/2$. Then, by (B.4), for all $|z| \ge R+1$, we have $\sup_{y \in B_{1/4}(z)} v_h(y) \le u_h(0)/2$. Hence, $v_h(y) \le u_h(0)/2$ for all $|y| \ge R+1$. Thus there exists a point $x_h$ such that where $u_h$ reaches its maximum value.

**Step 5.** *If $u_h$ attains a local maximum at $x_h \in \mathbb{R}^N$, then $\{x_h\}$ is bounded in $\mathbb{R}^N$ for $h$ sufficiently small.*

Assume that there exists a subsequence $\{x_{h_j}\} \subset \{h\}$ such that $|x_{h_j}| \to +\infty$. Take any $R > 0$ and fix it. Then, put $w_j(z) := u_{h_j}(h_j z + x_{h_j})$, for $j$ sufficiently large,

$$-\Delta w_j(z) + 2w_j(z) \le -\Delta w_j(z) + V(h_j z + x_{h_j})w_j(z) = f(w_j(z)), \ \text{for } |z| \le R . \tag{B.5}$$

Hence, for any $\varphi \in C_c^\infty(\mathbb{R}^N)$ with $\varphi \ge 0$, taking $R > 0$ such that $\mathrm{Supp}\varphi \subset B_R$,

$$\int_{\mathbb{R}^N} \nabla w_j(z) \cdot \nabla\varphi + 2w_j(z)\varphi \, dz \le \int_{\mathbb{R}^N} f(w_j(z))\varphi \, dz . \tag{B.6}$$

By $\|w_{h_j}\|_{H^1(\mathbb{R}^N)} \le C$, we have

$$w_j \rightharpoonup w \quad \text{weakly in } H^1(\mathbb{R}^N) , \tag{B.7}$$

$$w_j \to w \quad \text{in } L^p_{\mathrm{loc}}(\mathbb{R}^N) \ (2 \le p < 2^*) . \tag{B.8}$$

Since $x_{h_j}$ is a local maximum point, $w_j(0) = u_{h_j}(x_{h_j}) \ge \bar{u}$, hence $w(0) \ge \bar{u} > 0$, where $\bar{u} > 0$ satisfies $f(\bar{u}) = \bar{u}$. In particular $w \not\equiv 0$. By (B.6), for any $\varphi \in C_c^\infty(\mathbb{R}^N)$ with $\varphi \ge 0$,

$$\int_{\mathbb{R}^N} \nabla w \cdot \nabla\varphi + 2w\varphi \, dz \le \int_{\mathbb{R}^N} f(w)\varphi \, dz . \tag{B.9}$$

Hence, there exists $0 < t_0 < 1$ such that $\sup_{t>0} I[tw] = I[t_0 w]$.

$$\begin{aligned} c_* \le I[t_0 w] &= \int_{\mathbb{R}^N} \frac{1}{2} f(t_0 w)t_0 w - F(t_0 w) \, dz \\ &< \int_{\mathbb{R}^N} \frac{1}{2} f(w)w - F(w) \, dz \end{aligned}$$

$$\leq \varliminf_{j\to\infty} \int_{\mathbb{R}^N} \frac{1}{2} f(w_j)w_j - F(w_j)\,dz$$
$$= \lim_{j\to\infty} c_{h_j} = c_* .$$

This is impossible.

**Step 6.** *Assume $u_h$ attains a local maximum at $x_h \in \mathbb{R}^N$. Passing to a subsequence, we may assume $x_h \to x_0$ as $h \to 0$. Then $V(x_0) = \inf_{x\in\mathbb{R}^N} V(x)$. Moreover, putting $w_h(z) := u_h(hz + x_h)$, a limit $w$ of $w_h$ is a least energy solution of* (2.6).

Applying the argument in the proof of Theorem 2.1 in [22], we see that Step 6 follows.

**Step 7.** *A point where $u_h$ attains a local maximum is unique for sufficiently small $h$.*

Assume that there exists $\{h_j\} \subset \{h\}$ such that $u_{h_j}$ has two local maximum points $x_{h_j}$ and $\tilde{x}_{h_j}$.

CLAIM 2.

$$\frac{|x_{h_j} - \tilde{x}_{h_j}|}{h_j} \to +\infty \quad \textit{as } j \to \infty .$$

Assume that

$$\frac{|x_{h_j} - \tilde{x}_{h_j}|}{h_j} \leq C . \tag{B.10}$$

By Step 5, $\{x_{h_j}\}$ and $\{\tilde{x}_{h_j}\}$ are bounded in $\mathbb{R}^N$. Hence, passing to a subsequence, we may assume $x_{h_j} \to x_0$ and $\tilde{x}_{h_j} \to \tilde{x}_0$. By (B.10), we have $x_0 = \tilde{x}_0$. Put $w_j(z) := u_{h_j}(h_j z + x_{h_j})$. Then, we see that $w_{h_j} \to w$ in $C^2_{\text{loc}}(\mathbb{R}^N)$, where $w$ is a least energy solution of (2.6). Hence, taking any $R > 0$ and any $\varepsilon$, for $j$ sufficiently large, we have

$$\|w_j - w\|_{C^2(\bar{B}_R)} \leq \varepsilon . \tag{B.11}$$

Here, we use the following key Lemma.

LEMMA B.2 ([17, Lemma 4.2]). *Let $\phi \in C^2(\bar{B_a})$ be a radial function satisfying $\phi^{'}(0) = 0$ and $\phi^{''} < 0$ for $0 \leq r \leq a$. Then there exists a $\delta > 0$ such that if $\psi \in C^2(\bar{B_a})$ satisfies* (i) $\nabla\psi(0) = 0$ *and* (ii) $\|\psi - \phi\|_{C^2(\bar{B_a})} < \delta$, *then $\nabla\psi \neq 0$ for $x \neq 0$.*

Take $0 < a < b$ such that $w^{''}(r) < 0$ $(0 \leq r \leq a)$, $w(b) < \bar{u}$ and $C < b$ where $C$ is the constant of (B.10). By using (B.11) as $R := b$ and $\varepsilon := \min\{1/2 \min_{a\leq r\leq b} |w^{'}(r)|,\ \delta/2\}$ where $\delta$ is the constant of Lemma B.2, then for $j$ sufficiently large,

$$|\nabla w_j(z)| \geq |\nabla w(z)| - |\nabla w_j(z) - \nabla w(z)| \geq \min_{a\leq r\leq b} |w^{'}(r)| - \varepsilon > 0 ,$$

if $a \leq |z| \leq b$. On the other hand, by Lemma B.2, we have $\nabla w_j(z) \neq 0$ if $0 < |z| \leq a$. Hence $\nabla w_j(z) \neq 0$ if $0 < |z| \leq b$. But $|(x_{h_j} - \tilde{x}_{h_j})/h_j| \leq C < b$ and $\nabla w_j((\tilde{x}_{h_j} - x_{h_j})/h_j) = 0$. This is impossible.

By Claim 1, for any $R > 0$, we have $|x_{h_j} - \tilde{x}_{h_j}| \geq 2h_j R$ for $j$ sufficiently large. Put $w_j(y) := u_{h_j}(h_j y + x_{h_j})$ and $\tilde{w}_j(y) := u_{h_j}(h_j y + \tilde{x}_{h_j})$. Then,

$$
\begin{aligned}
c_{h_j} &= \frac{1}{h_j^N} \int_{\mathbb{R}^N} \frac{1}{2} f(u_{h_j}(x)) u_{h_j}(x) - F(u_{h_j}(x))\, dx \\
&\geq \frac{1}{h_j^N} \int_{B(x_{h_j}; h_j R)} \frac{1}{2} f(u_{h_j}(x)) u_{h_j}(x) - F(u_{h_j}(x))\, dx \\
&\quad + \frac{1}{h_j^N} \int_{B(\tilde{x}_{h_j}; h_j R)} \frac{1}{2} f(u_{h_j}(x)) u_{h_j}(x) - F(u_{h_j}(x))\, dx \\
&= \int_{B(0;R)} \frac{1}{2} f(w_j(y)) w_j(y) - F(w_j(y))\, dy + \int_{B(0;R)} \frac{1}{2} f(\tilde{w}_j(y)) - F(\tilde{w}_j(y))\, dy\,.
\end{aligned}
$$

As $j \to \infty$, we have

$$
c_* \geq \int_{B(0;R)} \frac{1}{2} f(w) w - F(w)\, dy + \int_{B(0;R)} \frac{1}{2} f(\tilde{w}) \tilde{w} - F(\tilde{w})\, dy\,.
$$

As $R \to \infty$, we see that $c_* \geq 2c_*$. This is impossible.

**Step 8.** *For any $\delta > 0$, there exists $R > 0$ such that for $j$ sufficiently large, $w_j(y) \leq \delta$ if $|y| \geq R$.*

Take any $\delta > 0$. Take $R > 0$ such that $w(R) \leq \delta/2$. By $w_j \to w$ in $C^2_{\mathrm{loc}}(\mathbb{R}^N)$, for $j$ sufficiently large,

$$
w_j(y) \leq w(y) + \frac{\delta}{2}
$$

if $|y| \leq R$. Hence, for $j$ sufficiently large, $w_j(R) \leq \delta$. If there exists $y_0 \in \mathbb{R}^N$ such that $w_j(y_0) > \delta$ and $|y_0| > R$, then, there exists a point $y_1$ where $u_{h_j}$ attains a local maximum such that $|y_1| > R$ by $\lim_{|y| \to \infty} w_j(y) = 0$. This is impossible because of Step 8.

**Step 9.** *For all $\delta > 0$, there exists $R > 0$ such that for $j$ sufficiently large,*

$$
w_j(y) \leq e^{\sqrt{1-\delta} R} e^{-\sqrt{1-\delta} |y|}
$$

*if $|y| \geq R$.*

Take any $\delta > 0$ and fix it. By Step 5, there exists $R > 0$ such that

$$
w_j(R) \leq 1 \text{ and } \frac{f(w_j(y))}{w_j(y)} \leq \delta \text{ if } |y| \geq R\,.
$$

Let $v(r)$ be a solution of the equation

$$
\begin{cases}
v''(r) - (1-\delta) v(r) = 0 \text{ in } (R, +\infty)\,, \\
v(R) = 1,\ v(+\infty) = 0\,.
\end{cases}
\tag{B.12}
$$

Then, we have $v(r) = \exp(-\sqrt{1-\delta}r)\exp(\sqrt{1-\delta}R)$.

$$
\begin{aligned}
&-\Delta(w_j - v)(y) + (1-\delta)(w_j - v)(y) \\
&= f(w_j(y)) - (V(h_j y + x_{h_j}) - 1 + \delta)w_j(y) + v''(|y|) + \frac{N-1}{|y|}v'(|y|) - (1-\delta)v(|y|) \\
&\le w_j(y)\left[\frac{f(w_j(y))}{w_j(y)} - \delta\right] + v''(|y|) - (1-\delta)v(|y|) \\
&\le 0 \quad \text{for } y \in (R, +\infty).
\end{aligned}
$$

By the weak maximum principle, we have $w_j(y) \le \exp(\sqrt{1-\delta}R)\exp(-\sqrt{1-\delta}|y|)$ if $|y| \ge R$.

**Step 10.** *For any $M > 0$, there exists $a(j, M) \in \mathbb{R}$ such that $a(j, M) \to 0$ as $j \to \infty$ and if $|x| \le M$,*

$$
u_{h_j}(x) \le \exp\left(-\frac{|x - x_{h_j}| + a(j, M)}{h_j}\right).
$$

By $\sup_{y\in\mathbb{R}^N} w_j(y) \le C$ and Step 9, we have

$$
u_{h_j}(h_j y + x_{h_j}) = w_j(y) \le C(\delta)e^{-(1-\delta)|y|},
$$

for all $y \in \mathbb{R}^N$. Hence for $j$ sufficiently large,

$$
\begin{aligned}
u_{h_j}(x) &\le C(\delta)\exp\left(-(1-\delta)\frac{|x - x_{h_j}|}{h_j}\right) \\
&= \exp\Big(-\frac{1}{h_j}\{|x - x_{h_j}| - \delta|x - x_{h_j}| - h_j \log C(\delta)\}\Big)
\end{aligned}
$$

for all $x \in \mathbb{R}^N$. Taking any $M > 0$, then, for $j$ sufficiently large,

$$
u_{h_j}(x) \le \exp\Big(-\frac{1}{h_j}\{|x - x_{h_j}| - M\delta - h_j \log C(\delta)\}\Big)
$$

if $|x| \le M$. Passing to a subsequence, we obtain $a(j, M) \in \mathbb{R}^N$ such that $a(j, M) \to 0$ as $j \to \infty$ and

$$
u_{h_j}(x) \le \exp\Big(-\frac{1}{h_j}\{|x - x_{h_j}| + a(j, M)\}\Big)
$$

for $j$ sufficiently large and $|x| \le M$.

**Step 11.** $w_j \to w$ in $H^1(\mathbb{R}^N)$.

Take any $R > 0$.

$$
\limsup_{j\to\infty} \|w_j\|^2_{H^1(\mathbb{R}^N)}
$$

$$
\begin{aligned}
&\le \limsup_{j\to\infty} \|w_j\|_{E_h}^2 \\
&= \limsup_{j\to\infty} \int_{\mathbb{R}^N} f(w_j)w_j \, dy \\
&= \limsup_{j\to\infty} \left\{ \int_{B_R} f(w_j)w_j \, dy + \int_{\mathbb{R}^N\setminus B_R} f(w_j)w_j \, dy \right\} \\
&\le \limsup_{j\to\infty} \left\{ \int_{B_R} f(w_j)w_j \, dy + e^{-\alpha R} \right\} \\
&\le \int_{B_R} f(w)w \, dy + e^{-\alpha R} ,
\end{aligned}
$$

where the second inequality follows by Step 9, and the third inequality holds by $w_j \to w$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$. As $R \to \infty$, it follows that

$$\limsup_{j\to\infty} \|w_j\|_{H^1(\mathbb{R}^N)} \le \|w\|_{H^1(\mathbb{R}^N)} .$$

On the other hand, by the weak lower semicontinuous of $\|\cdot\|_{H^1(\mathbb{R}^N)}$, we have

$$\|w\|_{H^1(\mathbb{R}^N)} \le \liminf_{j\to\infty} \|w_j\|_{H^1(\mathbb{R}^N)} .$$

Hence,

$$\lim_{j\to\infty} \|w_j\|_{H^1(\mathbb{R}^N)} = \|w\|_{H^1(\mathbb{R}^N)} .$$

By the weak convergence, Step 11 follows. □

Finally, we show Proposition 3.2. We use the next Lemma to prove Proposition 3.2. This result is well-known. For the proof, see [14, (3.17)].

LEMMA B.3. *Let $z \in \mathbb{R}^N$ and $u \in H^1(B_1(z))$ be a weak solution of*

$$-\Delta u = g \text{ in } B_1(z) ,$$

*where $g \in L^q(B_1(z))$ for $q > N$. Then, there exists a positive constant C which is independent of $z$ such that*

$$\sup_{B_{3/4}(z)} |\nabla u|^2 \le C\big(\|g\|^2_{L^q(B_1(z))} + \|\nabla u\|^2_{L^2(B_1(z))}\big) .$$

Now, we shall prove Proposition 3.2. Our proof consists of the well-known argument by using the Nehari manifold. But, remark that we use the rearrangement to gain a solution which is invariant with respect to the rearrangement.

PROOF OF PROPOSITION 3.2. First, we can show that there exists a positive least energy solution of (2.1) by the same argument as the proof of Proposition 2.1 Steps 1–3.

Now, our proof is carried out in four steps. In Step 1, we show that there exists a positive least energy solution which is invariant with respect to the rearrangement. Step 2 proves (ii), Step 3 proves (iii), and Step 4 proves (iv).

**Step 1.** *We claim that there exists a positive least energy solution which is invariant with respect to the rearrangement.*

We will prove that $v_h^*$ is a least energy solution, where we denote by $v_h^*$ the decreasing rearrangement of $v_h$ (see, Appendix A). By Proposition A.1 and Proposition A.3, we have

$$\begin{aligned}&\int_{\mathbb{R}^N} |\nabla v_h^*|^2 + V(hy)(v_h^*)^2\, dy - \int_{\mathbb{R}^N} f(v_h^*)v_h^*\, dy \\ \le &\int_{\mathbb{R}^N} |\nabla v_h|^2 + V(hy)(v_h)^2\, dy - \int_{\mathbb{R}^N} f(v_h)v_h\, dy = 0\,.\end{aligned}$$

Hence, there exists a constant $0 < t_1 \le 1$ such that $t_1 v_h^* \in N_h$. Then, we have

$$\begin{aligned} m_h \le J_h[t_1 v_h^*] &= \int_{\mathbb{R}^N} \frac{1}{2} f(t_1 v_h^*)t_1 v_h^* - F(t_1 v_h^*)\, dy \\ &\le \int_{\mathbb{R}^N} \frac{1}{2} f(v_h^*)v_h^* - F(v_h^*)\, dy \\ &= \int_{\mathbb{R}^N} \frac{1}{2} f(v_h)v_h - F(v_h)\, dy \\ &= J_h[v_h] = m_h\,. \end{aligned}$$

Hence, by (f4), we have $t_1 = 1$. Therefore, we obtain $v_h^* \in N_h$ and $m_h = J_h[v_h^*]$. Moreover, we see that $v_h^*$ is a least energy solution by the similar argument to Step 2 of the proof of Proposition 2.1.

By using the interior $L^p$ estimate, we have $v_h^* \in C^1(\mathbb{R}^N)$. Moreover, by the strong maximum principle, it follows that $v_h^* > 0$. Hence, we may assume that $v_h^* = v_h$.

**Step 2.** *We prove the part* (ii) *of Proposition* 4.2.

By $v_h^*(y) = v_h(y)$, we have $v_h(0) = \max_{y\in\mathbb{R}^N} v_h(y)$. Hence, by the strong maximum principle, there exists $\delta > 0$ such that for $h$ sufficiently small, $v_h(0) \ge \delta > 0$. By $\|v_h\|_{H^1(\mathbb{R}^N)} \le \|v_h\|_{E_h} \le C$, we see that

$$\begin{aligned} v_h &\rightharpoonup w \quad \text{weakly in } H^1(\mathbb{R}^N)\,, \\ v_h &\to w \quad \text{in } C^1_{\text{loc}}(\mathbb{R}^N)\,. \end{aligned}$$

Moreover, we see that $w$ is a least energy solution of (2.6). Take any $\delta > 0$. Take $R > 0$ such that $w(R) \le \delta/2$. By $v_h \to w$ in $C^1_{\text{loc}}(\mathbb{R}^N)$, for $h$ sufficiently small, $v_h(y) \le w(y) + \delta/2$ for $|y| \le R$. In particular, $v_h(R) \le \delta$. By $v_h^* = v_h$, $v_h(y) \le v_h(R) \le \delta$ for $|y| \ge R$. (ii) is obtained from the weak maximum principle by comparison with a suitable test function.

**Step 3.** *We show the part* (iii) *of Proposition* 3.2.

Take any $R > 0$.

$$\begin{aligned}
&\limsup_{h\to 0} \|v_h\|^2_{H^1(\mathbb{R}^N)} \\
&\le \limsup_{h\to 0} \|v_h\|^2_{E_h} \\
&= \limsup_{h\to 0} \int_{\mathbb{R}^N} f(v_h)v_h\, dy \\
&= \limsup_{h\to 0} \left\{ \int_{B_R} f(v_h)v_h\, dy + \int_{\mathbb{R}^N\setminus B_R} f(v_h)v_h\, dy \right\} \\
&\le \limsup_{h\to 0} \left\{ \int_{B_R} f(v_h)v_h\, dy + e^{-\alpha R} \right\} \\
&\le \int_{B_R} f(w)w\, dy + e^{-\alpha R},
\end{aligned}$$

where the second inequality follows by Step 2, and the third inequality holds by $v_h \to w$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$. As $R \to \infty$, it follows that

$$\limsup_{h\to 0} \|v_h\|_{H^1(\mathbb{R}^N)} \le \|w\|_{H^1(\mathbb{R}^N)} .$$

On the other hand, by the weak lower semicontinuous of $\|\cdot\|_{H^1(\mathbb{R}^N)}$, we have

$$\|w\|_{H^1(\mathbb{R}^N)} \le \liminf_{h\to 0} \|v_h\|_{H^1(\mathbb{R}^N)} .$$

Hence,

$$\lim_{h\to 0} \|v_h\|_{H^1(\mathbb{R}^N)} = \|w\|_{H^1(\mathbb{R}^N)} .$$

By the weak convergence, Step 3 follows.

**Step 4.** *We show the part* (iv) *of Proposition* 3.2.

Take for all $\beta > 0$. Let $z \in \mathbb{R}^N$.

$$-\Delta v_{h_j} = -V(h_j y)v_{h_j} + f(v_{h_j}) =: c_j(y) \text{ in } \mathbb{R}^N .$$

For $q := 2N/(N-2)p(> N)$, by Sobolev's embedding and $V(h_j y) \le V(\beta)$ for all $|y| \le \beta/h_j$, we have $\|c_j\|_{L^q(B_{\beta/h_j})} \le C$, uniformly for $j$. Hence it follows that

$$\sup_{y\in B_{1/2}(z)} |\nabla v_{h_j}(y)|^2 \le C\big(\|\nabla v_{h_j}\|^2_{L^2(B_1(z))} + \|c_j\|^2_{L^q(B_1(z))}\big),$$

by Lemma B.3. For all $|z| \le \beta/2h_j$, $\|c_j\|_{L^q(B_1(z))} \le \|c_j\|_{L^q(B_{\beta/h_j})} \le C$, hence $\|v'_{h_j}\|_{L^\infty(B_{\beta/2h_j})} \le C$ uniformly over $j$. □

ACKNOWLEDGMENT. I got many insightful comments and advices for this research from Professor Kazuhiro Kurata. Without his help this paper would not have been written. I am deeply grateful to him. Also, I thank the referee and the communicator for careful reading my manuscript and for giving useful comments.

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