TOKYO J. MATH.
VOL. 40, NO. 2, 2017
DOI: 10.3836/tjm/1502179240

Harmonic Analysis on the Space of p-adic Unitary Hermitian Matrices, Mainly for Dyadic Case

Yumiko HIRONAKA

Waseda University

Abstract. We are interested in harmonic analysis on p-adic homogeneous spaces based on spherical functions. In the present paper, we investigate the space X of unitary hermitian matrices of size m over a $\mathfrak{p}$-adic field k mainly for dyadic case, and give the unified description with our previous papers for non-dyadic case. The space becomes complicated for dyadic case, and the set of integral elements in X has plural Cartan orbits. We introduce a typical spherical function $\omega(x; z)$ on X, study its functional equations, which depend on m and the ramification index e of 2 in k, and give its explicit formula, where Hall-Littlewood polynomials of type C_n appear as a main term with different specialization according as the parity $m = 2n$ or $2n + 1$, but independent of e. By spherical transform, we show the Schwartz space $\mathcal{S}(K\backslash X)$ is a free Hecke algebra $\mathcal{H}(G, K)$-module of rank 2^n, and give parametrization of all the spherical functions on X and the explicit Plancherel formula on $\mathcal{S}(K\backslash X)$. The Plancherel measure does not depend on e, but the normalization of G-invariant measure on X depends.

0. Introduction

We have been interested in harmonic analysis on p-adic homogeneous spaces based on spherical functions. We have studied on the space of p-adic unitary hermitian matrices, mainly for odd residual case in [HK1] and [HK2], and in the present paper we study mainly for dyadic case and give the unified description including odd and even residual case. All the results for odd residual case have been proved in [HK1] (resp. [HK2]) for even (resp. odd) size matrices. When the matrix size is even, the space has a natural close relation to the theory of automorphic functions and classical theory of sesquilinear forms (e.g. [H3], [HS]), where we need not to distinguish the dyadic case.

We fix an unramified quadratic extension k' of $\mathfrak{p}$-adic field k, and consider hermitian and unitary matrices with respect to k'/k, and for $a \in M_{mn}(k')$ we denote by $a^* \in M_{nm}(k')$ its conjugate transpose. Let π be a prime element of k and $v_\pi(\ )$ the additive valuation on k and k'. Denote by $j_m \in GL_m(k)$ the matrix whose all anti-diagonal entries are 1 and others are 0,

---

Received February 25, 2016; revised November 22, 2016
2010 *Mathematics Subject Classification*: 11E85, 11E95, 11F70, 22E50, 33D52
Key words and phrases: Spherical functions, Plancherel formula, unitary groups, hermitian matrices, Hall-Littlewood polynomials, dyadic fields
This research is partially supported by Grant-in-Aid for Scientific Research (C): 24540031.

where and henceforth m is an integer such that $m \geq 2$. Set

$$G = U(j_m) = \left\{ g \in GL_m(k') \;\middle|\; g^* j_m g = j_m \right\} ,$$
$$X = \left\{ x \in G \;\middle|\; x^* = x,\ \Phi_{xj_m}(t) = \Phi_{j_m}(t) \right\} ,$$
$$g \cdot x = gxg^*, \quad (g \in G,\ x \in X),$$

where $\Phi_y(t)$ is the characteristic polynomial of matrix y. We note that X is a single $G(\overline{k})$-orbit over the algebraic closure $\overline{k}$ of k. We fix $K = G \cap GL_m(\mathcal{O}_{k'})$, which is a maximal compact open subgroup of G satisfying the Iwasawa decomposition $G = KB = BK$ with Borel subgroup B of G consisting of all the upper triangular matrices in G.

For dyadic case, i.e. $v_\pi(2) > 0$, there are K-orbits in X which have no diagonal element (cf. Theorem 1 below); while for the space of unramified hermitian matrices in $GL_m(k')$, each $GL_m(\mathcal{O}_{k'})$-orbit has a diagonal representative ([Ja]). It is known in general that the spherical functions on various p-adic groups Γ can be written in terms of the specialization of Hall-Littlewood polynomials of the corresponding root structure of Γ (cf. [M2, §10], also [Car, Theorem 4.4]). For the present space X, the main term of spherical functions can be written by using Hall-Littlewood polynomials of type C_n with different specialization according to the parity of m, independent of the residual characteristic (cf. Theorem 3 below). By using spherical functions we study the Schwartz space $\mathcal{S}(K\backslash X)$, and show its $\mathcal{H}(G, K)$-module structure, parametrization of all the spherical functions on X and Plancherel formula and Inversion formula on $\mathcal{S}(K\backslash X)$(cf. Theorem 4 below).

We will explain the results in some more details. Set

$$n = \left[\frac{m}{2}\right], \qquad e = v_\pi(2)\ (\geq 0) \tag{0.1}$$

and denote the corresponding groups $G,\ B,\ K$ and space X with subscript and superscript if necessary, as $G_n^{(ev)},\ X_n^{(ev)}$ for $m = 2n$, and $G_n^{(od)},\ X_n^{(od)}$ for $m = 2n+1$, etc. According to the parity of m, G has the root structure of type C_n for even m or type BC_n for odd m. We fix a unit $\epsilon \in k$ for which the set $\{1, \frac{1+\sqrt{\epsilon}}{2}\}$ forms an $\mathcal{O}_k$-basis for $\mathcal{O}_{k'}$, where $v_\pi(1-\epsilon) = 2e$ (cf. [Om, §64]). Set

$$\widetilde{\Lambda_n^+} = \left\{ \lambda \in \mathbb{Z}^n \;\middle|\; \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq -e \right\} , \tag{0.2}$$
$$\Lambda_n^+ = \left\{ \lambda \in \widetilde{\Lambda_n^+} \;\middle|\; \lambda_n \geq 0 \right\} ,$$

where $\Lambda_n^+ = \widetilde{\Lambda_n^+}$ if $e = 0$. For $\lambda \in \widetilde{\Lambda_n^+}$ such that $\lambda_r \geq 0 > \lambda_{r+1}$, define

$$x_\lambda^{(ev)} = Diag(\pi^{\lambda_1}, \ldots, \pi^{\lambda_r}, y_\lambda^{(ev)}, \pi^{-\lambda_r}, \ldots, \pi^{-\lambda_1}) \in X_n^{(ev)} ,$$

$$y_\lambda^{(ev)} = \begin{cases} \emptyset & \text{if } r = n \\ \begin{pmatrix} \pi^{\lambda_{r+1}}(1-\epsilon) & & & & & -\sqrt{\epsilon} \\ & \ddots & & & \cdot^{\cdot^{\cdot}} & \\ & & \pi^{\lambda_n}(1-\epsilon) & -\sqrt{\epsilon} & & \\ & & \sqrt{\epsilon} & \pi^{-\lambda_n} & & \\ & \cdot^{\cdot^{\cdot}} & & & \ddots & \\ \sqrt{\epsilon} & & & & & \pi^{-\lambda_{r+1}} \end{pmatrix} & \text{if } r < n, \end{cases}$$

$$x_\lambda^{(od)} = Diag(\pi^{\lambda_1}, \dots, \pi^{\lambda_r}, y_\lambda^{(od)}, \pi^{-\lambda_r}, \dots, \pi^{-\lambda_1}) \in X_n^{(od)},$$

$$y_\lambda^{(od)} = \begin{cases} 1 & \text{if } r = n \\ \begin{pmatrix} \pi^{\lambda_{r+1}}(1-\epsilon) & & & & & & -\sqrt{\epsilon} \\ & \ddots & & & & \cdot^{\cdot^{\cdot}} & \\ & & \pi^{\lambda_n}(1-\epsilon) & & -\sqrt{\epsilon} & & \\ & & & 1 & & & \\ & & \sqrt{\epsilon} & & \pi^{-\lambda_n} & & \\ & \cdot^{\cdot^{\cdot}} & & & & \ddots & \\ \sqrt{\epsilon} & & & & & & \pi^{-\lambda_{r+1}} \end{pmatrix} & \text{if } r < n, \end{cases}$$

and understand $x_\lambda = y_\lambda$ if $r = 0$, where and henceforth we write simply as x_λ or y_λ, if there is no danger of confusion. Here empty entries in matrices should be understood as 0.

THEOREM 1. (1) *The map* $\widetilde{\Lambda_n^+} \longrightarrow K\backslash X,\ \lambda \longmapsto K \cdot x_\lambda$ *is surjective. Further, it is bijective if m is even, $m = 3$, or $e = 1$.*
(2) *There are precisely two G-orbits in X represented by x_λ with $\lambda = \mathbf{0}$ and* $(1, 0, \dots, 0)$.

Each $x_\lambda,\ \lambda \in \Lambda_n^+$, gives a different K-orbit, since it gives a different $GL_m(\mathcal{O}_{k'})$-orbit in the space of hermitian matrices in $GL_m(k')$, where $m = 2n$ or $2n + 1$. Hence, when $e = 0$, it is enough to show that any K-orbit has a representative of the shape $x_\lambda,\ \lambda \in \Lambda_n^+$, which has been done in [HK1] for even m and [HK2] for odd m. For $e > 0$, we have to prove also the non-redundancy within the above representatives, which will be done as a corollary of the explicit formula of spherical functions on X for $n \geq 2$, i.e., $m \geq 4$ (cf. Theorem 3).

A spherical function on X is a K-invariant function on X which is a common eigenfunction with respect to the convolutive action of the Hecke algebra $\mathcal{H}(G, K)$, and a typical one is constructed by Poisson transform from relative invariants of a parabolic subgroup. We take the Borel subgroup B consisting of upper triangular matrices in G. For $x \in X$ and $s \in \mathbb{C}^n$, we consider the following integral

$$\omega(x; s) = \int_K \prod_{i=1}^n |d_i(k \cdot x)|^{s_i}\, dk, \tag{0.3}$$

where $|\ |$ is the absolute value on k normalized by $|\pi| = q^{-1}$, $q = \sharp(\mathcal{O}_k/(\pi))$, $d_i(y)$ is the determinant of the lower right i by i block of y, $1 \leq i \leq n$, and dk is the normalized Haar measure on K. Then the right hand side of (0.3) is absolutely convergent for $\mathrm{Re}(s_i) \geq 0$, $1 \leq i \leq n$, and continued to a rational function of $q^{s_1}, \ldots, q^{s_n}$, and we use the notation $\omega(x; s)$ in such sense. Since $d_i(x)$'s are relative B-invariants on X such that

$$d_i(p \cdot x) = \psi_i(p) d_i(x), \quad \psi_i(p) = N_{k'/k}(d_i(p)) \quad (p \in B,\ x \in X,\ 1 \leq i \leq n),$$

we see $\omega(x; s)$ is a spherical function on X which satisfies

$$f * \omega(x; s) = \lambda_s(f)\omega(x; s), \quad f \in \mathcal{H}(G, K)$$
$$\lambda_s(f) = \int_B f(p) \prod_{i=1}^{n} |\psi_i(p)|^{-s_i}\, \delta(p) dp,$$

where dp is the normalized left invariant measure on B with modulus character δ. The Weyl group W of G relative to B acts on rational characters of B, hence on s. It is convenient to introduce the new variable $z \in \mathbb{C}^n$ related to s by

$$s_i = -z_i + z_{i+1} - 1 + \frac{\pi\sqrt{-1}}{\log q}, \quad 1 \leq i \leq n-1$$
$$s_n = \begin{cases} -z_n - \dfrac{1}{2} & \text{if } m = 2n \\ -z_n - 1 + \dfrac{\pi\sqrt{-1}}{2\log q} & \text{if } m = 2n+1, \end{cases} \tag{0.4}$$

where, for the case $m = 2n$, we have slightly changed the relation between s and z from that in [HK1] (cf. Remark 2.1). Then $W = \langle S_n, \tau \rangle$ acts on z by permutation of indices as for the elements of S_n and $\tau(z_1, \ldots, z_n) = (z_1, \ldots, z_{n-1}, -z_n)$. Keeping the above relation (0.4), we denote $\omega(x; z) = \omega(x; s)$ and $\lambda_z = \lambda_s$. Then λ_z gives the Satake isomorphism

$$\lambda_z : \mathcal{H}(G, K) \xrightarrow{\sim} \mathbb{C}\left[q^{\pm 2z_1}, \ldots, q^{\pm 2z_n}\right]^W (= \mathcal{R}_0, \text{say}). \tag{0.5}$$

We will give the functional equation of $\omega(x; z)$ with respect to W. We set

$$\Sigma^+ = \Sigma_s^+ \sqcup \Sigma_\ell^+,$$
$$\Sigma_s^+ = \left\{ e_i + e_j,\ e_i - e_j \;\middle|\; 1 \leq i < j \leq n \right\}, \quad \Sigma_\ell^+ = \{2e_i \mid 1 \leq i \leq n\},$$

where $e_i \in \mathbb{Z}^n$ is the i-th unit vector, and define a pairing

$$\langle\ ,\ \rangle : \mathbb{Z}^n \times \mathbb{C}^n \longrightarrow \mathbb{C},\ \langle \alpha, z \rangle = \sum_{i=1}^{n} \alpha_i z_i.$$

THEOREM 2. *Assume $e \leq 1$ if m is odd.*
(1) *For any $\sigma \in W$, one has*

$$\omega(x; z) = \Gamma_\sigma^{(e)}(z) \cdot \omega(x; \sigma(z)).$$

where

$$\Gamma_\sigma^{(e)}(z) = \prod_\alpha \gamma_\alpha^{(e)}(z), \quad \gamma_\alpha^{(e)}(z) = \begin{cases} \dfrac{1-q^{-1+\langle \alpha, z\rangle}}{q^{\langle \alpha, z\rangle} - q^{-1}} & \alpha \in \Sigma_s^+ \\ q^{e\langle \alpha, z\rangle} & \alpha \in \Sigma_\ell^+, m = 2n \\ q^{e\langle \alpha, z\rangle} \dfrac{1-q^{-1+\langle \alpha, z\rangle}}{q^{\langle \alpha, z\rangle} - q^{-1}} & \alpha \in \Sigma_\ell^+, m = 2n+1\,, \end{cases}$$

and α runs over the set $\{\alpha \in \Sigma^+ \mid -\sigma(\alpha) \in \Sigma^+\}$.

(2) *The function $q^{-\langle e, z\rangle} G(z) \cdot \omega(x; z)$ is holomorphic and W-invariant, hence belongs to $\mathbb{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^W$. Here*

$$\langle e,\ z\rangle = e(z_1 + \cdots + z_n), \quad G(z) = \prod_\alpha \frac{1 + q^{\langle \alpha, z\rangle}}{1 - q^{-1+\langle \alpha, z\rangle}}\,,$$

and α runs over the set Σ_s^+ for $m = 2n$ and Σ^+ for $m = 2n+1$.

As for the explicit formula of $\omega(x; s)$ it suffices to give for x_λ by Theorem 1 (1).

THEOREM 3 (Explicit formula). *Assume $e \le 1$ if m is odd. For each $\lambda \in \widetilde{\Lambda_n^+}$, one has*

$$\omega(x_\lambda; z) = c_n\, q^{\langle \lambda, z_0\rangle} \cdot \frac{q^{\langle e, z\rangle}}{G(z)} \cdot Q_{\lambda+e}(z; \{t\})\,,$$

where $G(z)$ is given in Theorem 2, z_0 is the value in z-variable corresponding to $s = \mathbf{0}$,

$$c_n = \begin{cases} \dfrac{(1-q^{-2})^n}{w_m(-q^{-1})} & \text{if } m = 2n \\ \dfrac{(1+q^{-1})(1-q^{-2})^n}{w_m(-q^{-1})} & \text{if } m = 2n+1, \end{cases} \qquad w_m(t) = \prod_{i=1}^m (1 - t^i)\,,$$

$$Q_\mu(z; \{t\}) = \sum_{\sigma \in W} \sigma\left(q^{-\langle \mu, z\rangle} c(z, \{t\})\right), \quad c(z, \{t\}) = \prod_{\alpha \in \Sigma^+} \frac{1 - t_\alpha q^{\langle \alpha, z\rangle}}{1 - q^{\langle \alpha, z\rangle}}\,,$$

$$\{t\} = \{t_\alpha\} \quad \text{with} \quad t_\alpha = \begin{cases} -q^{-1} & \text{if } \alpha \in \Sigma_s^+ \\ q^{-1} & \text{if } \alpha \in \Sigma_\ell^+,\ m = 2n \\ -q^{-2} & \text{if } \alpha \in \Sigma_\ell^+,\ m = 2n+1\,. \end{cases}$$

We see the main part $Q_{\lambda+e}(z; \{t\})$ of $\omega(x_\lambda; z)$ belongs to $\mathcal{R} = \mathbb{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^W$ by Theorem 2. It is known that $Q_\mu(z; \{t\})$ is a constant multiple of Hall-Littlewood polynomial $P_\mu(z; \{t\})$ and the set $\{P_\mu(z; \{t\}) \mid \mu \in \Lambda_n^+\}$ forms a $\mathbb{C}$-basis for $\mathcal{R}$ (for more details, see Remark 3.2). Hence we see the non-redundancy for the representatives in Theorem 1-(1). The influence of the residual characteristic of the base field k in the explicit formula of $\omega(x_\lambda; z)$ appears as shifting $\lambda + e$ in Q_λ or P_λ and the factor $q^{\langle e, z \rangle}$.

We modify the spherical function by using the value at $x_{(-e)}$, $(-e) \in \widetilde{\Lambda_n^+}$ as

$$\Psi(x; z) = \frac{\omega(x; z)}{\omega(x_{(-e)}; z)} \in \mathbb{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^W (= \mathcal{R}),$$

and define the spherical Fourier transform on the Schwartz space $\mathcal{S}(K \backslash X)$ by

$$\widehat{\ }: \mathcal{S}(K\backslash X) \longrightarrow \mathcal{R},\ \varphi \longmapsto \widehat{\varphi}(z) = \int_X \varphi(x)\Psi(x; z)dx$$

where dx is a G-invariant measure on X. Then it satisfies

$$(f * \varphi)\widehat{\ } = \lambda_z(f)\widehat{\varphi}, \quad f \in \mathcal{H}(G, K),\ \varphi \in \mathcal{S}(K\backslash X),$$

where λ_z is the Satake isomorphism given in (0.5).

THEOREM 4. *Assume $e \le 1$ if m is odd.*
(1) *The above spherical Fourier transform is an $\mathcal{H}(G, K)$-module isomorphism and $\mathcal{S}(K\backslash X)$ becomes a free $\mathcal{H}(G, K)$-module of rank 2^n.*
(2) *All the spherical functions on X are parametrized by $z \in \left(\mathbb{C}/\frac{2\pi\sqrt{-1}}{\log q}\right)^n/W$ through λ_z, and the set $\left\{ \Psi(x; z+u) \;\middle|\; u \in \{0, \frac{\pi\sqrt{-1}}{\log q}\}^n \right\}$ forms a $\mathbb{C}$-basis of spherical functions corresponding to z.*
(3) (Plancherel formula) *Set a measure $d\mu(z)$ on $\mathfrak{a}^* = \left\{\sqrt{-1}\left(\mathbb{R}/\frac{2\pi}{\log q}\mathbb{Z}\right)\right\}^n$ by*

$$d\mu(z) = \frac{1}{2^n n!} \cdot \frac{w_n(-q^{-1})w_{m'}(-q^{-1})}{(1+q^{-1})^{m'}} \cdot \frac{1}{|c(z, \{t\})|^2}\, dz, \quad m' = \left[\frac{m+1}{2}\right],$$

where dz is the Haar measure on $\mathfrak{a}^*$. By the normalized G-invariant measure dx on X (explicitly given in Lemma 4.3), *one has*

$$\int_X \varphi(x)\overline{\psi(x)}dx = \int_{\mathfrak{a}^*} \widehat{\varphi}(z)\overline{\widehat{\psi}(z)}d\mu(z) \quad (\varphi, \psi \in \mathcal{S}(K\backslash X)).$$

(4) (Inversion formula) *For any $\varphi \in \mathcal{S}(K\backslash X)$, one has*

$$\varphi(x) = \int_{\mathfrak{a}^*} \widehat{\varphi}(z)\Psi(x; z)d\mu(z), \quad x \in X.$$

The spherical function $\Psi(x; z)$ and the G-invariant measure dx on X depend on m and $e = v_\pi(2)$, while the Plancherel measure $d\mu(z)$ depends only on m. A key point to establish Theorems 2, 3 and 4 for $e > 0$ is the functional equation of $\omega(x; z)$ for $n = 1$, i.e., $m = 2$ and 3 (cf. Proposition 2.3 and Proposition 2.4). If (2.14) in Proposition 2.4 is true for $e > 0$, then Theorems 1, 2, 3 and 4 hold for the same e (cf. Remark 2.5).

ACKNOWLEDGEMENTS. This paper was written when the author took a sabbatical period from Waseda University and stayed Mannheim University and Kyoto University. She would like to express her gratitude for the generosity of her institution and for the hospitality at Mannheim and Kyoto. She would like to express her gratitude also to the referee for the careful reading of the manuscript.

1. The space X

1.1. Let k' be an unramified quadratic extension of a $\mathfrak{p}$-adic field k and consider hermitian and unitary matrices with respect to k'/k, and for $a \in M_{mn}(k')$ we denote by $a^* \in M_{nm}(k')$ its conjugate transpose. Let π be a prime element of k and q the cardinality of the residue class field $\mathcal{O}_k/(\pi)$, and we normalize the absolute value on k by $|\pi| = q^{-1}$ and denote by $v_\pi(\ )$ the additive valuation on k, and on k' simultaneously. We set $e = v_\pi(2)$. We fix a unit $\epsilon \in \mathcal{O}_k^\times$ for which the set $\{1, \frac{1+\sqrt{\epsilon}}{2}\}$ forms an $\mathcal{O}_k$-basis for $\mathcal{O}_{k'}$ (cf. [Om, §64]). Then $v_\pi(1-\epsilon) = 2e$. We denote by N the norm map $N_{k'/k}$, and set

$$j_m = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \in M_m ,$$

where and henceforth empty entries in matrices should be understood as 0.

We consider the unitary group for $m \geq 2$

$$G = G(j_m) = \left\{ g \in GL_m(k') \,\middle|\, g^* j_m g = j_m \right\} ,$$

and the spaces of hermitian matrices in G

$$\widetilde{X} = \left\{ x \in G \,\middle|\, x^* = x \right\}, \quad X = \left\{ x \in \widetilde{X} \,\middle|\, \Phi_{xj_m}(t) = \Phi_{j_m}(t) \right\} , \tag{1.1}$$

where $\Phi_y(t)$ is the characteristic polynomial of a matrix y, and we see $\det(x) = 1$ for any $x \in X$. The group G acts on $\widetilde{X}$ and X by

$$g \cdot x = gxg^* = gxj_m g^{-1} j_m, \quad g \in G,\ x \in \widetilde{X} . \tag{1.2}$$

This action can be extended to the algebraic closure $\overline{k}$ of k, and the set $\widetilde{X}(\overline{k})$ is decomposed into $G(\overline{k})$-orbits as follows (cf. [HK1, Appendix A]) :

$$\widetilde{X}(\overline{k}) = \bigsqcup_{i=0}^{m} \left\{ x \in \widetilde{X}(\overline{k}) \,\middle|\, \Phi_{xj_m}(t) = (t-1)^i (t+1)^{m-i} \right\} . \tag{1.3}$$

Then $X(\overline{k})$ is a single $G(\overline{k})$-orbit containing 1_m and corresponding to $i = \left[\frac{m+1}{2}\right]$, and $X =$

$X(\overline{k}) \cap G$. It is easy to see the following:

$$\begin{aligned} &\text{If } m = 2, \text{ then } \widetilde{X} = \{j_2\} \sqcup \{-j_2\} \sqcup X\,. \\ &\text{If } m = 3, \text{ then } \widetilde{X} = \{j_3\} \sqcup \{-j_3\} \sqcup X \sqcup (-X)\,. \end{aligned} \tag{1.4}$$

We fix a maximal compact subgroup K of G by

$$K = G \cap M_m(\mathcal{O}_{k'})\,,$$

(cf. [Sa, §9]), and take a Borel subgroup B of G consisting of all the upper triangular matrices in G. Then the group G satisfies the Iwasawa decomposition $G = BK = KB$.

We are interested in Cartan decomposition of X, i.e., K-orbit decomposition of X. To state the results we prepare some notation. We set

$$n = \left[\frac{m}{2}\right] \tag{1.5}$$

and denote the corresponding groups $G,\ B,\ K$ and the space X with subscript and superscript if necessary, as $G_n^{(ev)},\ X_n^{(ev)}$ for $m = 2n$, and $G_n^{(od)},\ X_n^{(od)}$ for $m = 2n+1$, etc. We set

$$\begin{aligned} \widetilde{\Lambda_n^+} &= \left\{\lambda \in \mathbb{Z}^n \;\middle|\; \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq -e\right\}, \quad (e = v_\pi(2)) \\ \Lambda_n^+ &= \left\{\lambda \in \widetilde{\Lambda_n^+} \;\middle|\; \lambda_n \geq 0\right\} \left(= \widetilde{\Lambda_n^+} \text{ if } e = 0\right)\,, \end{aligned} \tag{1.6}$$

and for each $\lambda \in \widetilde{\Lambda_n^+}$ such that $\lambda_r \geq 0 > \lambda_{r+1}$, define $x_\lambda^{(ev)} \in X_n^{(ev)}$ for $m = 2n$ and $x_\lambda^{(od)} \in X_n^{(od)}$ for $m = 2n+1$ as follows

$$x_\lambda^{(ev)} = Diag(\pi^{\lambda_1}, \ldots, \pi^{\lambda_r}, y_\lambda^{(ev)}, \pi^{-\lambda_r}, \ldots, \pi^{-\lambda_1}), \tag{1.7}$$

$$y_\lambda^{(ev)} = \begin{cases} \emptyset & \text{if } r = n \\ \begin{pmatrix} \pi^{\lambda_{r+1}}(1-\epsilon) & & & & & -\sqrt{\epsilon} \\ & \ddots & & & \iddots & \\ & & \pi^{\lambda_n}(1-\epsilon) & -\sqrt{\epsilon} & & \\ & & \sqrt{\epsilon} & \pi^{-\lambda_n} & & \\ & \iddots & & & \ddots & \\ \sqrt{\epsilon} & & & & & \pi^{-\lambda_{r+1}} \end{pmatrix} & \text{if } r < n, \end{cases}$$

$$x_\lambda^{(od)} = Diag(\pi^{\lambda_1}, \ldots, \pi^{\lambda_r}, y_\lambda^{(od)}, \pi^{-\lambda_r}, \ldots, \pi^{-\lambda_1})\,, \tag{1.8}$$

$$y_\lambda^{(od)} = \begin{cases} 1 & \text{if } r = n \\ \begin{pmatrix} \pi^{\lambda_{r+1}}(1-\epsilon) & & & & & & -\sqrt{\epsilon} \\ & \ddots & & & & \iddots & \\ & & \pi^{\lambda_n}(1-\epsilon) & & -\sqrt{\epsilon} & & \\ & & & 1 & & & \\ & & \sqrt{\epsilon} & & \pi^{-\lambda_n} & & \\ & \iddots & & & & \ddots & \\ \sqrt{\epsilon} & & & & & & \pi^{-\lambda_{r+1}} \end{pmatrix} & \text{if } r < n, \end{cases}$$

and understand $x_\lambda = y_\lambda$ if $r = 0$, where and henceforth we write simply as x_λ or y_λ, if there is no danger of confusion.

THEOREM 1.1. (1) *The map* $\widetilde{\Lambda_n^+} \longrightarrow K_n \backslash X_n$, $\lambda \longmapsto K_n \cdot x_\lambda$ *is surjective.*
(2) *The above map is bijective if* $m = 2n$, $m = 3$, *or* $e = 1$.
(3) *There are precisely two* G_n*-orbits in* X_n *represented by* $x_0 = 1_m$ *and* $x_1 = Diag(\pi, 1_{m-2}, \pi^{-1})$. *For* $\lambda \in \widetilde{\Lambda_n^+}$, $x_\lambda \in G \cdot 1_m$ *if and only if* $|\lambda| = \sum_{i=1}^n \lambda_i$ *is even.*

We recall some classical results on unramified hermitian forms (cf. [Ja]). The group $GL_m(k')$ acts on the space $\mathcal{H}_m(k') = \left\{ x \in GL_m(k') \;\middle|\; x^* = x \right\}$ by $g \cdot x = gxg^*$, and

$$\mathcal{H}_m(k') = \bigsqcup_{\mu \in \Lambda_m} GL_m(\mathcal{O}_{k'}) \cdot \pi^\mu = GL_m(k') \cdot 1_m \sqcup GL_m(k') \cdot \pi^{(1,0,\cdots,0)}, \tag{1.9}$$

where $\Lambda_m = \{\mu \in \mathbb{Z}^m \mid \mu_1 \geq \cdots \geq \mu_m\}$, $\pi^\mu = Diag(\pi^{\mu_1}, \ldots, \pi^{\mu_m})$, and $\pi^\mu \in GL_m(k') \cdot 1_m$ if and only if $|\mu| = \sum_{i=1}^m \mu_i$ is even.

REMARK 1.2. As for (1), we have shown for even m in [HK1, §1] and for odd m with $e = 0$ in [HK2, §1]. In §1.2 (resp. §1.3), we will show the statement (1) for $m = 3$ (resp. general odd m). The non-redundancy of the representatives for $e = 0$ follows from (1.9). For $e > 0$, we see there are K-orbits without any diagonal element by Proposition 1.3 below, and non-redundancy for $m = 2, 3$ follows from this and the value $\ell(x - j_m)$ (cf. (1.11) below). We will see the non-redundancy for general m as a corollary of the explicit formula of spherical functions in §3 (See Remark 3.3). The property (3) is independent of the residual characteristic and we may prove in a similar way as in [HK1] and [HK2], so we omit the proof. We note here the stabilizer of $G(\overline{k})$ at $x = 1_m$ is isomorphic to $U(1_n)(\overline{k}) \times U(1_{m'})(\overline{k})$, $m' = \left[\frac{m+1}{2}\right]$, and explicitly given as follows ([HK1, (1.5)], [HK2, Proof of Theorem 1.1]):

$$\left\{ \begin{pmatrix} a & b \\ jbj & jaj \end{pmatrix} \in GL_{2n}(\overline{k}) \;\middle|\; a + bj, a - bj \in U(1_n)(\overline{k}) \right\}, \quad \text{or} \tag{1.10}$$

$$\left\{ \begin{pmatrix} A & b & C \\ d & f & dj \\ jCj & jb & jAj \end{pmatrix} \in GL_{2n+1}(\overline{k}) \;\middle|\; A - Cj \in U(1_n)(\overline{k}),\ \begin{pmatrix} A + Cj & \nu b \\ \nu^* d & f \end{pmatrix} \in U(1_{n+1})(\overline{k}) \right\},$$

where $j = j_n$ and $\nu \in \overline{k}$ such that $\nu\nu^* = 2$. Here, we may take ν within k' if $e = 0$, while for $e > 0$, we understand * as an extended automorphism of $\overline{k}$.

PROPOSITION 1.3. *If $x \in X \cap M_m(\mathcal{O}_{k'})$ satisfies $x \equiv j_m \pmod{(\pi)}$, then the orbit $K \cdot x$ has no diagonal element.*

PROOF. If $K \cdot x$ contains a diagonal element, it must contain 1_m. On the other hand, since any $k \in K$ fixes j_m, we have $1_m \equiv j_m \pmod{(\pi)}$, which is a contradiction. $\square$

For $a = (a_{ij}) \in M_m(k')$, $a \neq 0$, we set

$$-\ell(a) = \min\left\{ v_\pi(a_{ij}) \;\middle|\; 1 \le i, j \le m \right\}, \tag{1.11}$$

and say an entry of a to be *minimal* if its v_π-value is $-\ell(a)$.

LEMMA 1.4. (1) *Let $a \in M_m(\mathcal{O}_{k'})$ and $b \in M_m(k')$ such that $ab \neq 0$ and $ba \neq 0$. Then, one has*

$$\ell(ab) \le \ell(b), \quad \ell(ba) \le \ell(b),$$

and the equalities hold if $a \in GL_m(\mathcal{O}_{k'})$.
(2) *For any $g \in G$, one has $\ell(g) \ge 0$, and the equality holds if and only if $g \in K$.*

PROOF. (1) Let $a = (a_{ij}) \in M_m(\mathcal{O}_{k'})$ and $b = (b_{ij}) \in M_m(k')$. Then, we have

$$\begin{aligned} -\ell(ab) &= \min\left\{ v_\pi\Big(\sum_k a_{ik}b_{kj}\Big) \;\middle|\; 1 \le i, j \le m \right\} \ge \min\left\{ v_\pi(a_{ik}b_{kj}) \;\middle|\; 1 \le i, j, k \le m \right\} \\ &\ge \min\left\{ v_\pi(b_{kj}) \;\middle|\; 1 \le j, k \le m \right\} = -\ell(b), \end{aligned}$$

hence $\ell(ab) \le \ell(b)$, and similarly we have $\ell(ba) \le \ell(b)$. If $a \in GL_m(\mathcal{O}_{k'})$, we have the opposite inequalities and then $\ell(ab) = \ell(b) = \ell(ba)$.

The statement (2) follows from the fact $\det(g) \in \mathcal{O}_{k'}^\times$ and $K = M_m(\mathcal{O}_{k'}) \cap G$. $\square$

1.2. In this subsection we consider the case $m = 3$ and prove the following proposition.

PROPOSITION 1.5. *The set $\mathcal{R}_1^+ \sqcup \mathcal{R}_1^-$ is a set of complete representatives of $K_1\backslash X_1$, where*

$$\mathcal{R}_1^+ = \left\{ x_\ell = \begin{pmatrix} \pi^\ell & & \\ & 1 & \\ & & \pi^{-\ell} \end{pmatrix} \;\middle|\; \ell \ge 0 \right\},$$

$$\mathcal{R}_1^- = \left\{ x_{-r} = \begin{pmatrix} \pi^{-r}(1-\epsilon) & & -\sqrt{\epsilon} \\ & 1 & \\ \sqrt{\epsilon} & & \pi^r \end{pmatrix} \;\middle|\; 1 \le r \le e \right\}.$$

The set $\mathcal{R}_1^-$ is non-empty only if $e > 0$. In that case, for $x \in X_1$, $K_1 \cdot x$ has a representative in $\mathcal{R}_1^-$ if and only if $x \equiv j_3 \pmod{(\pi)}$.

We write down the group $K_1 = K_1^{(od)}$ explicitly for convenience.

LEMMA 1.6.

$$K_1 = K_{1,1} \sqcup K_{1,2}\,,$$

$$K_{1,1} := \left\{ g \in B_1 j_3 B_1 \cap K_1 \;\middle|\; g_{31} \in \mathcal{O}_{k'}^{\times} \right\} = \left\{ g \in K_1 \;\middle|\; g_{31} \in \mathcal{O}_{k'}^{\times} \right\}$$

$$= \left\{ \begin{pmatrix} \alpha & & \\ & u & \\ & & \alpha^{*-1} \end{pmatrix} \begin{pmatrix} 1 & -d^* & f \\ & 1 & d \\ & & 1 \end{pmatrix} \begin{pmatrix} & & 1 \\ & 1 & -b^* \\ 1 & b & c \end{pmatrix} \;\middle|\; \begin{array}{l} \alpha \in \mathcal{O}_{k'}^{\times},\ u \in \mathcal{O}_{k'}^{1} \\ b, c, d, f \in \mathcal{O}_{k'} \\ N(b) + c + c^* = N(d) + f + f^* = 0 \end{array} \right\},$$

$$K_{1,2} := \{ g \in K_1 \mid g_{31} \in (\pi) \}$$

$$= \left\{ \begin{pmatrix} \alpha & & \\ & u & \\ & & \alpha^{*-1} \end{pmatrix} \begin{pmatrix} 1 & & \\ b & 1 & \\ c & -b^* & 1 \end{pmatrix} \begin{pmatrix} 1 & d & f \\ & 1 & -d^* \\ & & 1 \end{pmatrix} \;\middle|\; \begin{array}{l} \alpha \in \mathcal{O}_{k'}^{\times},\ u \in \mathcal{O}_{k'}^{1} \\ b, c \in \pi\mathcal{O}_{k'},\ d, f \in \mathcal{O}_{k'} \\ N(b) + c + c^* = N(d) + f + f^* = 0 \end{array} \right\}.$$

PROOF OF PROPOSITION 1.5. The strategy is similar to [HK2, §1.2]. We take an element $x \in X_1$, write it as

$$x = \begin{pmatrix} a & b & c \\ b^* & d & f \\ c^* & f^* & g \end{pmatrix}, \qquad a, d, g \in k, \quad b, c, f \in k', \tag{1.12}$$

and show that the orbit $K_1 \cdot x$ has an element x_ℓ with $\ell \geq -e$ as in the statement.

By the fact $x \in G$ and $\Phi_{xj_3}(t) = (t^2 - 1)(t - 1)$, we obtain the following equations

$$ag + bf + c^2 = 1\,, \tag{1.13a}$$

$$af^* + b(c + d) = 0\,, \tag{1.13b}$$

$$a(c + c^*) + bb^* = 0\,, \tag{1.13c}$$

$$b^* g + (c + d) f = 0\,, \tag{1.13d}$$

$$bf + b^* f^* + d^2 = 1\,, \tag{1.13e}$$

$$(c + c^*) g + ff^* = 0\,, \tag{1.13f}$$

and

$$\begin{aligned} &(t^2 - 1)(t - 1) &(1.14)\\ &= (t - c)(t - c^*)(t - d) - (t - c)b^* f^* - (t - c^*)bf - (t - d)ag - aff^* - bb^* g\,. \end{aligned}$$

We recall that $N(\mathcal{O}_{k'}^{\times}) = \mathcal{O}_k^{\times}$ and $Tr(\mathcal{O}_{k'}) = \mathcal{O}_k$.

[Case 1] $a \neq 0$ and $v_\pi(a) \leq v_\pi(b)$, or $g \neq 0$ and $v_\pi(g) \leq v_\pi(f)$.

By the action of j_3, it suffices to consider the case $a \neq 0$. By the action of an element of K_1 of type

$$\begin{pmatrix} 1 & 0 & 0 \\ \lambda & 1 & 0 \\ \mu & -\lambda^* & 1 \end{pmatrix} \in K_1, \qquad \lambda, \mu \in k' \text{ such that } a\lambda + b^* = 0,\ N(\lambda) + \mu + \mu^* = 0\,, \tag{1.15}$$

we may assume $b = 0$ in (1.12). Then, by (1.13) and (1.14), we have

$$f = 0, \quad ag + c^2 = 1, \quad c + c^* = 0, \quad d = 1\,.$$

Thus, we may assume

$$x = \begin{pmatrix} a & 0 & -c_1\sqrt{\epsilon} \\ 0 & 1 & 0 \\ c_1\sqrt{\epsilon} & 0 & g \end{pmatrix}, \qquad \begin{array}{l} a, g, c_1 \in k,\ ag + c_1^2\epsilon = 1\,, \\ ag \neq 0,\ v_\pi(a) \geq v_\pi(g)\,, \end{array} \tag{1.16}$$

where $g \neq 0$ follows from $\epsilon \notin k^{\times 2}$. We consider two cases according to $v_\pi(g)$.

(i) If $v_\pi(g) \leq 0$, we may assume $g = \pi^{-\ell},\ \ell \geq 0$. Then $(\pi^\ell c_1)^2 = \pi^{2\ell} - ag^{-1} \in \mathcal{O}_k$, and

$$\begin{aligned} K_1 \cdot x \ni & \begin{pmatrix} 1 & 0 & \pi^\ell c_1\sqrt{\epsilon} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & 0 & -c_1\sqrt{\epsilon} \\ 0 & 1 & 0 \\ c_1\sqrt{\epsilon} & 0 & \pi^{-\ell} \end{pmatrix} \\ = & \begin{pmatrix} \pi^\ell & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pi^{-\ell} \end{pmatrix}. \end{aligned}$$

(ii) Assume that $v_\pi(g) > 0$. Since an element of k square modulo $4\pi\mathcal{O}_k$ is a square in k, the relation $c_1^2\epsilon \equiv 1(\pi^{v_\pi(ag)}\mathcal{O}_k)$ yields $e > 0$ and $v_\pi(g) \leq e$. Further $c_1 \equiv 1 \pmod{(\pi)}$, since $\epsilon \in 1 + 4\mathcal{O}_k$. Thus we see

$$K_1 \cdot x \ni x' = \begin{pmatrix} a & 0 & -(1+b)\sqrt{\epsilon} \\ 0 & 1 & 0 \\ (1+b)\sqrt{\epsilon} & 0 & \pi^r \end{pmatrix}, \qquad \begin{array}{l} 1 \leq r \leq e\,, \\ a, b \in \mathcal{O}_k, v_\pi(a) \geq r\,, \\ v_\pi(b) < r \text{ or } b = 0, \end{array} \tag{1.17}$$

By the equation

$$(t-1)(t^2-1) = \Phi_{x'j_3}(t) = (t-1)\left\{t^2 - (1+b)^2\epsilon - \pi^r a\right\}$$

and $\epsilon \in 1 + 4\mathcal{O}_k = 1 + (\pi^{2e})$, we have

$$2b + b^2 + \pi^r a \in \pi^{2e}\mathcal{O}_k\,,$$

which yields $b = 0$ and $a = \pi^{-r}(1-\epsilon)$. Hence

$$x' = \begin{pmatrix} \pi^{-r}(1-\epsilon) & 0 & -\sqrt{\epsilon} \\ 0 & 1 & 0 \\ \sqrt{\epsilon} & 0 & \pi^r \end{pmatrix} \ (=: x_{-r}), \quad 1 \le r \le e\,. \tag{1.18}$$

We note x_{-r} appears only if $e > 0$ and cannot be diagonalized by Proposition 1.3.

[Case 2] $ag = 0$. We may assume $a = 0$.
By (1.13) and (1.14), we see $b = 0$, $c = 1$, $d = -1$, and $2g + ff^* = 0$. Hence we see

$$K_1 \cdot x \ni x' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & h \\ 1 & h & -\frac{1}{2}h^2 \end{pmatrix}, \quad h = 0 \text{ or } \pi^\ell,\ (\ell \in \mathbb{Z})\,. \tag{1.19}$$

If $v_\pi(h) \le e$, then x' satisfies the assumption of Case1, and we have done. If $h \ne 0$ and $v_\pi(h) > e$, then $\frac{1}{2}h \in \mathcal{O}_k$. Taking $c \in \mathcal{O}_{k'}$ such that $\frac{h^2}{4} + c + c^* = 0$, one has

$$K_1 \cdot x \ni \begin{pmatrix} 1 & 0 & 0 \\ -\frac{h}{2} & 1 & 0 \\ c & \frac{h}{2} & 1 \end{pmatrix} \cdot x' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Hence we have only to consider $h = 0$ in (1.19). i.e.

$$x' = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix} = j_3 - 2yy^*, \quad y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{1.20}$$

For an element k in K_1 given by

$$\begin{aligned} k &= \begin{pmatrix} 1 & -1 & \frac{-1+\sqrt{\epsilon}}{2} \\ & 1 & 1 \\ & & 1 \end{pmatrix} \begin{pmatrix} & & 1 \\ & 1 & 1 \\ 1 & -1 & \frac{-1+\sqrt{\epsilon}}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{-1+\sqrt{\epsilon}}{2} & \frac{-1-\sqrt{\epsilon}}{2} & \frac{(-1+\sqrt{\epsilon})^2}{4} \\ 1 & 0 & \frac{1+\sqrt{\epsilon}}{2} \\ 1 & -1 & \frac{-1+\sqrt{\epsilon}}{2} \end{pmatrix}, \end{aligned}$$

one has

$$\begin{aligned} k \cdot x' &= j_3 - k \cdot (2yy^*) = j_3 - 2(ky)(ky)^* \\ &= j_3 - 2 \begin{pmatrix} \frac{-1-\sqrt{\epsilon}}{2} \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} \frac{-1+\sqrt{\epsilon}}{2} & 0 & -1 \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} \frac{\epsilon-1}{2} & 0 & -\sqrt{\epsilon} \\ 0 & 1 & 0 \\ \sqrt{\epsilon} & 0 & -2 \end{pmatrix},$$

which is K_1-equivalent to

$$x_{-e} = \begin{pmatrix} \pi^{-e}(1-\epsilon) & 0 & -\sqrt{\epsilon} \\ 0 & 1 & 0 \\ \sqrt{\epsilon} & 0 & \pi^e \end{pmatrix}.$$

If $e > 0$, the above x_{-e} cannot be diagonalized by Proposition 1.3 and one of the required representatives. If $e = 0$, x_{-0} reduces to Case 1-(i), and actually one has

$$K_1 \cdot x_{-0} \ni \begin{pmatrix} 1 & 0 & \sqrt{\epsilon} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} (1-\epsilon) & 0 & -\sqrt{\epsilon} \\ 0 & 1 & 0 \\ \sqrt{\epsilon} & 0 & 1 \end{pmatrix} = 1_3 .$$

[Case 3] $ag \neq 0$, $v_\pi(a) > v_\pi(b)$ and $v_\pi(g) > v_\pi(f)$. We may assume

$$v_\pi(a) \geq v_\pi(g) . \tag{1.21}$$

By (1.13b) and (1.13d), we have $aN(f) = N(b)g$, which implies

$$v_\pi(a) - v_\pi(g) = 2(v_\pi(b) - v_\pi(f)) \geq 0 . \tag{1.22}$$

We show $v_\pi(c) \geq 0$. If it was not, by (1.13a) and the assumption of Case 3, we had

$$v_\pi(bf) = 2v_\pi(c) \leq -2, \quad \text{hence} \quad v_\pi(f) < 0 .$$

Then

$$(cf^{-1})^2 = f^{-2} - (af^{-1})(gf^{-1}) - bf^{-1} \in \mathcal{O}_{k'} \quad (\text{by (1.13a)}),$$

hence $cf^{-1}, c^*f^{-1} \in \mathcal{O}_{k'}$ and so

$$(\pi) \ni (cf^{-1} + c^*f^{-1})gf^{*-1} = (c+c^*)gN(f)^{-1} = -1 \quad (\text{by (1.13f)}),$$

which is a contradiction. Hence $v_\pi(c) \geq 0$, and again by (1.13f) and the assumption of Case 3, we have

$$v_\pi(f) < v_\pi(g) \leq 2v_\pi(f), \quad \text{hence } v_\pi(f) > 0 .$$

Then, by (1.13e), $d \in \mathcal{O}_k^\times$ and $x \in M_{2n+1}(\mathcal{O}_{k'}) \cap X$, hence $\ell(x) = 0$. Set $r = v_\pi(f) > 0$. Then

$$\begin{aligned} &c + c^* + d = 1, \quad (\text{by (1.14)}) \\ &c + d = c^* + d \equiv 0 \pmod{(\pi^{r+1})}, \quad (\text{by (1.13d)}) \end{aligned}$$

hence

$$c \equiv c^* \equiv 1 \pmod{(\pi^{r+1})}, \quad c + c^* \equiv 2 \pmod{(\pi^{r+1})}\,.$$

Then by (1.13f)

$$2r = v_\pi(c + c^*) + v_\pi(g) \geq \min\{e, r+1\} + r + 1\,,$$

hence we see

$$r > e, \quad v_\pi(c + c^*) = e, \quad v_\pi(g) = 2r - e > r\,.$$

Setting $v_\pi(b) = m(\geq r)$, one has $v_\pi(a) = 2m - e$ by (1.13c). We may take the unit part of a off, and assume x becomes

$$x = \begin{pmatrix} \pi^{2m-e} & \pi^m u & c \\ \pi^m u^* & d & \pi^r v \\ c^* & \pi^r v^* & \pi^{2r-e} w \end{pmatrix}, \quad \begin{array}{l} m \geq r > e \\ c, u, v \in \mathcal{O}_{k'}^\times,\ d, w \in \mathcal{O}_k^\times \\ c \equiv c^* \equiv 1 \mod (\pi^{r+1}),\ d \equiv -1 \mod (\pi^{r+1}), \\ c + c^* + d = 1, \end{array} \tag{1.23}$$

and the set of equations (1.13) becomes

$$\begin{aligned}
&\pi^{2(m+r-e)} w + \pi^{m+r} uv + c^2 = 1\,, \\
&c + d = -\pi^{m+r-e} u^{-1} v^*\,, \\
&c + c^* = -\pi^e uu^*\,, \\
&c + d = -\pi^{m+r-e} u^* v^{-1} w\,, \\
&\pi^{m+r}(uv + u^*v^*) + d^2 = 1\,, \\
&c + c^* = -\pi^e vv^* w^{-1}\,.
\end{aligned}$$

Then together with (1.23), we have

$$\begin{aligned}
&w = (uu^*)^{-1} vv^*, \quad c + c^* = -\pi^e uu^*\,, \\
&d = 1 - (c + c^*) = 1 + \pi^e uu^*\,, \\
&c = 1 - (c^* + d) = 1 + \pi^{m+r-e} u^{*-1} v\,.
\end{aligned}$$

Now we have

$$x' := \begin{pmatrix} u^{-1} & & \\ & 1 & \\ & & u^* \end{pmatrix} \cdot x = \begin{pmatrix} \pi^{2m-e}(uu^*)^{-1} & \pi^m & c \\ \pi^m & d & \pi^r uv \\ c^* & \pi^r u^* v^* & \pi^{2r-e} vv^* \end{pmatrix}$$

$$= j_3 + \frac{\pi^e}{uu^*} \begin{pmatrix} \pi^{2(m-e)} & \pi^{m-e} uu^* & \pi^{m+r-2e} uv \\ \pi^{m-e} uu^* & (uu^*)^2 & \pi^{r-e} u^2 u^* v \\ \pi^{m+r-2e} u^* v^* & \pi^{r-e} uu^{*2} v^* & \pi^{2(r-e)} uu^* vv^* \end{pmatrix}$$

$$= j_3 + \frac{\pi^e}{uu^*} yy^*, \quad y = \begin{pmatrix} \pi^{m-e} \\ uu^* \\ \pi^{r-e}u^*v^* \end{pmatrix}.$$

We will show there exists some $k \in K_1$ for which the second entry $(ky)_2$ of ky is 0. Then $k \cdot x'$ has the shape $\begin{pmatrix} \cdot & 0 & \cdot \\ 0 & 1 & 0 \\ \cdot & 0 & \cdot \end{pmatrix}$, which reduces to Case 1 or Case 2 (if $e > 0$, x is equivalent to some $x_{-\ell}$, $1 \le \ell \le e$, by Proposition 1.3). Set

$$k = \begin{pmatrix} 1 & -\alpha^* & \beta \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -\gamma^* \\ 1 & \gamma & \delta \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \alpha & 1+\alpha\gamma & \alpha\delta - \gamma^* \\ \cdot & \cdot & \cdot \end{pmatrix}, \tag{1.24}$$

and solve

$$\big((ky)_2 = \big)\, \alpha\pi^{m-e} + (1+\alpha\gamma)uu^* + (\alpha\delta - \gamma^*)\pi^{r-e}u^*v^* = 0 \tag{1.25}$$

under the condition $\alpha, \beta, \gamma, \delta \in \mathcal{O}_{k'}$ and $N(\alpha) + \beta + \beta^* = N(\gamma) + \delta + \delta^* = 0$, which is equivalent that k of (1.24) becomes an element in K_1.

If (1.25) is satisfied, we see $1 + \alpha\gamma \in (\pi^{r-e}) \subset (\pi)$, and $\alpha, \gamma \in \mathcal{O}_{k'}^\times$. Writing $1 + \alpha\gamma = \pi^{r-e}\lambda$ with $\lambda \in \mathcal{O}_{k'}$, we have $\delta = \alpha^{-1}\gamma^* - \pi^{m-r}(u^*v^*)^{-1} - \alpha^{-1}\lambda u v^{*-1}$ by (1.25). Then, since $\alpha \neq 0$, the condition $N(\gamma) + \delta + \delta^* = 0$ is equivalent to

$$N(\alpha)N(\gamma) + \alpha\gamma + \alpha^*\gamma^* - \pi^{m-r}N(\alpha)((uv)^{-1} + (u^*v^*)^{-1}) - \alpha^*\lambda u v^{*-1} - \alpha\lambda u^* v^{-1} = 0,$$

then, since $1 + \alpha\gamma = \pi^{r-e}\lambda$, it becomes

$$\pi^{2(r-e)}N(\lambda) - \pi^{m-r}N(\alpha)((uv)^{-1} + (u^*v^*)^{-1}) - \alpha^*\lambda u v^{*-1} - \alpha\lambda^* u^* v^{-1} = 1. \tag{1.26}$$

Setting $\lambda = \alpha u^{-1}v^*\mu$ with $\mu \in \mathcal{O}_{k'}$, (1.26) is equivalent to

$$N(\alpha)\left(\pi^{2(r-e)}N(u)^{-1}N(v)N(\mu) - \pi^{m-r}((uv)^{-1} + (u^*v^*)^{-1}) - (\mu + \mu^*)\right) = 1. \tag{1.27}$$

Since $r > e$, we may choose $\mu \in \mathcal{O}_k$ for which the latter factor in the left hand side of (1.27) becomes a unit in $\mathcal{O}_k$, then for suitable $\alpha \in \mathcal{O}_{k'}^\times$ we establish the identity (1.27). Finally taking β such as $N(\alpha) + \beta + \beta^* = 0$, we obtain $k \in K_1$ for which $(ky)_2 = 0$, which establishes (1.25).

Thus we have shown that any K_1-orbit in X_1 has a representative in $\mathcal{R}_1^+ \sqcup \mathcal{R}_1^-$ and $K_1 \cdot x \cap \mathcal{R}_1^- \neq \emptyset$ if and only if $e > 0$ and $x \equiv j_3 \pmod{(\pi)}$. It is known that each x_ℓ, $\ell \ge 0$ gives a different $GL_3(\mathcal{O}_{k'})$-orbit in $\{x \in GL_3(k') \mid x^* = x\}$, hence it gives a different K_1-orbit in X_1. For x_{-r}, $1 \le r \le e$, $-r = \ell(x_{-r} - j_3)$ is an invariant of $K_1 \cdot x_{-r}$, since $h \cdot (x_{-r} - j_3) = h \cdot x_{-r} - j_3$ for any $h \in K_1$ (cf. Lemma 1.4 (1)). Hence $\mathcal{R}_1^+ \sqcup \mathcal{R}_1^-$ forms a set of complete representatives of $K_1 \backslash X_1$. $\square$

1.3. In this subsection we will show Theorem 1.1-(1) for the case $m = 2n+1$ with $n \geq 2$. Our strategy is similar to [HK2] (we have to be careful for $e > 0$.)

LEMMA 1.7. *Let $n \geq 2$. Then every $x \in X_n$ has a minimal entry except the $(n+1, n+1)$-entry.*

PROOF. Assume x had unique minimal entry $\pi^{-\ell}u,\ u \in \mathcal{O}_k^\times$ at $(n+1, n+1)$ and denote by E' the $(n+1, n+1)$-matrix unit in $M_{2n+1}(k')$. Then $\pi^\ell x \in M_{2n+1}(\mathcal{O}_{k'})$ and

$$\pi^{2\ell} 1_{2n+1} = (\pi^\ell x) j_{2n+1} (\pi^\ell x) j_{2n+1} \equiv u^2 E' \mod (\pi)\,,$$

which is impossible. □

The following lemmas can be shown in the same way as in [HK2], so we omit the details of proofs.

LEMMA 1.8. *Let $n \geq 2$ and assume that*

$(A1, n)$: $x \in X_n$ has a minimal entry in the diagonal except the $(n+1, n+1)$-entry. Then $K \cdot x$ contains a hermitian matrix of the type

$$\left(\begin{array}{c|c|c} \pi^\ell & 0 & 0 \\ \hline 0 & y & 0 \\ \hline 0 & 0 & \pi^{-\ell} \end{array}\right), \qquad y \in X_{n-1},\ \ell = \ell(x) \geq \ell(y)\,.$$

OUTLINE OF A PROOF. By the action of W, we may assume that the $(2n+1, 2n+1)$-entry of x is minimal, and then we may arrange it into the above shape by a suitable K-action. □

LEMMA 1.9. *Let $n \geq 2$ and assume one of the following conditions hold*:

$(A2, n)$: $x \in X_n$ has a minimal entry outside of the diagonal, the anti-diagonal, the $(n+1)$th row, and the $(n+1)$th column.

$(A3, n)$: x has a minimal entry and a non-minimal entry in the anti-diagonal except the $(n+1, n+1)$-entry.

Then $K \cdot x$ contains a hermitian matrix of type

$$\left(\begin{array}{c|c|c} \begin{matrix} \pi^\ell & 0 \\ 0 & \pi^\ell \end{matrix} & 0 & 0 \\ \hline 0 & y & 0 \\ \hline 0 & 0 & \begin{matrix} \pi^{-\ell} & 0 \\ 0 & \pi^{-\ell} \end{matrix} \end{array}\right), \qquad \begin{array}{l} y \in X_{n-2},\ \ell = \ell(x) \geq \ell(y)\ \ \textit{if}\ \ n \geq 3, \\ y = 1\ \ \textit{if}\ \ n = 2. \end{array}$$

OUTLINE OF A PROOF. Write $\ell = \ell(x)$. Under the condition $(A2, n)$, such minimal entries appear in pair, since x is hermitian. Then, through the action of W and $GL_2(\mathcal{O}_{k'})$, we may assume that the lower right 2 by 2 block of x is $\begin{pmatrix} \pi^{-\ell} & 0 \\ 0 & \pi^{-\ell} \end{pmatrix}$, then we may arrange it into the above shape by the suitable K-action. As for y, it is clear that $y = y^*$ and $\ell(y) \le \ell$. Considering the characteristic function of xj_{2n+1}, we see $y \in X_{n-2}$ if $n \ge 3$ and $y = 1$ if $n = 2$.

As for the condition $(A3, n)$, together with the action of W, we may assume the upper right 2 by 2 block of x is $\begin{pmatrix} a & \xi \\ b & c \end{pmatrix}$ such that $v_\pi(\xi) = -\ell$ and $a, b, c \in \pi^{-\ell+1}\mathcal{O}_{k'}$. Then, by the action of

$$h = \left(\begin{array}{c|c|c} \begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix} & 0 & 0 \\ \hline 0 & 1_{2n-3} & 0 \\ \hline 0 & 0 & \begin{matrix} 1 & -1 \\ 0 & 1 \end{matrix} \end{array}\right),$$

the $(1, 2n)$-entry becomes $a + b - c - \xi$, and its v_π-value is $\ell = \ell(x)$. Thus it reduces to the case $(A2, n)$. □

We have to consider the remaining case, which is the assumption $(A4, n)$ below. The statement for the non-dyadic case is a refinement of [HK2, Lemma 1.9].

LEMMA 1.10. *Let $x \in X_n$ with $n \ge 2$. Assume that*

$(A4, n)$: *any minimal entry of $x \in X_n$ stands in the anti-diagonal, the $(n+1)$th row, or the $(n+1)$th column, and that all the anti-diagonal entries except $(n+1, n+1)$ are minimal. Then $\ell(x) = 0$, and*
(i) *if k is non-dyadic, then $K \cdot x$ contains 1_{2n+1};*
(ii) *if k is dyadic, then $K \cdot x$ contains*

$$\begin{pmatrix} \pi^{\mu_1}(1-\epsilon) & & & & & & -\sqrt{\epsilon} \\ & \ddots & & & & \iddots & \\ & & \pi^{\mu_n}(1-\epsilon) & & -\sqrt{\epsilon} & & \\ & & & 1 & & & \\ & & \sqrt{\epsilon} & & \pi^{-\mu_n} & & \\ & \iddots & & & & \ddots & \\ \sqrt{\epsilon} & & & & & & \pi^{-\mu_1} \end{pmatrix}, \tag{1.28}$$

where $0 > \mu_1 \ge \mu_2 \ge \cdots \ge \mu_n \ge -e$.

PROOF. By the action of W and suitable K-action, we may assume x has the shape

$$x = \left(\begin{array}{cccc|c|cccc} & & & & \alpha & & & & \xi_1 \\ & * & & & 0 & & * & \iddots & \\ & & & & \vdots & & \iddots & * & \\ & & & & 0 & \xi_n & & & \\ \hline \alpha^* & 0 & \cdots & 0 & u & 0 & \cdots & 0 & \kappa^* \\ \hline & & & \xi_n^* & 0 & & & & \\ & * & \iddots & & \vdots & & & & \\ & \iddots & * & & 0 & & & * & \\ \xi_1^* & & & & \kappa & & & & \end{array}\right), \tag{1.29}$$

where $\alpha = 0$ or $v_\pi(\alpha) \leq v_\pi(\kappa)$. Write $\ell = \ell(x)$. Since $xj_{2n+1}xj_{2n+1} = 1_{2n+1}$, by its $(1, 2n+1)$-entry and $(2n+1, 1)$-entry, we have

$$\alpha\alpha^* \equiv \kappa\kappa^* \equiv 0 \pmod{(\pi^{-2\ell+1})},$$

hence α nor κ is not minimal, which means any entry outside of the anti-diagonal is non-minimal. Setting $u_i = \pi^\ell \xi_i \in \mathcal{O}_{k'}^\times$, $1 \leq i \leq n$, and $u_0 = \pi^\ell u \in \mathcal{O}_k$, we have

$$\begin{aligned} \pi^{2\ell} 1_{2n+1} &= \pi^{2\ell} x j_{2n+1} x j_{2n+1} = (\pi^\ell x) j_{2n+1} (\pi^\ell x) j_{2n+1} \\ &\equiv Diag(u_1^2, \ldots, u_n^2, u_0^2, u_n^{*2}, \ldots, u_1^{*2}) \pmod{(\pi)}, \end{aligned}$$

hence

$$\ell = 0, \quad \xi_i^2 \equiv \xi_i^{*2} \equiv u^2 \equiv 1 \pmod{(\pi)},$$

and

$$x \equiv \left(\begin{array}{ccc|c|ccc} & & & & & & \xi_1 \\ & & & & & \iddots & \\ & & & & \xi_n & & \\ \hline & & & u & & & \\ \hline & & \xi_n & & & & \\ & \iddots & & & & & \\ \xi_1 & & & & & & \end{array}\right) \pmod{(\pi)},$$

$$\xi_i \equiv \xi_i^* \equiv \pm 1 \pmod{(\pi)}, \quad u \equiv \pm 1 \pmod{(\pi)}.$$

First we consider the non-dyadic case, where $1 \not\equiv -1 \pmod{(\pi)}$. By the characteristic polynomial, we have

$$(t^2-1)^n(t+1) \equiv (t-u)\prod_{i=1}^{n}(t-\xi_i)^2 \pmod{(\pi)}, \quad n \geq 2\,,$$

hence we may assume $\xi_1 \not\equiv \xi_2 \pmod{(\pi)}$, after suitable W-action if necessary. For

$$h = \left(\begin{array}{cc|c|cc} 1 & & & & \\ & 1 & & & \\ \hline & & 1_{2n-3} & & \\ \hline 1 & & & 1 & \\ & -1 & & & 1 \end{array}\right) \in K\,,$$

we have

$$h \cdot x \equiv \left(\begin{array}{cc|c|cc} & & & & \xi_1 \\ & & & \xi_2 & \\ \hline & & C & & \\ \hline & \xi_2 & & & \xi_1-\xi_2 \\ \xi_1 & & & \xi_1-\xi_2 & \end{array}\right) \pmod{(\pi)}\,,$$

where $Cj_{2n-3} = Diag(\xi_3, \dots, \xi_n, u, \xi_n, \dots, \xi_3)$ and $\xi_1 - \xi_2 \not\equiv 0 \pmod{(\pi)}$. Since $h \cdot x$ satisfies $(A2, n)$ with $\ell = 0$, we see

$$K \cdot x \ni \left(\begin{array}{c|c|c} 1_2 & 0 & 0 \\ \hline 0 & y & 0 \\ \hline 0 & 0 & 1_2 \end{array}\right), \quad y \in X_{n-2},\ \ell(y) = 0,\ \text{or}\ \ y = 1\,.$$

Then, we see inductively $K \cdot x$ contains 1_{2n+1}.

Now we assume k is dyadic. Then $x \equiv j_{2n+1} \pmod{(\pi)}$ and x is not diagonalizable, by Proposition 1.3.

[Case 1] We assume that there is a non-zero entry of x outside of the anti-diagonal, $(n+1)$-th row and $(n+1)$-th column, i.e. within $*$-places of (1.29). Let ℓ be the minimal v_π-value within those entries. Then, after suitable K-action, we may assume the $(2n+1, 2n+1)$-entry of x is π^{ℓ}. Further, after suitable K-action, we may assume

$$x = \left(\begin{array}{cccc|c|cccc} c & & & & \alpha & & & & \xi_1 \\ & * & & & 0 & & * & \cdot^{\cdot^{\cdot}} & 0 \\ & & & & \vdots & & \cdot^{\cdot^{\cdot}} & * & \vdots \\ & & & & 0 & \xi_n & & & 0 \\ \hline \alpha^* & 0 & \cdots & 0 & u & 0 & \cdots & 0 & \kappa^* \\ \hline & & & \xi_n^* & 0 & & & & 0 \\ & * & \cdot^{\cdot^{\cdot}} & & \vdots & & & & \vdots \\ & \cdot^{\cdot^{\cdot}} & * & & 0 & & & * & 0 \\ \xi_1^* & 0 & \cdots & 0 & \kappa & 0 & \cdots & 0 & \pi^{\ell} \end{array}\right),$$

where $v_\pi(c) \leq \ell$, "$\alpha = 0$ or $v_\pi(\alpha) \leq v_\pi(\kappa)$", and "$\kappa = 0$ or $v_\pi(\kappa) < \ell$". Looking at the $(i, 1)$-entry with $i \neq n+1$ of $xj_{2n+1}xj_{2n+1} = 1_{2n+1}$, since x is hermitian, we have

$$x = \left(\begin{array}{cccc|c|cccc} c & 0 & \cdots & 0 & \alpha & 0 & \cdots & 0 & \xi_1 \\ 0 & & & & 0 & & * & \cdot^{\cdot^{\cdot}} & 0 \\ \vdots & & * & & \vdots & & \cdot^{\cdot^{\cdot}} & * & \vdots \\ 0 & & & & 0 & \xi_n & & & 0 \\ \hline \alpha^* & 0 & \cdots & 0 & u & 0 & \cdots & 0 & \kappa^* \\ \hline 0 & & & \xi_n^* & 0 & & & & 0 \\ \vdots & * & \cdot^{\cdot^{\cdot}} & & \vdots & & * & & \vdots \\ 0 & \cdot^{\cdot^{\cdot}} & * & & 0 & & & & 0 \\ \xi_1^* & 0 & \cdots & 0 & \kappa & 0 & \cdots & 0 & \pi^{\ell} \end{array}\right).$$

From the above x, we set

$$z = \begin{pmatrix} c & \alpha & \xi_1 \\ \alpha^* & u & \kappa^* \\ \xi_1^* & \kappa & \pi^{\ell}, \end{pmatrix} \tag{1.30}$$

and see z or $-z$ is an element of X_1 with $\ell(z) = 0$ (cf. (1.4)). By the action of $K_1 = U(j_3)(\mathcal{O}_{k'})$ through the embedding

$$\begin{array}{ccc} K_1 & \longrightarrow & K = K_n, \\ h = (h_{ij}) & \longmapsto & \widetilde{h} = \begin{pmatrix} h_{11} & & h_{12} & & h_{13} \\ & 1_{n-1} & & & \\ h_{21} & & h_{22} & & h_{23} \\ & & & 1_{n-1} & \\ h_{31} & & h_{32} & & h_{33} \end{pmatrix}, \end{array}$$

we may change z in x of (1.30) into $\pm x_{-r}^{(1)}$ for some r with $1 \le r \le e$, where the superscript (1) indicates the size $(m, n) = (3, 1)$. When $-x_{-r}^{(1)}$ appears as z, by the action of $K_0 := U(j_2)(\mathcal{O}_{k'})$ (tentative naming) through the embedding

$$\begin{array}{rcl} K_0 & \longrightarrow & K = K_n, \\ h = (h_{ij}) & \longmapsto & \widetilde{h} = \begin{pmatrix} h_{11} & & h_{12} \\ & 1_{2n-1} & \\ h_{21} & & h_{22} \end{pmatrix}, \end{array}$$

we may change $z = -x_{-r}^{(-1)}$ of x into

$$\begin{pmatrix} \pi^{-r_1}(1-\epsilon) & & -\sqrt{\epsilon} \\ & -1 & \\ \sqrt{\epsilon} & & \pi^{r_1} \end{pmatrix},$$

where r_1 might be changed from r but still $1 \le r_1 \le e$. Anyway, we see

$$K \cdot x \ni \begin{pmatrix} \pi^{-r}(1-\epsilon) & & -\sqrt{\epsilon} \\ & y & \\ \sqrt{\epsilon} & & \pi^{r} \end{pmatrix}, \quad \begin{array}{l} 1 \le r \le e, \\ y = y^* \in M_{2n-1}(\mathcal{O}_{k'}), \ y \equiv j_{2n-1} \pmod{(\pi)}. \end{array}$$

Since $\Phi_{xj_{2n+1}}(t) = (t^2-1)\Phi_{yj_{2n-1}}(t)$, we see $y \in X_{n-1}$, and y satisfies $(A4, n-1)$. By an inductive procedure, we see the K-orbit of x contains a matrix of type (1.28).

[Case 2] We consider the remaining situation of Case 1, i.e. any entry of $*$-places of x in (1.29) is 0. Then $\alpha = \kappa = 0$ follows from $xj_{2n+1}xj_{2n+1} = 1_{2n+1}$, and we have

$$z_1 = \begin{pmatrix} & & \xi_1 \\ & u & \\ \xi_1 & & \end{pmatrix}, \quad \xi_1 = \xi_1^* = \pm 1, \ u = \pm 1,$$

in stead of z of (1.30). By the same procedure as Case 1, we see $K \cdot x$ contain a matrix of type (1.28). $\square$

By Proposition 1.5, Lemma 1.8, Lemma 1.9 and Lemma 1.10, we see for every $x \in X$, $K \cdot x$ has a representative of shape x_λ for some $\lambda \in \widetilde{\Lambda_n^+}$, which completes the proof of Theorem 1.1-(1). $\square$

2. Spherical function on X

2.1. We consider $m = 2n$ or $2n+1$, and write $X = X_n,\ G = G_n,\ B = B_n,\ K = K_n$. For $g \in G$, we denote by $d_i(g)$ the determinant of the lower right i by i block of g. Then $d_i(x),\ 1 \le i \le n$, are relative B-invariants on X associated with rational character ψ_i of B, where

$$d_i(p \cdot x) = \psi_i(p) d_i(x), \quad \psi_i(p) = N(d_i(p)), \quad (x \in X,\ p \in B).$$

We set

$$X^{op} = \{x \in X \mid d_i(x) \neq 0,\ 1 \leq i \leq n\}\ ,$$

then $X^{op}(\overline{k})$ is a Zarisky open $B(\overline{k})$-orbit. For $x \in X$ and $s \in \mathbb{C}^n$, we consider the integral

$$\omega(x; s) = \int_K |\mathbf{d}(k \cdot x)|^s\, dk, \quad |\mathbf{d}(y)|^s = \begin{cases} \prod_{i=1}^n |d_i(y)|^{s_i} & \text{if } y \in X^{op} \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

where $|\ |$ is the absolute value on k normalized by $|\pi| = q^{-1}$, dk is the normalized Haar measure on K. The integral in (2.1) is absolutely convergent if $\mathrm{Re}(s_i) \geq 0,\ 1 \leq i \leq n$, and continued to a rational function of $q^{s_1}, \ldots, q^{s_n}$, and we use the notation $\omega(x; s)$ in such sense. We call $\omega(x; s)$ a spherical function on X, since it becomes an $\mathcal{H}(G, K)$-common eigenfunction on X (cf. [H1, §1], [H2, §1]). Indeed, $\mathcal{H}(G, K)$ is a commutative $\mathbb{C}$-algebra spanned by all the characteristic functions of double cosets $KgK,\ g \in G$, by definition, and we see

$$\begin{aligned}(f * \omega(x; s))(x) \left(= \int_G f(g)\omega(g^{-1} \cdot x; s)dg \right) \\ = \lambda_s(f)\omega(x; s), \quad (f \in \mathcal{H}(G, K))\,,\end{aligned} \tag{2.2}$$

where dg is the Haar measure on G normalized by $\int_K dg = 1$, and λ_s is the $\mathbb{C}$-algebra homomorphism (Satake transform) defined by

$$\begin{aligned} \lambda_s : \mathcal{H}(G, K) &\longrightarrow \mathbb{C}(q^{s_1}, \ldots, q^{s_n}), \\ f &\longmapsto \int_B f(p)\, |\psi(p)|^{-s}\, \delta(p) dp. \end{aligned} \tag{2.3}$$

Here $|\psi(p)|^{-s} = \prod_{i=1}^n |\psi_i(p)|^{-s_i}$, dp is the left-invariant measure on B such that $\int_{B \cap K} dp = 1$, and $\delta(p)$ is the modulus character of dp $(d(pq) = \delta(q)^{-1}dp)$.

It is convenient to introduce a new variable $z = (z_i) \in \mathbb{C}^n$ which is related to s by

$$\begin{aligned} s_i &= -z_i + z_{i+1} - 1 + \frac{\pi\sqrt{-1}}{\log q} \quad 1 \leq i \leq n-1 \\ s_n &= \begin{cases} -z_n - \dfrac{1}{2} & \text{if } m = 2n \\ -z_n - 1 + \dfrac{\pi\sqrt{-1}}{2\log q} & \text{if } m = 2n+1\,, \end{cases} \end{aligned} \tag{2.4}$$

and denote $\omega(x; z) = \omega(x; s)$ and $\lambda_z = \lambda_s$.

REMARK 2.1. For the case $m = 2n$, we have slightly changed the relation between s and z from that in [HK1], where we set $s_n = -z_n - \frac{1}{2} + \frac{\pi\sqrt{-1}}{\log q}$. By the explicit formula of $\omega(x; z)$ (Theorem 3.1), we will see the present $\omega(x; z)$ takes the same value as before on $G \cdot x_0$ and the multiple by (-1) on $G \cdot x_1$, and we will explain the reason of this change in Remark 3.4.

Keeping the relation (2.4), we see

$$|\psi(p)|^s = \delta^{\frac{1}{2}}(p)\prod_{i=1}^{n}|N(p_i)|^{z_i} \times \begin{pmatrix} 1 & \text{if } m = 2n \\ (-1)^{v_\pi(p_1\cdots p_n)} & \text{if } m = 2n+1 \end{pmatrix},$$

$$\lambda_z : \mathcal{H}(G, K) \xrightarrow{\sim} \mathbb{C}[q^{\pm 2z_1}, \ldots, q^{\pm 2z_n}]^W, \tag{2.5}$$

where p_i is the i-th diagonal entry of $p \in B$, $1 \leq i \leq n$, and W is the Weyl group of G with respect to the maximal k-split torus in B. Then $W \cong S_n \ltimes (\pm 1)^n$ acts on s and z through rational characters of B, where W is generated by S_n and τ, S_n acts on z by the permutation of indices and $\tau(z) = (z_1, \ldots, z_{n-1}, -z_n)$. The functional equation with respect to S_n is reduced to the case of unramified hermitian forms as follows. Define an embedding $K_0 = GL_n(\mathcal{O}_{k'}) \longrightarrow K = K_n$ by

$$K_0 \ni h \longmapsto \widetilde{h} = \left\{ \begin{array}{ll} \begin{pmatrix} j_n h^{*-1} j_n & \\ & h \end{pmatrix} & \text{if } m = 2n \\ \begin{pmatrix} j_n h^{*-1} j_n & & \\ & 1 & \\ & & h \end{pmatrix} & \text{if } m = 2n+1 \end{array} \right\} \in K .$$

Then, as is considered in [HK1] and [HK2], we have

$$\begin{aligned} \omega(x; s) &= \int_{K_0} dh \int_K |\mathbf{d}(k \cdot x)|^s \, dk \\ &= \int_{K_0}\int_K \left|\mathbf{d}(\widetilde{h}k \cdot x)\right|^s dkdh \\ &= \int_K \zeta_*^{(h)}(D(k \cdot x); s) dk , \end{aligned}$$

where $D(k \cdot x)$ is the lower n by n block of $k \cdot x$, and $\zeta_*^{(h)}(y; s)$ is a spherical function on hermitian matrices $\mathcal{H}_n(k')$. Since the behaviour of $\zeta_*^{(h)}(y; s)$ is independent of the residual characteristic (cf. [H1]), we may quote the the following from [HK1, Theorem 2.1] and [HK2, Theorem 2.1].

PROPOSITION 2.2. *The function $G_1(z) \cdot \omega(x; z)$ is invariant under the action of S_n on z, where*

$$G_1(z) = \prod_{1 \leq i < j \leq n} \frac{1 + q^{z_i + z_j}}{1 - q^{z_i - z_j - 1}} .$$

2.2. In this subsection we study $\omega(x; s)$ for the case $(m, n) = (2, 1)$ and show the following. As a result, we see again the set $\mathfrak{X}_1^{ev} = \left\{ x_\lambda \;\middle|\; \lambda \in \widetilde{\Lambda_1^+} \right\}$ forms a set of complete representatives of $K_1 \backslash X_1$, since $\omega(x_\lambda; z)$ takes different value for each λ for generic z.

PROPOSITION 2.3. *For $x_\lambda \in \mathfrak{X}_1^{ev}$, one has*

$$\omega(x_\lambda; z) = \frac{q^{-\frac{\lambda}{2}} q^{ez}}{1+q^{-1}} \left(\frac{q^{-(\lambda+e)z}(1-q^{2z-1})}{1-q^{2z}} + \frac{q^{(\lambda+e)z}(1-q^{-2z-1})}{1-q^{-2z}} \right).$$

In particular, for any $x \in X_1^{(ev)}$,

$$q^{-ez}\omega(x; z) \in \mathbb{C}[q^z + q^{-z}], \quad \omega(x; z) = q^{2ez}\,\omega(x; -z)\,. \tag{2.6}$$

We have proved for $e = 0$ in [HK1, Proposition 2.4], but we give a unified proof for $e \geq 0$ here. It is easy to see

$$\begin{aligned} K_1 &= K_{1,1} \sqcup K_{1,2}\,, \\ K_{1,1} &= \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{*-1} \end{pmatrix} \begin{pmatrix} 1 & v/\sqrt{\epsilon} \\ u\sqrt{\epsilon} & 1+uv \end{pmatrix} \;\middle|\; \alpha \in \mathcal{O}_{k'}^\times,\ u, v \in \mathcal{O}_k \right\}, \\ K_{1,2} &= \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{*-1} \end{pmatrix} \begin{pmatrix} \pi u\sqrt{\epsilon} & 1+\pi uv \\ 1 & v/\sqrt{\epsilon} \end{pmatrix} \;\middle|\; \alpha \in \mathcal{O}_{k'}^\times,\ u, v \in \mathcal{O}_k \right\}, \end{aligned}$$

and $vol(K_{1,1}) = \frac{1}{1+q^{-1}}$ and $vol(K_{1,2}) = \frac{q^{-1}}{1+q^{-1}}$ with respect to the measure on K_1 normalized by $vol(K_1) = 1$.

(1) The case $x_\lambda = Diag(\pi^\lambda, \pi^{-\lambda})$ with $\lambda \geq 0$.

For $h = \begin{pmatrix} 1 & v/\sqrt{\epsilon} \\ u\sqrt{\epsilon} & 1+uv \end{pmatrix} \in K_{1,1}$, we have

$$\begin{aligned} d_1(h \cdot x_\lambda) &= -\pi^\lambda u^2 \epsilon + \pi^{-\lambda}(1+uv)^2 = \pi^{-\lambda} N(1+uv-\pi^\lambda u\sqrt{\epsilon}) \\ &= \pi^{-\lambda} N\Big(1+uv+\pi^\lambda u - 2\pi^\lambda u \frac{1+\sqrt{\epsilon}}{2}\Big). \end{aligned}$$

If $u \in \pi\mathcal{O}_k$, then $1+uv+\pi^\lambda u \in \mathcal{O}_k^\times$. For $u \in \mathcal{O}_k^\times$ and $r > 0$, we have

$$vol\big(\big\{ v \in \mathcal{O}_k \;\big|\; 1+uv+\pi^\lambda u \equiv 0(\pi^r) \big\}\big) = vol\left(\left\{ v \in \mathcal{O}_k \;\middle|\; v+\pi^\lambda \equiv -u^{-1}(\pi^r) \right\}\right) = q^{-r},$$

and for $r \geq 0$

$$vol\big(\big\{ (u, v) \in \mathcal{O}_k^\times \times \mathcal{O}_k \;\big|\; v_\pi(1+uv+\pi^\lambda u) = r \big\}\big) = (1-q^{-1})^2 q^{-r}\,.$$

Hence we see

$$\int_{K_{1,1}} |d_1(h\cdot x_\lambda)|^s\, dh$$

$$= \frac{q^{\lambda s}}{1+q^{-1}}\left(q^{-1} + \sum_{r=0}^{e+\lambda-1}(1-q^{-1})^2 q^{-r-2rs} + \sum_{r\geq e+\lambda}(1-q^{-1})^2 q^{-r-2(e+\lambda)s}\right)$$

$$= \frac{q^{\lambda s}}{1+q^{-1}}\left(q^{-1} + \frac{(1-q^{-1})^2(1-q^{-(e+\lambda)-2(e+\lambda)s})}{1-q^{-1-2s}} + (1-q^{-1})q^{-(e+\lambda)(1+2s)}\right). \quad (2.7)$$

On the other hand, for $h = \begin{pmatrix} \pi u\sqrt{\epsilon} & 1+\pi uv \\ 1 & v/\sqrt{\epsilon} \end{pmatrix} \in K_{1,2}$, we have

$$d_1(h\cdot x_\lambda) = \pi^\lambda - \frac{\pi^{-\lambda}v^2}{\epsilon} = -\frac{\pi^{-\lambda}}{\epsilon}\cdot N(v+\pi^\lambda\sqrt{\epsilon})$$

$$= -\frac{\pi^{-\lambda}}{\epsilon}\cdot N\Big(v - \pi^\lambda + 2\pi^\lambda\frac{1+\sqrt{\epsilon}}{2}\Big).$$

Hence we see

$$\int_{K_{1,2}} |d_1(h\cdot x_\lambda)|^s\, dh$$

$$= \frac{q^{-1+\lambda s}}{1+q^{-1}}\left(\sum_{r=0}^{e+\lambda-1}(1-q^{-1})q^{-r-2rs} + \sum_{r\geq e+\lambda}(1-q^{-1})q^{-r-2(e+\lambda)s}\right)$$

$$= \frac{q^{-1+\lambda s}}{1+q^{-1}}\left(\frac{(1-q^{-1})(1-q^{-(e+\lambda)-2(e+\lambda)s})}{1-q^{-1-2s}} + q^{-(e+\lambda)-2(e+\lambda)s}\right). \quad (2.8)$$

By (2.7) and (2.8), we obtain for $s = -z - \frac{1}{2} \in \mathbb{C}$

$$\omega(x_\lambda; s) = \frac{q^{-\lambda z - \frac{\lambda}{2}}}{(1+q^{-1})(1-q^{2z})}\Big(1 - q^{2z-1} + q^{2(e+\lambda)z-1} - q^{2(e+\lambda+1)z}\Big)$$

$$= \frac{q^{-\frac{\lambda}{2}}}{(1+q^{-1})}\left(\frac{q^{-\lambda z}(1-q^{2z-1})}{1-q^{2z}} + \frac{q^{(2e+\lambda)z-1} - q^{(2e+\lambda+2)z}}{1-q^{2z}}\right)$$

$$= \frac{q^{-\frac{\lambda}{2}}q^{ez}}{1+q^{-1}}\left(\frac{q^{-(\lambda+e)z}(1-q^{2z-1})}{1-q^{2z}} + \frac{q^{(\lambda+e)z}(1-q^{(-2z-1)})}{1-q^{-2z}}\right). \quad (2.9)$$

(2) The case $x_\lambda = \begin{pmatrix} \pi^\lambda(1-\epsilon) & -\sqrt{\epsilon} \\ \sqrt{\epsilon} & \pi^{-\lambda} \end{pmatrix}$ with $-e \leq \lambda < 0$, only when $e > 0$.

Set $r = -\lambda$, then $1 \le r \le e$. For $h = \begin{pmatrix} 1 & v/\sqrt{\epsilon} \\ u\sqrt{\epsilon} & 1+uv \end{pmatrix} \in K_{1,1}$, we have

$$\begin{aligned} d_1(h \cdot x_\lambda) &= \big(\pi^{-r}u(1-\epsilon)\sqrt{\epsilon} + (1+uv)\sqrt{\epsilon}\big)\,(-u\sqrt{\epsilon}) + \big(-u\epsilon + \pi^r(1+uv)\big)\,(1+uv) \\ &= \pi^{-r}\big(\pi^{2r}(1+uv)^2 - 2\pi^r(1+uv)u\epsilon + u^2(\epsilon-1)\epsilon\big) \\ &= \pi^{-r} \cdot N(\pi^r(1+uv) - u\epsilon - u\sqrt{\epsilon}) \\ &= \pi^{-r} \cdot N\Big(\pi^r(1+uv) + u(1-\epsilon) - 2u\frac{1+\sqrt{\epsilon}}{2}\Big). \end{aligned}$$

Since $e > 0$ and $v_\pi(1-\epsilon) = 2e$, we see

$$v_\pi\Big(N\Big(\pi^r(1+uv) + u(1-\epsilon) - 2u\frac{1+\sqrt{\epsilon}}{2}\Big)\Big) = 2\min\{r + v_\pi(1+uv), e + v_\pi(u)\}.$$

We have

$$\begin{aligned} &vol\Big(\Big\{ (u,v) \in \mathcal{O}_k^2 \;\Big|\; v_\pi(1+uv) = 0\Big\} \\ &= vol\Big(\Big\{ (u,v) \in \mathcal{O}_k^2 \;\Big|\; uv \in (\pi)\Big\}\Big) + vol\Big(\Big\{ (u,v) \in \mathcal{O}_k^{\times 2} \;\Big|\; v \not\equiv -u^{-1}(\pi)\Big\}\Big) \\ &= 2q^{-1} - q^{-2} + (1-q^{-1})(1-2q^{-1}) = 1 - q^{-1} + q^{-2}, \end{aligned}$$

and, for $j > 0$,

$$\begin{aligned} &vol\Big(\Big\{ (u,v) \in \mathcal{O}_k^2 \;\Big|\; v_\pi(1+uv) = j\Big\}\Big) = vol\Big(\Big\{ (u,v) \in \mathcal{O}_k^{\times 2} \;\Big|\; v_\pi(1+uv) = j\Big\}\Big) \\ &= (1-q^{-1})^2 q^{-j}. \end{aligned}$$

Hence, for $\lambda = -r < 0$,

$$\begin{aligned} &\int_{K_{1,1}} |d_1(h \cdot x_\lambda)|^s \, dh \\ &= \frac{q^{rs}}{1+q^{-1}}\left((1-q^{-1}+q^{-2})q^{-2rs} + \sum_{j=1}^{e-r-1}(1-q^{-1})^2 q^{-j-2(r+j)s} + \sum_{j \ge e-r}(1-q^{-1})^2 q^{-j-2es}\right) \\ &= \frac{q^{rs}}{1+q^{-1}}\left((1-q^{-1}+q^{-2})q^{-2rs} + \frac{(1-q^{-1})^2(q^{-1-2(r+1)s} - q^{-(e-r)-2es})}{1-q^{-1-2s}} + (1-q^{-1})q^{-(e-r)-2es}\right). \end{aligned} \tag{2.10}$$

On the other hand, for $h = \begin{pmatrix} \pi u\sqrt{\epsilon} & 1+\pi uv \\ 1 & v/\sqrt{\epsilon} \end{pmatrix} \in K_{1,2}$, we have

$$\begin{aligned} d_1(h \cdot x_\lambda) &= (\pi^{-r}(1-\epsilon) + v) - (-\sqrt{\epsilon} + \pi^r v/\sqrt{\epsilon})v/\sqrt{\epsilon} \\ &= -\frac{\pi^{-r}}{\epsilon}\big(\pi^{2r}v^2 - 2\pi^r v\epsilon + \epsilon(\epsilon-1)\big) \\ &= -\frac{\pi^{-r}}{\epsilon} \cdot N(\pi^r v - \epsilon - \sqrt{\epsilon}) = -\frac{\pi^{-r}}{\epsilon} \cdot N\Big(\pi^r v + 1 - \epsilon - 2\frac{1+\sqrt{\epsilon}}{2}\Big), \end{aligned}$$

$$v_\pi(d_1(h\cdot x_\lambda)) = -r + 2\min\{v_\pi(v)+r, e\}\,.$$

Hence we obtain

$$\begin{aligned}
&\int_{K_{1,2}} |d_1(h\cdot x_\lambda)|^s\, dh \\
&= \frac{q^{-1+rs}}{1+q^{-1}}\left(\sum_{j=0}^{e-r-1}(1-q^{-1})q^{-j-2(r+j)s} + q^{-(e-r)-2es}\right)\\
&= \frac{q^{-1+rs}}{1+q^{-1}}\left(\frac{(1-q^{-1})(q^{-2rs}-q^{-(e-r)-2es})}{1-q^{-1-2s}} + q^{-(e-r)-2es}\right)\,. \qquad (2.11)
\end{aligned}$$

By (2.10) and (2.11), we obtain for $\lambda = -r$ with $1 \le r \le e$ and $s = -z-\frac{1}{2} \in \mathbb{C}$,

$$\begin{aligned}
\omega(x_\lambda; s) &= \frac{q^{-rz-\frac{r}{2}}}{(1+q^{-1})(1-q^{2z})}\left(q^{2rz+r} - q^{2(r+1)z+r-1} + q^{2ez+r-1} - q^{2(e+1)z+r}\right)\\
&= \frac{q^{ez+\frac{r}{2}}}{(1+q^{-1})(1-q^{2z})}\left(q^{(r-e)z} - q^{(r-e+2)z-1} + q^{(-r+e)z-1} - q^{(-r+e+2)z}\right)\\
&= \frac{q^{-\frac{\lambda}{2}}q^{ez}}{1+q^{-1}}\left(\frac{q^{-(\lambda+e)z}(1-q^{2z-1})}{1-q^{2z}} + \frac{q^{(\lambda+e)z}(1-q^{-2z-1})}{1-q^{-2z}}\right)\,. \qquad (2.12)
\end{aligned}$$

We have established the explicit formula of $\omega(x;s)$ by (2.9) and (2.12), from which the property (2.6) follows. □

2.3. In this subsection we study $\omega(x;s)$ for $(m,n) = (3,1)$ under the assumption $e \le 1$ and show the following. The odd residual case ($e = 0$) has been proved in [HK2, Proposition 2.3]. As a result, we see again the set $\mathfrak{X}_1^{od} = \left\{x_\lambda \;\middle|\; \lambda \in \widetilde{\Lambda_1^+}\right\}$ forms a set of complete representatives of $K_1\backslash X_1$, since $\omega(x_\lambda; z)$ takes different value for each λ for generic z.

PROPOSITION 2.4. *Assume $e \le 1$. Then, for $x_\lambda \in \mathfrak{X}_1^{od}$, one has*

$$\omega(x_\lambda; z) = \frac{\sqrt{-1}^{\lambda} q^{-\lambda} q^{ez}(1-q^{-1+2z})}{(1+q^{-3})(1+q^{2z})} \times \left(\frac{q^{-(\lambda+e)z}(1+q^{-2+2z})}{1-q^{2z}} + \frac{q^{(\lambda+e)z}(1+q^{-2-2z})}{1-q^{-2z}}\right)\,. \qquad (2.13)$$

In particular, for any $x \in X_1^{(od)}$,

$$\frac{q^{-ez}(1+q^{2z})}{1-q^{-1+2z}}\cdot\omega(x;z) \in \mathbb{C}[q^z+q^{-z}], \quad \omega(x;z) = q^{2ez}\frac{1-q^{-1+2z}}{q^{2z}-q^{-1}}\cdot\omega(x;-z)\,. \quad (2.14)$$

REMARK 2.5. We expect Proposition 2.4 holds for every $e \geq 0$. If the property (2.14) holds for $e > 0$, then all the statements in this paper hold for the same e. At the moment, since the calculation of (2.13) is troublesome, we have established only for $e = 0, 1$.

Recall the expression of K_1 given in Lemma 1.6. We see
the condition "$b, c \in \mathcal{O}_{k'}$ with $N(b) + c + c^* = 0$" is equivalent to
"$b \in \mathcal{O}_{k'}, c_1 \in \mathcal{O}_k$ with $N(b)+c_1 \in 2\mathcal{O}_k$", where $c = -\frac{N(b)+c_1}{2}+c_1\frac{1+\sqrt{\epsilon}}{2}$. (#)

LEMMA 2.6. *We normalize the Haar measures on k' by $vol(\mathcal{O}_{k'}) = 1$.*
(1) *For $r \in \mathbb{N}$ and $c_1 \in \mathcal{O}_k$ with $v_\pi(c_1) < r$, one has*

$$vol\big(\big\{ b \in \mathcal{O}_{k'} \bigm| N(b) + c_1 \in \pi^r\mathcal{O}_k\big\}\big) = \begin{cases} 0 & \text{if } v_\pi(c_1) \text{ is odd} \\ (1+q^{-1})q^{-r} & \text{if } v_\pi(c_1) \text{ is even.} \end{cases}$$

(2) *For any $c_1 \in \mathcal{O}_k$, one has*

$$\begin{aligned} &vol(\{b \in \mathcal{O}_{k'} \mid v_\pi(N(b) + c_1) = r\}) \\ &= \begin{cases} 0 & \text{for odd } r < v_\pi(c_1) \text{ and odd } v_\pi(c_1) < r \\ q^{-r} & \text{for even } r < v_\pi(c_1) \\ q^{-(r+1)} & \text{for odd } r = v_\pi(c_1) \\ (1-q^{-1}-q^{-2})q^{-r} & \text{for even } r = v_\pi(c_1) \\ (1-q^{-2})q^{-r} & \text{for even } v_\pi(c_1) < r\,. \end{cases} \end{aligned}$$

PROOF. (1) Set $S(c_1, r) = \{b \in \mathcal{O}_{k'} \mid N(b) + c_1 = r\}$. When $v_\pi(c_1)$ is odd, $S(c_1, r) = \emptyset$ and its volume is 0. When $c_1 \in \mathcal{O}_k^\times$, $S(c_1, r) \subset \mathcal{O}_{k'}^\times$. Since the norm map induces the surjective group homomorphism

$$\mathcal{O}_{k'}^\times/(\pi^r) \longrightarrow \mathcal{O}_k^\times/(\pi^r)\,,$$

which is $((1 + q^{-1})q^r : 1)$-map, we see $vol(S(c_1, r)) = (1 + q^{-1})q^r q^{-2r} = (1 + q^{-1})q^{-r}$. When $v_\pi(c_1) = 2t > 0$, we see

$$\begin{aligned} vol(S(c_1, r)) &= vol\Big(\Big\{\pi^t\xi \in \pi^t\mathcal{O}_{k'}^\times \Bigm| N(\xi) + \pi^{-2t}c_1 \in \pi^{r-2t}\mathcal{O}_k\Big\}\Big) \\ &= q^{-2t} \cdot (1 + q^{-1})q^{-(r-2t)} = (1 + q^{-1})q^{-r}\,. \end{aligned}$$

As for (2), the result is clear except for the case $r = v_\pi(c_1)$ is even. We see

$$\begin{aligned} &vol(\{b \in \mathcal{O}_{k'} \mid v_\pi(N(b) + c_1) = r\}) \\ &= vol(\pi^{r/2+1}\mathcal{O}_{k'}) + vol\Big(\Big\{\pi^{r/2}\xi \in \pi^{r/2}\mathcal{O}_{k'}^\times \Bigm| N(\xi) - \pi^{-r}c_1 \notin \pi\mathcal{O}_k\Big\}\Big) \\ &= q^{-r-2} + q^{-r}(1 - q^{-2} - q^{-1}(1 + q^{-1})) \\ &= (1 - q^{-1} - q^{-2})q^{-r}\,. \end{aligned}$$

□

LEMMA 2.7. *By the Haar measures on k and k' normalized by $vol(\mathcal{O}_k) = vol(\mathcal{O}_{k'}) = 1$, one has*

$$vol(\{(b, c_1) \in \mathcal{O}_{k'} \times \mathcal{O}_k \mid N(b) + c_1 \in 2\mathcal{O}_k\}) = q^{-e},$$
$$vol(\{(b, c_1) \in \pi\mathcal{O}_{k'} \times \pi\mathcal{O}_k \mid N(b) + c_1 \in 2\pi\mathcal{O}_k\}) = q^{-(3+e)},$$

and by the Haar measure on G_1 normalized by $vol(K_1) = 1$, one has

$$vol(K_{1,1}) = \frac{1}{1+q^{-3}}, \quad vol(K_{1,2}) = \frac{q^{-3}}{1+q^{-3}}.$$

PROOF. By Lemma 2.6, we obtain

$$\begin{aligned}
&vol(\left\{(b, c_1) \in \mathcal{O}_{k'} \times \mathcal{O}_k \;\middle|\; N(b) + c_1 \in \pi^e\mathcal{O}_k\right\}) \\
&= (1-q^{-1})(1+q^{-1})q^{-e} + \sum_{t=1}^{[\frac{e-1}{2}]} (1-q^{-1})q^{-2t}(1+q^{-1})q^{-e} + q^{-e} \times \left\{ \begin{array}{ll} q^{-e} & \text{if } 2 \mid e \\ q^{-(e+1)} & \text{if } 2 \nmid e \end{array} \right\} \\
&= (1-q^{-2})q^{-e} + q^{-e} \times \left\{ \begin{array}{ll} q^{-2} - q^{-e} & \text{if } 2 \mid e \\ q^{-2} - q^{-(e+1)} & \text{if } 2 \nmid e \end{array} \right\} + q^{-e} \times \left\{ \begin{array}{ll} q^{-e} & \text{if } 2 \mid e \\ q^{-(e+1)} & \text{if } 2 \nmid e \end{array} \right\} \\
&= q^{-e},
\end{aligned}$$

$$\begin{aligned}
&vol\left(\left\{(b, c_1) \in \pi\mathcal{O}_{k'} \times \pi\mathcal{O}_k \;\middle|\; N(b) + c_1 \in \pi^{e+1}\mathcal{O}_k\right\}\right) \\
&= \sum_{t=1}^{[\frac{e}{2}]} (1-q^{-1})q^{-2t}(1+q^{-1})q^{-(e+1)} + q^{-(e+1)} \times \left\{ \begin{array}{ll} q^{-(e+2)} & \text{if } 2 \mid e \\ q^{-(e+1)} & \text{if } 2 \nmid e \end{array} \right\} \\
&= q^{-(e+1)} \times \left\{ \begin{array}{ll} q^{-2} - q^{-(e+2)} & \text{if } 2 \mid e \\ q^{-2} - q^{-(e+1)} & \text{if } 2 \nmid e \end{array} \right\} + q^{-(e+1)} \times \left\{ \begin{array}{ll} q^{-(e+2)} & \text{if } 2 \mid e \\ q^{-(e+1)} & \text{if } 2 \nmid e \end{array} \right\} \\
&= q^{-(e+3)}.
\end{aligned}$$

Now we see the volume of $K_{1,1}$ and $K_{1,2}$ as above, by the explicit description of K_1 in Lemma 1.6. □

In the rest of this subsection we assume $e = 1$, i.e., 2 is a prime element in k. As for the calculation of $|d_1(h \cdot x_\lambda)|$, $\lambda \geq -1$, only the third row of $h \in K_1$ is concerned.

(1) The case $\lambda \geq 0$.

For $h = \begin{pmatrix} 1 & & \\ b & 1 & \\ c & -b^* & 1 \end{pmatrix} \begin{pmatrix} 1 & d & f \\ & 1 & -d^* \\ & & 1 \end{pmatrix} \in K_{1,2}$, we have

$$\begin{aligned} d_1(h \cdot x_\lambda) &= \pi^\lambda N(c) + N(cd - b^*) + \pi^{-\lambda} N(1 + cf + b^* d^*) \\ &= \pi^{-\lambda} \left(\pi^{2\lambda} N(c) + \pi^\lambda N(cd - b^*) + N(1 + cf + b^* d^*) \right) \in \pi^{-\lambda} \mathcal{O}_k^\times , \end{aligned}$$

hence we obtain

$$\int_{K_{1,2}} |d_1(h \cdot x_\lambda)|^s \, dh = \frac{q^{-3+\lambda s}}{1 + q^{-3}} . \tag{2.15}$$

For $h = \begin{pmatrix} & & 1 \\ & 1 & -b^* \\ 1 & b & c \end{pmatrix} \in K_{1,1}$, we have

$$d_1(h \cdot x_\lambda) = \pi^\lambda + bb^* + \pi^{-\lambda} cc^* = \pi^{-\lambda} (\pi^{2\lambda} + \pi^\lambda N(b) + N(c)) .$$

Here, since we have by (#)

$$\begin{aligned} c &= c_0 + c_1 \frac{1 + \sqrt{\epsilon}}{2} = -\frac{N(b)}{2} + \frac{c_1}{2} \sqrt{\epsilon} \quad (c_0, \ c_1 \in \mathcal{O}_k) , \\ N(c) &= \frac{1}{4} (N(b)^2 - c_1^2 \epsilon) , \end{aligned} \tag{2.16}$$

we see

$$\begin{aligned} d_1(h \cdot x_\lambda) &= \pi^{-\lambda} \left((\pi^\lambda + \frac{N(b)}{2})^2 - \frac{c_1^2 \epsilon}{4} \right) \\ &= \pi^{-\lambda} \cdot N\Big(\pi^\lambda + \frac{N(b)}{2} - c_1 \frac{\sqrt{\epsilon}}{2} \Big) \\ &= \pi^{-\lambda} \cdot N\Big(\pi^\lambda + \frac{N(b) + c_1}{2} - c_1 \frac{1 + \sqrt{\epsilon}}{2} \Big) . \end{aligned}$$

By Lemma 2.7, we have

$$\int_{K_{1,1}} |d_1(h \cdot x_\lambda)|^s \, dh = \frac{q^{1+\lambda s}}{1 + q^{-3}} \sum_{r \geq 0} \mu(\lambda, r) q^{-2rs} ,$$

where

$$\begin{aligned} &\mu(\lambda, r) = vol(\{ (b, c_1) \in \mathcal{O}_{k'} \times \mathcal{O}_k \mid N(b) + c_1 \in 2\mathcal{O}_k, \ v_\pi(y_\lambda) = r \}) , \\ &y_\lambda = \pi^\lambda + \frac{N(b) + c_1}{2} - c_1 \frac{1 + \sqrt{\epsilon}}{2} . \end{aligned}$$

For simplicity of notation, we set $t = q^{-1}$ and $X = q^{-s}$. We calculate the value $\mu(\lambda, r)$ case by case by using Lemma 2.6.
The case even r with $0 \le r \le \lambda - 1$:

$$\begin{aligned}\mu(\lambda, r) &= vol(\{(b, c_1) \mid v_\pi(c_1) = r,\ v_\pi(N(b) + c_1) \ge r+1\}) \\ &\quad + vol(\{(b, c_1) \mid v_\pi(c_1) = v_\pi(N(b) + c_1) = r+1\}) \\ &= (1-t)t^r \cdot (1+t)t^{r+1} + (1-t)t^{r+1}t^{r+2} = (1-t^3)t^{2r+1}.\end{aligned}$$

The case odd r with $0 \le r \le \lambda - 1$:

$$\begin{aligned}\mu(\lambda, r) &= vol(\{(b, c_1) \mid v_\pi(c_1) = v_\pi(N(b) + c_1) = r+1\}) \\ &\quad + vol(\{(b, c_1) \mid v_\pi(c_1) \ge r+2,\ v_\pi(N(b) + c_1) = r+1\}) \\ &= (1-t)t^{r+1}(1-t-t^2)t^{r+1} + (1-t^2)t^{2r+3} = (1-t)t^{2r+2}.\end{aligned}$$

The case $r = \lambda$ is even:

$$\begin{aligned}\mu(\lambda, \lambda) &= vol(\{(b, c_1) \mid v_\pi(c_1) = \lambda,\ v_\pi(N(b) + c_1) \ge \lambda+1\}) \\ &\quad + vol(\left\{(b, c_1) \;\middle|\; v_\pi(c_1) = v_\pi(N(b) + 2\pi^\lambda + c_1) = \lambda+1\right\}) \\ &\quad + vol(\left\{(b, c_1) \;\middle|\; v_\pi(c_1) \ge \lambda+2,\ v_\pi(N(b) + 2\pi^\lambda) = \lambda+1\right\}) \\ &= (1-t^2)t^{2\lambda+1} + (1-2t)t^{\lambda+1}t^{\lambda+2} + t^{\lambda+2}t^{\lambda+2} = (1-t^3)t^{2\lambda+1}.\end{aligned}$$

The case $r = \lambda$ is odd:

$$\begin{aligned}\mu(\lambda, \lambda) &= vol(\left\{(b, c_1) \;\middle|\; v_\pi(c_1) = v_\pi(N(b) + 2\pi^\lambda + c_1) = \lambda+1\right\}) \\ &\quad + vol(\left\{(b, c_1) \;\middle|\; v_\pi(c_1) \ge \lambda+2,\ v_\pi(N(b) + 2\pi^\lambda) = \lambda+1\right\}) \\ &= (1-2t)t^{\lambda+1}(1-t-t^2)t^{\lambda+1} + t^{\lambda+2}(1-t^2)t^{\lambda+1} + t^{\lambda+2}(1-t-t^2)t^{\lambda+1} \\ &= (1-t)t^{2\lambda+2}.\end{aligned}$$

The case $r = \lambda + 1$ and λ is even:

$$\begin{aligned}\mu(\lambda, r) &= vol(\left\{(b, c_1) \;\middle|\; v_\pi(c_1) = \lambda+1,\ v_\pi(N(b) + 2\pi^\lambda + c_1) \ge \lambda+2\right\}) \\ &\quad + vol(\left\{(b, c_1) \;\middle|\; v_\pi(c_1) \ge \lambda+2,\ v_\pi(N(b) + 2\pi^\lambda + c_1) = \lambda+2\right\}) \\ &= t^{\lambda+2}t^{\lambda+2} + 0 = t^{2\lambda+4}.\end{aligned}$$

The case $r = \lambda + 1$ and λ is odd:

$$\begin{aligned}\mu(\lambda, r) &= vol(\left\{(b, c_1) \;\middle|\; v_\pi(c_1) = \lambda+1,\ v_\pi(N(b) + 2\pi^\lambda + c_1) \ge \lambda+2\right\}) \\ &\quad + vol(\left\{(b, c_1) \;\middle|\; v_\pi(c_1) \ge \lambda+2,\ v_\pi(N(b) + 2\pi^\lambda + c_1) = \lambda+2\right\}) \\ &= (t^{\lambda+2}t^{\lambda+3} + (1-2t)t^{\lambda+1}(1+t)t^{\lambda+2}) + t^{\lambda+2}(1-t^2)t^{\lambda+2} \\ &= (1-t^2-t^3)t^{2\lambda+3}.\end{aligned}$$

The case $r \ge \lambda + 2$ and λ is even: $\mu(\lambda, r) = 0$.

The case $r \ge \lambda + 2$ and λ is odd:

$$\begin{aligned}
\mu(\lambda, r) &= vol(\{(b, c_1) \mid v_\pi(c_1) = r,\ v_\pi(N(b) + 2\pi^\lambda + c_1) \ge r + 1\}) \\
&\quad + vol(\{(b, c_1) \mid v_\pi(c_1) \ge r + 1,\ v_\pi(N(b) + 2\pi^\lambda + c_1) = r + 1\}) \\
&= (1 - t)t^r(1 + t)t^{r+1} + t^{r+1}(1 - t^2)t^{r+1} \\
&= (1 + t)(1 - t^2)t^{2r+1}.
\end{aligned}$$

By these data, we obtain the value $\int_{K_{1,1}} |d_1(h \cdot x)|^s \, dh$ as follows:

If λ is even,

$$\frac{X^{-\lambda}}{1 + t^3}\left(\frac{(1 - t^3)(1 - t^{2\lambda+4}X^{2\lambda+4})}{1 - t^4X^4} + \frac{(1 - t)(t^3X^2 - t^{2\lambda+3}X^{2\lambda+2})}{1 - t^4X^4} + t^{2\lambda+3}X^{2\lambda+2}\right);$$

if λ is odd,

$$\begin{aligned}
&\frac{X^{-\lambda}}{1 + t^3}\left(\frac{(1 - t^3)(1 - t^{2\lambda+2}X^{2\lambda+2})}{1 - t^4X^4} + \frac{(1 - t)(t^3X^2 - t^{2\lambda+5}X^{2\lambda+4})}{1 - t^4X^4}\right. \\
&\qquad \left. + (1 - t^2 - t^3)t^{2\lambda+2}X^{2\lambda+2} + \frac{(1 + t)(1 - t^2)t^{2\lambda+4}X^{2\lambda+4})}{1 - t^2X^2}\right).
\end{aligned}$$

Together with (2.15), we continue the calculation, where we recall the relation $s = -z - 1 + \frac{\pi\sqrt{-1}}{2\log q}$, $t = q^{-1}$ and $X = q^{-s}$. If λ is even,

$$\begin{aligned}
\omega(x_\lambda; s) &= \frac{(1 + t^3X^2)X^{-\lambda}}{(1 + t^3)(1 - t^4X^4)}\left\{1 - t^4X^2 + t^{2\lambda+4}X^{2\lambda+2}(1 - X^2)\right\} \\
&= \frac{(1 - q^{-1+2z})(\sqrt{-1})^\lambda q^{-\lambda-\lambda z}}{(1 + q^{-3})(1 - q^{4z})}\left\{1 + q^{-2+2z} - q^{-2+(2\lambda+2)z}(1 + q^{2+2z})\right\} \qquad (2.17) \\
&= \frac{(\sqrt{-1})^\lambda q^{-\lambda}q^z(1 - q^{-1+2z})}{(1 + q^{-3})(1 + q^{2z})}\left\{\frac{q^{-(\lambda+1)z}(1 + q^{-2+2z})}{1 - q^{2z}} + \frac{q^{(\lambda+1)z}(1 + q^{-2-2z})}{1 - q^{-2z}}\right\}. \qquad (2.18)
\end{aligned}$$

If λ is odd,

$$\begin{aligned}
\omega(x_\lambda; s) &= \frac{(1 + t^3X^2)X^{-\lambda}}{(1 + t^3)(1 - t^4X^4)}\left\{1 - t^4X^2 - t^{2\lambda+4}X^{2\lambda+2}(1 - X^2)\right\} \\
&= \frac{(1 - q^{-1+2z})(\sqrt{-1})^\lambda q^{-\lambda-\lambda z}}{(1 + q^{-3})(1 - q^{4z}))}\left\{1 + q^{-2+2z} - q^{-2+(2\lambda+2)z}(1 + q^{2+2z})\right\},
\end{aligned}$$

which is the same with (2.17), and we obtain the same expression (2.18) for odd λ. Thus we have proved the formula (2.13) for $e = 1$ and $\lambda \ge 0$.

(2) We consider the remaining case for $e = 1$, i.e. $\lambda = -1$, $\pi = 2$, and

$$x_{-1} = \begin{pmatrix} \frac{1-\epsilon}{2} & & -\sqrt{\epsilon} \\ & 1 & \\ \sqrt{\epsilon} & & 2 \end{pmatrix}.$$

For $h = \begin{pmatrix} 1 & & \\ b & 1 & \\ c & -b^* & 1 \end{pmatrix} \begin{pmatrix} 1 & d & f \\ & 1 & -d^* \\ & & 1 \end{pmatrix} \in K_{1,2}$, we have

$$\begin{aligned} d_1(h \cdot x_{-1}) &= \frac{1-\epsilon}{2} N(c) + (1 + b^*d^* + cf)c^*\sqrt{\epsilon} + N(b - c^*d^*) \\ &\quad + (-\sqrt{\epsilon}c + 2(1 + b^*d^* + cf))(1 + bd + c^*f^*) \\ &= \frac{1-\epsilon}{2} N(c) + N(b - c^*d^*) + (c^* - c)\sqrt{\epsilon} + (b^*c^*d^* - bcd)\sqrt{\epsilon} \\ &\quad + N(c)(f\sqrt{\epsilon} - f^*\sqrt{\epsilon}) + 2N(1 + bd + c^*f^*). \end{aligned}$$

Since $b, c \in \pi\mathcal{O}_{k'}$ satisfying $N(b) + c + c^* = 0$, we see $v_\pi(c - c^*) \geq 2$ and $v_\pi(h \cdot x_{-1}) = v_\pi(2N(1 + bd + c^*f^*)) = 1$. Hence

$$\int_{K_{1,2}} |d_1(h \cdot x_\lambda)|^s \, dh = \frac{q^{-3-s}}{1 + q^{-3}}. \tag{2.19}$$

For $h = \begin{pmatrix} & & 1 \\ & 1 & -b^* \\ 1 & b & c \end{pmatrix} \in K_{1,1}$, we have

$$d_1(h \cdot x_{-1}) = \frac{1-\epsilon}{2} + (c - c^*)\sqrt{\epsilon} + N(b) + 2N(c).$$

Here, since we have (2.16) and

$$(c - c^*)\sqrt{\epsilon} = c_1\epsilon,$$

we see

$$\begin{aligned} d_1(h \cdot x_{-1}) &= \frac{1}{2}\left\{(N(b) + 1)^2 - (c_1 - 1)^2\epsilon\right\} \\ &= \frac{1}{2} N(N(b) + 1 - (c_1 - 1)\sqrt{\epsilon}) \\ &= \frac{1}{2} N\Big(N(b) + c_1 - 2(c_1 - 1)\frac{1 + \sqrt{\epsilon}}{2}\Big). \end{aligned}$$

By Lemma 2.7, we have

$$\int_{K_{1,1}} |d_1(h \cdot x_{-1})|^s \, dh = \frac{q}{1 + q^{-3}} \sum_{r \geq 1} \mu(r) q^{-(2r-1)s},$$

where

$$\mu(r) = vol(\{(b, c_1) \in \mathcal{O}_{k'} \times \mathcal{O}_k \mid N(b) + c_1 \in 2\mathcal{O}_k,\ v_\pi(y) = r\}),$$
$$y = N(b) + c_1 - 2(c_1 - 1)\frac{1+\sqrt{\epsilon}}{2}.$$

Then we have

$$\begin{aligned}\mu(1) &= vol\big(\big\{(b, c_1) \,\big|\, c_1 \in \mathcal{O}_k^\times,\ c_1 \notin 1 + \pi\mathcal{O}_k,\ v_\pi(N(b) + c_1) \geq 1\big\}\big)\\ &\quad + vol(\{(b, c_1) \mid c_1 \in 1 + \pi\mathcal{O}_k,\ v_\pi(N(b) + c_1) = 1\})\\ &\quad + vol(\{(b, c_1) \mid c_1 \in \pi\mathcal{O}_k,\ b \in \pi\mathcal{O}_{k'}\})\\ &= (1 - 2t)(1 + t)t + t(1 - t^2)t + tt^2\\ &= (1 - t^2 - t^3)t;\end{aligned}$$

and for $r \geq 2$,

$$\begin{aligned}\mu(r) &= vol\Big(\Big\{(b, c_1) \,\Big|\, c_1 \in 1 + \pi^{r-1}\mathcal{O}_k^\times,\ v_\pi(N(b) + c_1) \geq r\Big\}\Big)\\ &\quad + vol\big(\big\{(b, c_1) \,\big|\, c_1 \in 1 + \pi^r\mathcal{O}_k,\ v_\pi(N(b) + c_1) = r\big\}\big)\\ &= (1 - t)t^{r-1}(1 + t)t^2 + t^r(1 - t^2)t^r\\ &= (1 + t)(1 - t^2)t^{2r-1}.\end{aligned}$$

Hence we have

$$\int_{K_{1,1}} |d_1(h \cdot x)|^s\, dh = \frac{1}{1 + t^3}\left\{(1 - t^2 - t^3)X + \frac{(1 + t)(1 - t^2)t^2X^3}{1 - t^2X^2}\right\},$$

and together with (2.19), we obtain

$$\begin{aligned}\omega(x_{-1}; s) &= \frac{(1 - t^2)(1 + t^3X^2)X}{(1 + t^3)(1 - t^2X^2)}\\ &= \frac{(1 - q^{-2})(\sqrt{-1})^{-1}q^{1+z}(1 - q^{-1+2z})}{(1 + q^{-3})(1 + q^{2z})},\end{aligned} \tag{2.20}$$

which coincides with (2.13) for $e = 1$ and $\lambda = -1$. Thus we have established the explicit formula of $\omega(x; s)$ by (2.18) and (2.20), from which the property (2.14) follows. □

2.4. In this subsection we give the functional equation with respect to τ for general n.

THEOREM 2.8. *Assume $e \leq 1$ if m is odd. For general size n, the spherical function satisfies the functional equation*

$$\omega(x; z) = q^{2ez_n}\begin{pmatrix} 1 & \text{if } m = 2n \\ \dfrac{1 - q^{-1+2z_n}}{q^{2z_n} - q^{-1}} & \text{if } m = 2n + 1 \end{pmatrix} \times \omega(x; \tau(z)),$$

where $\tau(z) = (z_1, \dots, z_{n-1}, -z_n)$.

For $n = 1$ the statement has been shown in Proposition 2.2 and Proposition 2.3. Hereafter we assume $n \geq 2$, i.e., $m \geq 4$, and set

$$w_\tau = \begin{pmatrix} 1_{n-1} & & \\ & j_r & \\ & & 1_{n-1} \end{pmatrix} \in K, \quad r = m - 2(n-1) \in \{2, 3\}.$$

Then the standard parabolic subgroup P of G attached to τ is given as follows, keeping r as above,

$$\begin{aligned} P &= B \cup B w_\tau B \\ &= \left\{ \begin{pmatrix} A & & \\ & h & \\ & & j_{n-1} A^{*-1} j_{n-1} \end{pmatrix} \begin{pmatrix} 1_{n-1} & \alpha j_r & B j_{n-1} \\ & 1_r & -\alpha^* j_{n-1} \\ & & 1_{n-1} \end{pmatrix} \middle| \begin{array}{l} A \in B_{n-1}(k') \\ h \in G_1 = U(j_r) \\ \alpha \in M_{n-1,r}(k') \\ B \in M_{n-1}(k') \\ B + B^* + \alpha j_r \alpha^* = 0 \end{array} \right\}, \end{aligned} \tag{2.21}$$

where $B_{n-1}(k')$ is the Borel subgroup of $GL_{n-1}(k')$ consisting of all the upper triangular matrices. Here $d_i(x)$ is a relative P-invariant for $1 \leq i \leq n-1$, but $d_n(x)$ is not. We enlarge the group and the space and consider the action of $P' = P \times GL_1(k')$ on $X' = X \times V$ with $V = M_{r1}(k')$:

$$(p, t) \star (x, v) = (p \cdot x, \rho(p) v t^{-1}), \quad (p, t) \in P', \ (x, v) \in X',$$

where $\rho(p) = h \in U(j_r)$ for the decomposition of p as in (2.21). Set

$$g(x, v) = \det\left[\left(\begin{array}{c|c} v^* j_r & \\ \hline & 1_{n-1} \end{array} \right) \cdot x_{(n-1+r)} \right],$$

where $x_{(n-1+r)}$ is the lower $(n-1+r) \times (n-1+r)$-block of x, and the matrix inside of [ ] is of size n. Though we have slightly changed the definition of $g(x, v)$ when $m = 2n$, we have the following similar results as in [HK1] and [HK2].

LEMMA 2.9. (1) *The function $g(x, v)$ is a relative P'-invariant on X' associated by the character*

$$P' \ni (p, t) \longmapsto N(d_{n-1}(p)) N(t)^{-1} = \psi_{n-1}(p) N(t)^{-1},$$

and satisfies $g(x, v_0) = d_n(x)$ where $v_0 = {}^t(1, 0)$ or ${}^t(1, 0, 0)$, according to the parity of m.

(2) *For $x \in X^{op}$, there is $D_1(x) \in X_1$ satisfying*

$$g(x, v) = (d_{n-1}(x) D_1(x))[v].$$

Here, for diagonal x, $D_1(x) = Diag(x_n^{-1}, x_n)$ or $Diag(x_n^{-1}, x_0, x_n)$, according to the parity of m, where x_n is the n-th diagonal entry and x_0 is the $(n+1)$-th diagonal entry for odd m.

By the embedding from K_1 to $K = K_n$ defined by

$$K_1 \ni h \longmapsto \widetilde{h} = \begin{pmatrix} 1_{n-1} & & \\ & h & \\ & & 1_{n-1} \end{pmatrix},$$

we see

$$\begin{aligned}
\omega(x; s) &= \int_{K_1} dh \int_K |\mathbf{d}(k \cdot x)|^s \, dk \\
&= \int_{K_1} \int_K \left|\mathbf{d}(\widetilde{h}k \cdot x)\right|^s dkdh \\
&= \int_K \prod_{i<n} |d_i(k \cdot x)|^{s_i} \int_{K_1} \left|d_n(\widetilde{h}k \cdot x)\right|^{s_n} dhdk .
\end{aligned}$$

Since we obtain, for $y \in X^{op}$

$$\begin{aligned}
&d_n(\widetilde{h} \cdot y) = g((\widetilde{h}, 1) \star (y, h^{-1}v_0)) = g(y, h^{-1}v_0) = d_{n-1}(y)D_1(y)[h^{-1}v_0] \\
&= d_{n-1}(y)h^{*-1}D_1(y)[v_0] = d_{n-1}(y)d_1(j_r h^{*-1} \cdot D_1(y)) = d_{n-1}(y)d_1(hj_r \cdot D_1(y)) ,
\end{aligned}$$

we have

$$\omega(x; s) = \int_K \prod_{i \le n-2} |d_i(k \cdot x)|^{s_i} \cdot |d_{n-1}(k \cdot x)|^{s_{n-1}+s_n} \, \omega^{(1)}(D_1(k \cdot x); s_n)dk .$$

Hence we obtain, for $m = 2n$, by the property (2.6), where $q^{-ez}\omega^{(1)}(y, z)$ is holomorphic and τ-invariant,

$$\begin{aligned}
\omega(x; s) &= q^{2ez_n}\omega(x; s_1, \dots, s_{n-2}, s_{n-1} + 2s_n + 1, -s_n - 1) \\
&= q^{2ez_n}\omega(x; \tau(z)) ;
\end{aligned}$$

and for $m = 2n + 1$, by the property (2.14), where $\frac{q^{-ez}(1+q^{2z})}{1-q^{-1+2z}} \cdot \omega^{(1)}(y; z)$ is holomorphic and τ-invariant,

$$\begin{aligned}
&\frac{1 + q^{2z_n}}{1 - q^{-1+2z_n}} \times \omega(x; s) \\
&= q^{2ez_n} \frac{1 + q^{-2z_n}}{1 - q^{-1-2z_n}} \times \omega\Big(x; s_1, \dots, s_{n-2}, s_{n-1} + 2s_n + 2 + \frac{\pi\sqrt{-1}}{\log q}, -s_n - 2 - \frac{\pi\sqrt{-1}}{\log q}\Big) ,
\end{aligned}$$

thus

$$\omega(x; s) = \frac{q^{2ez_n}(1 - q^{-1+2z_n})}{q^{2z_n} - q^{-1}}\omega(x; \tau(z)) ,$$

which completes the proof of Theorem 2.8. $\square$

2.5. To describe the functional equation with respect to W, we prepare some notation. Set

$$\Sigma = \left\{ \pm e_i \pm e_j,\ 2e_i \;\middle|\; 1 \le i, j \le n,\ i \ne j \right\}, \quad \Sigma^+ = \Sigma_s^+ \cup \Sigma_\ell^+ ,$$
$$\Sigma_s^+ = \left\{ e_i + e_j,\ e_i - e_j \;\middle|\; 1 \le i < j \le n \right\}, \quad \Sigma_\ell^+ = \{ 2e_i \mid 1 \le i \le n \} ,$$

where e_i is the i-th unit vector in $\mathbb{Z}^n$, $1 \le i \le n$. We note here that Σ is the set of roots of $G_n^{(ev)}$ and $\Sigma \cup \{ e_i \mid 1 \le i \le n \}$ is the set of roots of $G_n^{(od)}$. We consider the pairing

$$\mathbb{Z}^n \times \mathbb{C}^n \ni (t, z) \longmapsto \langle t,\ z \rangle = \sum_{i=1}^{n} t_i z_i \in \mathbb{C} ,$$

which satisfies

$$\langle \alpha,\ z \rangle = \langle \sigma(\alpha),\ \sigma(z) \rangle , \quad (\alpha \in \Sigma,\ z \in \mathbb{C}^n,\ \sigma \in W) .$$

THEOREM 2.10. *Assume $e \le 1$ if m is odd. The spherical function $\omega(x; z)$ satisfies the following functional equation*

$$\omega(x; z) = \Gamma_\sigma^{(e)}(z) \cdot \omega(x; \sigma(z)), \qquad (\sigma \in W) ,$$

where

$$\Gamma_\sigma^{(e)}(z) = \prod_{\alpha \in \Sigma^+(\sigma)} \gamma_\alpha^{(e)}(z), \quad \Sigma^+(\sigma) = \left\{ \alpha \in \Sigma^+ \;\middle|\; -\sigma(\alpha) \in \Sigma^+ \right\} ,$$

$$\gamma_\alpha^{(e)}(z) = \begin{cases} \dfrac{1 - q^{-1+\langle \alpha,\, z \rangle}}{q^{\langle \alpha,\, z \rangle} - q^{-1}} & \text{if } \alpha \in \Sigma_s^+ , \\ q^{e\langle \alpha,\, z \rangle} & \text{if } \alpha \in \Sigma_\ell^+,\ m = 2n , \\ \dfrac{q^{e\langle \alpha,\, z \rangle}(1 - q^{-1+\langle \alpha,\, z \rangle})}{q^{\langle \alpha,\, z \rangle} - q^{-1}} & \text{if } \alpha \in \Sigma_\ell^+,\ m = 2n + 1 . \end{cases}$$

OUTLINE OF A PROOF. The Weyl group W is generated by $\{ \sigma_i = (i\ i+1) \in S_n \mid 1 \le i \le n-1 \}$ and τ. As for the gamma factor, we have $\Gamma_{\sigma_i}^{(e)}(z) = \gamma_{e_i - e_{i+1}}^{(e)}(z)$ by Proposition 2.2, which is independent of e, and $\Gamma_\tau^{(e)}(z) = \gamma_{2e_n}^{(e)}(z)$ by Theorem 2.8. Then, by the cocycle relation of gamma factors, we obtain the results. (Of course $\Gamma_\sigma^{(0)}(z)$ is the same as $\Gamma_\sigma(z)$ in [HK1] or [HK2], according to the parity of m.) □

The following theorem can be proved in the same way as in [HK1, Theorem 2.7] based on Theorem 2.10, where the function $G(z)$ below is the same as in [HK1] or [HK2], according to the parity of m.

THEOREM 2.11. *Assume $e \le 1$ if m is odd. The function $q^{-\langle e,\, z \rangle} G(z) \cdot \omega(x; z)$ is holomorphic on $\mathbb{C}^n$ and W-invariant, in particular it is an element in $\mathbb{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^W$.*

Here $\langle e, z\rangle = e(z_1 + \cdots + z_n)$ *and*

$$G(z) = \prod_{\alpha} \frac{1 + q^{\langle \alpha, z\rangle}}{1 - q^{-1+\langle \alpha, z\rangle}},$$

where α *runs over the set* Σ_s^+ *for* $m = 2n$ *and* Σ^+ *for* $m = 2n+1$. *In particular, each Gamma factor in Theorem* 2.10 *is given as*

$$\Gamma_\sigma^{(e)}(z) = \frac{q^{\langle e, z\rangle}}{G(z)} \cdot \frac{G(\sigma(z))}{q^{\langle e, \sigma(z)\rangle}}, \quad \sigma \in W\,. \tag{2.22}$$

3. The explicit formula for $\omega(x; z)$

As for the explicit formula of $\omega(x; z)$, it suffices to determine at a representative of each K-orbit, hence at x_λ, $\lambda \in \widetilde{\Lambda_n^+}$ by Theorem 1.1-(1).

THEOREM 3.1. *Assume* $e \leq 1$ *if* m *is odd. For each* $\lambda \in \widetilde{\Lambda_n^+}$, *one has the explicit formula*

$$\omega(x_\lambda; z) = c_n\, q^{\langle \lambda, z_0\rangle} \cdot \frac{q^{\langle e, z\rangle}}{G(z)} \cdot Q_{\lambda+e}(z; \{t\}),$$

where $\lambda + e = (\lambda_1 + e, \ldots, \lambda_n + e) \in \Lambda_n^+$, $G(z)$ *is given in Theorem* 2.11 (*depending on the parity of* m), $z_0 \in \mathbb{C}^n$ *is the value in* z-*variable corresponding to* $\mathbf{0} \in \mathbb{C}^n$ *in* s-*variable*,

$$z_{0,i} = \begin{cases} -(n-i+\frac{1}{2}) + (n-i)\frac{\pi\sqrt{-1}}{\log q} & \text{if } m = 2n \\ -(n-i+1) + (n-i+\frac{1}{2})\frac{\pi\sqrt{-1}}{\log q} & \text{if } m = 2n+1, \end{cases} \qquad (1 \leq i \leq n),$$

$$c_n = \begin{cases} \dfrac{(1-q^{-2})^n}{w_m(-q^{-1})} & \text{if } m = 2n \\ \dfrac{(1+q^{-1})(1-q^{-2})^n}{w_m(-q^{-1})} & \text{if } m = 2n+1, \end{cases} \qquad w_m(t) = \prod_{i=1}^{m}(1-t^i),$$

$$Q_\mu(z; \{t\}) = \sum_{\sigma \in W} \sigma\left(q^{-\langle \mu, z\rangle} c(z; \{t\})\right), \quad c(z; \{t\}) = \prod_{\alpha \in \Sigma^+} \frac{1 - t_\alpha q^{\langle \alpha, z\rangle}}{1 - q^{\langle \alpha, z\rangle}},$$

$$\{t\} = \{t_\alpha\} \quad \textit{with} \quad t_\alpha = \begin{cases} -q^{-1} & \text{if } \alpha \in \Sigma_s^+ \\ q^{-1} & \text{if } \alpha \in \Sigma_\ell^+,\ m = 2n \\ -q^{-2} & \text{if } \alpha \in \Sigma_\ell^+,\ m = 2n+1\,. \end{cases} \tag{3.1}$$

REMARK 3.2. We see the main part $Q_{\lambda+e}(z; \{t\})$ of $\omega(x_\lambda; z)$ is contained in $\mathcal{R} = \mathbb{C}[q^{\pm z_1}, \ldots, q^{\pm z_n}]^W$ by Theorem 2.11, and related to Hall-Littlewood polynomial

$P_{\lambda+e}(z;\{t\})$ of type C_n as follows (cf. [M2], in general):

$$P_\mu(z;\{t\}) = \frac{1}{W_\mu(\{t\})} \cdot Q_\mu(z;\{t\}), \quad \mu \in \Lambda_n^+, \tag{3.2}$$

where $W_\mu(\{t\})$ is the Poincaré polynomial of the stabilizer W_μ of W at μ, and with the present choice of t_α, it is given precisely as follows

$$W_\mu(\{t\}) = \frac{\widetilde{w}_\mu(-q^{-1})}{(1+q^{-1})^{m'}}, \qquad m' = \left[\frac{m+1}{2}\right],$$

$$\widetilde{w}_\mu(t) = \begin{cases} w_{n_0}(t)^2 \prod_{\ell \geq 1} w_{n_\ell}(t) & \text{if } n = 2m \\ w_{n_0+1}(t) w_{n_0}(t) \prod_{\ell \geq 1} w_{n_\ell}(t) & \text{if } n = 2m+1,\ n_0 > 0 \\ \prod_{\ell \geq 1} w_{n_\ell}(t) & \text{if } n = 2m+1,\ n_0 = 0, \end{cases} \tag{3.3}$$

with $n_\ell = n_\ell(\mu) = \sharp\{i \mid \mu_i = \ell\}$. It is known (cf. [M2], [HK1, Proposition B.3]) that the set $\left\{ P_\mu(z;\{t\}) \mid \mu \in \Lambda_n^+ \right\}$ forms an orthogonal $\mathbb{C}$-basis for $\mathcal{R}$ for each $t_\alpha \in \mathbb{R}$, $|t_\alpha| < 1$, and $P_{\mathbf{0}}(z;\{t\}) = 1$; and we will use this property in §4. The explicit formula can be rewritten by using $P_\mu(z;\{t\})$ as

$$\omega(x_\lambda; z) = \frac{(1-q^{-1})^n}{w_m(-q^{-1})} \cdot \frac{q^{\langle e, z\rangle}}{G(z)} \cdot q^{\langle \lambda, z_0\rangle}\, \widetilde{w}_{\lambda+e}(-q^{-1}) \cdot P_{\lambda+e}(z;\{t\}), \quad (\lambda \in \widetilde{\Lambda_n^+}). \tag{3.4}$$

REMARK 3.3. The influence of the residual characteristic of the base field k in the explicit formula of $\omega(x_\lambda; z)$ appears as shifting $\lambda + e$ in Q_λ or P_λ and the factor $q^{\langle e, z\rangle}$.

Since $\omega(x_\lambda; z)$ takes a different value at each $\lambda \in \widetilde{\Lambda_n^+}$ for generic z, we see each x_λ represents a different K-orbit in X, which completes the Cartan decomposition of X (i.e. Theorem 1.1-(2)). As we noted in Remark 2.5, if (2.14) in Proposition 2.4 holds for $e (> 0)$, one has the explicit formula for odd size m for the same e, and the Cartan decomposition follows also.

REMARK 3.4. The vector z_0 in Theorem 3.1 can be regarded as *a generalization of the dual Weyl vector* as follows, and this is the reason we changed the relation between s and z for $m = 2n$ from that in [HK1](cf. Remark 2.1). We remarked about this interpretation already in [HK2, Remark3.3]. For $v \in \mathbb{Z}^n$, set

$$\{t\}^{\mathrm{ht}(v)} = \prod_{\beta \in \Sigma^+} t_\beta^{\langle v, \beta^\vee\rangle/2}, \qquad \beta^\vee = \frac{2\beta}{\langle \beta, \beta\rangle}. \tag{3.5}$$

This is the generalization of the height of roots when $v \in \Sigma$ ([M1]), while it can be rewritten by using z_0 as

$$\{t\}^{\mathrm{ht}(v)} = q^{\langle v, z_0\rangle}.$$

We prove Theorem 3.1 in the same way as in the case $e = 0$ ([HK1], [HK2]) by using a general expression formula given in [H2] (or in [H1]) of spherical functions on homogeneous spaces, which is based on functional equations of finer spherical functions corresponding to B-orbits in X and some data depending only on the group G. We have to check the assumptions there, but it has no problem since it is independent of the residual characteristic, and we omit it.

Recall $X^{op} = \{x \in X \mid d_i(x) \neq 0,\ 1 \leq i \leq n\}$ and the Borel subgroup B of G consisting of the upper triangular matrices in G. According to the B-orbit decomposition

$$X^{op} = \bigsqcup_{u \in \mathcal{U}} X_u, \qquad \mathcal{U} = (\mathbb{Z}/2\mathbb{Z})^n \ ,$$
$$X_u = \left\{x \in X^{op} \;\middle|\; v_\pi(d_i(x)) \equiv u_1 + \cdots + u_i \pmod 2,\ 1 \leq i \leq n\right\} ,$$

we define finer spherical functions

$$\omega_u(x; s) = \int_K |\mathbf{d}(k \cdot x)|_u^s \, dk, \quad |\mathbf{d}(y)|_u^s = \begin{cases} \prod_{i=1}^n |d_i(y)|^{s_i} & \text{if } y \in X_u \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

Then, for each $\lambda \in \widetilde{\Lambda_n^+}$ and generic z, we have the following identity:

$$(\omega_u(x_\lambda; s))_{u \in \mathcal{U}} = c^{-1} \sum_{\sigma \in W} \gamma(\sigma(z)) B^{(e)}(\sigma, z) \, (\delta_u(x_\lambda; \sigma(z)))_{u \in \mathcal{U}} \ , \tag{3.6}$$

where

$c := \Sigma_{w \in W} [U\sigma U : U]$ (U is the Iwahori subgroup of K associated with B),

$$\gamma(z) := \begin{cases} \prod_{\alpha \in \Sigma_s^+} \dfrac{1 - q^{-2+2\langle \alpha, z\rangle}}{1 - q^{2\langle \alpha, z\rangle}} \cdot \prod_{\alpha \in \Sigma_\ell^+} \dfrac{1 - q^{-1+\langle \alpha, z\rangle}}{1 - q^{\langle \alpha, z\rangle}} & \text{if } m = 2n \\ \prod_{\alpha \in \Sigma_s^+} \dfrac{1 - q^{-2+2\langle \alpha, z\rangle}}{1 - q^{2\langle \alpha, z\rangle}} \cdot \prod_{\alpha \in \Sigma_\ell^+} \dfrac{(1 + q^{-2+\langle \alpha, z\rangle})(1 - q^{-1+\langle \alpha, z\rangle})}{1 - q^{2\langle \alpha, z\rangle}} & \text{if } m = 2n+1 \ , \end{cases}$$

$$\delta_u(x_\lambda; z) := \int_U |\mathbf{d}(\nu \cdot x_\lambda)|_u^s \, d\nu = |\mathbf{d}(x_\lambda)|_u^s = \begin{cases} q^{\langle \lambda, z_0\rangle} q^{-\langle \lambda, z\rangle} & \text{if } x_\lambda \in X_u \\ 0 & \text{otherwise}, \end{cases} \tag{3.7}$$

and $B^{(e)}(\sigma, z)$ is a matrix of size 2^n determined by the functional equation

$$(\omega_u(x_\lambda; z))_{u \in \mathcal{U}} = B^{(e)}(\sigma, z) \, (\omega_u(x_\lambda; \sigma(z)))_{u \in \mathcal{U}} \ .$$

We note here c and $\gamma(z)$ are determined by the group $G = U(j_m)$ ([Car, Theorem 4.4]) and $\gamma(z)$ coincides with $c(\lambda)$ there for the character $\lambda(p) = (-1)^{v_\pi(p_1 \cdots p_n)} \prod_{i=1}^n |N(p_i)|^{z_i}$, where p_i is the i-th diagonal entry of $p \in B$. We don't need to calculate the constant c in advance, since it is determined by the property $\omega(x; s) \mid_{s=0} = P_{\mathbf{0}}(z) = 1$. We have to be careful the second equality in (3.7) especially when $\lambda \notin \Lambda_n^+$, where we should consider the integral associated with the decomposition

$$U = (U \cap B)U_N, \quad U_N = \left\{ u \in U \;\middle|\; {}^t u \in B,\ u \equiv 1_m \pmod{(\pi)} \right\}.$$

We explain how $B^{(e)}(\sigma, z)$ is obtained by Theorem 2.10. A character $\chi = (\chi_1, \dots, \chi_n)$ of $\mathcal{U}$ can be regarded as a character of $\left(k^\times / N(k'^\times)\right)^n$, which is isomorphic to $\mathcal{U}$ via $v_\pi(\ )$, and we may consider the following integral for any $\chi \in \widehat{\mathcal{U}}$

$$\int_K \prod_{i=1}^n \chi_i(d_i(k \cdot x))\, |d_i(k \cdot x)|^{s_i}\, dk\,.$$

Then we see the above integral is equal to

$$\sum_{u \in \mathcal{U}} \chi(u)\omega_u(x; s) = \sum_{u \in \mathcal{U}} \chi(u)\omega_u(x; z) = \omega(x; z_\chi)\,,$$

where $z_{\chi,i} = z_i$ or $z_i + \frac{\pi\sqrt{-1}}{\log q}$ suitably, and the following functional equation holds by Theorem 2.10

$$\omega(x; z_\chi) = \Gamma_\sigma^{(e)}(z_\chi)\, \omega(x; \sigma(z_\chi)), \qquad (\sigma \in W)\,.$$

We may take $\sigma\chi \in \widehat{\mathcal{U}}$ such that $\omega(x; \sigma(z_\chi)) = \omega(x; \sigma(z)_{\sigma\chi})$, where $(\sigma\chi)(u) = \chi(\sigma^{-1}(u)),\ u \in \mathcal{U}$. Thus we obtain

$$\big(\chi(u)\big)_{\chi,u}\big(\omega_u(x; z)\big)_{u \in \mathcal{U}} = \widetilde{\Gamma^{(e)}}(\sigma, z)\big((\sigma\chi)(u)\big)_{\chi,u}\big(\omega_u(x; \sigma(z))\big)_{u \in \mathcal{U}}\,, \tag{3.8}$$

where $\widetilde{\Gamma^{(e)}}(\sigma, z)$ is the diagonal matrix with $\Gamma_\sigma^{(e)}(z_\chi)$ as the χ-diagonal entry, and

$$B^{(e)}(\sigma, z) = \big(\chi(u)\big)_{\chi,u}^{-1}\, \widetilde{\Gamma^{(e)}}(\sigma, z)\, \big((\sigma\chi)(u)\big)_{\chi,u}\,. \tag{3.9}$$

We set the first row for $(\chi, u) \in \widehat{\mathcal{U}} \times \mathcal{U}$ as the trivial character $\mathbf{1}$. Then the first entry in the left hand side of (3.8) is equal to $\omega(x; z)$, and we note $z_{\mathbf{1}} = z$ and $(\sigma\mathbf{1})(u) = 1,\ u \in \mathcal{U}$. Hence we have by (3.6), (3.8), and (3.9),

$$\begin{aligned}
\omega(x; z) &= c^{-1} \sum_{\sigma \in W} \gamma(\sigma(z))\Gamma_\sigma^{(e)}(z) q^{\langle \lambda, z_0 \rangle} q^{-\langle \lambda, \sigma(z) \rangle} \\
&= \frac{q^{\langle \lambda, z_0 \rangle}}{c} \cdot \frac{q^{\langle e, z \rangle}}{G(z)} \sum_{\sigma \in W} \sigma\left(\gamma(z)G(z)q^{-\langle \lambda + e, z \rangle}\right) \quad \text{(by (2.22))} \\
&= \frac{q^{\langle \lambda, z_0 \rangle}}{c} \cdot \frac{q^{\langle e, z \rangle}}{G(z)} \sum_{\sigma \in W} \sigma\left(q^{-\langle \lambda + e, z \rangle} \prod_{\alpha \in \Sigma^+} \frac{1 - t_\alpha q^{\langle \alpha, z \rangle}}{1 - q^{\langle \alpha, z \rangle}}\right),
\end{aligned} \tag{3.10}$$

where t_α is given as in (3.1). By (3.10), we have, taking $\lambda = (-e)$ and $z = z_0$,

$$c = \frac{1}{G(z_0)} \sum_{\sigma \in W} \sigma \left(\prod_{\alpha} \frac{1 - t_\alpha q^{\langle \alpha, z_0 \rangle}}{1 - q^{\langle \alpha, z_0 \rangle}} \right),$$

which is the same as in case $e = 0$, and $c^{-1} = c_n$ in (3.1). □

4. The structure of the Schwartz space

We keep the assumption that $e \leq 1$ if m is odd. We define the Schwartz space

$$\mathcal{S}(K \backslash X) = \{ \varphi : X \longrightarrow \mathbb{C} \mid \text{ left } K\text{-invariant, compactly supported} \},$$

and study $\mathcal{H}(G, K)$-module structure and Plancherel formula about it. Based on the explicit formula in §3, we modify the spherical function by using the value at $x_{(-e)}$ as

$$\Psi(x; z) = \omega(x; z)/\omega(x_{(-e)}; z) \in \mathcal{R} = \mathbb{C}\big[q^{\pm z_1}, \ldots, q^{\pm z_n}\big]^W. \tag{4.1}$$

Then, we have (cf. (2.5) and (3.4))

$$f * \Psi(x; z) = \lambda_z(f)\Psi(x; z), \quad (f \in \mathcal{H}(G, K)), \tag{4.2}$$

$$\lambda_z : \mathcal{H}(G, K) \xrightarrow{\sim} \mathcal{R}_0 = \mathbb{C}\big[q^{\pm 2z_1}, \ldots, q^{\pm 2z_n}\big]^W,$$

$$\Psi(x_\lambda; z) = q^{\langle \lambda + e, z_0 \rangle} \frac{\widetilde{w}_{\lambda+e}(-q^{-1})}{\widetilde{w}_{\mathbf{0}}(-q^{-1})} \cdot P_{\lambda+e}(z; \{t\}), \quad (\lambda \in \widetilde{\Lambda_n^+}). \tag{4.3}$$

Here the value $\Psi(x_\lambda; z)$ for dyadic case, i.e., the case $e = v_\pi(2) > 0$, coincides with the value $\Psi(x_{\lambda+e}; z)$ for odd residual case. Hence all the results of this section are parallel to odd residual case ([HK1, §4] or [HK2, §4]).

We define the spherical Fourier transform on $\mathcal{S}(K \backslash X)$ by

$$\begin{array}{rccl} F: & \mathcal{S}(K\backslash X) & \longrightarrow & \mathcal{R} \\ & \varphi & \longmapsto & F(\varphi)(z) = \int_X \varphi(x)\Psi(x; z)dx, \end{array} \tag{4.4}$$

where dx is a G-invariant measure on X, and we fix the normalization of dx later. The Hecke algebra $\mathcal{H}(G, K)$ acts on $\mathcal{S}(K \backslash X)$ by convolution product

$$f * \varphi(x) = \int_G f(g)\varphi(g^{-1} \cdot x)dg, \quad (f \in \mathcal{H}(G, K),\ \varphi \in \mathcal{S}(K \backslash X)),$$

where dg is the Haar measure on G, and on $\mathcal{R}$ through Satake isomorphism λ_z. Then the map F is compatible with $\mathcal{H}(G, K)$-action as follows

$$F(f * \varphi)(z) = \lambda_z(f)\varphi(z), \qquad (f \in \mathcal{H}(G, K),\ \varphi \in \mathcal{S}(K \backslash X)). \tag{4.5}$$

The space $\mathcal{S}(K\backslash X)$ is spanned by $\left\{ ch_\lambda \,\middle|\, \lambda \in \widetilde{\Lambda_n^+} \right\}$, where ch_λ is the characteristic function of $K \cdot x_\lambda$, and we have by (4.3)

$$F(ch_\lambda)(z) = q^{\langle \lambda+e,\, z_0 \rangle} \frac{\widetilde{w}_{\lambda+e}(-q^{-1})}{\widetilde{w}_{\mathbf{0}}(-q^{-1})} \cdot v(K \cdot x_\lambda) \cdot P_{\lambda+e}(z; \{t\}), \tag{4.6}$$

where $v(K \cdot x_\lambda)$ the volume of $K \cdot x_\lambda$ with respect to dx. Since the set $\left\{ P_\mu(z; \{t\}) \,\middle|\, \mu \in \Lambda_n^+ \right\}$ forms a $\mathbb{C}$-basis for $\mathcal{R}$, the map F is an $\mathcal{H}(G, K)$-module isomorphism. Thus we see the following.

THEOREM 4.1. *Assume $e \leq 1$ if m is odd. The spherical Fourier transform F gives an $\mathcal{H}(G, K)$-module isomorphism*

$$\mathcal{S}(K\backslash X) \xrightarrow{\sim} \mathbb{C}\left[q^{\pm z_1}, \dots, q^{\pm z_n}\right]^W (= \mathcal{R}),$$

where $\mathcal{R}$ is regarded as $\mathcal{H}(G, K)$-module via λ_z. In particular $\mathcal{S}(K\backslash X)$ is a free $\mathcal{H}(G, K)$-module of rank 2^n.

Each spherical functions on X is associated with some λ_z like as (4.2), and it is determined by the class of z in $\left(\mathbb{C}/\frac{2\pi\sqrt{-1}}{\log q}\mathbb{Z}\right)^n / W$. The dimension of spherical functions associated with the same λ_z is at most 2^n by Theorem 4.1, and we can give a basis as below.

COROLLARY 4.2. *Assume $e \leq 1$ if m is odd. All the spherical functions on X are parametrized by eigenvalues $z \in \left(\mathbb{C}/\frac{2\pi\sqrt{-1}}{\log q}\mathbb{Z}\right)^n / W$ through Satake isomorphism λ_z. The set $\left\{ \Psi(x; z+u) \,\middle|\, u \in \left\{0, \frac{\pi\sqrt{-1}}{\log q}\right\}^n \right\}$ forms a basis of spherical functions on X corresponding to z.*

We will give the Plancherel formula on $\mathcal{S}(K\backslash X)$. Recall the notation $c(z; \{t\})$, $P_\mu(z; \{t\})$ and $\widetilde{w}_\mu(-q^{-1})$ given in Theorem 3.1 and Remark 3.2. We define an inner product on $\mathcal{R}$ by

$$\langle P,\, Q \rangle_{\mathcal{R}} = \int_{\mathfrak{a}^*} P(z)\overline{Q(z)}d\mu(z), \quad (P, Q \in \mathcal{R}), \tag{4.7}$$

where

$$\mathfrak{a}^* = \left\{ \sqrt{-1}\left(\mathbb{R}\Big/\frac{2\pi}{\log q}\mathbb{Z}\right) \right\}^n,$$

$$d\mu(z) = \frac{1}{n!2^n} \cdot \frac{\widetilde{w}_{\mathbf{0}}(-q^{-1})}{(1+q^{-1})^{m'}} \cdot \frac{1}{|c(z; \{t\})|^2} dz, \quad m' = \left[\frac{m+1}{2}\right], \tag{4.8}$$

and dz is the Haar measure on $\mathfrak{a}^*$. In the following, for simplicity we write

$$P_{\lambda+e} = P_{\lambda+e}(z; \{t\}) \in \mathcal{R}, \quad \widetilde{w}_\lambda = \widetilde{w}_\lambda(-q^{-1}) \in \mathbb{R}, \qquad (\lambda \in \widetilde{\Lambda_n^+}).$$

Then, by [HK1, Proposition B.3], we have

$$\left\langle P_{\lambda+e},\ P_{\mu+e}\right\rangle_{\mathcal{R}} = \delta_{\lambda,\mu}\frac{\widetilde{w}_{\mathbf{0}}}{\widetilde{w}_{\lambda+e}}, \qquad (\lambda,\mu\in\widetilde{\Lambda_n^+})\,, \tag{4.9}$$

and by using (4.6),

$$\langle F(ch_\lambda),\ F(ch_\lambda)\rangle_{\mathcal{R}} = \delta_{\lambda,\mu}\, q^{2\langle\lambda+e,\,\mathrm{Re}(z_0)\rangle}\frac{\widetilde{w}_{\lambda+e}}{\widetilde{w}_{\mathbf{0}}}\cdot v(K\cdot x_\lambda)^2\,. \tag{4.10}$$

Since there are precisely two G-orbits in X represented by x_0 and x_1 (Theorem 1.1-(3)), we may normalize the G-invariant measure dx on each orbit by fixing the volume of $K\cdot x_0$ and $K\cdot x_1$, where $x_0 = 1_m$ and $x_1 = x_{\langle 1\rangle}$ with $\langle 1\rangle = (1,0,\dots,0)\in\Lambda_n^+$.

LEMMA 4.3. *By the normalization of the G-invariant measure dx on X given as*

$$v(K\cdot x_0) = q^{-2\langle e,\,\mathrm{Re}(z_0)\rangle}\,\frac{\widetilde{w}_{\mathbf{0}}(-q^{-1})}{\widetilde{w}_e(-q^{-1})},\quad v(K\cdot x_1) = q^{-2\langle\langle 1\rangle+e,\,\mathrm{Re}(z_0)\rangle}\,\frac{\widetilde{w}_{\mathbf{0}}(-q^{-1})}{\widetilde{w}_{\langle 1\rangle+e}(-q^{-1})}\,, \tag{4.11}$$

one has

$$v(K\cdot x_\lambda) = q^{-2\langle\lambda+e,\,\mathrm{Re}(z_0)\rangle}\,\frac{\widetilde{w}_{\mathbf{0}}(-q^{-1})}{\widetilde{w}_{\lambda+e}(-q^{-1})},\quad \lambda\in\widetilde{\Lambda_n^+}\,.$$

PROOF. For any $f\in\mathcal{H}(G,K)$ and $\mu\in\widetilde{\Lambda_n^+}$, we may write

$$f * ch_\mu = \sum_{\nu\in\widetilde{\Lambda_n^+}} a^\mu_\nu(f)\,ch_\nu,\quad (a^\mu_\nu(f)\in\mathbb{C})\,. \tag{4.12}$$

For $f\in\mathcal{H}(G,K)$ and $\lambda\in\widetilde{\Lambda_n^+}$, we have

$$\begin{aligned}(f*\Psi(\ ;z))(x_\lambda) &= \sum_{\mu\in\widetilde{\Lambda_n^+}}\Psi(x_\mu;z)(f*ch_\mu)(x_\lambda) = \sum_{\mu\in\widetilde{\Lambda_n^+}}\Psi(x_\mu;z)a^\mu_\lambda(f)\\ &= \sum_{\mu\in\widetilde{\Lambda_n^+}} q^{\langle\mu+e,\,z_0\rangle}\frac{\widetilde{w}_{\mu+e}}{\widetilde{w}_{\mathbf{0}}}\,a^\mu_\lambda(f)\cdot P_{\mu+e}\,,\end{aligned} \tag{4.13}$$

where we used (4.3) and the summation over $\widetilde{\Lambda_n^+}$ is essentially a finite sum, since the support of f is compact. On the other hand, by (4.2) and (4.3), we have

$$\begin{aligned}(f*\Psi(\ ;z))(x_\lambda) &= \lambda_z(f)\Psi(x_\lambda;z)\\ &= q^{\langle\lambda+e,\,z_0\rangle}\frac{\widetilde{w}_{\lambda+e}}{\widetilde{w}_{\mathbf{0}}}\cdot\lambda_z(f)P_{\lambda+e}\,.\end{aligned} \tag{4.14}$$

Taking the inner product of (4.13) and (4.14) with $P_{\mu+e}$, we have by (4.9)

$$q^{\langle\mu+e,\,z_0\rangle}a^\mu_\lambda(f) = q^{\langle\lambda+e,\,z_0\rangle}\frac{\widetilde{w}_{\lambda+e}}{\widetilde{w}_{\mathbf{0}}}\left\langle\lambda_z(f)P_{\lambda+e},\ P_{\mu+e}\right\rangle_{\mathcal{R}},\quad (f\in\mathcal{H}(G,K),\ \lambda,\mu\in\widetilde{\Lambda_n^+})\,. \tag{4.15}$$

Applying the spherical transform F to each side of (4.12), we have

$$v(K\cdot x_\mu)q^{\langle \mu+e,\, z_0\rangle}\frac{\widetilde{w}_{\mu+e}}{\widetilde{w}_{\mathbf{0}}}\lambda_z(f)P_{\mu+e} = \sum_{\nu\in\widetilde{\Lambda_n^+}} a_\nu^\mu(f)v(K\cdot x_\nu)q^{\langle \nu+e,\, z_0\rangle}\frac{\widetilde{w}_{\nu+e}}{\widetilde{w}_{\mathbf{0}}}P_{\nu+e}\,,$$

taking the inner product of each side of the above identity with $P_{\lambda+e}$, we obtain

$$v(K\cdot x_\mu)q^{\langle \mu+e,\, z_0\rangle}\frac{\widetilde{w}_{\mu+e}}{\widetilde{w}_{\mathbf{0}}}\left\langle\lambda_z(f)P_{\mu+e},\ P_{\lambda+e}\right\rangle_{\mathcal{R}} = a_\lambda^\mu(f)v(K\cdot x_\lambda)q^{\langle\lambda+e,\, z_0\rangle}\,,$$
$$(f\in\mathcal{H}(G,K),\ \lambda,\mu\in\widetilde{\Lambda_n^+})\,. \quad (4.16)$$

Now assume $|\lambda|\equiv|\mu|\pmod 2$ and take $f_1\in\mathcal{H}(G,K)$ to be the characteristic function of Kg_1K such that $x_\lambda = g_1\cdot x_\mu$. Then

$$a_\lambda^\mu(f_1)\neq 0,\quad \left\langle\lambda_z(f_1)P_{\lambda+e},\ P_{\mu+e}\right\rangle_{\mathcal{R}} = \left\langle\lambda_z(f_1)P_{\mu+e},\ P_{\lambda+e}\right\rangle_{\mathcal{R}}\,, \quad (4.17)$$

Hence we obtain by (4.15) and (4.16)

$$\frac{v(K\cdot x_\lambda)}{v(K\cdot x_\mu)} = q^{2\langle\mu-\lambda,\, z_0\rangle}\frac{\widetilde{w}_{\mu+e}}{\widetilde{w}_{\lambda+e}} = q^{2\langle\mu-\lambda,\, \mathrm{Re}(z_0)\rangle}\frac{\widetilde{w}_{\mu+e}}{\widetilde{w}_{\lambda+e}},\quad \text{if } |\lambda|\equiv|\mu| \pmod 2)\,. \quad (4.18)$$

Since $x_\lambda\in G\cdot x_0$ if and only if $|\lambda|\equiv 0\pmod 2$ for $\lambda\in\widetilde{\Lambda_n^+}$, under the normalization of dx as in (4.11), we obtain the volume $v(K\cdot x_\lambda)$ by (4.18), which completes the proof. $\square$

We take the normalization as in Lemma 4.3. Then by (4.10), we see

$$\int_X ch_\lambda(x)\overline{ch_\mu(x)}dx = \delta_{\lambda,\mu}v(K\cdot x_\lambda) = \int_{\mathfrak{a}^*}F(ch_\lambda)(z)\overline{F(ch_\mu)(z)}d\mu(z),\quad (\lambda,\mu\in\widetilde{\lambda_n^+})\,.$$

Since $\mathcal{S}(K\backslash X)$ is spanned by the set $\left\{ch_\lambda \;\middle|\; \lambda\in\widetilde{\Lambda_n^+}\right\}$, we obtain the following theorem.

THEOREM 4.4 (Plancherel formula on $\mathcal{S}(K\backslash X)$). *Assume $e\leq 1$ if m is odd. For any $\varphi,\psi\in\mathcal{S}(K\backslash X)$, one has*

$$\int_X\varphi(x)\overline{\psi(x)}dx = \int_{\mathfrak{a}^*}F(\varphi)(z)\overline{F(\psi)(z)}d\mu(z)\,,$$

where dx is normalized as in Lemma 4.3, *and $\mathfrak{a}^*$ and $d\mu(z)$ are given in* (4.8).

COROLLARY 4.5 (Inversion formula). *Assume $e\leq 1$ if m is odd. For any $\varphi\in\mathcal{S}(K\backslash X)$ and $x\in X$, one has*

$$\varphi(x) = \int_{\mathfrak{a}^*}F(\varphi)(z)\Psi(x;z)d\mu(z)\,.$$

PROOF. For any $\varphi \in \mathcal{S}(K\backslash X)$ and $x \in X$, we have by Theorem 4.4

$$\begin{aligned}
\varphi(x) &= \frac{1}{v(K \cdot x)} \int_X \varphi(y)\overline{ch_{K\cdot x}(y)}dx \\
&= \frac{1}{v(K \cdot x)} \int_{\mathfrak{a}^*} F(\varphi)(z)\overline{F(ch_{K\cdot x})(z)}d\mu(z) \\
&= \int_{\mathfrak{a}^*} F(\varphi)(z)\overline{\Psi(x;z)}d\mu(z) \\
&= \int_{\mathfrak{a}^*} F(\varphi)(z)\Psi(x;z)d\mu(z)\,.
\end{aligned}$$

□

References

[Bo] A. BOREL, Linear Algebraic Groups, Second enlarged edition, Graduate Texts in Mathematics **126**, Springer, 1991.

[Car] P. CARTIER, Representations of $\mathfrak{p}$-adic groups — A survey, Proc. Symp. Pure Math. **33-1** (1979), 111–155.

[H1] Y. HIRONAKA, Spherical functions and local densities on hermitian forms, J. Math. Soc. Japan **51** (1999), 553–581.

[H2] Y. HIRONAKA, Spherical functions on p-adic homogeneous spaces, "Algebraic and Analytic Aspects of Zeta Functions and L-functions" – Lectures at the French-Japanese Winter School (Miura, 2008)–, MSJ Memoirs **21** (2010), 50–72.

[H3] Y. HIRONAKA, Spherical functions on $U(2n)/(U(n) \times U(n))$ and hermitian Siegel series, "Geometry and Analysis of Automorphic Forms of Several Variables", Series on Number Theory and Its Applications **7**, World Scientific, 2011, 120–159.

[HK1] Y. HIRONAKA and Y. KOMORI, Spherical functions on the space of p-adic unitary hermitian matrices, Int. J. Number Theory **10** (2014), 513–558; Math arXiv:1207.6189

[HK2] Y. HIRONAKA and Y. KOMORI, Spherical functions on the space of p-adic unitary hermitian matrices II, the case of odd size, Commentarii Math. Univ. Sancti Pauli **63** (2014); Math arXiv:1403.3748

[HS] Y. HIRONAKA and F. SATO, The Siegel series and spherical functions on $O(2n)/(O(n) \times O(n))$, "Automorphic forms and zeta functions" – Proceedings of the conference in memory of Tsuneo Arakawa –, World Scientific, 2006, 150–169.

[Ja] R. JACOBOWITZ, Hermitian forms over local fields, Amer. J. Math. **84** (1962), 441–465.

[M1] I. G. MACDONALD, The Poincaré series of a Coxeter group, Math. Ann. **199** (1972), 161–174.

[M2] I. G. MACDONALD, Orthogonal polynomials associated with root systems, Séminaire Lotharingien de Combinatoire **45** (2000), Article B45a.

[Om] O. T. O'MEARA, Introduction to quadratic forms, Grund. Math. Wiss. **117**, Springer-Verlag, 1973.

[Sa] I. SATAKE, Theory of spherical functions on reductive algebraic groups over p-adic fields, Publ. Math. I.H.E.S. **18** (1963), 5–69.

[Se] J. P. SERRE, Galois cohomology, Springer-Verlag, 1997, (English translation of "Cohomologie galoisienne", 1964).

Present Address:
DEPARTMENT OF MATHEMATICS, FACULTY OF EDUCATION AND INTEGRATED SCIENCES,
WASEDA UNIVERSITY,
NISHI-WASEDA, TOKYO 169–8050, JAPAN.
e-mail: hironaka@waseda.jp