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Second Sectional Classes of Polarized Three-folds

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Abstract. Let X be a smooth complex projective variety of dimension 3 and L an ample line bundle on X. In this paper we study the second sectional class $\mathrm{cl}_2(X, L)$ of (X, L). First we show the inequality $\mathrm{cl}_2(X, L) \geq L^3 - 1$, and we characterize (X, L) with $-1 \leq \mathrm{cl}_2(X, L) - L^3 \leq 3$. Furthermore the classification of pairs (X, L) with small second sectional classes is obtained. We also classify (X, L) with $2L^3 \geq \mathrm{cl}_2(X, L)$.

1. Introduction

Let (X, L) be a polarized manifold of dimension n defined over the field of complex numbers, that is, let X be a smooth projective variety of dimension n defined over the field of complex numbers and L an ample line bundle on X. If L is very ample, then the *class* $m(X, L)$ of (X, L) is defined as follows. Let $\phi_{|L|} : X \hookrightarrow \mathbf{P}^N$ be the embedding defined by $|L|$ and $X^\vee$ the dual variety of X. The class $m(X, L)$ of (X, L) is defined by

$$m(X, L) = \begin{cases} \deg X^\vee, & \text{if } X^\vee \text{ is a hypersurface of } (\mathbf{P}^N)^\vee, \\ 0, & \text{if } X^\vee \text{ is not a hypersurface of } (\mathbf{P}^N)^\vee. \end{cases}$$

The class of (X, L), when L is very ample, has been extensively studied ([16], [19], [24], [17], [20], [18], [1], [23]). In [10], we defined the ith sectional class $\mathrm{cl}_i(X, L)$ of (X, L) for every integer i with $0 \leq i \leq n$ (see Definition 2.4). If L is very ample, then $\mathrm{cl}_n(X, L) = m(X, L)$, and in this case, there exists a sequence of smooth subvarieties $X \supset X_1 \supset \cdots \supset X_{n-i}$ such that $X_j \in |L_{j-1}|$ and $\dim X_j = n - j$ for every integer j with $1 \leq j \leq n - i$, where $L_j = L|_{X_j}$ and $L_0 = L$. Then, $\mathrm{cl}_i(X, L) = m(X_{n-i}, L_{n-i})$. The ith sectional class is a natural generalization of the class. Moreover, we stress that this $\mathrm{cl}_i(X, L)$ is defined for merely ample divisors (not necessarily very ample). By the definition of $\mathrm{cl}_i(X, L)$, we can expect that $\mathrm{cl}_i(X, L)$ has properties similar to those of the class of (Y, H) such that $\dim Y = i$ and that H is a very ample divisor on Y.

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If $i = 2$, $n \geq 3$, and L is very ample, then there are some results in [24], [20], and [18], but there are no results for the case in which L is ample. In this paper, we study the case in which $i = 2$, $\dim X = 3$, and L is ample. First we show the inequality $\mathrm{cl}_2(X, L) - L^3 \geq -1$ (see Theorem 3.1), and we characterize polarized 3-folds (X, L) with $-1 \leq \mathrm{cl}_2(X, L) - L^3 \leq 3$ (see Theorem 3.2). Moreover we get the classification of pairs (X, L) with $0 \leq \mathrm{cl}_2(X, L) \leq 4$ (see Theorem 3.3). Finally we also classify (X, L) with $2L^3 \geq \mathrm{cl}_2(X, L)$ as a generalization for the case of 3-folds in [20, Theorem 2] (see Theorem 3.4).

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2. Preliminaries

DEFINITION 2.1. Let X be a smooth projective variety of dimension n and $\mathcal{E}$ a vector bundle on X. For every integer j with $j \geq 0$, the *jth Segre class $s_j(\mathcal{E})$ of $\mathcal{E}$* is defined by the following equation: $c_t(\mathcal{E}^\vee)s_t(\mathcal{E}) = 1$, where $c_t(\mathcal{E}^\vee) = \sum_{k\geq 0} c_k(\mathcal{E}^\vee)t^k$, which is called the Chern polynomial of $\mathcal{E}^\vee$, and $s_t(\mathcal{E}) = \sum_{j\geq 0} s_j(\mathcal{E})t^j$.

REMARK 2.1. Let X be a smooth projective variety and $\mathcal{E}$ a vector bundle on X. Let $\tilde{s}_j(\mathcal{E})$ be the Segre class which is defined in [13, Chapter 3]. Then $s_j(\mathcal{E}) = \tilde{s}_j(\mathcal{E}^\vee)$.

DEFINITION 2.2 ([9, Definition 2.1.3]). Let X be a smooth projective variety of dimension n and $\mathcal{E}$ an ample vector bundle on X with rank $\mathcal{E} = r$. We assume that $r \leq n$. For every integer p with $0 \leq p \leq n - r$, we set

$$C_p^{n,r}(X, \mathcal{E}) := \sum_{k=0}^{p} c_k(X)s_{p-k}(\mathcal{E}^\vee).$$

DEFINITION 2.3 ([9, Definition 3.1 (i)]). Let (X, L) be a polarized manifold of dimension n, and i an integer with $0 \leq i \leq n$. The *ith sectional Euler number $e_i(X, L)$ of (X, L)* is defined as follows:

$$e_i(X, L) := C_i^{n,n-i}(X, L^{\oplus n-i})L^{n-i}.$$

REMARK 2.2. By the definition of $e_i(X, L)$,

$$e_i(X, L) = \sum_{l=0}^{i} (-1)^l \binom{n-i+l-1}{l} c_{i-l}(X)L^{n-i+l}.$$

The definition of the ith sectional Euler number in Definition 2.3 is the same as that in [8, Definition 3.1 (1)].

DEFINITION 2.4 ([10, Definitions 2.8 and 2.9]). Let (X, L) be a polarized manifold of dimension n and i an integer with $0 \leq i \leq n$. The *ith sectional class* of (X, L) is defined as follows:

$$\mathrm{cl}_i(X,L)=\begin{cases} e_0(X,L)\,, & \text{if } i=0\,,\\ (-1)\{e_1(X,L)-2e_0(X,L)\}\,, & \text{if } i=1\,,\\ (-1)^i\{e_i(X,L)-2e_{i-1}(X,L)+e_{i-2}(X,L)\}\,, & \text{if } 2\le i\le n\,. \end{cases}$$

PROPOSITION 2.1. *Let (X, L) be a polarized manifold of dimension n. For any integer i with $0 \le i \le n$,*

$$cl_i(X,L)=\sum_{t=0}^{i}(-1)^{i-t}\binom{n-i+t+1}{t}c_{i-t}(X)L^{n-i+t}\,.$$

PROOF. By Definition 2.4 and Remark 2.2,

$$\begin{aligned} cl_i(X,L) &= (-1)^i\Bigg(\sum_{l=0}^{i}(-1)^l\binom{n-i+l-1}{l}c_{i-l}(X)L^{n-i+l}\\ &\quad -2\sum_{l=0}^{i-1}(-1)^l\binom{n-i+l}{l}c_{i-1-l}(X)L^{n-i+l+1}\\ &\quad +\sum_{l=0}^{i-2}(-1)^l\binom{n-i+l+1}{l}c_{i-2-l}(X)L^{n-i+l+2}\Bigg)\\ &= (-1)^i\Bigg(\sum_{l=2}^{i}(-1)^l\Bigg\{\binom{n-i+l-1}{l}+2\binom{n-i+l-1}{l-1}\\ &\quad +\binom{n-i+l-1}{l-2}\Bigg\}c_{i-l}(X)L^{n-i+l}\\ &\quad +c_i(X)L^{n-i}-(n-i)c_{i-1}(X)L^{n-i+1}-2c_{i-1}(X)L^{n-i+1}\Bigg)\\ &= (-1)^i\Bigg(\sum_{l=2}^{i}(-1)^l\binom{n-i+l+1}{l}c_{i-l}(X)L^{n-i+l}\\ &\quad -(n-i+2)c_{i-1}(X)L^{n-i+1}+c_i(X)L^{n-i}\Bigg)\\ &= (-1)^i\Bigg(\sum_{l=0}^{i}(-1)^l\binom{n-i+l+1}{l}c_{i-l}(X)L^{n-i+l}\Bigg)\,. \end{aligned}$$

This establishes the assertion. □

REMARK 2.3. (i) By [2, Lemma 1.6.4], $\mathrm{cl}_i(X, L) = c_i(J_1(L))L^{n-i}$, where $J_1(L)$ is the first Jet bundle of L.
(ii) By Proposition 2.1,

$$\mathrm{cl}_2(X, L) = c_2(X)L^{n-2} + nK_X L^{n-1} + \frac{1}{2}(n^2 + n)L^n . \tag{1}$$

REMARK 2.4 ([10, Remark 2.3]). Assume that L is very ample. There exists a sequence of smooth subvarieties $X \supset X_1 \supset \cdots \supset X_{n-i}$ such that $X_j \in |L_{j-1}|$ and $\dim X_j = n - j$ for every integer j with $1 \le j \le n - i$, where $L_j = L|_{X_j}$ and $L_0 = L$. In particular, X_{n-i} is a smooth projective variety of dimension i and L_{n-i} is a very ample line bundle on X_{n-i}. Then, $\mathrm{cl}_i(X, L)$ is equal to the class of (X_{n-i}, L_{n-i}).

PROPOSITION 2.2. *Let (X, L) be a polarized manifold of dimension $n \ge 2$, (Y, H) a polarized manifold such that (X, L) is a composite of simple blowing ups of (Y, H) and γ the number of its simple blowups. For every integer i with $0 \le i \le n$,*

$$\mathrm{cl}_i(X, L) = \begin{cases} \mathrm{cl}_0(Y, H) - \gamma\,, & \textit{if } i = 0\,, \\ \mathrm{cl}_1(Y, H) - 2\gamma\,, & \textit{if } i = 1\,, \\ \mathrm{cl}_i(Y, H)\,, & \textit{if } 2 \le i \le n-1 \textit{ or } i = n \ge 2\,. \end{cases}$$

PROOF. See [10, Corollary 2.3]. □

NOTATION 2.1. (i) (See [3, §3].) Let (X, L) be a polarized manifold of dimension $n \ge 3$. Assume that (X, L) is a hyperquadric fibration over a smooth curve C. Let $f : X \to C$ be its morphism. We put $\mathcal{E} := f_*(L)$. Then, $\mathcal{E}$ is a locally free sheaf of rank $n + 1$ on C. Let $\pi : \mathbf{P}_C(\mathcal{E}) \to C$ be the projective bundle. Then, $X \in |2H(\mathcal{E}) + \pi^*(B)|$ for some $B \in \mathrm{Pic}(C)$ and $L = H(\mathcal{E})|_X$, where $H(\mathcal{E})$ is the tautological line bundle of $\mathbf{P}_C(\mathcal{E})$. We put $e := \deg \mathcal{E}$ and $b := \deg B$. Then,

$$\begin{aligned} K_X &= ((-n + 1)H(\mathcal{E}) + \pi^*(K_C + \det(\mathcal{E}) + B))|_X \\ &= (-n + 1)L + f^*(K_C + \det(\mathcal{E}) + B)\,, \end{aligned}$$

and

$$g(X, L) = 2g(C) - 1 + e + b\,.$$

(ii) (See [5, (13.10)].) Let (M, A) be a $\mathbf{P}^2$-bundle over a smooth curve C, and $A|_F = \mathcal{O}_{\mathbf{P}^2}(2)$ for any fiber F of $M \to C$. Let $f : M \to C$ be the fibration and $\mathcal{E} := f_*(K_M + 2A)$. Then, $\mathcal{E}$ is a locally free sheaf of rank 3 on C, and $M \cong \mathbf{P}_C(\mathcal{E})$ such that $H(\mathcal{E}) = K_M + 2A$. In this case, $A = 2H(\mathcal{E}) + f^*(B)$ for a line bundle B on C, and by the canonical bundle formula, $K_M = -3H(\mathcal{E}) + f^*(K_C + \det\mathcal{E})$. Here, we set $e := \deg \mathcal{E}$ and $b := \deg B$.

REMARK 2.5. Let (X, L) be a polarized manifold of dimension 3. If $K_M + A$ is nef, then $\kappa(K_X + L) = \kappa(K_M + A) \geq 0$, where (M, A) is the reduction of (X, L) (see [8, Definition 2.4]). So by [2, Proposition 7.2.2, Theorems 7.2.4, 7.3.2, and 7.3.4] or [5, Chapter II, Theorems (11.2), (11.7), and (11.8)] we see that if $\kappa(K_X + L) = -\infty$, then (X, L) is one of the following types.

(I) $(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1))$.
(II) $(\mathbf{Q}^3, \mathcal{O}_{\mathbf{Q}^3}(1))$.
(III) A scroll over a smooth projective curve.
(IV) $K_X \sim -2L$, that is, (X, L) is a Del Pezzo manifold.
(V) A hyperquadric fibration over a smooth curve.
(VI) A classical scroll over a smooth projective surface.
(VII) Let (M, A) be the reduction of (X, L).
(VII.1) $(M, A) = (\mathbf{Q}^3, \mathcal{O}_{\mathbf{Q}^3}(2))$.
(VII.2) $(M, A) = (\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(3))$.
(VII.3) M is a $\mathbf{P}^2$-bundle over a smooth curve C, and for any fiber F' of $M \to C$, $(F', A_{F'}) \cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(2))$.

Here, we calculate $\mathrm{cl}_2(X, L)$ and L^3 of (X, L) above.
(I) If $(X, L) \cong (\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1))$, then $\mathrm{cl}_2(X, L) = 0$ and $L^3 = 1$ by [12, Example 2.1 (i)].
(II) If $(X, L) \cong (\mathbf{Q}^3, \mathcal{O}_{\mathbf{Q}^3}(1))$, then $\mathrm{cl}_2(X, L) = 2$ and $L^3 = 2$ by [12, Example 2.1 (ii)].
(III) If (X, L) is a scroll over a smooth curve, then $\mathrm{cl}_2(X, L) = L^3$ by [12, Example 2.1 (ix)].
(IV) If (X, L) is a Del Pezzo manifold, then $\mathrm{cl}_2(X, L) = 12$ and $L^3 \leq 8$ by [12, Example 2.1 (vii)].
(V) Assume that (X, L) is a hyperquadric fibration over a smooth curve C. Then, $L^3 = 2e+b$, and by [12, Example 2.1 (viii)], $\mathrm{cl}_2(X, L) = 8e + 8b + 4(g(C) - 1)$.
(VI) Assume that (X, L) is a classical scroll over a smooth projective surface S. Then, there exists an ample vector bundle $\mathcal{E}$ of rank 2 on S such that $(X, L) \cong (\mathbf{P}_S(\mathcal{E}), H(\mathcal{E}))$, where $H(\mathcal{E})$ is the tautological line bundle of $\mathbf{P}_S(\mathcal{E})$. Then, $L^3 = H(\mathcal{E})^3 = c_1(\mathcal{E})^2 - c_2(\mathcal{E})$. By [12, Example 2.1 (x)],

$$\mathrm{cl}_2(X, L) = c_2(S) + 3c_1(\mathcal{E})^2 + 2K_S c_1(\mathcal{E}).$$

(VII.1) If $(M, A) = (\mathbf{Q}^3, \mathcal{O}_{\mathbf{Q}^3}(2))$, then $\mathrm{cl}_2(X, L) = \mathrm{cl}_2(M, A) = 40$ and $L^3 \leq A^3 = 16$ by [12, Example 2.1 (iv)].
(VII.2) If $(M, A) = (\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(3))$, then $\mathrm{cl}_2(X, L) = \mathrm{cl}_2(M, A) = 72$ and $L^3 \leq A^3 = 27$ by [12, Example 2.1 (v)].
(VII.3) Assume that (M, A) is the type (VII.3). We use Notation 2.1. Then, $\mathrm{cl}_2(X, L) = \mathrm{cl}_2(M, A) = 36e + 47b$ and $L^3 \leq A^3 = 8e + 12b$ by [12, Example 2.1 (vi)].

3. Main results

THEOREM 3.1. *Let* (X, L) *be a polarized manifold of dimension* 3. *Then,* $\mathrm{cl}_2(X, L) \geq L^3 - 1$ *holds.*[1]

PROOF. (A) Assume that $\kappa(K_X + L) \geq 0$. Let (M, A) be the reduction of (X, L). Then, $K_M + A$ is nef, and according to the result of Höring [14, 1.5 Theorem], $h^0(K_X + L) = h^0(K_M + A) > 0$.

CLAIM 3.1. *If* $h^0(K_X + L) > 0$, *then a* $\mathbf{Q}$*-twisted bundle* $\Omega_X\langle L\rangle$ *is generically nef.*

PROOF. If $\Omega_X\langle L\rangle$ is not generically nef, then by [14, 3.1 Theorem], there exist smooth projective varieties X' and Y, a birational morphism $\mu : X' \to X$, and a fiber space $\lambda : X' \to Y$ such that $m := \dim Y < 3$, a general fiber F_λ of λ is rationally connected, and $h^0(D) = 0$ for any Cartier divisor D on F_λ with $D \sim_{\mathbf{Q}} K_{F_\lambda} + j\,(\mu^*(L))_{F_\lambda}$ and $j \in [0, 3-m] \cap \mathbf{Q}$. However, this is impossible because $h^0(K_{F_\lambda} + (\mu^* L)_{F_\lambda}) > 0$ under the assumption. □

By [14, 2.11 Corollary],

$$c_2(X)L + 2K_X L^2 + 3L^3 \geq 0\,.$$

Hence, by (1) in Remark 2.3 (ii),

$$\begin{aligned} \mathrm{cl}_2(X, L) &= c_2(X)L + 3K_X L^2 + 6L^3 \\ &\geq -2K_X L^2 - 3L^3 + 3K_X L^2 + 6L^3 \\ &= (K_X + 2L)L^2 + L^3\,. \end{aligned} \tag{2}$$

Since $\kappa(K_X + L) \geq 0$, $K_X + 2L$ is nef. Hence, $\mathrm{cl}_2(X, L) \geq L^3$.
(B) Assume that $\kappa(K_X + L) = -\infty$. Then, (X, L) is one of the types in Remark 2.5.
(a) If $(X, L) \cong (\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1))$, then $\mathrm{cl}_2(X, L) = L^3 - 1$ by Remark 2.5.
(b) If $(X, L) \cong (\mathbf{Q}^3, \mathcal{O}_{\mathbf{Q}^3}(1))$, then $\mathrm{cl}_2(X, L) = L^3$ by Remark 2.5.
(c) If (X, L) is a scroll over a smooth curve, then $\mathrm{cl}_2(X, L) = L^3$ by Remark 2.5.
(d) If (X, L) is a Del Pezzo manifold, then $\mathrm{cl}_2(X, L) \geq L^3 + 4$ by Remark 2.5.
(e) If $(M, A) = (\mathbf{Q}^3, \mathcal{O}_{\mathbf{Q}^3}(2))$, then $\mathrm{cl}_2(X, L) \geq L^3 + 24$ by Remark 2.5.
(f) If $(M, A) = (\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(3))$, then $\mathrm{cl}_2(X, L) \geq L^3 + 45$ by Remark 2.5.
(g) Assume that (M, A) is the type (VII.3). We use Notation 2.1. Note that

$$8e + 12b = A^3\,, \tag{3}$$

$$2g(C) - 2 + e + 2b = 0\,, \tag{4}$$

$$g(X, L) = 1 + 2e + 2b\,. \tag{5}$$

[1] In particular, $\mathrm{cl}_2(X, L) \geq 0$.

Here, we set $A^3 = 4m$. Then, m is an integer with $m \geq 1$. From (3) and (4),

(6) $$b = 4(1 - g(C)) - m$$

and

(7) $$e = 6(g(C) - 1) + 2m.$$

Therefore, by Remark 2.5,

$$\mathrm{cl}_2(X, L) - L^3 \geq \mathrm{cl}_2(M, A) - A^3 = 28e + 35b = 21m + 28(g(C) - 1).$$

CLAIM 3.2. *Either $g(C) \geq 1$ or $m \geq 2$ holds.*

PROOF. If $g(C) = 0$ and $m = 1$, then $e = -4$ and $b = 3$, but then, by (5), $g(X, L) = -1 < 0$, which is impossible. Hence, $g(C) \geq 1$ or $m \geq 2$. □

By Claim 3.2,

$$\mathrm{cl}_2(X, L) - L^3 \geq 21m + 28(g(C) - 1) \geq 14.$$

(h) Assume that (X, L) is a hyperquadric fibration over a smooth curve C.

LEMMA 3.1. $\mathrm{cl}_2(X, L) - L^3 \geq 4$ *holds.*

PROOF. Here, we use Notation 2.1. By Remark 2.5,

$$\mathrm{cl}_2(X, L) - L^3 = 6e + 7b + 4(g(C) - 1).$$

Here, $L^3 = 2e + b > 0$, and [3, (3.3)] implies that $2e + 4b \geq 0$.

CLAIM 3.3. $6e + 7b \geq 5$ *holds.*

PROOF. Assume that $b \geq 0$. If $2e + b = 1$ and $b = 0$, then $2e = 1$. However, this is impossible because e is an integer. Hence, $2e + b \geq 2$ or $b > 0$. Then, $6e + 7b = 3(2e + b) + 4b \geq 6$.

Assume that $b < 0$. Then, $6e + 7b = 3(2e + 4b) - 5b \geq 5$. This establishes the assertion. □

By Claim 3.3,

$$\mathrm{cl}_2(X, L) - L^3 = 6e + 7b + 4(g(C) - 1) \geq 1.$$

(h.1) If $\mathrm{cl}_2(X, L) - L^3 = 1$, then $g(C) = 0$, $b = -1$, and $2e + 4b = 0$. Thus, $(g(C), b, e) = (0, -1, 2)$. However, $g(X, L) = b + e + 2g(C) - 1 = 0$, which is impossible because $\kappa(K_X + 2L) \geq 0$ in this case.

(h.2) If $\mathrm{cl}_2(X, L) - L^3 = 2$, then $g(C) = 0$ and $6e + 7b = 6$. If $b < 0$, then $2e + 4b = 0$. Hence, $6 = 6e + 7b = -12b + 7b = -5b$, which is impossible. Hence, we can assume that $b \geq 0$. Then, $b = 0$ and $2e+b = 2$. Hence, $e = 1$. In this case, $g(X, L) = b+e+2g(C)-1 =$

0, which also does not occur.
(h.3) If $\mathrm{cl}_2(X, L) - L^3 = 3$, then $g(C) = 0$ and $6e + 7b = 7$. If $b < 0$, then $2e + 4b = 0$. Hence, $7 = 6e + 7b = -12b + 7b = -5b$, which is impossible. Hence, we can assume that $b \geq 0$. Then, $b = 1$ and $2e+b = 1$. Hence, $e = 0$. In this case, $g(X, L) = b+e+2g(C)-1 = 0$, which is also impossible. This establishes the assertion of Lemma 3.1. □

(i) Assume that (X, L) is a classical scroll over a smooth projective surface S. There exists an ample vector bundle $\mathcal{E}$ of rank 2 on S such that $(X, L) \cong (\mathbf{P}_S(\mathcal{E}), H(\mathcal{E}))$.

LEMMA 3.2. $\mathrm{cl}_2(X, L) - L^3 \geq 5$ *holds unless* $(S, \mathcal{E}) \cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1) \oplus \mathcal{O}_{\mathbf{P}^2}(1))$.

PROOF. By Remark 2.5,

$$\mathrm{cl}_2(X, L) - L^3 = c_2(S) + c_2(\mathcal{E}) + 4(g(S, c_1(\mathcal{E})) - 1).$$

(i.1) If $\kappa(S) \geq 0$, then $c_2(S) \geq 0$ by [21, Proposition 2] and $g(S, c_1(\mathcal{E})) \geq 2$. Hence, we get $\mathrm{cl}_2(X, L) - L^3 \geq 5$.
(i.2) Assume that $\kappa(S) = -\infty$. First, we prove the following.

CLAIM 3.4. *If* $S \not\cong \mathbf{P}^2$, *then* $c_2(S) \geq 4(1 - q(S))$ *holds.*

PROOF. Let S' be the relatively minimal model of S and u the number of its blowups. Then, by Noether's formula,

$$\begin{aligned} 12(1 - q(S)) = 12\chi(\mathcal{O}_S) &= c_2(S) + K_S^2 \\ &= c_2(S) + K_{S'}^2 - u \\ &= c_2(S) - u + 8(1 - q(S)). \end{aligned}$$

Hence,

$$c_2(S) = u + 4(1 - q(S)) \geq 4(1 - q(S)).$$

This establishes the assertion of Claim 3.4. □

By Claim 3.4,

$$\begin{aligned} \mathrm{cl}_2(X, L) - L^3 &= c_2(S) + c_2(\mathcal{E}) + 4(g(S, c_1(\mathcal{E})) - 1) \\ &\geq 4(1 - q(S)) + c_2(\mathcal{E}) + 4(g(S, c_1(\mathcal{E})) - 1) \\ &= 4(g(S, c_1(\mathcal{E})) - q(S)) + c_2(\mathcal{E}) . \end{aligned}$$

Here, $g(S, c_1(\mathcal{E})) \geq 2q(S)$ because $(S, c_1(\mathcal{E}))$ is not a scroll over a smooth curve (see [7, Lemma 1.16]).
(i.2.1) If $q(S) \geq 1$, then $g(S, c_1(\mathcal{E})) - q(S) \geq q(S) \geq 1$. Hence, $\mathrm{cl}_2(X, L) - L^3 \geq 5$.
(i.2.2) Assume that $q(S) = 0$. If $g(S, c_1(\mathcal{E})) \geq 1$, then $\mathrm{cl}_2(X, L) - L^3 \geq 5$ holds. Hence, we can assume that $g(S, c_1(\mathcal{E})) = 0$. Then, $K_S + c_1(\mathcal{E})$ is not nef, and we see from [25, Theorem 1] that $(S, \mathcal{E}) \cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1) \oplus \mathcal{O}_{\mathbf{P}^2}(1))$. This establishes the assertion of Lemma 3.2. □

Note that if $(S, \mathcal{E}) \cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1) \oplus \mathcal{O}_{\mathbf{P}^2}(1))$, then $\mathrm{cl}_2(X, L) = 3$ and $L^3 = 3$. Hence, in this case, $\mathrm{cl}_2(X, L) = L^3$. □

THEOREM 3.2. *Let (X, L) be a polarized manifold of dimension 3. Then, the following hold.*

(i) *If* $\mathrm{cl}_2(X, L) = L^3 - 1$, *then* $(X, L) \cong (\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1))$.
(ii) *If* $\mathrm{cl}_2(X, L) = L^3$, *then (X, L) is one of the following.*
  (ii.1) *A scroll over a smooth curve.*
  (ii.2) $(\mathbf{Q}^3, \mathcal{O}_{\mathbf{Q}^3}(1))$.
  (ii.3) $(\mathbf{P}_{\mathbf{P}^2}(\mathcal{E}), H(\mathcal{E}))$, *where* $\mathcal{E} \cong \mathcal{O}_{\mathbf{P}^2}(1) \oplus \mathcal{O}_{\mathbf{P}^2}(1)$.
(iii) *There exists no polarized manifold of dimension 3 with* $1 \leq \mathrm{cl}_2(X, L) - L^3 \leq 3$.

PROOF. If $\kappa(K_X + L) \geq 0$, then by (A) in the proof of Theorem 3.1,

$$\mathrm{cl}_2(X, L) - L^3 \geq 2g(X, L) - 2\,.$$

Assume that $\mathrm{cl}_2(X, L) - L^3 \leq 3$. Then, $g(X, L) \leq 2$. Assuming that $\kappa(K_X + L) \geq 0$, we have $g(X, L) \geq 2$. Hence, $g(X, L) = 2$. By the classification of polarized manifolds with $g(X, L) = 2$, (X, L) is one of the following three types.

(I) $\mathcal{O}(K_X) = \mathcal{O}_X$ and $L^3 = 1$.
(II) X is a double covering of $\mathbf{P}^3$ whose branch locus is a smooth hypersurface of degree 6 in $\mathbf{P}^3$ and $L = \pi^*(\mathcal{O}_{\mathbf{P}^3}(1))$, where $\pi : X \to \mathbf{P}^3$ is its double covering.
(III) (X, L) is a simple blowup of a polarized manifold (M, A) of type (II) above.

(I) If (X, L) satisfies $\mathcal{O}(K_X) = \mathcal{O}_X$ and $L^3 = 1$, then $\mathrm{cl}_2(X, L) = c_2(X)L + 3K_X L^2 + 6L^3 = c_2(X)L + 6$. According to the result of Miyaoka, $c_2(X)L \geq 0$ (see [22, Theorem 6.6]). Hence, $\mathrm{cl}_2(X, L) \geq 6 = L^3 + 5$.
(II) If (X, L) is type (II), then by [11, Lemma 3.4],

$$\begin{aligned} e_2(X, L) &= 4 - \frac{1}{6}(5 + (-5)^3) = 24, \\ e_1(X, L) &= 3 - \frac{1}{6}(5 + (-5)^2) = -2, \\ e_0(X, L) &= 2 - \frac{1}{6}(5 + (-5)) = 2. \end{aligned}$$

Therefore,

$$\mathrm{cl}_2(X, L) = 24 - 2(-2) + 2 = 30 = L^3 + 28.$$

(III) Assume that (X, L) is type (III). Then, $\mathrm{cl}_2(X, L) = \mathrm{cl}_2(M, A)$ by Proposition 2.2. Note also that $L^3 = A^3 - u$, where u is the number of its blowups. Hence,

$$\mathrm{cl}_2(X, L) = \mathrm{cl}_2(M, A) = A^3 + 28 = L^3 + 28 + u \geq L^3 + 28.$$

Therefore, if $\kappa(K_X + L) \geq 0$, then $\mathrm{cl}_2(X, L) \geq L^3 + 4$ holds. Hence, if $\mathrm{cl}_2(X, L) \leq L^3 + 3$, then $\kappa(K_X + L) = -\infty$. By (B) in the proof of Theorem 3.1, we get the assertion. $\square$

THEOREM 3.3. *Let (X, L) be a polarized manifold of dimension 3. Then the following hold.*

(i) *If* $\mathrm{cl}_2(X, L) = 0$, *then* $(X, L) \cong (\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1))$.
(ii) *If* $\mathrm{cl}_2(X, L) = 1$, *then (X, L) is a scroll over a smooth projective curve with* $L^3 = 1$.
(iii) *If* $\mathrm{cl}_2(X, L) = 2$, *then (X, L) is either* $(\mathbf{Q}^3, \mathcal{O}_{\mathbf{Q}^3}(1))$ *or a scroll over a smooth projective curve with* $L^3 = 2$.
(iv) *If* $\mathrm{cl}_2(X, L) = 3$, *then (X, L) is either a scroll over a smooth projective curve with* $L^3 = 3$ *or* $(\mathbf{P}_{\mathbf{P}^2}(\mathcal{E}), H(\mathcal{E}))$, *where* $\mathcal{E} \cong \mathcal{O}_{\mathbf{P}^2}(1) \oplus \mathcal{O}_{\mathbf{P}^2}(1)$.
(v) *If* $\mathrm{cl}_2(X, L) = 4$, *then (X, L) is a scroll over a smooth projective curve with* $L^3 = 4$.

PROOF. Assume that $\kappa(K_X + L) \geq 0$. By the proof of Theorem 3.2, $\mathrm{cl}_2(X, L) \geq L^3 + 4 \geq 5$. Hence, this is impossible. This enables us to assume that $\kappa(K_X + L) = -\infty$. By (B) in the proof of Theorem 3.1, we get the assertion. $\square$

In [20, Theorem 2], Lanteri and Turrini have studied (X, L) such that $\dim X = n$, L is very ample and $2L^n \geq \mathrm{cl}_2(X, L)$. In the following result, we treat the case in which $\dim X = 3$, L is *ample* in general, and $2L^3 \geq \mathrm{cl}_2(X, L)$.

THEOREM 3.4. *Let (X, L) be a polarized manifold of dimension 3. If* $2L^3 \geq \mathrm{cl}_2(X, L)$, *then (X, L) is one of the following.*

(i) $(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1))$. *In this case*, $2L^3 = \mathrm{cl}_2(X, L) + 2$.
(ii) $(\mathbf{Q}^3, \mathcal{O}_{\mathbf{Q}^3}(1))$. *In this case*, $2L^3 = \mathrm{cl}_2(X, L) + 2$.
(iii) *A scroll over a smooth projective curve. In this case*, $L^3 = \mathrm{cl}_2(X, L)$.
(iv) *A Del Pezzo manifold with* $6 \leq L^3 \leq 8$. *In this case*, $\mathrm{cl}_2(X, L) = 12$.
(v) *A classical scroll over a smooth projective surface S. Then, there exists an ample vector bundle of rank two on S such that $X = \mathbf{P}_S(\mathcal{E})$ and $L = H(\mathcal{E})$, where $H(\mathcal{E})$ denotes the tautological line bundle. $(S, \mathcal{E})$ is one of the following.*

(v.1) $S \cong \mathbf{P}^2$ *and* $\mathcal{E} \cong \mathcal{O}_{\mathbf{P}^2}(1) \oplus \mathcal{O}_{\mathbf{P}^2}(1)$. *In this case*, $2L^3 = \mathrm{cl}_2(X, L) + 3 = 6$.
(v.2) $S \cong \mathbf{P}^2$ *and* $\mathcal{E} \cong \mathcal{O}_{\mathbf{P}^2}(1) \oplus \mathcal{O}_{\mathbf{P}^2}(2)$. *In this case*, $2L^3 = \mathrm{cl}_2(X, L) + 2 = 14$.
(v.3) $S \cong \mathbf{P}^2$ *and* $\mathcal{E} \cong T_{\mathbf{P}^2}$. *In this case*, $2L^3 = \mathrm{cl}_2(X, L) = 12$.
(v.4) $S \cong \mathbf{P}^1 \times \mathbf{P}^1$ *and* $\mathcal{E} \cong (p_1^*\mathcal{O}_{\mathbf{P}^1}(1) \otimes p_2^*\mathcal{O}_{\mathbf{P}^1}(1))^{\oplus 2}$, *where* $p_i : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$ *is the ith projection for* $i = 1, 2$. *In this case*, $2L^3 = \mathrm{cl}_2(X, L) = 12$.

PROOF. (I) The case in which $\kappa(K_X + L) \geq 0$. By (2) in the proof of Theorem 3.1, $\mathrm{cl}_2(X, L) \geq (K_X + L)L^2 + 2L^3 \geq 2L^3$. If $\mathrm{cl}_2(X, L) = 2L^3$, then $(K_X + L)L^2 = 0$. Thus,

$\kappa(K_X + L) = 0$ and $K_X + L = \mathcal{O}_X$ by [2, Lemma 3.3.2]. In particular, $h^i(\mathcal{O}_X) = 0$ for every $i > 0$, and $\chi(\mathcal{O}_X) = 1$. By the Hirzebruch-Riemann-Roch theorem,

$$\begin{aligned} h^0(K_X + L) &= -\chi(-L) \\ &= \frac{1}{6}L^3 + \frac{1}{4}K_X L^2 + \frac{1}{12}(K_X^2 + c_2(X))L - \chi(\mathcal{O}_X) \\ &= \frac{1}{12}c_2(X)L - 1\,. \end{aligned}$$

Since $h^0(K_X + L) = 1$, $c_2(X)L = 24$. Hence,

$$\begin{aligned} \mathrm{cl}_2(X, L) &= c_2(X)L + 3K_X L^2 + 6L^3 \\ &= 3L^3 + 24\,. \end{aligned}$$

However, this is impossible because we have assumed that $\mathrm{cl}_2(X, L) = 2L^3$. Therefore, $\mathrm{cl}_2(X, L) \geq 2L^3 + 1$ if $\kappa(K_X + L) \geq 0$.

(II) The case in which $\kappa(K_X + L) = -\infty$. Then, (X, L) is one of the types in Remark 2.5.

(a) If $(X, L) \cong (\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1))$, then $\mathrm{cl}_2(X, L) = 2L^3 - 2 < 2L^3$ by Remark 2.5.

(b) If $(X, L) \cong (\mathbf{Q}^3, \mathcal{O}_{\mathbf{Q}^3}(1))$, then $\mathrm{cl}_2(X, L) = 2L^3 - 2 < 2L^3$ by Remark 2.5.

(c) If (X, L) is a scroll over a smooth curve, then $\mathrm{cl}_2(X, L) = 2L^3 - L^3 < 2L^3$ by Remark 2.5.

(d) Assume that (X, L) is a Del Pezzo manifold. If $2L^3 \geq \mathrm{cl}_2(X, L)$, then $6 \leq L^3 \leq 8$ by Remark 2.5.

(e) If $(M, A) = (\mathbf{Q}^3, \mathcal{O}_{\mathbf{Q}^3}(2))$, then $\mathrm{cl}_2(X, L) \geq 2L^3 + 8$ by Remark 2.5.

(f) If $(M, A) = (\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(3))$, then $\mathrm{cl}_2(X, L) \geq 2L^3 + 18$ by Remark 2.5

(g) Assume that (M, A) is the type (VII.3). We use Notation 2.1. By Remark 2.5, $\mathrm{cl}_2(X, L) - 2L^3 \geq 36e + 47b - 16e - 24b = 20e + 23b$. By (6) and (7), $\mathrm{cl}_2(X, L) - 2L^3 \geq 20e + 23b = 28(g(C) - 1) + 17m$. By Claim 3.2, $\mathrm{cl}_2(X, L) - 2L^3 \geq 6$.

(h) Let (X, L) be a hyperquadric fibration over a smooth projective curve C. Assume that $2L^3 \geq \mathrm{cl}_2(X, L)$. Then,

$$\begin{aligned} 0 \geq \mathrm{cl}_2(X, L) - 2L^3 &= 8e + 8b + 4(g(C) - 1) - 2(2e + b) \\ &= 4e + 6b + 4(g(C) - 1). \end{aligned} \tag{8}$$

Note that the following hold (see [3, (3.3) and (3.4)]).

$$2e + 4b \geq 0, \tag{9}$$

$$2e + b > 0. \tag{10}$$

(a) If $b \geq 0$, then by (10), we have $4e + 6b > 0$. Hence, $g(C) = 0$ and $0 \geq 4e + 6b - 4$. Then, $2 \geq 2e + 3b = 2e + b + 2b > 2b$. Therefore, $b = 0$ from the assumption. Hence, $2 \geq 2e + 3b = 2e$, that is, $e \leq 1$. By (10) and $b = 0$, $e > 0$. Therefore, $e = 1$. Thus, $(g(C), b, e) = (0, 0, 1)$. However, $g(X, L) = b + e + 2g(C) - 1 = 0 + 1 - 1 = 0$, which is

impossible.
(b) If $b < 0$, then by (9), $4e + 6b \geq -2b \geq 2$. Hence, $0 \geq 4e + 6b + 4(g(C) - 1) \geq 2 + 4(g(C) - 1)$ and $g(C) = 0$. Since $4e + 6b \leq 4$ by (8), we see $-2b \leq 4$, that is, $b \geq -2$. Therefore, $b = -1$ or -2.
(b.1) Assume that $b = -2$. Then, $4e + 6b \leq 4$ implies $e \leq 4$. On the other hand, $e \geq 4$ by (9). Hence, $e = 4$. Therefore, $(g(C), b, e) = (0, -2, 4)$. However, then $g(X, L) = b + e + 2g(C) - 1 = 1$, which is impossible because (X, L) is a hyperquadric fibration over C.
(b.2) Assume that $b = -1$. Then, $4e + 6b \leq 4$ implies $e \leq 2$. On the other hand, $e \geq 2$ by (9). Hence, $e = 2$. Therefore, $(g(C), b, e) = (0, -1, 2)$. However, $g(X, L) = b + e + 2g(C) - 1 = 0$, which is impossible because (X, L) is a hyperquadric fibration over C.
(i) Let (X, L) be a classical scroll over a smooth projective surface S. Then, there exists an ample vector bundle of rank two on S such that $X = \mathbf{P}_S(\mathcal{E})$ and $L = H(\mathcal{E})$, where $H(\mathcal{E})$ denotes the tautological line bundle. Then, $\mathrm{cl}_2(X, L) = c_2(S) + 3c_1(\mathcal{E})^2 + 2K_S c_1(\mathcal{E})$ and $L^3 = c_1(\mathcal{E})^2 - c_2(\mathcal{E})$. Assume that $2L^3 \geq \mathrm{cl}_2(X, L)$. Then,

$$0 \geq c_2(S) + 2c_2(\mathcal{E}) - c_1(\mathcal{E})^2 + 4(g(S, c_1(\mathcal{E})) - 1). \tag{11}$$

(a) Assume that $\kappa(S) \geq 0$. Then,

$$c_2(S) = 12\chi(\mathcal{O}_S) - K_S^2 \geq 0\,, \tag{12}$$

$$4(g(S, c_1(\mathcal{E})) - 1) - c_1(\mathcal{E})^2 = 2K_S c_1(\mathcal{E}) + c_1(\mathcal{E})^2 > 0\,, \tag{13}$$

$$c_2(\mathcal{E}) > 0\,. \tag{14}$$

However, by (11), (12), (13), and (14), this is impossible.
(b) Assume that $\kappa(S) = -\infty$.
(b.1) If $S \cong \mathbf{P}^2$, then $c_2(S) = 3$. We put $c_1(\mathcal{E}) = \mathcal{O}_{\mathbf{P}^2}(a)$. By (11),

$$\begin{aligned} 0 &\geq c_2(S) + 2c_2(\mathcal{E}) - c_1(\mathcal{E})^2 + 4(g(S, c_1(\mathcal{E})) - 1) \\ &= 3 + 2c_2(\mathcal{E}) - 6a + a^2\,. \end{aligned}$$

If $c_2(\mathcal{E}) \geq 4$, then

$$0 \geq 3 + 8 - 6a + a^2 = a^2 - 6a + 11 = (a - 3)^2 + 2 > 0\,,$$

which is impossible. Hence, $c_2(\mathcal{E}) \leq 3$. From the result of Ishihara [15, Corollary (4.7)], $\mathcal{E}$ is one of the following.

(i) $\mathcal{E} \cong \mathcal{O}_{\mathbf{P}^2}(1) \oplus \mathcal{O}_{\mathbf{P}^2}(1)$.
(ii) $\mathcal{E} \cong \mathcal{O}_{\mathbf{P}^2}(1) \oplus \mathcal{O}_{\mathbf{P}^2}(2)$.
(iii) $\mathcal{E} \cong T_{\mathbf{P}^2}$.
(iv) $\mathcal{E} \cong \mathcal{O}_{\mathbf{P}^2}(1) \oplus \mathcal{O}_{\mathbf{P}^2}(3)$.

If $\mathcal{E}$ is case (i) (resp., (ii), (iii), (iv)), then $c_2(\mathcal{E}) = 1$ (resp., 2, 3, 3) and $a = 2$ (resp., 3, 3, 4). Hence, $\mathrm{cl}_2(X, L) - 2L^3 = -3$ (resp. -2, 0, 1). Therefore, if $S \cong \mathbf{P}^2$ and $2L^3 \geq \mathrm{cl}_2(X, L)$,

then $\mathcal{E}$ is isomorphic to either $\mathcal{O}_{\mathbf{P}^2}(1) \oplus \mathcal{O}_{\mathbf{P}^2}(1)$, $\mathcal{O}_{\mathbf{P}^2}(1) \oplus \mathcal{O}_{\mathbf{P}^2}(2)$, or $T_{\mathbf{P}^2}$.
(b.2) Assume that $S \not\cong \mathbf{P}^2$. Then, there exists a $\mathbf{P}^1$-bundle S' over a smooth curve and a birational morphism $\pi : S \to S'$. Let u be the number of blowups of π. Since $K_S + c_1(\mathcal{E})$ is nef by [25],

$$\begin{aligned}
0 &\leq (K_S + c_1(\mathcal{E}))^2 \\
&= K_S^2 + 2K_S c_1(\mathcal{E}) + c_1(\mathcal{E})^2 \\
&= 8(1 - q(S)) - u + 4(g(S, c_1(\mathcal{E})) - 1) - c_1(\mathcal{E})^2 \\
&= 4(g(S, c_1(\mathcal{E})) - 2q(S) + 1) - u - c_1(\mathcal{E})^2 .
\end{aligned}$$

On the other hand, since $c_2(S) = u + 4(1 - q(S))$,

$$\begin{aligned}
\mathrm{cl}_2(X, L) - 2L^3 &= c_2(S) + 2c_2(\mathcal{E}) - c_1(\mathcal{E})^2 + 4(g(S, c_1(\mathcal{E})) - 1) \\
&= u + 2c_2(\mathcal{E}) - c_1(\mathcal{E})^2 + 4(g(S, c_1(\mathcal{E})) - 2q(S) + 1) + 4q(S) - 4 \\
&\geq 2u + 2c_2(\mathcal{E}) + 4q(S) - 4 .
\end{aligned}$$

If $2L^3 \geq \mathrm{cl}_2(X, L)$, then $(q(S), c_2(\mathcal{E}), u) = (0, 1, 0), (0, 1, 1), (0, 2, 0)$.
(b.2.1) If $(q(S), c_2(\mathcal{E}), u) = (0, 1, 0)$, then S is a Hirzebruch surface, which by [15, Corollary (2.11)] is impossible.
(b.2.2) If $(q(S), c_2(\mathcal{E}), u) = (0, 2, 0)$, then from [15, Corollary (2.11)], $S \cong \mathbf{P}^1 \times \mathbf{P}^1$ and $\mathcal{E} \cong (p_1^*\mathcal{O}_{\mathbf{P}^1}(1) \otimes p_2^*\mathcal{O}_{\mathbf{P}^1}(1))^{\oplus 2}$, where $p_i : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$ is the ith projection for $i = 1, 2$.
(b.2.3) If $(q(S), c_2(\mathcal{E}), u) = (0, 1, 1)$, then $(K_S + c_1(\mathcal{E}))^2 = 0$ and $K_S + c_1(\mathcal{E})$ is not ample. By a list of [6, Main Theorem], this case cannot occur because S is a blowup of a $\mathbf{P}^1$-bundle over $\mathbf{P}^1$. □

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