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A Lower Bound of the Dimension of the Vector Space Spanned by the Special Values of Certain Functions

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Abstract. Let K be a number field. Fix a finite set of analytic functions $\mathbf{f}_\infty := \{f_{1,\infty}(x), \dots, f_{s,\infty}(x)\}$ defined on $\{x \in \mathbb{C} \mid |x| > 1\}$ (resp. $\mathbb{C}_p$-valued functions $\mathbf{f}_p := \{f_{1,p}(x), \dots, f_{s,p}(x)\}$ defined on $\{x \in \mathbb{C}_p \mid |x|_p > 1\}$). For $\beta \in K$, we denote the K-vector space spanned by $f_{1,\infty}(\beta), \dots, f_{s,\infty}(\beta)$ by $V_K(\mathbf{f}_\infty, \beta)$ (resp. $f_{1,p}(\beta), \dots, f_{s,p}(\beta)$ by $V_K(\mathbf{f}_p, \beta)$). In this article, under some assumptions for $\mathbf{f}_\infty$ (resp. $\mathbf{f}_p$), we give an estimation of a lower bound of the dimension of $V_K(\mathbf{f}_\infty, \beta)$ (resp. $V_K(\mathbf{f}_p, \beta)$) (see Theorem 2.4 for Archimedean case and Theorem 8.6 for p-adic case). Applying our estimation, we give a lower bound of the dimension of the K-vector space spanned by the special values of the Lerch functions over a number field in $\mathbb{C}$ (see Theorem 1.1 and Remark 1.2) and the p-adic analog of the above result (see Theorem 1.3 and Remark 1.4). Furthermore, we also give a lower bound of the K-vector space spanned by the special values of certain p-adic functions related with p-adic Hurwitz zeta function (see Theorem 1.5).

1. Introduction

Fix a prime number p. Let $\overline{\mathbb{Q}}$ be an algebraic closure of the rational number field. We denote by $\mathbb{C}_p$ the completion of an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. We denote the normalized valuation of $\mathbb{C}_p$ by $|\cdot|_p$ with $|p|_p = p^{-1}$. We fix embeddings $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ and sometimes regard $\overline{\mathbb{Q}}$ as a subfield of $\mathbb{C}$ or $\mathbb{C}_p$ by the fixed embeddings.

Let us fix a finite set of analytic functions $\mathbf{f}_\infty := \{f_{1,\infty}(x), \dots, f_{s,\infty}(x)\}$ defined on $D_\infty := \{x \in \mathbb{C} \mid |x| > 1\}$. For an algebraic number field K and an element $\beta \in D_\infty(K) := \{x \in K \mid |x| > 1\}$, we denote the K-vector space spanned by $f_{1,\infty}(\beta), \dots, f_{s,\infty}(\beta)$ by $V_K(\mathbf{f}_\infty, \beta)$. We study the following type of estimation of a lower bound of the dimension of $V_K(\mathbf{f}_\infty, \beta)$.

(Type A)$_\infty$

There exists a subset W_∞ of $D_\infty(\overline{\mathbb{Q}}) \times \mathcal{A}_{\mathbb{Q}}$ and a function $F^{(\infty)} : W_\infty \longrightarrow \mathbb{R}_{\geq 0}$ satisfying the following inequality:

$$\dim_K V_K(\mathbf{f}_\infty, \beta) \geq F^{(\infty)}(\beta, K) \text{ for all } (\beta, K) \in W_\infty,$$

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where $D_\infty(\overline{\mathbb{Q}}) := \{x \in \overline{\mathbb{Q}} \mid |x| > 1\}$ and $\mathcal{A}_\mathbb{Q}$ is the set of all algebraic number fields.

We also study the following p-adic analog of the estimation of (**Type A**)$_\infty$. Let us fix a finite set of $\mathbb{C}_p$-valued functions $\mathbf{f}_p := \{f_{1,p}(x), \ldots, f_{s,p}(x)\}$ defined on $D_p := \{x \in \mathbb{C}_p \mid |x|_p > 1\}$. For an algebraic number field K and an element $\beta \in D_p(K) := \{x \in K \mid |x|_p > 1\}$, we denote by $V_K(\mathbf{f}_p, \beta)$ the K-vector space spanned by $f_{1,p}(\beta), \ldots, f_{s,p}(\beta)$.

(**Type A**)$_p$

There exists a subset W_p of $D_p(\overline{\mathbb{Q}}) \times \mathcal{A}_\mathbb{Q}$ and a function $F^{(p)} : W_p \longrightarrow \mathbb{R}_{\geq 0}$ satisfying the following inequality:

$$\dim_K V_K(\mathbf{f}_p, \beta) \geq F^{(p)}(\beta, K) \text{ for all } (\beta, K) \in W_p\,.$$

where $D_p(\overline{\mathbb{Q}}) := \{x \in \overline{\mathbb{Q}} \mid |x|_p > 1\}$.

In this article, for a set $\mathbf{f}_\infty := \{f_{1,\infty}(x), \ldots, f_{s,\infty}(x)\}$ (resp. $\mathbf{f}_p := \{f_{1,p}(x), \ldots, f_{s,p}(x)\}$) with some assumptions, we give a statement of type (**Type A**)$_\infty$ (cf. Theorem 2.4) (resp. (**Type A**)$_p$ (cf. Theorem 8.6)). Using Theorem 2.4 (resp. Theorem 8.6), we give a lower bound of the dimension of the vector space spanned by the special values of the Lerch function (resp. the p-adic Lerch function and the function relating with the p-adic Hurwitz zeta function) for Archimedean case (resp. p-adic case). Firstly, we shall state our result on estimation of a lower bound of the dimension of the vector space spanned by the special values of the Lerch function for the Archimedean case. The definition of the Lerch function is as follows:

$$\Phi : \mathbb{N} \times \left(\mathbb{C} \setminus \mathbb{Z}_{\leq 0}\right) \times D_\infty \longrightarrow \mathbb{C} \ \ (s, x_1, x) \mapsto \Phi(s, x_1, x) := \sum_{m=0}^{\infty} \frac{x^{-m-1}}{(m + x_1)^s}\,. \tag{1}$$

We give a (**Type A**)$_\infty$ estimation of a lower bound of the dimension of the vector space spanned by the special values of the Lerch function. We define the denominator function as follows:

$$\mathrm{den} : \overline{\mathbb{Q}} \longrightarrow \mathbb{N}\ \gamma \mapsto \min\{n \in \mathbb{N} \mid n\gamma \text{ is an algebraic integer}\}\,. \tag{2}$$

We denote the completion with respect to the usual absolute value of K by K_∞. Our result is as follows:

THEOREM 1.1. *Let r be a natural number, $s_1, \ldots, s_r$ natural numbers and $a_1, \ldots, a_r$ rational numbers satisfying $0 < a_1 < \cdots < a_r \leq 1$. We put*

$$A := \mathrm{l.c.m.}\{\mathrm{den}(a_i)\}_{1 \leq i \leq r}\,,$$

$$M := \mathrm{l.c.m.}\{\mathrm{den}(a_{i'} - a_i)\}_{1 \leq i, i' \leq r,\ i \neq i'}\,,$$

$$S := \max_{1 \leq i \leq r} s_i\,,$$

$$s := \sum_{i=1}^{r} s_i\,.$$

We denote the set $\{(\beta, K) \in D_\infty(\overline{\mathbb{Q}}) \times \mathcal{A}_{\mathbb{Q}} \mid \beta \in K\}$ *by* W_∞ *and define the following four functions*:

$$f^{(\infty)} : D_\infty(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\geq 0} \text{ by } \beta \mapsto S \log A + S \sum_{\substack{q:\text{prime} \\ q|A}} \frac{\log q}{q-1} + S(M+A) + \log \operatorname{den}(\beta)\,,$$

$$g^{(\infty)} : D_\infty(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\geq 0} \text{ by } \beta \mapsto \log \max\{1, |\beta|\} + (s \log s + (2s+1)\log 2)\,,$$

$$h^{(\infty)} : D_\infty(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\geq 0} \text{ by } \beta \mapsto s \log |\beta|\,,$$

$$F^{(\infty)} : W_\infty \longrightarrow \mathbb{R}_{\geq 0} \text{ by } (\beta, K) \mapsto \frac{[K_\infty : \mathbb{R}](g^{(\infty)}(\beta) + h^{(\infty)}(\beta))}{[K : \mathbb{Q}(\beta)] \sum_{\tau \in I_{\mathbb{Q}(\beta)}} (f^{(\infty)}(\tau\beta) + g^{(\infty)}(\tau\beta))}\,.$$

Then we obtain the following inequality:

$$\dim_K \left(K + \sum_{v_1=1}^{s_1} K \Phi(v_1, a_1, \beta) + \cdots + \sum_{v_r=1}^{s_r} K \Phi(v_r, a_r, \beta) \right) \geq F^{(\infty)}(\beta, K)$$

for all $(\beta, K) \in W_\infty$.

REMARK 1.2. In [11, Theorem 0.2], the second author gave a criterion of linear independence of special values of the Lerch function over the rational number field. In [8, Theorem 2.1], N. Hirata, M. Ito and Y. Washio gave a criterion of linear independence of special values of the polylogarithm functions over algebraic number fields. Theorem 1.1 is a generalization of the both results [11, Theorem 0.2] and [8, Theorem 2.1].

Secondly, we describe the p-adic case. The definition of the p-adic Lerch function is as follows:

$$\hat{\Phi}_p : \mathbb{N} \times \left(\mathbb{C}_p \setminus \mathbb{Z}_{\leq 0}\right) \times D_p \longrightarrow \mathbb{C}_p \ \ (s, x_1, x) \mapsto \hat{\Phi}_p(s, x_1, x) := \sum_{m=0}^{\infty} \frac{x^{-m-1}}{(m + x_1)^s}\,.$$

Note that the s-th p-adic polylogarithm function is defined by

$$\mathrm{Li}_p(s, x) := \hat{\Phi}_p(s, 1, x) = \sum_{m=0}^{\infty} \frac{x^{-m-1}}{(m+1)^s} \ \text{ for } s \in \mathbb{N} \text{ and } x \in D_p\,.$$

For an algebraic number field K, we denote the completion with respect to the p-adic absolute value of K in $\mathbb{C}_p$ by K_p. A p-adic analog of Theorem 1.1 is as follows:

THEOREM 1.3. *Let* r *be a natural number*, $s_1, \ldots, s_r$ *natural numbers and* $a_1, \ldots, a_r$ *rational numbers satisfying* $0 < a_1 < \cdots < a_r \leq 1$. *We denote the set* $\{(\beta, K) \in D_p(\overline{\mathbb{Q}}) \times \mathcal{A}_{\mathbb{Q}} \mid \beta \in K\}$ *by* W_p *and use the same notation as in Theorem* 1.1. *We define the following* *four functions*:

$$f^{(p)} : D_p(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\geq 0} \ \ by\ \beta \mapsto s \log A + s \sum_{\substack{q:\text{prime} \\ q|A}} \frac{\log q}{q-1} + S(M+A) + \log \operatorname{den}(\beta)\,,$$

$$g^{(p)} : D_p(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\geq 0} \ \ by\ \beta \mapsto \log \max\{1, |\beta|\} + (s \log s + (2s+1)\log 2)\,,$$

$$h^{(p)} : D_p(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\geq 0} \ \ by\ \beta \mapsto s \log |\beta|_p\,,$$

$$F^{(p)} : W_p \longrightarrow \mathbb{R}_{\geq 0} \ \ by\ (\beta, K) \mapsto \frac{[K_p : \mathbb{Q}_p](h^{(p)}(\beta) + s \log |\beta|_p)}{[K : \mathbb{Q}(\beta)] \sum_{\tau \in I_{\mathbb{Q}(\beta)}} (f^{(p)}(\tau\beta) + g^{(p)}(\tau\beta))}\,.$$

Then we obtain the following inequality:

$$\dim_K \left(K + \sum_{v_1=1}^{s_1} K \hat{\Phi}_p(v_1, a_1, \beta) + \cdots + \sum_{v_r=1}^{s_r} K \hat{\Phi}_p(v_r, a_r, \beta) \right) \geq F^{(p)}(\beta, K)$$

for all $(\beta, K) \in W_p$.

REMARK 1.4. P. Bel gave a **(Type A)**$_p$ estimation of lower bound of the dimension of the K-vector space spanned by the special values of p-adic polylogarithm functions $\{\mathrm{Li}_p(1, x), \ldots, \mathrm{Li}_p(s, x)\}$ in [3, Theorem 3]. Theorem 1.3 is a generalization of [3, Theorem 3].

We also obtain a lower bound of the dimension of the vector space spanned by the special values of the following p-adic function. To state our result, we define the function $\hat{\zeta_p}$ by

$$\hat{\zeta}_p : \mathbb{N} \times \mathbb{Z}_p \times (\mathbb{C}_p \setminus D_p(1, 1^-) \cup \{1\}) \times D_p \longrightarrow \mathbb{C}_p\,,$$

$$(s, x_1, x_2, x) \mapsto \hat{\zeta_p}(s, x_1, x_2, x) := \frac{\epsilon(x_2)}{s-1} \frac{1}{x^{s-1}} + \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{(m+1)!} B_{m+1}(x_1, x_2)(s)_m \frac{1}{x^{s+m}}\,,$$

where $D_p(1, 1^-) := \{x \in \mathbb{C}_p \mid |x-1|_p < 1\}$,

$$\epsilon(x_2) = \begin{cases} 0 & \text{if } x_2 \neq 1 \\ 1 & \text{if } x_2 = 1\,, \end{cases} \quad (s)_m = \begin{cases} s(s+1)\cdots(s+m-1) & \text{if } m \geq 1 \\ 1 & \text{if } m = 0\,, \end{cases}$$

and $B_k(x_1, x_2)$ are defined by the following generating function:

$$\frac{t e^{x_1 t}}{x_2 e^t - 1} = \sum_{k=0}^{\infty} B_k(x_1, x_2) \frac{t^k}{k!}$$

We give the following estimation of a lower bound of the dimension of the vector space spanned by the special values of $\hat{\zeta_p}(s, x_1, x_2, z)$:

THEOREM 1.5. *We use the notations as before. Let r be a natural number, $s_1, \dots, s_r$ natural numbers, $a_1, \dots, a_r \in \mathbb{Q} \cap \mathbb{Z}_p$ satisfying $0 < a_1 < \cdots < a_r \leq 1$ and $\alpha \in \{\alpha \in \overline{\mathbb{Q}} \mid |\alpha| = 1\}$ satisfying $|\alpha - 1|_p \geq 1$. Let W_p be the set $D_p(\mathbb{Q}) \times \mathcal{A}_{\mathbb{Q}}$. We put the following numbers*:

$$B(b) := \text{l.c.m.}\{\text{den}(b + a_i)\}_{1 \leq i \leq r} \text{ for } b \in D_p(\mathbb{Q}) ,$$

$$M := \text{l.c.m.}\{\text{den}(a_{i'} - a_i)\}_{1 \leq i, i' \leq r,\ i \neq i'} ,$$

$$S := \max_{1 \leq i \leq r} s_i ,$$

$$T := \min_{1 \leq i \leq r} s_i ,$$

$$s := \sum_{i=1}^{r} s_i .$$

and define the following four functions:

$$f^{(p)} : D_p(\mathbb{Q}) \longrightarrow \mathbb{R}_{\geq 0} \text{ by } b \mapsto S + M(s + r - T - 1) + \sum_{\substack{q:\text{prime} \\ q \mid B(b)}} \frac{\log q}{q - 1} + \log \text{den}(\alpha) ,$$

$$g^{(p)} : D_p(\mathbb{Q}) \longrightarrow \mathbb{R}_{\geq 0} \text{ by } b \mapsto \log \max\{1, |\alpha|\} + s \log 2 ,$$

$$h^{(p)} : D_p(\mathbb{Q}) \longrightarrow \mathbb{R}_{\geq 0} \text{ by } b \mapsto \sum_{\substack{q:\text{prime} \\ q \mid B(b)}} \frac{\log q}{q - 1} - \frac{\log p}{p - 1} + \log \text{den}(\alpha) - \log \max\{1, |\alpha|_p\} ,$$

$$F^{(p)} : W_p \longrightarrow \mathbb{R}_{\geq 0} \text{ by } (b, K) \mapsto \frac{[K_p : \mathbb{Q}_p](h^{(p)}(b) + T \log |b|_p)}{[K : \mathbb{Q}](f^{(p)}(b) + g^{(p)}(b))} .$$

Then we obtain the following inequality:

$$\dim_K \left(K + \sum_{v_1 = 1}^{s_1 + 1} K \hat{\zeta}_p(v_1, a_1, \alpha, b) + \cdots + \sum_{v_r = 1}^{s_r + 1} K \hat{\zeta}_p(v_r, a_r, \alpha, b) \right) \geq F^{(p)}(b, K)$$

for all $(b, K) \in W_p$.

REMARK 1.6. When $r = 1$, P. Bel in [2, Theorem 3.1] also gave a **(Type A)**$_p$ estimation of the dimension of the vector space spanned by the special values of $\{\hat{\zeta}_p(2, a, 1, b), \dots, \hat{\zeta}_p(s + 1, a, 1, b)\}$ for $s \in \mathbb{N}$ and $a \in \mathbb{Q} \cap \mathbb{Z}_p$:

$$\dim_K \left(K + \sum_{v=2}^{s+1} K \hat{\zeta}(v, a, 1, b) \right) \geq F_1^{(p)}(b, K) \text{ for all } (b, K) \in W_p$$

where $F_1^{(p)}(b, K)$ is defined by the same way in Theorem 1.5 for $\alpha = 1$. In Theorem 1.5, r is general but we exclude the case $\alpha = 1$. Thus, Theorem 1.5 is not regarded as a complete generalization of [2, Theorem 3.1]. (see Remark 10.10 for the reason why we exclude $\alpha = 1$ in Theorem 1.5.)

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NOTATIONS. Let $\mathbb{N} := \{1, 2, 3, \dots, \}$ be the set of natural numbers. We denote by $|\cdot|$ the Archimedean absolute value on $\mathbb{C}$. We fix a prime number p. We also denote the ring of integers of $\mathbb{C}_p$ by $\mathcal{O}_{\mathbb{C}_p}$. We denote the set of algebraic integers in $\overline{\mathbb{Q}}$ by $\overline{\mathbb{Z}}$. We regard all the algebraic number fields as subfields of $\overline{\mathbb{Q}}$. We fix embeddings $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ and sometimes regard $\overline{\mathbb{Q}}$ as a subfield of $\mathbb{C}$ (resp. $\mathbb{C}_p$) by the fixed embedding ι_∞ (resp. ι_p).

For an algebraic number field K, we denote the ring of integers of K by $\mathcal{O}_K$ and the set of all embedding K into $\mathbb{C}$ (resp. $\mathbb{C}_p$) by I_K (resp. $I_K^{(p)}$). For an algebraic number field K, we denote the set of algebraic number fields containing K by $\mathcal{A}_K$. We denote the completion with respect to the usual absolute value of K by K_∞ and the completion with respect to the p-adic absolute value of K in $\mathbb{C}_p$ by K_p. For an algebraic extension K of $\mathbb{Q}$, we denote $\{x \in K \mid |x| > 1\}$ by $D_\infty(K)$ and $\{x \in \mathbb{C} \mid |x| > 1\}$ by D_∞. We also denote $\{x \in K \mid |x|_p > 1\}$ by $D_p(K)$ and $\{x \in \mathbb{C}_p \mid |x|_p > 1\}$ by D_p.

We define the denominator function, $\mathrm{den} : \overline{\mathbb{Q}} \longrightarrow \mathbb{N}$ the same as (2). For a natural number n, we denote the least common multiple of $1, 2, \dots, n$ by d_n. We put $\mu_n(b) := \prod_{\substack{q:\text{prime} \\ q|b}} q^{[n/(q-1)]}$ for natural numbers b and n. In this article, we use X as a variable of infinite Laurent series and x as a parameter of function on some $U \subset \mathbb{C}$ or $U \subset \mathbb{C}_p$.

Part I. Archimedean case

2. A criterion of linear independence of special values of Archimedean functions

Let $\mathbf{f}_\infty := \{f_{1,\infty}(x), \dots, f_{s,\infty}(x)\}$ be a finite set of analytic functions defined on D_∞. For an algebraic number field K and an element $\beta \in D_\infty(K)$, we denote the K-vector space spanned by $f_{1,\infty}(\beta), \dots, f_{s,\infty}(\beta)$ by $V_K(\mathbf{f}_\infty, \beta)$. In this section, for $\mathbf{f}_\infty$ with some assumptions (see Assumptions 2.2 and 2.3), we give an estimation of a lower bound of $\dim_K V_K(\mathbf{f}_\infty, \beta)$. Firstly, we recall a result of Marcovecchio (cf. [13]) on a lower bound of the dimension of the vector space spanned by certain complex numbers over a number field, which is based on a result in Siegel's article [18]. Let K be an algebraic number field. We put

$$\delta_K := [K : \mathbb{Q}]/[K_\infty : \mathbb{R}]. \tag{3}$$

We denote the complex conjugation by F. For a natural number s and a vector $\mathbf{x} = (x_1, \dots, x_s) \in K^s$, we denote

$$h_0(\mathbf{x}) = \frac{1}{[K:\mathbb{Q}]} \sum_{\substack{\tau \in I_K \\ \tau \neq id_K, F \circ id_K}} \log |\tau \mathbf{x}| ,$$

and $|\tau \mathbf{x}| = \max_{1 \le v \le s} |\tau x_v|$. For an $(s+1)$-variable linear form $L(X_0, \dots, X_s) = \sum_{v=0}^{s} A_v X_v \in K[X_0, \dots, X_s]$, we use the following notations:

$$||L|| := \max_{0 \le v \le s} |A_v| ,$$

$$L(\boldsymbol{\theta}) := L(\theta_0, \theta_1, \dots, \theta_s) \text{ for } \boldsymbol{\theta} = (\theta_0, \theta_1, \dots, \theta_s) \in \mathbb{C}^{s+1} ,$$

$$\tau L := \sum_{v=0}^{s} \tau(A_v) X_v \text{ for } \tau \in I_K ,$$

$$\mathbf{L} := (A_0, \dots, A_s) .$$

From now on throughout the article, we fix a natural number $s \in \mathbb{N}$. The following lemma was proved by Marcovecchio in [13].

LEMMA 2.1 ([13, Proposition 4.1]). *Let K be an algebraic number field. Let $\boldsymbol{\theta} := (1, \theta_1, \dots, \theta_s) \in \mathbb{C}^{s+1}$. Suppose that, for all $n \in \mathbb{N}$ there exist $(s+1)$ linear forms*

$$L_w^{(n)}(X_0, \dots, X_s) = \sum_{v=0}^{s} A_{v,w}^{(n)} X_v \quad (0 \le w \le s; n \in \mathbb{N}),$$

with coefficients $\{A_{v,w}^{(n)}\}_{1 \le v,w \le s} \subset \mathcal{O}_K$ satisfying

$$\det((A_{v,w}^{(n)})_{0 \le v,w \le s}) \neq 0 .$$

Assume also that there exist $\rho, c, c' \in \mathbb{R}$ satisfying $c, c', \rho + c' > 0$ and the following relations for all $0 \le w \le s$:

$$\limsup_n \frac{\log ||L_w^{(n)}||}{n} \le c ,$$

$$\limsup_n \frac{h_0(\mathbf{L}_w^{(n)})}{n} \le c' ,$$

$$\limsup_n \frac{\log |L_w^{(n)}(\boldsymbol{\theta})|}{n} \le -\rho .$$

Then we have

$$\dim_K (K + K\theta_1 + \cdots + K\theta_s) \ge \frac{c + \rho}{c + \delta_K c'} .$$

Using Lemma 2.1, we axiomatize the estimation of a lower bound of the vector space spanned by the special values of certain functions. Let L be an algebraic number field, s a natural number and

$$\theta_v : D_\infty \longrightarrow \mathbb{C}$$

analytic functions for all $0 \le v \le s$ with $\theta_0(x) = 1$ for all $x \in D_\infty$. We consider a family of polynomials

$$\{A^{(n)}_{v,w}(x)\}_{0\le v,w\le s, n\in\mathbb{N}} \subset L[x]\,,$$

to approximate $\{\theta_v(x)\}_{0\le v\le s}$. Introduce a family of functions $\{\mathcal{R}^{(n)}_w(x)\}_{n\in\mathbb{N}}$ on D_∞, defined by

$$\mathcal{R}^{(n)}_w(x) = \sum_{v=0}^{s} A^{(n)}_{v,w}(x)\theta_v(x)\,,$$

for $0 \le w \le s$. We put $\Delta_n(x) = \det((A^{(n)}_{v,w}(x))_{0\le v,w\le s}) \in L[x]$. We make the following assumptions on $\{A^{(n)}_{v,w}(x)\}_{0\le v,w\le s,n\in\mathbb{N}} \subset L[x]$:

ASSUMPTION 2.2. Suppose that there exists a non-empty subset $V_\infty \subset D_\infty(\overline{\mathbb{Q}})$ satisfying the following assumptions:

(4) There exists an integer l such that

$\mathcal{R}^{(n)}_w(x) = o(x^{-ns+w+l})$ $(x \to \infty)$ for all $n \in \mathbb{N}$ and $0 \le w \le s$.

(5) We have $\Delta_n(\beta) \neq 0$ for all $\beta \in V_\infty$ and infinitely many $n \in \mathbb{N}$.

(6) There exists a family of functions $\{D_n : V_\infty \longrightarrow \overline{\mathbb{Z}} \setminus \{0\}\}_{n\in\mathbb{N}}$ satisfying

$D_n(\beta) \in L(\beta)$ and $D_n(\beta)A^{(n)}_{v,w}(\beta) \in \mathcal{O}_{L(\beta)}$ for all $\beta \in V_\infty$, $0 \le v, w \le s$ and $n \in \mathbb{N}$.

ASSUMPTION 2.3. We use the notations as above. Suppose that $\tau(\beta) \in V_\infty$ for all $\beta \in V_\infty$, $\tau \in I_{\mathbb{Q}(\beta)}$ and there exist some constants $c_1, c_2, c_3 > 0$ such that the following conditions hold for any sufficiently large n.

(7) There exists a function $f^{(\infty)} : V_\infty \longrightarrow \mathbb{R}_{>0}$ satisfying $|\tau D_n(\beta)| \le n^{c_1+o(1)}e^{nf^{(\infty)}(\tau\beta)}$

for all $\beta \in V_\infty$ and $\tau \in I_{\mathbb{Q}(\beta)}$.

(8) There exists a function $g^{(\infty)} : V_\infty \longrightarrow \mathbb{R}_{>0}$ satisfying $|\tau A^{(n)}_{v,w}(\beta)| \le n^{c_2+o(1)}e^{ng^{(\infty)}(\tau\beta)}$

for all $\beta \in V_\infty$ and $\tau \in I_{\mathbb{Q}(\beta)}$.

(9) There exists a family of functions $h^{(\infty)}$:

$V_\infty \longrightarrow \mathbb{R}_{>0}$ satisfying $|\mathcal{R}^{(n)}_w(\beta)| \le n^{c_3+o(1)}e^{-nh^{(\infty)}(\beta)}$

for all $\beta \in V_\infty$.

Under Assumptions 2.2 and 2.3, we obtain the following **(Type A)**$_\infty$ estimation of a lower bound of the dimension of the vector space spanned by the special values of $\{\theta_v(x)\}_{0\le v\le s}$.

THEOREM 2.4. *Let s be a natural number and*

$$\theta_v : D_\infty \longrightarrow \mathbb{C}$$

be analytic functions for $0 \le v \le s$ with $\theta_0(x) = 1$ for all $x \in D_\infty$. We assume Assumptions 2.2 and 2.3 for $\{\theta_v(x)\}_{0\le v\le s}$. We assume that the functions $f^{(\infty)}$ and $g^{(\infty)}$ in (7) *and* (8) *respectively satisfy the following relations*:

$$f^{(\infty)}(\tau\beta) = f^{(\infty)}(F \circ \tau\beta) \tag{10}$$

$$g^{(\infty)}(\tau\beta) = g^{(\infty)}(F \circ \tau\beta)$$

for all $\beta \in V_\infty$ and $\tau \in I_{L(\beta)}$. We denote the set $\{(\beta, K) \in V_\infty \times \mathcal{A}_L \mid \beta \in K\}$ by W_∞. We define the function

$$F^{(\infty)} : W_\infty \longrightarrow \mathbb{R}_{\ge 0} \ \ (\beta, K) \mapsto \frac{[K_\infty : \mathbb{R}](g^{(\infty)}(\beta) + h^{(\infty)}(\beta))}{[K : \mathbb{Q}(\beta)] \sum_{\tau \in I_{\mathbb{Q}(\beta)}} (f^{(\infty)}(\tau\beta) + g^{(\infty)}(\tau\beta))} .$$

Then we obtain the following estimations of the dimension of the vector space spanned by the special values of $\{\theta_v(x)\}_{0\le v\le s}$:

$$\dim_K (K + K\theta_1(\beta) + \cdots + K\theta_s(\beta)) \ge F^{(\infty)}(\beta, K)$$

for all $(\beta, K) \in W_\infty$.

PROOF. We fix $(\beta, K) \in W_\infty$ and put

$$\boldsymbol{\theta} := (1, \theta_1(\beta), \dots, \theta_s(\beta)) ,$$

$$A^{(n)}_{v,w} := D_n(\beta) A^{(n)}_{v,w}(\beta) ,$$

$$L^{(n)}_w(X_0, \dots, X_s) := \sum_{v=0}^{s} A^{(n)}_{v,w} X_q \ \text{ for } \ 0 \le w \le s, ; n \in \mathbb{N} ,$$

$$\Delta^{(n)} := \det \begin{pmatrix} A^{(n)}_{0,0} & A^{(n)}_{0,1} & \dots & A^{(n)}_{0,s} \\ A^{(n)}_{1,0} & A^{(n)}_{1,1} & \dots & A^{(n)}_{1,s} \\ \vdots & \vdots & \ddots & \vdots \\ A^{(n)}_{s,0} & A^{(n)}_{s,1} & \dots & A^{(n)}_{s,s} \end{pmatrix} .$$

Using the condition (5), we get

$$\Delta^{(n)} \ne 0 . \tag{11}$$

Using the condition (6), we get

$$\{L^{(n)}_w(X_0, \dots, X_s)\}_{0\le w\le s} \subset \mathcal{O}_K[X_0, \dots, X_s] . \tag{12}$$

Using the conditions (7) and (8) in assumption 2.3, we obtain:

$$\limsup_n \frac{\log ||\tau L_w^{(n)}||}{n} \leq f^{(\infty)}(\tau\beta) + g^{(\infty)}(\tau\beta) \tag{13}$$

for any $\tau \in I_K$. Using the inequality (13), we also obtain:

$$\limsup_n \frac{h_0(\mathbf{L}_w^{(n)})}{n} \leq \frac{1}{[K:\mathbb{Q}]} \sum_{\substack{\tau \in I_K \\ \tau \neq id_K, F \circ id_K}} (f^{(\infty)}(\tau\beta) + g^{(\infty)}(\tau\beta)) . \tag{14}$$

Using the conditions (7) and (9), we obtain:

$$\limsup_n \frac{\log |\tau L_w^{(n)}(\boldsymbol{\theta})|}{n} \leq -(h^{(\infty)}(\beta) - f^{(\infty)}(\tau\beta)) \tag{15}$$

for all $\tau \in I_L$. Since (11) and (12) are satisfied, we can use Lemma 2.1 for (13), (14) and (15). Then we have the following inequality:

$$\dim_K (K + K\theta_1(\beta) + \cdots + K\theta_s(\beta)) \geq$$
$$\frac{[K_\infty : \mathbb{R}](g^{(\infty)}(\beta) + h^{(\infty)}(\beta))}{[K_\infty : \mathbb{R}](f^{(\infty)}(\beta) + g^{(\infty)}(\beta)) + \sum_{\tau \in I_K,\ \tau \neq id_K, F\circ id_K} (f^{(\infty)}(\tau\beta) + g^{(\infty)}(\tau\beta))} .$$

Since we have the relation (10), we have the following equality:

$$\frac{[K_\infty : \mathbb{R}](g^{(\infty)}(\beta) + h^{(\infty)}(\beta))}{[K_\infty : \mathbb{R}](f^{(\infty)}(\beta) + g^{(\infty)}(\beta)) + \sum_{\substack{\tau \in I_K \\ \tau \neq id_K, F\circ id_K}} (f^{(\infty)}(\tau\beta) + g^{(\infty)}(\tau\beta))}$$
$$= \frac{[K_\infty : \mathbb{R}](g^{(\infty)}(\beta) + h^{(\infty)}(\beta))}{[K : \mathbb{Q}(\beta)] \sum_{\tau \in I_{\mathbb{Q}(\beta)}} (f^{(\infty)}(\tau\beta) + g^{(\infty)}(\tau\beta))} .$$

This completes the proof of Theorem 2.4. □

Using Theorem 2.4, we obtain the following criterion of linear independence of special values of $\{\theta_v(x)\}_{0 \leq v \leq s}$.

COROLLARY 2.5. *Under the same assumption of Theorem* 2.4, *we obtain the following criterion of linear independence of special values of* $\{\theta_v(x)\}_{1 \leq v \leq s}$.

Suppose $(\beta, K) \in W_\infty$ *satisfies*

$$s[K : \mathbb{Q}(\beta)] \sum_{\tau \in I_{\mathbb{Q}(\beta)}} (f^{(\infty)}(\tau\beta) + g^{(\infty)}(\tau\beta)) < [K_\infty : \mathbb{R}](g^{(\infty)}(\beta) + h^{(\infty)}(\beta)) .$$

Then we obtain:

$$\dim_K (K + K\theta_1(\beta) + \cdots + K\theta_s(\beta)) = s + 1 .$$

3. A Padé approximation of the Lerch function

Let r be a natural number. From here to the last section, we fix the following numbers:

$$s_1, \dots, s_r \text{ : natural numbers},$$
$$s := \sum_{i=1}^{r} s_i ,$$
$$a_1, \dots, a_r \text{ : rational numbers satisfying } 0 < a_1 < \dots < a_r \le 1 .$$

In this section, we give a Padé approximation of the Lerch function which is a generalization of that in [11]. Let $\Phi(v, x_1, x)$ be the Lerch function defined in (1). For a positive integer n and an r tuple of non-negative integers $\mathbf{w} = (w_1, \dots, w_r)$ with $0 \le w_i \le s_i$ for all $1 \le i \le r$, we put

$$Q^{(n)}_{\mathbf{w}}(u) := \frac{u(u-1)\cdots(u-\sigma_{n,w}+2)}{\prod_{i=1}^{r}\left[(u+a_i)_n^{s_i}(u+n+a_i)^{w_i}\right]},$$
$$\mathcal{R}^{(n)}_{\mathbf{w}}(x) := \sum_{m=0}^{\infty} Q^{(n)}_{\mathbf{w}}(m)x^{-m-1},$$

where $w = \sum_{i=1}^{r} w_i$ and $\sigma_{n,w} = ns + w$. We define a family of rational numbers $\{b^{(n)}_{i,j,v_i,\mathbf{w}}\}_{1\le i\le r, 1\le v_i\le s_i, 0\le j\le n}$ by

$$Q^{(n)}_{\mathbf{w}}(u) = \sum_{i=1}^{r}\left(\sum_{v_i=1}^{s_i}\sum_{j=0}^{n}\frac{b^{(n)}_{i,j,v_i,\mathbf{w}}}{(u+a_i+j)^{v_i}}\right), \tag{16}$$

and a family of polynomials $\{A^{(n)}_{i,v_i,\mathbf{w}}(x), P^{(n)}_{\mathbf{w}}(x)\}_{1\le i\le r, 0\le v_i\le s_i} \subset \mathbb{Q}[x]$ by

$$A^{(n)}_{i,v_i,\mathbf{w}}(x) = \sum_{j=0}^{n} b^{(n)}_{i,j,v_i,\mathbf{w}}x^j , \tag{17}$$

$$P^{(n)}_{\mathbf{w}}(x) = \sum_{i=1}^{r}\sum_{j=1}^{n}\sum_{v_i=1}^{s_i}\sum_{l=0}^{j-1} b^{(n)}_{i,j,v_i,\mathbf{w}}\frac{x^{j-1-l}}{(l+a_i)^{v_i}} . \tag{18}$$

By the definition of $A^{(n)}_{i,v_i,\mathbf{w}}(x)$, we obtain

$$\deg_x A^{(n)}_{i,v_i,\mathbf{w}}(x) \le \begin{cases} n-1 & \text{for } w_i < v_i , \\ n & \text{for } w_i \ge v_i . \end{cases} \tag{19}$$

REMARK 3.1. For $1 \le i \le r$ and $\mathbf{w} = (w_1, \dots, w_r) \in \prod_{i=1}^{r}\{0, 1, \dots, s_i\}$, we have the following equality:

$$\deg A^{(n)}_{i,w_i,\mathbf{w}}(x) = n$$

for every $n \in \mathbb{N}$. In fact, the coefficient of x^n of the polynomial $A^{(n)}_{i,w_i,\mathbf{w}}(x) \in \mathbb{Q}[x]$ is

$$b^{(n)}_{i,n,w_i,\mathbf{w}} = \frac{u(u-1)\cdots(u-\sigma_{n,w}+2)}{\prod_{i'=1}^{r}\left[\left(\prod_{j=0}^{n-1}(u+a_{i'}+j)^{s_{i'}}\right)(u+a_{i'}+n)^{w_{i'}}\right]}(u+a_i+n)^{w_i}\Bigg|_{u=-a_i-n}$$
$$\neq 0\,.$$

Using the rational functions $\{A^{(n)}_{i,v_i,\mathbf{w}}(x),\ P^{(n)}_{\mathbf{w}}(x)\}_{1\le i\le r, 0\le v_i\le s_i} \subset \mathbb{Q}[x]$, we obtain the following Padé approximation of the Lerch functions that was proved in [11, Proposition 2.1].

LEMMA 3.2. *Under the notation above, we have*

$$\mathcal{R}^{(n)}_{\mathbf{w}}(x) = o(x^{-\sigma_{n,w}+1}) \tag{20}$$

and the following Padé approximation:

$$\mathcal{R}^{(n)}_{\mathbf{w}}(x) = \sum_{i=1}^{r}\left(\sum_{v_i=1}^{s_i} A^{(n)}_{i,v_i,\mathbf{w}}(x)\Phi(v_i, a_i, x^{-1})\right) - P^{(n)}_{\mathbf{w}}(x)\,. \tag{21}$$

PROOF. From the definition of $\mathcal{R}^{(n)}_{\mathbf{w}}(x)$, we have the relation (20). The relation (21) follows from the following calculation:

$$\begin{aligned}
\mathcal{R}^{(n)}_{\mathbf{w}}(x) &= \sum_{m=0}^{\infty}\sum_{i=1}^{r}\sum_{v_i=0}^{s_i}\sum_{j=0}^{n}\frac{b^{(n)}_{i,j,v_i,\mathbf{w}}}{(m+a_i+j)^{v_i}}x^{-m-1} \\
&= \sum_{i=1}^{r}\sum_{v_i=0}^{s_i}\sum_{j=0}^{n} b^{(n)}_{i,j,v_i,\mathbf{w}}\sum_{m=0}^{\infty}\frac{x^{-m-1}}{(m+a_i+j)^{v_i}} \\
&= \sum_{i=1}^{r}\sum_{v_i=0}^{s_i}\sum_{j=0}^{n} b^{(n)}_{i,j,v_i,\mathbf{w}}\left(x^j\Phi(v_i,a_i,x) - \sum_{l=0}^{j-1}\frac{x^{j-l-1}}{(m+a_i)^{v_i}}\right) \\
&= \sum_{i=1}^{r}\left(\sum_{v_i=1}^{s_i} A^{(n)}_{i,v_i,\mathbf{w}}(x)\Phi(v_i,a_i,x)\right) - P^{(n)}_{\mathbf{w}}(x)\,.
\end{aligned}$$

This gives the proof of the lemma. □

4. Some estimations

LEMMA 4.1. *Let β be a non-zero complex number. There exists $c > 0$ such that the inequality*

(22)
$$\max_{1\le i\le r, 1\le v_i\le s_i}\{|A^{(n)}_{i,v_i,\mathbf{w}}(\beta)|, |P^{(n)}_{\mathbf{w}}(\beta)|\} \le n^c \max\{1, |\beta|^n\}\exp\{n(s\log s + (2s+1)\log 2)\}\,,$$

holds for any sufficiently large $n \in \mathbb{N}$.

PROOF. We fix an enough large natural number k satisfying the following conditions:

$$|a_{i_1} - a_{i_2}| > \frac{2}{k} \text{ and } 1 > |a_{i_1} - a_{i_2}| + \frac{2}{k} \text{ for all } 1 \le i_1, i_2 \le r, i_1 \ne i_2 \,. \tag{23}$$

Firstly, we give an upper bound of $\{b^{(n)}_{i,j,v_i,\mathbf{w}}\}_{1\le j\le n, 1\le v_i\le s_i}$ for a fixed i. Using the definition of $b^{(n)}_{i,j,v_i,\mathbf{w}}$ given by (16), we get

$$b^{(n)}_{i,j,v_i,\mathbf{w}} = \frac{1}{2\pi\sqrt{-1}} \int_{|u+j+a_i|=\frac{1}{k}} Q^{(n)}_{\mathbf{w}}(u)(u+a_i+j)^{v_i-1} du \,. \tag{24}$$

From the equality (24) and the definition of $Q^{(n)}_{\mathbf{w}}(u)$, we obtain

$$\begin{aligned} |b^{(n)}_{i,j,v_i,\mathbf{w}}| &\le k^{-v_i} \sup\nolimits_{|u+a_i+j|=\frac{1}{k}} |Q^{(n)}_{\mathbf{w}}(u)| \\ &\le k^{-v_i} \sup\nolimits_{|u+j+a_i|=\frac{1}{k}} \left| \frac{(u+\sigma_{n,w}-2)_{\sigma_{n,w}-1}}{\prod_{i'=1}^{r} (u+a_{i'})_n^{s_{i'}} (u+n+a_{i'})^{w_{i'}}} \right| . \end{aligned} \tag{25}$$

We give an upper bound of $\left| \frac{(u+\sigma_{n,w}-2)_{\sigma_{n,w}-1}}{\prod_{i'=1}^{r} (u+a_{i'})_n^{s_{i'}} (u+n+a_{i'})^{w_{i'}}} \right|$. We have the following inequalities for $u \in \left\{u \in \mathbb{C} \mid |u+j+a_i| = \frac{1}{k}\right\}$:

$$\begin{aligned} |(u-\sigma_{n,w}+2)_{\sigma_{n,w}-1}| &= |(u+j+a_i-j-a_i)\cdots(u+j+a_i-\sigma_{n,w}-j-a_i+2)| \\ &\le (j+1)\cdots(\sigma_{n,w}+j+3) \,. \end{aligned} \tag{26}$$

Estimating a lower bound of $|(u+a_{i'})_n|$ and $|u+n+a_{i'}|$ for $u \in \{u \in \mathbb{C} \mid |u+j+a_i| = \frac{1}{k}\}$, we give a lower bound of $|u+j+a_i+(a_{i'}-a_i)+(l-j)|$ for $1 \le i' \le r$, $0 \le l \le n$:

(In the case of $i' = i$)

$$|u+j+a_i+(a_{i'}-a_i)+(l-j)| \ge \begin{cases} \frac{1}{k} & \text{if } l = j-1, j, j+1\,, \\ j-l-1 & \text{if } j-1 > l\,, \\ l-j-1 & \text{if } l > j+1. \end{cases} \tag{27}$$

(In the case of $i' > i$)

$$|u+j+a_i+(a_{i'}-a_i)+(l-j)| \ge \begin{cases} \frac{1}{k} & \text{if } l = j-1, j\,, \\ j-l-1 & \text{if } j-1 > l\,, \\ l-j & \text{if } l > j\,. \end{cases} \tag{28}$$

(In the case of $i > i'$)

$$|u + j + a_i + (a_{i'} - a_i) + (l - j)| \geq \begin{cases} \frac{1}{k} & \text{if } l = j, j + 1\,, \\ j - l & \text{if } j > l\,, \\ l - j - 1 & \text{if } l > j + 1\,. \end{cases} \tag{29}$$

From the inequalities (27), (28) and (29), we have the following estimation for all $1 \leq i' \leq r$:

$$\begin{aligned} |(u + a_{i'})_n| &= \prod_{l=0}^{n-1} |u + l + a_{i'}| \\ &= \prod_{l=0}^{n-1} |u + j + a_i + (a_{i'} - a_i) + (l - j)| \\ &\geq \frac{(n-j)!j!}{k^3 n^3}\,. \end{aligned} \tag{30}$$

We also have the inequality:

$$|u + n + a_{i'}| = |u + j + a_i + (a_{i'} - a_i) + (n - j)| \geq \frac{1}{k}\,. \tag{31}$$

From the inequalities (25), (26), (30) and (31), we obtain

$$\begin{aligned} & k^{-v_i} \sup_{|u+j+a_i| = \frac{1}{k}} \left| \frac{(u - \sigma_{n,w} + 2)_{\sigma_{n,w} - 1}}{\prod_{i'=1}^{r} (u + a_{i'})_n{}^{s_{i'}} (u + n + a_{i'})^{w_{i'}}} \right| \\ & \leq n^{c_1} \frac{(j+1) \cdots (\sigma_{n,w} + j + 3)}{((n-j)!j!)^s} \\ & = n^{c_1} \frac{(j + \sigma_{n,w} + 3)!}{j!(n!)^s} \binom{n}{j}^s\,, \end{aligned} \tag{32}$$

where c_1 is a positive constant. By using the inequality $\binom{n}{j} \leq 2^n$ for the inequality (32), we get

$$\begin{aligned} \frac{(j + \sigma_{n,w} + 3)!}{j!(n!)^s} \binom{n}{j}^s &= \frac{(\sigma_{n,w} + 3)!}{(n!)^s} \binom{j + \sigma_{n,w} + 3}{j} \binom{n}{j}^s \\ &\leq n^{c_2} \frac{(\sigma_{n,w} + 3)!}{(n!)^s} 2^{2ns+n}\,, \end{aligned} \tag{33}$$

for some positive constant c_2. By using the Stirling formula for the inequality (33), we obtain

$$\begin{aligned} \frac{(\sigma_{n,w} + 3)!}{(n!)^s} 2^{2ns+n} &\leq n^{c_3} \frac{(\sigma_{n,w} + 3)^{(\sigma_{n,w}+3)} e^{-(\sigma_{n,w}+3)}}{n^{ns} e^{-ns}} 2^{2ns+n} \\ &\leq n^{c_4} \exp\{n(s \log s + (2s + 1) \log 2)\}\,, \end{aligned} \tag{34}$$

where c_3, c_4 are some positive constants. From the inequality (34), we conclude that

$$|b^{(n)}_{i,j,v_i,\mathbf{w}}| \leq n^{c_5} \exp\{n(s \log s + (2s+1) \log 2)\}\,, \tag{35}$$

for some positive constant c_5. From the definition (17), (18) and inequality (35), we obtain the desired estimation:

$$\max_{1 \leq i \leq r, 1 \leq v_i \leq s_i} \{|A^{(n)}_{i,v_i,\mathbf{w}}(\beta)|, |P^{(n)}_{\mathbf{w}}(\beta)|\} \leq \max\{1, |\beta|^n\} n^c \exp\{n(s \log s + (2s+1)\log 2)\}$$

for some constant $c > 0$. This completes the proof of Lemma 4.1. □

LEMMA 4.2. *Let $\beta \in D_\infty$. Then there exists $C > 0$, satisfying*

$$|\mathcal{R}^{(n)}_{\mathbf{w}}(\beta)| \leq C|\beta|^{-ns}\,, \tag{36}$$

for all $n \in \mathbb{N}$.

PROOF. Let m be a positive integer. Since there is a trivial inequality $|Q^{(n)}_{\mathbf{w}}(m)| \leq 1$ for $m \geq \sigma_{n,w} - 1$, we the following estimation:

$$\begin{aligned} |\mathcal{R}^{(n)}_{\mathbf{w}}(\beta)| &\leq \sum_{m=\sigma_{n,w}-1}^{\infty} |Q^{(n)}_{\mathbf{w}}(m)||\beta|^{-m-1} \\ &\leq |\beta|^{-\sigma_{n,w}} \sum_{m=0}^{\infty} |\beta|^{-m}\,. \end{aligned}$$

Since $|\beta| > 1$, the sum $\sum_{m=0}^{\infty} |\beta|^{-m}$ converges, we obtain the desired estimation. □

For a non-zero algebraic number β, we construct an integer $D_n(\beta) = D_n$ which satisfies $D_n A^{(n)}_{i,v_i,\mathbf{w}}(\beta) \in \mathcal{O}_{\mathbb{Q}(\beta)}$ and $D_n P^{(n)}_{\mathbf{w}}(\beta) \in \mathcal{O}_{\mathbb{Q}(\beta)}$. Before stating next lemma, we prepare some notations:

$$\begin{aligned} A &:= \text{l.c.m.}\{\text{den}(a_i)\}_{1 \leq i \leq r}\,, \\ b_i &:= a_i \text{den}(a_i) \text{ for } 1 \leq i \leq r\,, \\ b &:= \max_{1 \leq i \leq r} b_i\,, \\ M &:= \text{l.c.m.}\{\text{den}(a_{i'} - a_i)\}_{1 \leq i, i' \leq r,\, i \neq i'}\,, \\ e_{i',i} &:= M(a_{i'} - a_i) \text{ for all } 1 \leq i, i' \leq r\,, \\ e &:= \max_{1 \leq i, i' \leq r} e_{i',i}\,, \\ S &:= \max_{1 \leq i \leq r} s_i\,. \end{aligned}$$

We give the following lemma.

LEMMA 4.3. *We use the notation as above. Let β be a non-zero algebraic number. Then we have the following relations*:

$$S!\mu_n(A)^s A^{s(n+1)} d_{e+Mn}^{s_i-v_i} \mathrm{den}(\beta)^n A_{i,v_i,\mathbf{w}}^{(n)}(\beta) \in \mathcal{O}_{\mathbb{Q}(\beta)}$$

$$S!\mu_n(A)^s A^{s(n+1)} d_{e+Mn}^{S} \mathrm{den}(\beta)^n d_{b+(n-1)A}^{S} P_{\mathbf{w}}^{(n)}(\beta) \in \mathcal{O}_{\mathbb{Q}(\beta)}$$

for all $1 \le i \le r$, $1 \le v_i \le s_i$.

PROOF. We construct an integer which is divisible by the denominator of $b_{i,j,v_i,\mathbf{w}}^{(n)}$ for $1 \le i \le r$, $1 \le v_i \le s_i$. According to the equation (16), we get

(37)

$$b_{i,j,v_i,\mathbf{w}}^{(n)} = \begin{cases} \frac{1}{(s_i-v_i)!}\left(\frac{d}{du}\right)^{s_i-v_i} Q_{\mathbf{w}}^{(n)}(u)(u+a_i+j)^{s_i}|_{u=-a_i-j} & \text{for } 0 \le j \le n-1, \\ & \quad 1 \le v_i \le s_i, \\ \frac{1}{(w_i-v_i)!}\left(\frac{d}{du}\right)^{w_i-v_i} Q_{\mathbf{w}}^{(n)}(u)(u+a_i+n)^{w_i}|_{u=-a_i-n} & \text{for } j = n, \\ & \quad 1 \le v_i \le w_i, \\ 0 & \text{for } j = n, v_i > w_i. \end{cases}$$

First we calculate $\frac{1}{(s_i-v_i)!}\left(\frac{d}{du}\right)^{s_i-v_i} Q_{\mathbf{w}}^{(n)}(u)(u+a_i+j)^{s_i}|_{u=-a_i-j}$ for $0 \le j \le n-1$, $1 \le v_i \le s_i$. For $c \ge 0$, we put

$$R_{i,\mathbf{w}}^{(n)}(u) = (u-\sigma_{n,w}+2)(u-\sigma_{n,w}+3)\cdots(u-\sigma_{n,w}+s_i),$$

$$Q_{i,c,n,\mathbf{w}}(u) = \frac{(u-c)(u-c-1)\cdots(u-c-(n-1))}{(u+a_i)(u+a_i+1)\cdots(u+a_i+j-1)(u+a_i+j+1)\cdots(u+a_i+n)},$$

$$Q_{i,c,n-1,\mathbf{w}}(u) = \frac{(u-c)(u-c-1)\cdots(u-c-(n-2))}{(u+a_i)(u+a_i+1)\cdots(u+a_i+j-1)(u+a_i+j+1)\cdots(u+a_i+n-1)},$$

$$S_{i',c,n,\mathbf{w}}(u) = \frac{(u-c)(u-c-1)\cdots(u-c-n))}{(u+a_{i'})(u+a_{i'}+1)\cdots(u+a_{i'}+n)} \text{ for and } i' \ne i,$$

$$S_{i',c,n-1,\mathbf{w}}(u) = \frac{(u-c)(u-c-1)\cdots(u-c-(n-2))}{(u+a_{i'})(u+a_{i'}+1)\cdots(u+a_{i'}+n-1)} \text{ and } i' \ne i.$$

From the equality

$$Q_{\mathbf{w}}^{(n)}(u)(u+a_i+j)^{s_i}$$
$$= \frac{u(u-1)\cdots(u-\sigma_{n,w}+2)}{\prod_{i'=1,i'\ne i}^{r}\left(\prod_{j=0}^{n-1}(u+a_{i'}+j)^{s_{i'}}(u+a_{i'}+n)^{w_{i'}}\right)\times\left(\prod_{j'=0,\,j'\ne j}^{n-1}(u+a_i+j')^{s_i}(u+a_i+n)^{w_i}\right)},$$

we can express $Q_{\mathbf{w}}^{(n)}(u)(u+a_i+j)^{s_i}$ for $1 \le i \le r$, $0 \le j \le n$ as follows:

$$Q_{\mathbf{w}}^{(n)}(u)(u+a_i+j)^{s_i} := R_{i,\mathbf{w}}^{(n)}(u) \prod_{m_i=1}^{s_i} Q_{i,c_{m_i},\mathbf{w}}^{(n)}(u) \times \prod_{i'=1,i'\ne i}^{r} \left(\prod_{m_{i'}=1}^{s_{i'}} S_{i',c_{m_{i'}},\mathbf{w}}^{(n)}(u)\right)$$

where $Q^{(n)}_{i,c_{m_i},\mathbf{w}}(u)$ stands for either $Q_{i,c,n,\mathbf{w}}(u)$ or $Q_{i,c,n-1,\mathbf{w}}(u)$ and $S^{(n)}_{i',c_m,\mathbf{w}}(u)$ stands for either $S_{i',c,n,\mathbf{w}}(u)$ or $S_{i',c,n-1,\mathbf{w}}(u)$. Hence, we get

$$\begin{aligned}&\left(\frac{d}{du}\right)^{s_i-v_i} Q^{(n)}_{\mathbf{w}}(u)(u+a_i+j)^{s_i}|_{u=-a_i-j}\\ &=\sum_{l_0+l_1+\cdots+l_s=s_i-v_i}\frac{(s_i-v_i)!}{l_0!\cdots l_s!}\times\left(\frac{d}{du}\right)^{l_0} R_{i,\mathbf{w}}{}^{(n)}(u)\times\prod_{m_i=1}^{s_i}\left(\frac{d}{du}\right)^{l_{m_i}} Q^{(n)}_{i,c_{m_i},\mathbf{w}}(u)\\ &\times\prod_{i'=1,i'\neq i}^{r}\left(\prod_{m_{i'}=1}^{s_{i'}}\left(\frac{d}{du}\right)^{l_{m_{i'}}} S^{(n)}_{i',c_{m_{i'}},\mathbf{w}}(u)\right)|_{u=-a_i-j}\,.\end{aligned}$$

The same argument of the proof of [11, Lemma 3.3, p.184] , we have

$$\mu_n(\mathrm{den}(a_i))\mathrm{den}(a_i)^n d_n^l\left(\frac{d}{du}\right)^l Q^{(n)}_{i,c_{m_i},\mathbf{w}}(u)\bigg|_{u=-a_i-j}\in\mathbb{Z}\text{ for }0\le l\le s_i-v_i\,, \tag{38}$$

where $Q^{(n)}_{i,c_{m_i},\mathbf{w}}(u)$ stands for either $Q_{i,c,n,\mathbf{w}}(u)$ or $Q_{i,c,n-1,\mathbf{w}}(u)$.

We can express

$$\begin{aligned}S_{i',c,n,\mathbf{w}}(u)&=\frac{(u-c)(u-c-1)\cdots(u-c-n))}{(u+a_{i'})(u+a_{i'}+1)\cdots(u+a_{i'}+n)}\\ &=1+\frac{B_{i',0,\mathbf{w}}}{(u+a_{i'})}+\frac{B_{i',1,\mathbf{w}}}{(u+a_{i'}+1)}+\cdots+\frac{B_{i',n,\mathbf{w}}}{(u+a_{i'}+n)}\,,\end{aligned}$$

where

$$B_{i',l,\mathbf{w}}=(-1)^{n+l+1}\frac{(a_{i'}+l+c)\cdots(a_{i'}+l+c+n)}{l!(n-l)!}\,.$$

Substituting $b_{i'}/\mathrm{den}(a_{i'})$ for $a_{i'}$, we get

$$B_{i',l,\mathbf{w}}=\frac{(-1)^{n+l+1}\mathrm{den}(a_{i'})^{-n-1}\prod_{k=0}^{n}(b_{i'}+\mathrm{den}(a_{i'})(c+l+k))}{l!(n-l)!}\,.$$

Since

$$\frac{\prod_{k=0}^{n}(b_{i'}+\mathrm{den}(a_{i'})(c+l+k))}{l!(n-l)!}=\frac{\prod_{k=0}^{n}(b_{i'}+\mathrm{den}(a_{i'})(c+l+k))}{(n+1)!}\frac{n!(n+1)}{l!(n-l)!}\,,$$

we obtain

$$\mathrm{den}(a_{i'})^{n+1}\mu_n(\mathrm{den}(a_{i'}))B_{i',l,\mathbf{w}}\in\mathbb{Z}\quad\text{for }n\in\mathbb{N}\,. \tag{39}$$

Using the relation (39), the equation

$$\begin{aligned}\left(\frac{d}{du}\right)^l S_{i',c,n,\mathbf{w}}(u)\bigg|_{u=-a_i-j}&=\frac{d}{du}^{l}1+(-1)^l l!\bigg[\frac{B_{i',0,\mathbf{w}}}{(a_{i'}-a_i-j)^{l+1}}\\ &+\frac{B_{i',1,\mathbf{w}}}{(a_{i'}-a_i+1-j)^{l+1}}+\cdots+\frac{B_{i',n,\mathbf{w}}}{(a_{i'}-a_i+n-j)^{l+1}}\bigg]\end{aligned} \tag{40}$$

for $0 \le l \le s_i - v_i$, and the definition of $e_{i',i}$, we obtain

$$\mathrm{den}(a_{i'})^{n+1}\mu_n(\mathrm{den}(a_{i'}))d_{e_{i',i}+Mn}^{l+1}\left(\frac{d}{du}\right)^l S_{i',c,n,\mathbf{w}}(u)\Big|_{u=-a_i-j} \in \mathbb{Z}\,, \tag{41}$$

for $0 \le l \le s_i - v_i$. Similarly, we obtain

$$\mathrm{den}(a_{i'})^{n+1}\mu_n(\mathrm{den}(a_{i'}))d_{e_{i',i}+Mn}^{l+1}\left(\frac{d}{du}\right)^l S_{i',c,n-1,\mathbf{w}}(u)\Big|_{u=-a_i-j} \in \mathbb{Z}\,,$$

for $0 \le l \le s_i - v_i$. Thus we obtain

$$S!\mu_n(A)^s A^{s(n+1)} d_{e+Mn}^{s_i-v_i} b_{i,j,v_i,\mathbf{w}}^{(n)} \in \mathbb{Z}\,. \tag{42}$$

We conclude that $S!\mu_n(A)^s A^{s(n+1)} d_{e+Mn}^{s_i-v_i}\mathrm{den}(\beta)^n A_{i,v_i,\mathbf{w}}^{(n)}(\beta) \in \mathcal{O}_{\mathbb{Q}(\beta)}$. From the definition (18) we get the equation

$$\begin{aligned} P_{\mathbf{w}}^{(n)}(\beta) &= \sum_{i=1}^{r}\sum_{j=0}^{n}\sum_{v_i=1}^{s_i} b_{i,j,v_i,\mathbf{w}}^{(n)}\left(\frac{\beta^{j-1}}{a_i^{v_i}} + \cdots + \frac{1}{(j-1+a_i)^{v_i}}\right) \\ &= \sum_{i=1}^{r}\sum_{j=0}^{n}\sum_{v_i=1}^{s_i} b_{i,j,v_i,\mathbf{w}}^{(n)}\left(\frac{\mathrm{den}(a_i)^{v_i}\beta^{j-1}}{b_i^{v_i}} + \cdots + \frac{\mathrm{den}(a_i)^{v_i}}{(\mathrm{den}(a_i)(j-1)+b_i)^{v_i}}\right). \end{aligned}$$

Since we have $S!\mu_n(A)^s A^{s(n+1)} d_{e+Mn}^{s_i-v_i} b_{i,j,v_i,\mathbf{w}}^{(n)} \in \mathbb{Z}$, then we conclude that

$$S!\mu_n(A)^s A^{s(n+1)} d_{e+Mn}^{S}\mathrm{den}(\beta)^n d_{b+(n-1)A}^{S} P_{\mathbf{w}}^{(n)}(\beta) \in \mathcal{O}_{\mathbb{Q}(\beta)}\,.$$

This completes the proof of Lemma 4.3. □

We denote $(0,\dots,0) \in \prod_{i=1}^r \{0,\dots,s_i\}$ by $\mathbf{0}$. Hereafter, we fix a subset

$$\{\mathbf{w}_{i,j} := (w_{i,j}^{(1)},\dots,w_{i,j}^{(r)})\}_{1\le i\le r, 1\le j\le s_i} \subset \prod_{i=1}^{r}\{0,\dots,s_i\}$$

satisfying

$$w_{i,j}^{(k)} = \begin{cases} 0 & \text{if } k \ne i\,, \\ j & \text{if } k = i\,. \end{cases}$$

We put the determinant of the following $(s+1)\times(s+1)$-matrix

$$
(43)\qquad \begin{pmatrix}
-P^{(n)}_{\mathbf{0}}(x) & A^{(n)}_{1,1,\mathbf{0}}(x) & \dots & A^{(n)}_{1,s_1,\mathbf{0}}(x) & \cdots & A^{(n)}_{r,1,\mathbf{0}}(x) & \dots & A^{(n)}_{r,s_r,\mathbf{0}}(x) \\
-P^{(n)}_{\mathbf{w}_{1,1}}(x) & A^{(n)}_{1,1,\mathbf{w}_{1,1}}(x) & \dots & A^{(n)}_{1,s_1,\mathbf{w}_{1,1}}(x) & \cdots & A^{(n)}_{r,1,\mathbf{w}_{1,1}}(x) & \dots & A^{(n)}_{r,s_r,\mathbf{w}_{1,1}}(x) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
-P^{(n)}_{\mathbf{w}_{1,s_1}}(x) & A^{(n)}_{1,1,\mathbf{w}_{1,s_1}}(x) & \dots & A^{(n)}_{1,s_1,\mathbf{w}_{1,s_1}}(x) & \cdots & A^{(n)}_{r,1,\mathbf{w}_{1,s_1}}(x) & \dots & A^{(n)}_{r,s_r,\mathbf{w}_{1,s_1}}(x) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
-P^{(n)}_{\mathbf{w}_{r,1}}(x) & A^{(n)}_{1,1,\mathbf{w}_{r,1}}(x) & \dots & A^{(n)}_{1,s_1,\mathbf{w}_{r,1}}(x) & \cdots & A^{(n)}_{r,1,\mathbf{w}_{r,1}}(x) & \dots & A^{(n)}_{r,s_r,\mathbf{w}_{r,1}}(x) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
-P^{(n)}_{\mathbf{w}_{r,s_r}}(x) & A^{(n)}_{1,1,\mathbf{w}_{r,s_r}}(x) & \dots & A^{(n)}_{1,s_1,\mathbf{w}_{r,s_r}}(x) & \cdots & A^{(n)}_{r,1,\mathbf{w}_{r,s_r}}(x) & \dots & A^{(n)}_{r,s_r,\mathbf{w}_{r,s_r}}(x)
\end{pmatrix}
$$

by $\Delta^{(n)}(x)$ for every $n \in \mathbb{N}$. Then we have the following lemma.

LEMMA 4.4. *Let $\Delta^{(n)}(x)$ be as above. Then $\Delta^{(n)}(x)$ are non-zero rational numbers for all $n \in \mathbb{N}$.*

PROOF. For $\mathbf{w} \in \prod_{i=1}^{r}\{0,\dots,s_i\}$, we define $\mathcal{R}^{(n)}_{\mathbf{w}}(x)$, $c^{(n)}_{0,\mathbf{w}}$ and $b^{(n)}$ by

$$\mathcal{R}^{(n)}_{\mathbf{w}}(x) := \sum_{i=1}^{r}\left(\sum_{v_i=1}^{s} A^{(n)}_{i,v_i,\mathbf{w}}(x)\Phi(v_i,a_i,x)\right) - P^{(n)}_{\mathbf{w}}(x) = \frac{c^{(n)}_{0,\mathbf{w}}}{x^{\sigma_{n,w}}} + \cdots,$$

and define

$$b^{(n)} := \prod_{i=1}^{r}\left(\prod_{j=1}^{s_i} b^{(n)}_{i,n,j,\mathbf{w}_{i,j}}\right).$$

Then, we obtain

$$\Delta^{(n)}(x) = b^{(n)}c^{(n)}_{0,\mathbf{0}} \in \mathbb{Q},$$

(cf. [11, Lemma 3.4]). By Remark 3.1 and the definition of $c^{(0)}_{0,\mathbf{0}}$, we obtain $b^{(n)} \neq 0$ and $c^{(n)}_{0,\mathbf{0}} \neq 0$. This completes the proof of Lemma 4.4. □

5. Proof of Theorem 1.1

Theorem 1.1 is proved by using the results of Sections 2 and 3. We use the notations of the previous sections. Fix a set of rational numbers $\{a_1,\dots,a_r\} \subset \mathbb{Q}$ satisfying $0 < a_1 < \cdots < a_r \le 1$. We use Theorem 2.4 for

$$\theta_{(i,v_i)}(x) = \Phi(v_i,a_i,x) : D_\infty \longrightarrow \mathbb{C},\ \ 1 \le i \le r,\ 1 \le v_i \le s_i .$$

PROOF OF THEOREM 1.1. We denote the set $\{(\beta, K) \in D_\infty(\overline{\mathbb{Q}}) \times \mathcal{A}_{\mathbb{Q}} \mid \beta \in K\}$ by W_∞ and fix an element $(\beta, K) \in W_\infty$. Let the polynomials $\{A^{(n)}_{i,v_i,\mathbf{w}}(x)\}_{1\le i\le r, 1\le v_i\le s_i, \mathbf{w}\in\{\mathbf{0},\mathbf{w}_{i,j}\}} \cup \{P^{(n)}_{\mathbf{w}}(x)\}_{\mathbf{w}\in\{\mathbf{0},\mathbf{w}_{i,j}\}}$ be defined by (17) and (18). Then, from the equality (21), we have

$$\mathcal{R}^{(n)}_{\mathbf{w}}(x) = \sum_{i=1}^{r}\sum_{v_i=1}^{s} A^{(n)}_{i,v_i,\mathbf{w}}(x)\Phi(v_i, a_i, x) - P^{(n)}_{\mathbf{w}}(x) = o(x^{-\sigma_{n,w}+1}) .$$

The above relation shows that $\{A^{(n)}_{i,v_i,\mathbf{w}}(x), P^{(n)}_{\mathbf{w}}(x)\}_{1\le i\le r, 1\le v_i\le s_i, \mathbf{w}\in\{\mathbf{0},\mathbf{w}_{i,j}\}}$ satisfy (4) in Assumption 2.2.

We define the following functions

$$D^{(\infty)}_n : D_\infty(\overline{\mathbb{Q}}) \longrightarrow \mathbb{N} \text{ by } \beta \mapsto S!\mu_n(A)^s A^{s(n+1)} d^S_{e+Mn} \mathrm{den}(\beta)^n d^S_{b+(n-1)A} ,$$

$$f^{(\infty)} : D_\infty(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\ge 0} \text{ by } \beta \mapsto s\log A + s \sum_{\substack{q:\text{prime} \\ q|A}} \frac{\log q}{q-1} + S(M+A) + \log \mathrm{den}(\beta) ,$$

$$g^{(\infty)} : D_\infty(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\ge 0} \text{ by } \beta \mapsto \log\max\{1, |\beta|\} + (s\log s + (2s+1)\log 2) ,$$

$$h^{(\infty)} : D_\infty(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\ge 0} \text{ by } \beta \mapsto s\log|\beta| .$$

We note that $f^{(\infty)}$ and $g^{(\infty)}$ satisfy the relation (10). From Lemma 4.4, we have $\Delta^{(n)}(x) \neq 0$. This shows that $\{A^{(n)}_{i,v_i,\mathbf{w}}(x), P^{(n)}_{\mathbf{w}}(x)\}_{1\le i\le r, 1\le v_i\le s_i, \mathbf{w}\in\{\mathbf{0},\mathbf{w}_{i,j}\}}$ satisfies the assumption (4). From Lemma 4.3, the family of functions $\{D_n : D_\infty(\overline{\mathbb{Q}}) \longrightarrow \mathbb{N}\}_{n\in\mathbb{N}}$ satisfy the assumption (6). From Lemma 4.3, $f^{(\infty)}$ satisfies the assumption (7). From Lemma 4.1 (22), $g^{(\infty)}$ satisfies the assumption (8). From Lemma 4.2, $h^{(\infty)}$ satisfies the assumption (9). We define the following function:

$$F^{(\infty)} : W_\infty \longrightarrow \mathbb{R}_{\ge 0} \text{ by } (\beta, K) \mapsto \frac{[K_\infty : \mathbb{R}](g^{(\infty)}(\beta) + h^{(\infty)}(\beta))}{[K : \mathbb{Q}(\beta)] \sum_{\tau\in I_{\mathbb{Q}(\beta)}} (f^{(\infty)}(\tau\beta) + g^{(\infty)}(\tau\beta))} .$$

Using Theorem 2.4, we obtain the following inequality:

$$\dim_K\left(K + \sum_{v_1=1}^{s_1} K\Phi(v_1, a_1, \beta) + \cdots + \sum_{v_r=1}^{s_r} K\Phi(v_r, a_r, \beta)\right) \ge F^{(\infty)}(\beta, K)$$

for all $(\beta, K) \in W_\infty$. This completes the proof of Theorem 1.1. □

Part II. p-adic case

6. Lower bound of the dimension of vector space spanned by p-adic numbers

In this section, we recall an estimation of a lower bound of the dimension of the vector space over a number field spanned by p-adic numbers. The method is based on that of Siegel (cf. [18]) and a p-adic analog of Lemma 2.1. The following Lemma was proved by Pierre Bel in [2].

LEMMA 6.1 ([2, lemma 4.1]). *Let K be an algebraic number field. Let $\theta := (1, \theta_1, \dots, \theta_s) \in \mathbb{C}_p^{s+1}$. Suppose that there exist $(s+1)$ linear forms*

$$L_w^{(n)}(X_0, \dots, X_s) = \sum_{v=0}^{s} A_{v,w}^{(n)} X_v \quad (0 \le w \le s; n \in \mathbb{N}),$$

for all $n \in \mathbb{N}$ with coefficients $\{A_{v,w}^{(n)}\}_{0 \le v,w \le s} \subset \mathcal{O}_K$, which satisfy $\det((A_{v,w}^{(n)})_{0\le v,w\le s}) \neq 0$. Suppose there exist positive real numbers $\{c_\tau\}_{\tau \in I_K}$ and ρ satisfying the following conditions:

$$\limsup_n \frac{\log ||\tau L_w^{(n)}||}{n} \le c_\tau \text{ for each } \tau \in I_K \text{ and } 0 \le w \le s,$$

$$\limsup_n \frac{\log |L_w^{(n)}(\boldsymbol{\theta})|_p}{n} \le -\rho \text{ for all } 0 \le w \le s.$$

Then we have

$$\dim_K (K + K\theta_1 + \cdots + K\theta_s) \ge \frac{[K_p : \mathbb{Q}_p]\rho}{\sum_{\tau \in I_K} c_\tau}. \tag{44}$$

REMARK 6.2. Under the assumption of Lemma 6.1, we assume that $\{A_{v,w}^{(n)}\}_{0\le v,w\le s} \subset \mathbb{Z}$ and there exists $c > 0$ satisfying $\limsup_n \dfrac{\log ||\tau L_w^{(n)}||}{n} \le c$ for all $\tau \in I_K$ and $0 \le w \le s$. Then, for an algebraic number field K satisfying $[K : \mathbb{Q}] = [K_p : \mathbb{Q}_p]$, the conclusion (44) becomes

$$\dim_K (K + K\theta_1 + \cdots + K\theta_s) \ge \frac{\rho}{c}. \tag{45}$$

We remark that the right hand side of (45) does not depend on K satisfying $[K : \mathbb{Q}] = [K_p : \mathbb{Q}_p]$.

7. A construction of Padé approximation of formal Laurent series

In this section, we give a construction of Padé approximation of formal Laurent series due to the method of Rivoal (cf. [17, Proposition 4]). Throughout this section, we fix a subfield L of $\mathbb{C}$.

DEFINITION 7.1. Let A be a real number and $f : \mathbb{R}_{>A} \longrightarrow \mathbb{C}$ a function. We say $f(x)$ has an asymptotic expansion at $x = \infty$ over L if there exists a sequence $\{a_k(f)\}_{k\in\mathbb{Z}_{\ge 0}} \subset L$ which satisfies the following condition for all $N \in \mathbb{Z}_{\ge 0}$:

$$f(x) = \sum_{k=0}^{N-1} a_k(f) x^{-k} + O\big(x^{-N}\big) \ (x \to \infty). \tag{46}$$

We note that the coefficients $\{a_k(f)\}_{k\in\mathbb{Z}_{\geq 0}}$ are determined uniquely from f. In fact

$$a_N(f) = \begin{cases} \lim_{x\to\infty} f(x) & \text{if } N = 0\,, \\ \lim_{x\to\infty} x^N\left(f(x) - \sum_{k=0}^{N-1} a_k(f)x^{-k}\right) & \text{if } N > 0\,. \end{cases} \tag{47}$$

We define a ring M_L^A as follows:

$$M_L^A := \{f : \mathbb{R}_{>A} \longrightarrow \mathbb{C} \mid f \text{ has an asymptotic expansion over } L \text{ at } x = \infty\}\,.$$

From the equality (47), we can define the following L-homomorphism:

$$\pi_A : M_L^A \longrightarrow L[[\tfrac{1}{X}]], \quad f(x) \mapsto \sum_{k=0}^{\infty} a_k(f)\frac{1}{X^k}\,. \tag{48}$$

REMARK 7.2. Let A, A' be real numbers. Suppose $A < A'$, there is a natural ring homomorphism $\phi_{A,A'} : M_L^A \longrightarrow M_L^{A'}$ and the set $\{M_L^A, \phi_{A,A'}\}_{A\in\mathbb{R}}$ becomes a direct system of rings. We denote the ring of direct limit of the above direct system by $M_L := \varinjlim_{A\in\mathbb{R}} M_L^A$. Note that, since there is the equality (47), the coefficients $\{a_k(f)\}_{k\in\mathbb{Z}_{\geq 0}}$ in the equalities (46) depends only on the image of $[f] \in M_L$ and are determined uniquely for $[f] \in M_L$. We can also define an L-algebra homomorphism

$$\pi : M_L \longrightarrow L[[\tfrac{1}{X}]], \ \ [f] \mapsto \sum_{k=0}^{\infty} a_k(f)\frac{1}{X^k}\,.$$

By definition of π_A and π, we have the following commutative diagram for all $A \in \mathbb{R}$,

$$\begin{array}{ccc} M_L^A & \xrightarrow{\pi_A} & L[[\tfrac{1}{X}]] \\ {\scriptstyle i_A}\downarrow & & \| \\ M_L & \xrightarrow[\pi]{} & L[[\tfrac{1}{X}]]\,, \end{array}$$

where i_A is the canonical homomorphism.

We denote $\pi_A(f)$ by $\hat{f}(X)$ for $f \in M_L^A$. If $L \subset \overline{\mathbb{Q}}$, there is a natural homomorphism $\iota_p \circ \iota_\infty^{-1} : L[[\frac{1}{X}]] \longrightarrow \iota_p \circ \iota_\infty^{-1}(L)[[\frac{1}{X}]]$.

EXAMPLE 7.3

1. For a natural number s and an element $\alpha \in \overline{\mathbb{Q}} \setminus \mathbb{Z}_{\leq 0}$, we have $\Phi(s, \alpha, x) \in M^1_{\mathbb{Q}(\alpha)}$. The formal Laurent series

$$\hat{\Phi}_p(s, \alpha, X) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s}\frac{1}{X^{m+1}} \in \mathbb{Q}(\alpha)\left[\left[\tfrac{1}{X}\right]\right]$$

converges on D_p.

2. We fix a pair of algebraic numbers $(\alpha_1, \alpha_2) \in (\overline{\mathbb{Q}} \cap \mathbb{R}_{>0}) \times \{x_2 \in \overline{\mathbb{Q}} \mid |x_2| = 1\}$. Let s be a positive integer and assume $s \geq 2$ if $\alpha_2 = 1$. Then we have $\Phi(s, x+\alpha_1, \alpha_2) \in M^0_{\mathbb{Q}(\alpha_1,\alpha_2)}$. We denote the formal Laurent series obtained by the asymptotic expansion of $\Phi(s, x+\alpha_1, \alpha_2)$ with respect to the parameter x at $x = \infty$ by $\hat{\zeta}_p(s, \alpha_1, \alpha_2, X) \in \mathbb{Q}(\alpha_1, \alpha_2)[[\frac{1}{X}]]$. Then we have the following identity:

$$\hat{\zeta}_p(s, X+\alpha_1, \alpha_2) = \frac{\epsilon(\alpha_2)}{s-1}\frac{1}{X^{s-1}} + \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{(m+1)!} B_{m+1}(\alpha_1, \alpha_2)(s)_m \frac{1}{X^{s+m}} \tag{49}$$

where $\epsilon(\alpha_2) = \begin{cases} 0 & \text{if } \alpha_2 \neq 1 \\ 1 & \text{if } \alpha_2 = 1, \end{cases}$ and $B_k(\alpha_1, \alpha_2)$ are defined by the following generating function:

$$\frac{te^{\alpha_1 t}}{\alpha_2 e^t - 1} = \sum_{k=0}^{\infty} B_k(\alpha_1, \alpha_2)\frac{t^k}{k!}.$$

The identity (49) was proved by Katsurada in [10, Theorem 1].

From the later Lemma 10.13, if $\alpha_1 \in \mathbb{Q}_{>0} \cap \mathbb{Z}_p$ and $1 \leq |\alpha_2 - 1|_p$ or $\alpha_2 = 1$ then $\hat{\zeta}_p(s, \alpha_1, \alpha_2, X)$ converges on D_p. Note that $\hat{\zeta}_p(s, \alpha_1, \alpha_2, X)$ depends only on $s, \alpha_1 + X, \alpha_2$ in this case.

PROPOSITION 7.4 (cf. [17, Proposition 4]). *Let l, s be natural numbers. Let $f_1(x), \dots, f_s(x) \in M^A_L$. Suppose there exists a family of polynomials $\{P^{(n)}_v(x)\}_{0\leq v\leq s} \subset L[x]$ which satisfy the following properties*:

Let $\mathcal{R}(x) = P_0(x) + \sum_{v=1}^{s} P_v(x) f_v(x)$. Then $\mathcal{R}(x)$ satisfies

$$\mathcal{R}(x) = o(x^{-l+1}) \ \ (x \to \infty).$$

Then, we have $\deg P_0(x) \leq \max_{1\leq v\leq s} \deg P_v(x)$, $\mathcal{R}(x) \in M^A_L$ *and* $\hat{\mathcal{R}}(X) \in L\big[\big[\frac{1}{X}\big]\big]$ *satisfying*

$$\hat{\mathcal{R}}(X) = P_0(X) + \sum_{v=1}^{s} P_v(X)\hat{f}_v(X) \in \left(\tfrac{1}{X}\right)^l.$$

PROOF. Put $q_v = \deg P_v(x)$, $q = \max_{1\leq v\leq s} q_v$ and define $b_{v,j}$ by $P_v(x) = \sum_{j=0}^{q_v} b_{v,j}x^j$ for $1 \leq v \leq s$. From the definition of M^A_L, there exists $\{a_k(f_v)\}_{k\in\mathbb{Z}_{\geq 0}} \subset L$ satisfying the following condition for $f_v(x)$:

$$f_v(x) = \sum_{k=0}^{N} a_k(f_v)x^{-k} + o(x^{-N}) \ (x \to \infty) \text{ for all } N \in \mathbb{Z}_{\geq 0}. \tag{50}$$

By using (50) for any $N \geq q$, we obtain

$$(51) \qquad \sum_{v=1}^{s} P_v(x) f_v(x) = Q(x) + \sum_{v=1}^{s} \Big[\sum_{j=0}^{q_v} \sum_{\substack{0 \leq k \leq N \\ -N+q \leq j-k<0}} b_{v,j} a_k(f_v) x^{j-k} \Big] + o(x^{-N+q}) ,$$

where $Q(x)$ is a polynomial with coefficients in L which satisfies $\deg Q(x) \leq q$. Using the equality (51) in the case of $N = l + q - 1$ and the assumption

$$\mathcal{R}(x) = P_0(x) + \sum_{v=1}^{s} P_v(x) f_v(x) = o(x^{-l+1}) ,$$

we obtain $P_0(x) = -Q(x)$ and

$$\mathcal{R}(x) = \sum_{k=0}^{N} a_k(\mathcal{R}) x^{-k} + o(x^{-N}) \ (x \to \infty) \text{ for all } N \in \mathbb{Z}_{\geq 0} ,$$

where

$$a_N(\mathcal{R}) = \begin{cases} 0 & \text{if } N < l , \\ \sum_{v=1}^{s} \sum_{j=0}^{n} b_{v,j} a_{N+j}(f_v) & \text{if } N \geq l . \end{cases}$$

This shows that $\deg P_0(x) \leq \max_{1 \leq v \leq s} \deg P_v(x)$, $\mathcal{R}(x) \in M_L^A$ and $\hat{\mathcal{R}}(X) \in \left(\frac{1}{X}\right)^l$. That is the statement of Proposition 7.4. □

8. A criterion of linear independence of special values of p-adic functions

Let $\mathbf{f}_p := \{f_{1,p}(x), \ldots, f_{s,p}(x)\}$ be a finite set of $\mathbb{C}_p$-valued functions defined on D_p. For an algebraic number field K and an element $\beta \in D_p(K)$, we denote the K-vector space spanned by $f_{1,p}(\beta), \ldots, f_{s,p}(\beta)$ by $V_K(\mathbf{f}_p, \beta)$. In this section, for $\mathbf{f}_p$ with some assumptions (see Assumptions 8.1, 8.2 and 8.3), we give an estimation of a lower bound of $\dim_K V_K(\mathbf{f}_p, \beta)$ by using the method in Sections 6 and 7. Let L be an algebraic number field, A a real number, s a natural number and

$$\theta_v : \mathbb{R}_{>A} \longrightarrow \mathbb{C}$$

be elements of M_L^A for $0 \leq v \leq s$ with $\theta_0(x) = 1$ for all $x \in \mathbb{R}_{>A}$. Firstly, we assume the following important assumption:

ASSUMPTION 8.1. We assume that $\hat{\theta}_{v,p}(X) := \iota_p \circ \iota_\infty^{-1}(\hat{\theta}_v(X) \in L\left[\left[\frac{1}{X}\right]\right]$ converges on D_p for all $1 \leq v \leq s$. Namely, $\hat{\theta}_{v,p}(X)$ can be regarded as the functions on D_p for all $1 \leq v \leq s$.

We consider a family of polynomials

$$\{A_{v,w}^{(n)}(x)\}_{0 \leq v,w \leq s, n \in \mathbb{N}} \subset L[x] ,$$

to approximate $\{\theta_v(x)\}_{0\le v\le s}$. Introduce a family of functions on $\mathbb{R}_{>A}$, $\{\mathcal{R}_w^{(n)}(x)\}_{n\in\mathbb{N}}$ defined by

$$\mathcal{R}_w^{(n)}(x) = \sum_{v=0}^{s} A_{v,w}^{(n)}(x)\theta_v(x)\,,$$

for $0 \le w \le s$. We put $\Delta_n(x) = \det((A_{v,w}^{(n)}(x))_{0\le v,w\le s}) \in L[x]$. We assume the following assumptions on $\{A_{v,w}^{(n)}(x)\}_{0\le v,w\le s,n\in\mathbb{N}} \subset L[x]$:

ASSUMPTION 8.2. Suppose that there exist an integer l and a non-empty subset $V_p \subset D_p(\overline{\mathbb{Q}})$ satisfying the following conditions:

(52) $\mathcal{R}_w^{(n)}(x) = o(x^{-ns+w+l})$ $(x \to \infty)$ for all $n \in \mathbb{N}$ and $0 \le w \le s$.

(53) We have $\Delta_n(\beta) \ne 0$ for all $\beta \in V_p$ and infinitely many $n \in \mathbb{N}$.

(54) There exists a family of functions $\{D_n : V_p \longrightarrow \overline{\mathbb{Z}} \setminus \{0\}\}_{n\in\mathbb{N}}$ satisfying

$D_n(\beta) \in \mathcal{O}_{L(\beta)}$ and $D_n(\beta)A_{v,w}^{(n)}(\beta) \in \mathcal{O}_{L(\beta)}$

for all $\beta \in V_p$, $0 \le v, w \le s$ and $n \in \mathbb{N}$.

ASSUMPTION 8.3. We use the notations as above. Suppose that $\tau(\beta) \in V_p$ for all $\beta \in V_p$, $\tau \in I_{\mathbb{Q}(\beta)}^{(p)}$ and there exist some constants $c_1, c_2, c_3 > 0$ such that the following conditions hold for enough large n.

(55) There exists function $f^{(p)} : V_p \longrightarrow \mathbb{R}_{>0}$ satisfying $|\tau D_n(\beta)| \le n^{c_1+o(1)}e^{nf^{(p)}(\tau\beta)}$

for all $\beta \in V_p$ and $\tau \in I_{\mathbb{Q}(\beta)}^{(p)}$.

(56) There exists a function $g^{(p)} : V_p \longrightarrow \mathbb{R}_{>0}$ satisfying $|\tau A_{v,w}^{(n)}(\beta)| \le n^{c_2+o(1)}e^{ng^{(p)}(\tau\beta)}$

for all $\beta \in V_p$ and $\tau \in I_{\mathbb{Q}(\beta)}^{(p)}$.

(57) There exist functions $\{E_n : V_p \longrightarrow \mathcal{O}_{\mathbb{C}_p} \setminus \{0\}\}_{n\in\mathbb{N}}$ satisfying

$|E_n(\beta)\hat{\mathcal{R}}_w^{(n)}(\beta)|_p \le n^{c_3+o(1)}|\beta|_p^{-ns}$ for all $\beta \in V_p$.

(58) There exists a function $h^{(p)} : V_p \longrightarrow \mathbb{R}_{>0}$ satisfying

$|D_n(\beta)/E_n(\beta)|_p \le n^{c_4+o(1)}e^{-nh^{(p)}(\beta)}$ for all $\beta \in V_p$.

REMARK 8.4. We use the notations as above. Without assuming (58), we have the following estimation by using Proposition 7.4:

$$|D_n(\beta)\hat{\mathcal{R}}_w^{(n)}(\beta)|_p \le n^{c_1+o(1)}|\beta|_p^{-ns}$$

for all $\beta \in V_p$, $0 \le w \le s$ and $n \in \mathbb{N}$. The assumption (58) is important to improve the estimation.

REMARK 8.5. We define $c_{v,m}$ by $\hat{\theta}_{v,p}(X) = \sum_{m=0}^{\infty} c_{v,m} \dfrac{1}{X^m}$ and suppose that there exists $c > 0$ satisfying $|c_{v,m}|_p < m^{c+o(1)}$ $(m \to \infty)$ for all $0 \le v \le s$. We also assume that there exists a family of functions $\{E_n : V_p \longrightarrow \mathcal{O}_{\mathbb{C}_p} \setminus \{0\}\}_{n \in \mathbb{N}}$ satisfying

$$E_n(\beta) A_{v,w}^{(n)}(x) \in \mathcal{O}_{\mathbb{C}_p}[x] \text{ for all } \beta \in V_p \text{ and } n \in \mathbb{N}\,.$$

Then there exists $c_3 > 0$ satisfying

$$|E_n(\beta)\hat{\mathcal{R}}_w^{(n)}(\beta)|_p \le n^{c_3} |\beta|_p^{-ns} \text{ for all } \beta \in V_p \text{ and } n \in \mathbb{N}\,.$$

Under Assumptions 8.1, 8.2 and 8.3, we obtain the following **(Type A)**$_p$ estimation of lower bound of the vector space spanned by the special values of $\{\hat{\theta}_{v,p}(x)\}_{0 \le v \le s}$.

THEOREM 8.6. *Let L be an algebraic number field, A a real number, s a natural number and*

$$\theta_v : \mathbb{R}_{>A} \longrightarrow \mathbb{C}\,,$$

be elements of M_L^A for $0 \le v \le s$ with $\theta_0(x) = 1$ for all $x \in \mathbb{R}_{>A}$. We assume Assumptions 8.1, 8.2 and 8.3 for $\{\theta_v(x)\}_{0 \le v \le s}$. We denote the set $\{(\beta, K) \in V_p \times \mathcal{A}_L \mid \beta \in K\}$ by W_p. We define a function

$$F^{(p)} : W_p \longrightarrow \mathbb{R}_{\ge 0}, \quad (\beta, K) \mapsto \frac{[K_p : \mathbb{Q}_p](h^{(p)}(\beta) + s \log |\beta|_p)}{[K : \mathbb{Q}(\beta)] \sum_{\tau \in I_{\mathbb{Q}(\beta)}} (f^{(p)}(\tau\beta) + g^{(p)}(\tau\beta))}\,.$$

Then we obtain the following estimations of dimension of vector space spanned by the special values of $\{\hat{\theta}_v(x)\}_{0 \le v \le s}$:

$$\dim_K \Big(K + K\hat{\theta}_{1,p}(\beta) + \cdots + K\hat{\theta}_{s,p}(\beta) \Big) \ge F^{(p)}(\beta, K)$$

for all $(\beta, K) \in W_p$.

Theorem 8.6 is a p-adic analog of Theorem 2.4. Since we can deduce Theorem 8.6 from Lemma 6.1 by using an argument similar to that used in the proof of Theorem 2.4, we omit the proof of Theorem 8.6.

COROLLARY 8.7. *Under the same assumption of Theorem 8.6, we obtain the following criterion of linear independence of special values of $\{\hat{\theta}_{v,p}(x)\}_{0 \le v \le s}$.*

Suppose $(\beta, K) \in W_p$ satisfies

$$s[K_p : \mathbb{Q}_p](h^{(p)}(\beta) + s \log |\beta|_p) < [K : \mathbb{Q}(\beta)] \sum_{\tau \in I_{\mathbb{Q}(\beta)}} (f^{(p)}(\tau\beta) + g^{(p)}(\tau\beta))\,.$$

Then we obtain:

$$\dim_K \Big(K + K\hat{\theta}_{1,p}(\beta) + \cdots + K\hat{\theta}_{s,p}(\beta) \Big) = s + 1\,.$$

In the next section, we prove Theorem 1.3.

9. Proof of Theorem 1.3

We use the notations of the previous section. Before starting to prove Theorem 1.3, we show the following lemma.

LEMMA 9.1. *Under the notation as above, we define a constant function*

$$E_n : D_p \longrightarrow \mathbb{Z} \setminus \{0\}, \quad \beta \mapsto S!\mu_n(A)^s A^{s(n+1)} d^S_{b+(n-1)A} d^S_{e+Mn} .$$

Then there exists a positive number c satisfying the following inequality:

$$|E_n(\beta)\hat{\mathcal{R}}^{(n)}_{w,p}(\beta)|_p \le n^c |\beta|_p^{-ns} \text{ for any sufficiently large } n . \tag{59}$$

PROOF. From the relation (42) and the definition of $A^{(n)}_{i,v_i,\mathbf{w}}(x)$ and $P^{(n)}_{\mathbf{w}}(x)$, we have the following relation:

$$\begin{aligned} &E_n(\beta)A^{(n)}_{i,v_i,\mathbf{w}}(x) \in \mathbb{Z}_p[x] \ \text{ for all } 1 \le i \le r,\ 1 \le v_i \le s_i , \\ &E_n(\beta)P^{(n)}_{\mathbf{w}}(x) \in \mathbb{Z}_p[x] . \end{aligned} \tag{60}$$

Let a be a positive rational number. We put $a_1 = a\mathrm{den}(a)$. Then we have the following inequality:

$$\left| \frac{1}{(m+a)^v} \right|_p \le (\mathrm{den}(a)m + a_1)^v = m^{v+o(1)} \ (m \to \infty) . \tag{61}$$

From Remark 8.5, we obtain the desired estimation. □

Fix a set of rational numbers $\{a_1, \ldots, a_r\} \subset \mathbb{Q}$ satisfying $0 < a_1 < \cdots < a_r \le 1$. To prove Theorem 1.3, we use Theorem 8.6 for

$$\theta_{(i,v_i)}(x) = \Phi(v_i, a_i, x) : \mathbb{R}_{>1} \longrightarrow \mathbb{C}, \quad \text{for } 1 \le i \le r,\ 1 \le v_i \le s_i .$$

PROOF OF THEOREM 1.3. We define the following functions:

$$D_n : D_p \longrightarrow \mathbb{N} \text{ by } \beta \mapsto S!\mu_n(A)^s A^{s(n+1)} d^S_{e+Mn} \mathrm{den}(\beta)^n d^S_{b+(n-1)A} ,$$

$$E_n : D_p \longrightarrow \mathbb{Z} \setminus \{0\} \text{ by } \beta \mapsto S!\mu_n(A)^s A^{s(n+1)} d^S_{b+(n-1)A} d^S_{e+Mn} ,$$

$$f^{(p)} : D_p \longrightarrow \mathbb{R}_{\ge 0} \text{ by } \beta \mapsto s\log A + s \sum_{\substack{q:\text{prime} \\ q|A}} \frac{\log q}{q-1} + S(M+A) + \log \mathrm{den}(\beta) ,$$

$$g^{(p)} : D_p \longrightarrow \mathbb{R}_{\ge 0} \text{ by } \beta \mapsto \log\max\{1, |\beta|\} + (s\log s + (2s+1)\log 2) ,$$

$$h^{(p)} : D_p \longrightarrow \mathbb{R}_{\ge 0} \text{ by } \beta \mapsto s\log|\beta|_p .$$

From Lemma 4.4, we have $\Delta^{(n)}(\beta) \ne 0$ for all $\beta \in D_p(\mathbb{Q})$. This shows that $\{A^{(n)}_{i,v_i,\mathbf{w}}(x)\}_{1\le i\le r, 1\le v_i\le s_i, \mathbf{w}\in\{\mathbf{0},\mathbf{w}_{i,j}\}} \cup \{P^{(n)}_{\mathbf{w}}(x)\}_{\mathbf{w}\in\{\mathbf{0},\mathbf{w}_{i,j}\}}$ satisfies (54) in Assumption 8.2. From Lemma 4.3, the family of functions $\{D_n : V_p \longrightarrow \mathbb{N}\}_{n\in\mathbb{N}}$ satisfy (55) in Assumption 8.2.

From Lemma 4.3, $f^{(p)}$ satisfies (56) in Assumption 8.3. From Lemma 4.1 (22), $g^{(p)}$ satisfies (57) in Assumption 8.3. From Lemma 4.2, $h^{(p)}$ satisfies (59) in Assumption 8.3. For $W_p := \{(\beta, K) |\, \beta \in K\}$, we define the following function:

$$F^{(p)} : W_p \longrightarrow \mathbb{R}_{\geq 0} \text{ by } (\beta, K) \mapsto \frac{[K_p : \mathbb{Q}_p](h^{(p)}(\beta) + s \log|\beta|_p)}{[K : \mathbb{Q}(\beta)] \sum_{\tau \in I_{\mathbb{Q}(\beta)}} (f^{(p)}(\tau\beta) + g^{(p)}(\tau\beta))} .$$

Using Theorem 8.6, we obtain:

$$\dim_K \left(K + \sum_{v_1=1}^{s_1} K \hat{\Phi}_p(v_1, a_1, \beta) + \cdots + \sum_{v_r=1}^{s_r} K \hat{\Phi}_p(v_r, a_r, \beta) \right) \geq F^{(p)}(\beta, K)$$

for all $(\beta, K) \in W_p$. The above inequality is the one that we want to prove in Theorem 1.3.

□

10. Proof of Theorem 1.5

10.1. A simultaneous Padé approximation of the Lerch function. To prove Theorem 1.5, we give a Padé approximation of the Lerch function that is different from that of Section 3. Let r be a natural number. From here to the last section, we fix r natural numbers $s_1, \dots, s_r$, and r rational numbers $a_1, \dots, a_r$ satisfying $0 < a_1 < \cdots < a_r \leq 1$ and put the following numbers:

$$s := \sum_{i=1}^{r} s_i ,$$

$$A := \text{l.c.m.}\{\text{den}(a_i)\}_{1 \leq i \leq r} ,$$

$$M := \text{l.c.m.}\{\text{den}(a_{i'} - a_i)\}_{1 \leq i, i' \leq r,\, i \neq i'} ,$$

$$e_{i',i} := M(a_{i'} - a_i) \text{ for all } 1 \leq i, i' \leq r ,$$

$$e := \max_{1 \leq i, i' \leq r} |e_{i',i}| ,$$

$$S := \max_{1 \leq i \leq r} s_i ,$$

$$T := \min_{1 \leq i \leq r} s_i .$$

In this section, we give a Padé approximation of the Lerch function $\{\Phi(v_i, x + a_i, x_1)\}_{1 \leq i \leq r, 1 \leq v_i \leq s_i + 1}$ with variable x.

For a positive integer n and $\mathbf{w} := (w_1, \dots, w_r) \in \prod_{i=1}^{r} \{0, \dots, s_i + 1\}$, we put

$$H_{\mathbf{w}}^{(n)}(x, u) := (n!)^{s+r-1} \frac{u(u+1)\cdots(u+n)}{\prod_{i=1}^{r} \left[(u + x + a_i)_n^{s_i+1} (u + x + a_i + n)^{w_i} \right]} ,$$

$$\mathcal{H}_{\mathbf{w}}^{(n)}(x, x_1) := \sum_{m=0}^{\infty} H_{\mathbf{w}}^{(n)}(x, m) x_1^{-m-1} .$$

We define a family of rational functions $\{d^{(n)}_{i,j,v_i,\mathbf{w}}(x)\}_{1\le i\le r, 0\le j\le n, 1\le v_i\le s_i+1}$ by

$$H^{(n)}_{\mathbf{w}}(x,u)=\sum_{i=1}^{r}\sum_{v_i=1}^{s_i+1}\sum_{j=0}^{n}\frac{d^{(n)}_{i,j,v_i,\mathbf{w}}(x)}{(u+x+a_i+j)^{v_i}}\,, \tag{62}$$

and a family of polynomials $\{D^{(n)}_{i,v_i,\mathbf{w}}(x,x_1),\, Q^{(n)}_{\mathbf{w}}(x,x_1)\}_{1\le i\le r, 1\le v_i\le s_i+1}\subset \mathbb{Q}(x)[x_1]$ by

$$D^{(n)}_{i,v_i,\mathbf{w}}(x,x_1)=\sum_{j=0}^{n} d^{(n)}_{i,j,v_i,\mathbf{w}}(x)x_1^{j}\,, \tag{63}$$

$$Q^{(n)}_{\mathbf{w}}(x,x_1)=\sum_{i=1}^{r}\sum_{j=1}^{n}\sum_{v_i=1}^{s_i+1}\sum_{l=0}^{j-1} d^{(n)}_{i,j,v_i,\mathbf{w}}(x)\frac{x_1^{j-1-l}}{(l+a_i)^{v_i}}\,. \tag{64}$$

In the following, unless we mention, we denote $\mathbf{w}$ as an element of $\prod_{i=1}^{r}\{0,\dots,s_i+1\}$.

REMARK 10.1

1. By the same argument of the proof of Theorem 1 in [17], we can show that

$$D^{(n)}_{i,v_i,\mathbf{w}}(x,x_1)\in\mathbb{Q}[x,x_1] \text{ and } Q^{(n)}_{\mathbf{w}}(x,x_1)\in\mathbb{Q}[x,x_1]\,.$$

2. By the same argument as is given in Remark 3.1, we can show the following:

$$\deg_{x_1} D^{(n)}_{i,w_i,\mathbf{w}}(x,x_1)=n \text{ for all } n\in\mathbb{N} \text{ with } \mathbf{w} \text{ satisfying } w_i\ge 1\,.$$

3. Since we have the following equality:

$$d^{(n)}_{i,j,v_i,\mathbf{w}}(x)=\begin{cases}\dfrac{(-1)^{s_i-v_i+1}}{(s_i-v_i+1)!}\left(\dfrac{d}{du}\right)^{s_i-v_i+1} H^{(n)}_{\mathbf{w}}(x,-u-x-a_i)(-u+j)^{s_i+1}|_{u=j} & \\ \qquad\qquad \text{for } 0\le j\le n-1,\ 1\le v_i\le s_i+1\,, & \\ \dfrac{(-1)^{w_i-v_i}}{(w_i-v_i)!}\left(\dfrac{d}{du}\right)^{w_i-v_i} H^{(n)}_{\mathbf{w}}(x,-u-x-a_i)(-u+n)^{w_i}|_{u=n} & \\ \qquad\qquad \text{for } j=n,\ 1\le v_i\le w_i\,, & \\ 0 \qquad\qquad \text{for } j=n,\ v_i>w_i\,, & \end{cases}$$

we have

$$\deg_{x} D^{(n)}_{i,v_i,\mathbf{w}}(x,x_1)=n+1 \text{ for all } 1\le i\le r,\ 1\le v_i\le s_i+1 \text{ and } \mathbf{w}\,. \tag{65}$$

For simplicity, we denote $D^{(n)}_{i,v_i,\mathbf{w}}(x,1)$ by $D^{(n)}_{i,v_i,\mathbf{w}}(x)$ and $Q^{(n)}_{\mathbf{w}}(x,1)$ by $Q^{(n)}_{\mathbf{w}}(x)$. From the definition of $\mathcal{H}^{(n)}_{\mathbf{w}}(x,\alpha)$ and the same argument of the proof of Lemma 3.2 and that of [17, Corollary 2], we obtain the following proposition.

PROPOSITION 10.2 (cf. [17, Corollary 2], [2, Corollary 5.2]). *Let* $\alpha \in \overline{\mathbb{Q}}$ *satisfying* $|\alpha| \geq 1$. *Under the notation as above, we put* $w = \sum_{i=1}^{r} w_i$. *Then we obtain* $\mathcal{H}_{\mathbf{w}}^{(n)}(x, \alpha) = o(x^{-(ns+w+n(r-1)-3)})$ $(x \to \infty)$ *and the following Padé approximation of the Lerch function*:

$$\mathcal{H}_{\mathbf{w}}^{(n)}(x,\alpha) = \sum_{i=1}^{r}\sum_{v_i=1}^{s_i+1} \begin{cases} D_{i,v_i,\mathbf{w}}^{(n)}(x,\alpha)\Phi(v_i, x+a_i, \alpha) - Q_{\mathbf{w}}^{(n)}(x,\alpha) & \text{if } \alpha \neq 1\,, \\ D_{i,v_i,\mathbf{w}}^{(n)}(x)\Phi(v_i, x+a_i, 1) - Q_{\mathbf{w}}^{(n)}(x) & \text{if } \alpha = 1\,. \end{cases} \tag{66}$$

10.2. Some estimations. We denote $(0, \ldots, 0) \in \prod_{i=1}^{r}\{0, \ldots, s_i + 1\}$ by $\mathbf{0}$. Hereafter, we fix a subset

$$\{\mathbf{w}_{i,j} := (w_{i,j}^{(1)}, \ldots, w_{i,j}^{(r)})\}_{1\leq i\leq r, 1\leq j\leq s_i+1} \subset \prod_{i=1}^{r}\{0, \ldots, s_i+1\}$$

satisfying

$$w_{i,j}^{(k)} = \begin{cases} 0 & \text{if } k \neq i\,, \\ j & \text{if } k = i\,. \end{cases}$$

We denote the determinant of $(s + r + 1) \times (s + r + 1)$ matrix

$$\begin{pmatrix} -Q_{\mathbf{0}}^{(n)}(x,x_1) & D_{1,1,\mathbf{0}}^{(n)} & \cdots & D_{1,s_1+1,\mathbf{0}}^{(n)} & \cdots & D_{r,1,\mathbf{0}}^{(n)} & \cdots & D_{r,s_r+1,\mathbf{0}}^{(n)} \\ -Q_{\mathbf{w}_{1,1}}^{(n)}(x,x_1) & D_{1,1,\mathbf{w}_{1,1}}^{(n)} & \cdots & D_{1,s_1+1,\mathbf{w}_{1,1}}^{(n)} & \cdots & D_{r,1,\mathbf{w}_{1,1}}^{(n)} & \cdots & D_{r,s_r+1,\mathbf{w}_{1,1}}^{(n)} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -Q_{\mathbf{w}_{1,s_1+1}}^{(n)}(x,x_1) & D_{1,1,\mathbf{w}_{1,s_1+1}}^{(n)} & \cdots & D_{1,s_1+1,\mathbf{w}_{1,s_1+1}}^{(n)} & \cdots & D_{r,1,\mathbf{w}_{1,s_1+1}}^{(n)} & \cdots & D_{r,s_r+1,\mathbf{w}_{1,s_1+1}}^{(n)} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -Q_{\mathbf{w}_{r,1}}^{(n)}(x,x_1) & D_{1,1,\mathbf{w}_{r,1}}^{(n)} & \cdots & D_{1,s_1+1,\mathbf{w}_{r,1}}^{(n)} & \cdots & D_{r,1,\mathbf{w}_{r,1}}^{(n)} & \cdots & D_{r,s_r,\mathbf{w}_{r,1}}^{(n)} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -Q_{\mathbf{w}_{r,s_r+1}}^{(n)}(x,x_1) & D_{1,1,\mathbf{w}_{r,s_r+1}}^{(n)} & \cdots & D_{1,s_1+1,\mathbf{w}_{r,s_r+1}}^{(n)} & \cdots & D_{r,1,\mathbf{w}_{r,s_r+1}}^{(n)} & \cdots & D_{r,s_r+1,\mathbf{w}_{r,s_r+1}}^{(n)} \end{pmatrix}$$

by $\Delta^{(n)}(x, x_1)$ for $n \in \mathbb{N}$ where we denote $D_{i,j,\mathbf{w}}^{(n)}(x, x_1)$ by $D_{i,j,\mathbf{w}}^{(n)}$.

Under the notation as above, we have the following lemma that corresponds to the assumption (54).

LEMMA 10.3 (cf. [2, Proposition 5.9], [2, Proposition 5.10]). *Let* $\Delta^{(n)}(x, x_1)$ *be as above. Then* $\Delta^{(n)}(x, \alpha)$ *has zero only at* $x \in \{-a_1, \ldots, -a_r\}$ *for* $\alpha \in \{\alpha \in \overline{\mathbb{Q}} \mid |\alpha| = 1\} \setminus \{1\}$ *and* $n \in \mathbb{N}$.

Proof of Lemma 10.3 is based on that of [2, Proposition 5.9]. Before proving Lemma 10.3, we give some preparatory lemmas.

LEMMA 10.4. *Let n be a natural number. Then* $\Delta^{(n)}(x, x_1)$ *is divisible by* $\prod_{i=1}^{r}(x + a_i)^{s_i+1}$.

LEMMA 10.5. *Let n be a natural number and fix $x_1 \in \mathbb{C}$ satisfying $|x_1| > 1$. Then we have the following relation*:

$$\lim_{\mathrm{Re}(x)\to\infty} \frac{\Delta^{(n)}(x, x_1)}{x^{s+r}} < \infty .$$

Especially, we have $\deg_x \Delta^{(n)}(x, x_1) \le s + r$.

Note that from Lemma 10.4 and Lemma 10.5, there exists a polynomial $Q(x_1) \in \mathbb{Q}[x_1]$ satisfying

$$\Delta^{(n)}(x, x_1) = Q(x_1) \prod_{i=1}^{r} (x + a_i)^{s_i+1} . \tag{67}$$

LEMMA 10.6. *Let n be a natural number. We have $\Delta^{(n)}(x, x_1) \neq 0$ and the following inequality*:

$$\deg_{x_1} \Delta^{(n)}(x, x_1) \le n(s + r) - 2 .$$

LEMMA 10.7. *Let n be a natural number. Then $\Delta^{(n)}(x, x_1)$ is divisible by x_1^n.*

LEMMA 10.8. *Let n be a natural number. Then $\Delta^{(n)}(x, x_1)$ is divisible by $(x_1 - 1)^{(s+r-1)n-2}$.*

REMARK 10.9. Lemmas 10.4, 10.5, 10.6, 10.7 and 10.8 are generalization of Lemmas 5.11, 5.13, 5.12, 5.15 and 5.16 in [2] respectively. Since Lemmas 10.4, 10.5, 10.6, 10.7 and 10.8 can be proved by the same method of Lemmas 5.11, 5.13, 5.12, 5.15 and 5.16 in [2], we omit the proof of them.

PROOF OF LEMMA 10.3. From the equality (68), Lemma 10.6, Lemma 10.7 and Lemma 10.8, there exists an element $\delta \in \mathbb{Q}^*$ satisfying

$$\Delta^{(n)}(x, x_1) = \delta x_1^n (x_1 - 1)^{(s+r-1)n-2} \prod_{i=1}^{r} (x + a_i)^{s_i+1} . \tag{68}$$

The equality (68) shows Lemma 10.3. □

REMARK 10.10. We explain the reason why we exclude $\alpha = 1$ in Theorem 1.5 for $r \ge 2$ (cf. Remark 1.6). For a set

$$\{\mathbf{w}_1, \dots, \mathbf{w}_{s+1}\} \subset \prod_{i=1}^{r} \{0, \dots, s_i + 1\} ,$$

satisfying $\mathbf{w}_i \neq \mathbf{w}_j$ for $i \neq j$. We denote the following determinant of $(s + 1) \times (s + 1)$

matrix

(69)

$$\begin{pmatrix} -Q^{(n)}_{\mathbf{w}_1}(x) & D^{(n)}_{1,2,\mathbf{w}_1}(x) & \dots & D^{(n)}_{1,s_1+1,\mathbf{w}_1}(x) & \cdots & D^{(n)}_{r,2,\mathbf{w}_1}(x) & \dots & D^{(n)}_{r,s_r+1,\mathbf{w}_1}(x) \\ -Q^{(n)}_{\mathbf{w}_2}(x) & D^{(n)}_{1,2,\mathbf{w}_2}(x) & \dots & D^{(n)}_{1,s_1+1,\mathbf{w}_2}(x) & \cdots & D^{(n)}_{r,2,\mathbf{w}_2}(x) & \dots & D^{(n)}_{r,s_r+1,\mathbf{w}_2}(x) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -Q^{(n)}_{\mathbf{w}_{s_1+1}}(x) & D^{(n)}_{1,2,\mathbf{w}_{s_1+1}}(x) & \dots & D^{(n)}_{1,s_1+1,\mathbf{w}_{s_1+1}}(x) & \cdots & D^{(n)}_{r,2,\mathbf{w}_{s_1+1}}(x) & \dots & D^{(n)}_{r,s_r+1,\mathbf{w}_{s_1+1}}(x) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -Q^{(n)}_{\mathbf{w}_{s-s_r+2}}(x) & D^{(n)}_{1,2,\mathbf{w}_{s-s_r+2}}(x) & \dots & D^{(n)}_{1,s_1+1,\mathbf{w}_{s-s_r+2}}(x) & \cdots & D^{(n)}_{r,2,\mathbf{w}_{s-s_r+2}}(x) & \dots & D^{(n)}_{r,s_r+1,\mathbf{w}_{s-s_r+2}}(x) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -Q^{(n)}_{\mathbf{w}_{s+1}}(x) & D^{(n)}_{1,2,\mathbf{w}_{s+1}}(x) & \dots & D^{(n)}_{1,s_1+1,\mathbf{w}_{s+1}}(x) & \cdots & D^{(n)}_{r,2,\mathbf{w}_{s+1}}(x) & \dots & D^{(n)}_{r,s_r+1,\mathbf{w}_{s+1}}(x) \end{pmatrix},$$

by $\tilde{\Delta}^{(n)}(x)$ for every $n \in \mathbb{N}$. For $r \geq 2$, we will show the following:

(70)

$$\tilde{\Delta}^{(n)}(x) = 0 \text{ for any sufficiently large } n \text{ and any } \{\mathbf{w}_1, \dots, \mathbf{w}_{s+1}\} \subset \prod_{i=1}^{r} \{0, \dots, s_i + 1\}.$$

Then the assumption (55) for $\{-Q^{(n)}_{\mathbf{w}_j}(x, x_1)\}_{1 \leq j \leq s+1} \cup \{D^{(n)}_{i,v_i,\mathbf{w}_j}(x)\}_{1 \leq i \leq r, 1 \leq v_i \leq s_i+1, 1 \leq j \leq s+1}$ is no longer satisfied for any $\{\mathbf{w}_1, \dots, \mathbf{w}_{s+1}\} \subset \prod_{i=1}^{r} \{0, \dots, s_i + 1\}$. For this reason, we have to exclude $\alpha = 1$ in Theorem 1.5 for $r \geq 2$. We shall prove (70). Fix a set $\{\mathbf{w}_1, \dots, \mathbf{w}_{s+1}\} \subset \prod_{i=1}^{r} \{0, \dots, s_i + 1\}$ satisfying $\mathbf{w}_i \neq \mathbf{w}_j$ for $i \neq j$. By the same argument as proof of Lemma 10.4 (cf. [2, Proposition 5.11]), we obtain

$$\prod_{i=1}^{r} (x + a_i)^{s_i} \,|\, \tilde{\Delta}^{(n)}(x) \text{ for all } n \in \mathbb{N}.$$

Especially we have

$$\deg \tilde{\Delta}^{(n)}(x) \geq s. \tag{71}$$

Next, we show the following:

$$\lim_{\mathrm{Re}(x) \to \infty} \frac{\tilde{\Delta}^{(n)}(x)}{x^s} = 0 \text{ for enough large } n. \tag{72}$$

Let i and j be integers satisfying $0 \leq i \leq r - 1$ and $2 \leq j \leq s_{i+1} + 1$. By adding the $\left(j + \sum_{l=1}^{i} s_l\right)$-th column of the matrix (69) multiplied by $\Phi(j, x + a_i, 1)$ to the first column of the matrix (69), we obtain the matrix (73) below. Note that if $i = 0$, we mean $\sum_{l=1}^{i} s_l = 0$.

(73)

$$\begin{pmatrix}
\mathcal{H}^{(n)}_{\mathbf{w}_1}(x) & D^{(n)}_{1,2,\mathbf{w}_1}(x) & \cdots & D^{(n)}_{1,s_1+1,\mathbf{w}_1}(x) & \cdots & D^{(n)}_{r,2,\mathbf{w}_1}(x) & \cdots & D^{(n)}_{r,s_r+1,\mathbf{w}_1}(x) \\
\mathcal{H}^{(n)}_{\mathbf{w}_2}(x) & D^{(n)}_{1,2,\mathbf{w}_2}(x) & \cdots & D^{(n)}_{1,s_1+1,\mathbf{w}_2}(x) & \cdots & D^{(n)}_{r,2,\mathbf{w}_2}(x) & \cdots & D^{(n)}_{r,s_r+1,\mathbf{w}_2}(x) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\mathcal{H}^{(n)}_{\mathbf{w}_{s_1+1}}(x) & D^{(n)}_{1,2,\mathbf{w}_{s_1+1}}(x) & \cdots & D^{(n)}_{1,s_1+1,\mathbf{w}_{s_1+1}}(x) & \cdots & D^{(n)}_{r,2,\mathbf{w}_{s_1+1}}(x) & \cdots & D^{(n)}_{r,s_r+1,\mathbf{w}_{s_1+1}}(x) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\mathcal{H}^{(n)}_{\mathbf{w}_{s-s_r+2}}(x) & D^{(n)}_{1,2,\mathbf{w}_{s-s_r+2}}(x) & \cdots & D^{(n)}_{1,s_1+1,\mathbf{w}_{s-s_r+2}}(x) & \cdots & D^{(n)}_{r,2,\mathbf{w}_{s-s_r+2}}(x) & \cdots & D^{(n)}_{r,s_r+1,\mathbf{w}_{s-s_r+2}}(x) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\mathcal{H}^{(n)}_{\mathbf{w}_{s+1}}(x) & D^{(n)}_{1,2,\mathbf{w}_{s+1}}(x) & \cdots & D^{(n)}_{1,s_1+1,\mathbf{w}_{s+1}}(x) & \cdots & D^{(n)}_{r,2,\mathbf{w}_{s+1}}(x) & \cdots & D^{(n)}_{r,s_r+1,\mathbf{w}_{s+1}}(x)
\end{pmatrix}.$$

Since the determinant of (69) is equal to that of (73) by definition, the determinant of (73) coincides with $\tilde{\Delta}^{(n)}(x)$. Thus, it suffices to show that the determinant of (73) is zero. Denote the $(1,q)$-th cofactor matrix of the matrix (73) by $\tilde{\Delta}^{(n)}_q(x)$. We calculate the cofactor expansion of the matrix (73) at the first row, we obtain:

$$\tilde{\Delta}^{(n)}(x) = \sum_{q=1}^{s+1}(-1)^{q+1}\mathcal{H}^{(n)}_{\mathbf{w}_q}(x)\tilde{\Delta}^{(n)}_q(x)\,.$$

From the definition of $\mathcal{H}^{(n)}_{\mathbf{w}_q}(x)$, we have

$$\mathcal{H}^{(n)}_{\mathbf{w}_q}(x)\tilde{\Delta}^{(n)}_q(x) = (n!)^{s+r-1}\sum_{m=0}^{\infty}\frac{(m)_{n+1}\tilde{\Delta}^{(n)}_q(x)}{\prod_{i=1}^{r}\left[(m+x+a_i)_n^{s_i+1}(m+x+a_i+n)^{w_{q,i}}\right]}\,,$$

where $w_{q,i}$ is the i-th factor of $\mathbf{w}_q$. Since we have $\deg\tilde{\Delta}^{(n)}_q(x) \le s(n+1)$ (see Remark 10.1 3 (65)),

$$\begin{aligned}\left|\frac{(m)_{n+1}\tilde{\Delta}^{(n)}_q(x)}{x^s\prod_{i=1}^{r}\left[(m+x+a_i)_n^{s_i+1}(m+x+a_i+n)^{w_{q,i}}\right]}\right| &\le \left|\frac{\tilde{\Delta}^{(n)}_q(x)}{x^{s(n+1)}}\frac{(m)_{n+1}}{\prod_{i=1}^{r}(m+x+a_i)_n}\right| \\ &\le \left|\frac{\tilde{\Delta}^{(n)}_q(x)}{x^{s(n+1)}}\frac{(m)_{n+1}}{(m+x)^{n+3}}\frac{1}{x^{n(r-1)-3}}\right|\,,\end{aligned}$$

and $r \ge 2$, we obtain $\lim_{\mathrm{Re}(x)\to\infty}\dfrac{\tilde{\Delta}^{(n)}(x)}{x^s} = 0$ for enough large n. From the relations (71) and (72), we obtain (70).

We have the following lemma that corresponds to Assumption (54).

LEMMA 10.11 (cf. [2, Proposition 5.5]). *Let α be a non-zero algebraic number. Then we obtain the following relations*:

(74) $$d_n^{s_i+1}d_{e+Mn}^{s+r-s_i-1}p^{[n/(p-1)]}p^{\mathrm{ord}_p(\mathrm{den}(a_i))}\max\{1,|\alpha|_p\}^n D^{(n)}_{i,v_i,\mathbf{w}}(x,\alpha) \in \mathcal{O}_{\mathbb{C}_p}[x]\,,$$

$$d_n^{s_i+1} d_{e+Mn}^{s+r-s_i-1} p^{[n/(p-1)]} p^{\mathrm{ord}_p(\mathrm{den}(a_i))} \max\{1, |\alpha|_p\}^n Q_{\mathbf{w}}^{(n)}(x, \alpha) \in \mathcal{O}_{\mathbb{C}_p}[x] .$$

Let b be a rational number satisfying $b + a_i \neq 0$ for all $1 \leq i \leq r$. Then we also obtain the following relations:

$$d_n^{s_i+1} d_{e+Mn}^{s+r-s_i-1} \mu_n(\mathrm{den}(b + a_i))\mathrm{den}(b)\mathrm{den}(\alpha)^n D_{i,v_i,\mathbf{w}}^{(n)}(b, \alpha) \in \mathcal{O}_{\mathbb{Q}(\alpha)} , \tag{75}$$

$$d_n^{s_i+1} d_{e+Mn}^{s+r-s_i-1} \mu_n(\mathrm{den}(b + a_i))\mathrm{den}(b)\mathrm{den}(\alpha)^n Q_{\mathbf{w}}^{(n)}(b, \alpha) \in \mathcal{O}_{\mathbb{Q}(\alpha)} .$$

PROOF. From the equality (62), we have

(76)

$$d_{i,j,v_i,\mathbf{w}}^{(n)}(x) = \begin{cases} \dfrac{1}{(s_i - v_i + 1)!}\left(\dfrac{d}{du}\right)^{s_i - v_i + 1} H_{\mathbf{w}}^{(n)}(x, -u - x - a_i)(-u + j)^{s_i+1}|_{u=j} \\ \qquad \text{for } 0 \leq j \leq n-1, 1 \leq v_i \leq s_i + 1 , \\ \dfrac{1}{(w_i - v_i)!}\left(\dfrac{d}{du}\right)^{w_i - v_i} H_{\mathbf{w}}^{(n)}(x, -u - x - a_i)(-u + n)^{w_i}|_{u=n} \\ \qquad \text{for } j = n, 1 \leq v_i \leq w_i , \\ 0 \qquad \text{for } j = n, v_i > w_i . \end{cases}$$

We give natural numbers which are divisible by the denominator of $d_{i,j,v_i,\mathbf{w}}^{(n)}(x)$. Firstly, we calculate

$$\frac{1}{(s_i - v_i + 1)!}\left(\frac{d}{du}\right)^{s_i - v_i + 1} H_{\mathbf{w}}^{(n)}(x, -u - x - a_i)(-u + j)^{s_i+1}|_{u=j}$$

for $0 \leq j \leq n - 1, 1 \leq v_i \leq s_i + 1$. We have the following equality

(77)

$$\begin{aligned} &H_{\mathbf{w}}^{(n)}(x, -u - x - a_i)(-u + j)^{s_i+1} \\ &= \frac{(n!)^{s+r-1}(-u - x - a_i)(-u - x - a_i + 1)\cdots(-u - x - a_i + n)}{\prod_{i' \neq i}\left(\prod_{j=0}^{n-1}(-u + (a_{i'} - a_i) + j)^{s_{i'}+1}(-u + (a_{i'} - a_i) + n)^{w_{i'}}\right)\left(\prod_{j'=0, j' \neq j}^{n-1}(-u + j')^{s_i+1}(-u + n)^{w_i}\right)} \\ &= \frac{(n!)^{s_i}(-u - x - a_i)\cdots(-u - x - a_i + n)}{\prod_{j'=0, j' \neq j}^{n-1}(-u + j')^{s_i+1}(-u + n)^{w_i}} \prod_{i' \neq i} I_{i',n}(u)^{w_{i'}} I_{i',n-1}(u)^{s_{i'}+1-w'_i} , \end{aligned}$$

where the functions $I_{i',n}(u)$ and $I_{i',n-1}(u)$ are as follows:

$$I_{i',n}(u) := \frac{n!}{(-u + a_{i'} - a_i)\cdots(-u + a_{i'} - a_i + n)} \text{ for } i' \neq i , \tag{78}$$

$$I_{i',n-1}(u) := \frac{n!}{(-u + a_{i'} - a_i)\cdots(-u + a_{i'} - a_i + n - 1)} \text{ for } i' \neq i . \tag{79}$$

From the proof of Bel [2, p.204] , we have the following equality:

$$\frac{(n!)^{s_i}(-u - x - a_i)\cdots(-u - x - a_i + n)}{\prod_{j'=0, j' \neq j}^{n-1}(-u + j')^{s_i+1}(-u + n)^{w_i}} = F(u)G(u)^{s_i}H(u) \tag{80}$$

where

$$F(u)=\frac{(-u-x-a_i)_n}{(-u)_{n+1}}(-u+j),\ G(u)=\frac{n!}{(-u)_{n+1}}(-u+j),$$

and

$$H(u)=(n-u)^{s_i+1-w_i}(-u-x-a_i+n).$$

Also we have the equalities:

(81)

$$\frac{n!}{(-u+a_{i'}-a_i)\cdots(-u+a_{i'}-a_i+n)}=\sum_{j'=0}^{n}(-1)^{j'}\frac{n!}{j'!(n-j')!}\frac{1}{-u+a_i'-a_i+j'},$$

(82)

$$\frac{n!}{(-u+a_{i'}-a_i)\cdots(-u+a_{i'}-a_i+n-1)}=\sum_{j'=0}^{n-1}(-1)^{j'}\frac{n!}{j'!(n-j'-1)!}\frac{1}{-u+a_i'-a_i+j'}.$$

For a non-negative integer ν, we denote $\frac{1}{\nu!}\left(\frac{d}{du}\right)^{\nu}$ by ∂_ν. From the equality (77), we obtain

(83)
$$\begin{aligned}&\frac{1}{(s_i-v_i+1)!}\left(\frac{d}{du}\right)^{s_i-v_i+1}H_{\mathbf{w}}^{(n)}(x,-u-x-a_i)(-u+j)^{s_i+1}|_{u=j}\\&=\sum_{\boldsymbol{\nu}}\partial_{\nu_0}(F)|_{u=j}\prod_{k_1=1}^{s_i}\partial_{\nu_{k_1}}(G)|_{u=j}\partial_{\nu_{s_i+1}}(H)|_{u=j}\\&\times\prod_{i'\neq i}\prod_{k_2=1}^{w_{i'}}\partial_{\nu_{k_2}}(I_{i',n})|_{u=j}\prod_{k_3=1}^{s_{i'}+1-w_{i'}}\partial_{\nu_{k_3}}(I_{i',n-1})|_{u=j}.\end{aligned}$$

Here the sum $\sum_{\boldsymbol{\nu}}$ stands for all possible summation arising from the Leibniz rule. Note that the 'index' of $\partial_{\nu_{s_i+1}}$ is ν_{s_i+1} and it is not $\nu_{s_i}+1$. As for the second case of (76), we obtain a similar presentation to (83). The argument to deduce the presentation for the second case is the same as (77) to (82) for the first case. Finally, by applying the same argument as the proof of [2, Proposition 5.5] to these representations for (76), we conclude (74) and (75). □

For a rational number b satisfying $b+a_i\neq 0$ for all $1\leq i\leq r$, we denote l.c.m.$\{\mathrm{den}(b+a_i)\}_{1\leq i\leq r}$ by $B(b)$. We define the following functions:

$$D_n: D_p(\mathbb{Q})\longrightarrow \mathbb{Z}\setminus\{0\} \text{ by } b\mapsto d_n^{S+1}d_{e+Mn}^{s+r-T-1}\mu_n(B(b))\mathrm{den}(b)\mathrm{den}(\alpha)^n,$$

$$f^{(p)}: D_p(\mathbb{Q})\longrightarrow \mathbb{R}_{\geq 0} \text{ by } b\mapsto S+M(s+r-T-1)+\sum_{\substack{q:\text{prime}\\ q|B(b)}}\frac{\log q}{q-1}+\log\mathrm{den}(\alpha).$$

Then from Proposition 10.11, $\{D_n\}_{n\in\mathbb{N}}$ satisfies the assumption (56) and there exists $c_1>0$

such that

(84) $$|D_n(b)| \le n^{c_1} e^{nf^{(p)}(b)}\,.$$

The inequality (84) corresponds to the assumption (56). We have the following estimation that corresponds to Assumption (57).

LEMMA 10.12. *Let β be a complex number satisfying $\beta + a_i \notin \mathbb{Z}_{\le 0}$ for all $1 \le i \le r$. If $n \in \mathbb{N}$ is enough large, then there exists $c > 0$ which is independent of n and satisfies the following inequality*:

(85) $$\max_{1\le i\le r, 1\le v_i \le s_i+1} \{|D^{(n)}_{i,v_i,\mathbf{w}}(\beta,\alpha)|, |Q^{(n)}_{\mathbf{w}}(\beta,\alpha)|\} \le n^c \max\{1, |\alpha|^n\} \exp(ns\log 2)\,.$$

PROOF. We fix an enough large natural number k satisfying the following conditions:

(86) $$|a_{i_1} - a_{i_2}| > \frac{2}{k} \text{ and } 1 > |a_{i_1} - a_{i_2}| + \frac{2}{k} \text{ for all } 1 \le i_1, i_2 \le r, i_1 \ne i_2\,.$$

Firstly, we give an upper bound of $\{|d^{(n)}_{i,j,v_i,\mathbf{w}}(\beta)|\}_{1\le j\le n, 1\le v_i\le s_i+1}$ for a fixed i.

We fix $1 \le i \le r$, $1 \le v_i \le s_i + 1$ and $1 \le j \le n$. Using the definition of $d^{(n)}_{i,j,v_i,\mathbf{w}}(\beta)$ given by (62), we get

(87) $$d^{(n)}_{i,j,v_i,\mathbf{w}}(\beta) = \frac{1}{2\pi\sqrt{-1}} \int_{|u+j+\beta+a_i|=\frac{1}{k}} H^{(n)}_{\mathbf{w}}(x,u)(u+\beta+a_i+j)^{v_i-1} du\,.$$

From the equality (87) and the definition of $Q^{(n)}_{\mathbf{w}}(u)$, we obtain

(88)

$$\begin{aligned} |d^{(n)}_{i,j,v_i,\mathbf{w}}(\beta)| &\le k^{-v_i} \sup_{|u+\beta+a_i+j|=\frac{1}{k}} |H^{(n)}_{\mathbf{w}}(x,u)| \\ &\le k^{-v_i} \sup_{|u+\beta+a_i+j|=\frac{1}{k}} \left| \frac{(n!)^{s+r-1}(u)_{n+1}}{\prod_{i'=1}^{r} \left[(u+\beta+a_{i'})_n^{s_{i'}+1} (u+\beta+n+a_{i'})^{w_{i'}} \right]} \right|\,. \end{aligned}$$

We give an upper bound of $\left| \frac{(n!)^{s+r-1}(u)_{n+1}}{\prod_{i'=1}^{r} \left[(u+\beta+a_{i'})_n^{s_{i'}+1} (u+\beta+n+a_{i'})^{w_{i'}} \right]} \right|$. We have the following inequalities for $u \in \{u \in \mathbb{C} \mid |u+\beta+a_i+j| = \frac{1}{k}\}$:

(89) $$\begin{aligned} |(u)_{n+1}| &= |(u+\beta+a_i+j-\beta-a_i-j)\cdots(u+\beta+a_i+j+n-\beta-a_i-j)| \\ &\le (f+j)!(f+n-j)! \end{aligned}$$

where f is a natural number satisfying $\beta + a_i + \frac{1}{k} \le f$. Estimating a lower bound of $|(u+\beta+a_{i'})_n|$ and $|u+\beta+a_{i'}+n|$ for $u \in \{u \in \mathbb{C} \mid |u+\beta+a_i+j| = \frac{1}{k}\}$, we give a lower bound of $|u+\beta+a_i+j+(a_{i'}-a_i)+(l-j)|$ for $1 \le i' \le r$, $0 \le l \le n$:

(In the case of $i' = i$)

$$|u+\beta+a_i+j+(a_{i'}-a_i-\beta)+(l-j)| \geq \begin{cases} \frac{1}{k} & \text{if } l = j-1, j, j+1\,, \\ j-l-1 & \text{if } j-1 > l\,, \\ l-j-1 & \text{if } l > j+1\,. \end{cases} \tag{90}$$

(In the case of $i' > i$)

$$|u+\beta+a_i+j+(a_{i'}-a_i)+(l-j)| \geq \begin{cases} \frac{1}{k} & \text{if } l = j-1, j\,, \\ j-l-1 & \text{if } j-1 > l\,, \\ l-j & \text{if } l > j\,. \end{cases} \tag{91}$$

(In the case of $i' < i$)

$$|u+\beta+a_i+j+(a_{i'}-a_i)+(l-j)| \geq \begin{cases} \frac{1}{k} & \text{if } l = j, j+1\,, \\ j-l & \text{if } j > l\,, \\ l-j-1 & \text{if } l > j+1\,. \end{cases} \tag{92}$$

From the inequalities (92), (91) and (92), we have the following estimation for $1 \leq i' \leq r$:

$$\begin{aligned} |(u+\beta+a_{i'})_n| &= \prod_{l=0}^{n-1} |u+\beta+a_{i'}+l| \\ &= \prod_{l=0}^{n-1} |u+\beta+a_i+j+(a_{i'}-a_i)+(l-j)| \\ &\geq \frac{(n-j)!j!}{k^3n^3}\,. \end{aligned} \tag{93}$$

We also have the inequality:

$$|u+\beta+a_{i'}+n| = |u+\beta+a_i+j+(a_{i'}-a_i)+(n-j)| \geq \frac{1}{k}\,, \tag{94}$$

for $1 \leq i' \leq r$. From the inequalities (88), (89), (93) and (94), we obtain

$$\begin{aligned} & k^{-v_i} \sup_{|u+\beta+a_i+j|=\frac{1}{k}} \left| \frac{(n!)^{s+r-1}(u)_{n+1}}{\prod_{i'=1}^{r} (u+a_{i'})_n{}^{s_{i'}+1}(u+n+a_{i'})^{w_{i'}}} \right| \\ & \leq n^{c_0} \frac{(n!)^{s+r-1} j!(n-j)!}{((n-j)!j!)^{s+r}} \\ & = n^{c_1} \binom{n}{j}^{s+r-1} \\ & \leq n^{c_2} 2^{sn} \end{aligned} \tag{95}$$

where c_0, c_1, c_2 are positive constants. From the inequality (95) and the definition of $D^{(n)}_{i,v_i,\mathbf{w}}(\beta, \alpha)$ and $Q^{(n)}_{\mathbf{w}}(\beta, \alpha)$, we obtain the desired estimation. □

We define the following function on $D_p(\mathbb{Q})$:

$$g^{(p)} : D_p(\mathbb{Q}) \longrightarrow \mathbb{R}_{\geq 0} \text{ by } b \mapsto \log \max\{1, |\alpha|\} + s \log 2 \,.$$

Then there exists $c_2 > 0$ such that

(96)

$$\max_{1\leq i\leq r, 1\leq v_i\leq s_i+1, \mathbf{w}\in\{\mathbf{0},\mathbf{w}_{i,j}\}} \{|D^{(n)}_{i,v_i,\mathbf{w}}(b, \alpha)|, |Q^{(n)}_{\mathbf{w}}(b, \alpha)|\} \leq n^{c_2} e^{n g^{(p)}(b)} \text{ for all } b \in D_p(\mathbb{Q}) \,.$$

10.3. Proof of Theorem 1.5. We use the notations of the previous section. Before starting to prove Theorem 1.5, we show the following Lemma relating with the convergence of $\hat{\zeta}_p(v, a, \alpha, x)$.

LEMMA 10.13. *Let $a \in \mathbb{Q} \cap \mathbb{Z}_p$ and $\alpha \in \{\alpha \in \overline{\mathbb{Q}} \mid |\alpha| = 1\}$ satisfies $\alpha = 1$ or $1 \leq |\alpha - 1|_p$. Let v be a natural number satisfying $v \geq 2$ (resp. $v \geq 1$) if $\alpha = 1$ (resp. $1 \leq |\alpha - 1|_p$). Then $\hat{\zeta}_p(v, a, \alpha, x)$ converges on D_p.*

PROOF. We claim that the set $\{|B_k(a, \alpha)|_p\}_{k\in\mathbb{N}}$ is bounded for $(a, \alpha) \in \big(\mathbb{Q} \cap \mathbb{Z}_p\big) \times \{\alpha \in \overline{\mathbb{Q}} \mid |\alpha| = 1\}$. Firstly, we assume $\alpha = 1$. Then we have the equalities

$$B_k(a, 1) = \sum_{i=0}^{k} \binom{k}{i} B_i a^{k-i} \,.$$

Using Theorem of Clausen-Von Staudt giving an upper bound of p-adic absolute value of Bernoulli numbers and the assumption for a, we obtain that the set $\{|B_k(a, 1)|_p\}_{k\in\mathbb{N}}$ is bounded. Secondly, we assume $1 \leq |\alpha - 1|_p$. Note that, from the definition of $B_k(a, \alpha)$, we have the following equality:

$$\frac{Te^{aT}}{\alpha e^T - 1} = \sum_{k=0}^{\infty} B_k(a, \alpha) \frac{T^k}{k!} = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} \binom{k}{i} B_{i,\alpha} a^{k-i} \right) \frac{T^{k+1}}{k!}$$

where $B_{i,\alpha}$ is defined by the generating function $\dfrac{1}{\alpha e^T - 1} = \sum_{k=0}^{\infty} \dfrac{B_{k,\alpha}}{k!} T^k$. Since $1 \leq |\alpha - 1|_p$, we have that the set $\{|B_{k,\alpha}|_p\}_{k\in\mathbb{N}}$ is bounded [12, p.24]. This completes the proof of this lemma. □

By Lemma 10.13, if we assume that $a_1, \dots, a_r \in \mathbb{Z}_p \cap \mathbb{Q}$ and $\alpha \in \{\alpha \in \overline{\mathbb{Q}} \mid |\alpha| = 1\}$ satisfies $\alpha = 1$ or $|\alpha - 1|_p \leq 1$, then $\hat{\zeta}_p(v_i, a_i, \alpha, x)$ converges on D_p. We define the following functions:

(97) $E_n : D_p(\mathbb{Q}) \longrightarrow \mathcal{O}_{\mathbb{C}_p} \setminus \{0\}$ by $b \mapsto d_n^{S+1} d_{e+Mn}^{s+r-T-1} p^{[n/(p-1)]} p^{\mathrm{ord}_p(A)} \max\{1, |\alpha|_p\}^n$.

Then, from Lemma 10.13, the set of coefficients of $\hat{\zeta}_p(v_i, a_i, \alpha, x)$ is bounded for all $1 \le i \le r$, $1 \le v_i \le s_i$. Then, from the relation (74), there exists $c_3 > 0$ such that

$$|E_n(b)\hat{\mathcal{R}}^{(n)}_{w,p}(b,\alpha)|_p \le n^{c_3}|b|_p^{-nT} \text{ for some constant } c_3 > 0\,, \tag{98}$$

(cf. Remark 8.5). The inequality (98) corresponds to (59) in Assumption 8.3. We define $h^{(p)} : D_p(\mathbb{Q}) \longrightarrow \mathbb{R}_{\ge 0}$ by

$$h^{(p)}(b) = \sum_{\substack{q:\text{prime}\\ q|A}} \frac{\log q}{q-1} - \frac{\log p}{p-1} + \log \operatorname{den}(\alpha) - \log \max\{1, |\alpha|_p\}\,.$$

PROOF OF THEOREM 1.5. We use the notations as before. Fix the following set:

$$\{(a_1,\alpha),\dots,(a_r,\alpha)\} \subset \left(\mathbb{Q}\cap\mathbb{Z}_p\right) \times \{\alpha \in \overline{\mathbb{Q}} \mid |\alpha| = 1\} \text{ satisfying } 0 < a_1 < \cdots < a_r \le 1$$

and use Theorem 8.6 for

$$\theta_{(i,v_i)}(x) = \Phi(v_i, x + a_i, \alpha) : \mathbb{R}_{>0} \longrightarrow \mathbb{C},\quad 1 \le i \le r,\ 1 \le v_i \le s_i + 1\,.$$

From Section 10.1, we defined the following functions:

$$D^{(n)}_{i,v_i,\mathbf{w}}(x,x_1) = \sum_{j=0}^{n} d^{(n)}_{i,j,v_i,\mathbf{w}}(x) x_1^j\,,$$

$$Q^{(n)}_{\mathbf{w}}(x,x_1) = \sum_{i=1}^{r}\sum_{j=0}^{n}\sum_{v_i=1}^{s_i+1}\sum_{l=0}^{j-1} d^{(n)}_{i,j,v_i,\mathbf{w}}(x) \frac{x_1^{j-1-l}}{(l+a_i)^{v_i}}\,,$$

$$\mathcal{H}^{(n)}_{\mathbf{w}}(x,x_1) := \sum_{m=0}^{\infty} H^{(n)}_{\mathbf{w}}(x,m) x_1{}^{-m-1}\,.$$

In Lemma 10.13, the set $\{\theta_{(i,v_i)}(x)\}_{1\le i\le r,\ 1\le v_i \le s_i+1}$ satisfies Assumption 8.1. From Section 10.2, we defined the following five functions:

$D_n : D_p(\mathbb{Q}) \longrightarrow \mathbb{N}$ by $b \mapsto d_n^{S+1} d_{e+Mn}^{s+r-T-1} \mu_n(B(b)) \operatorname{den}(b) \operatorname{den}(\alpha)^n$,

$E_n : D_p(\mathbb{Q}) \longrightarrow \mathbb{Z}\setminus\{0\}$ by $b \mapsto d_n^{S+1} d_{e+Mn}^{s+r-T-1} p^{[n/(p-1)]} p^{\operatorname{ord}_p(A)} \max\{1, |\alpha|_p\}^n$,

$f^{(p)} : D_p(\mathbb{Q}) \longrightarrow \mathbb{R}_{\ge 0}$ by $b \mapsto S + M(s+r-T-1) + \sum_{\substack{q:\text{prime}\\ q|B(b)}} \frac{\log q}{q-1} + \log \operatorname{den}(\alpha)$,

$g^{(p)} : D_p(\mathbb{Q}) \longrightarrow \mathbb{R}_{\ge 0}$ by $b \mapsto \log \max\{1, |\alpha|\} + s \log 2$,

$h^{(p)} : D_p(\mathbb{Q}) \longrightarrow \mathbb{R}_{\ge 0}$ by $b \mapsto \sum_{\substack{q:\text{prime}\\ q|A}} \frac{\log q}{q-1} - \frac{\log p}{p-1} + \log \operatorname{den}(\alpha) - \log \max\{1, |\alpha|_p\}$.

In the lemmas in Section 10.1 and 10.2, we can easily check that the functions $\mathcal{R}_{\mathbf{w}}^{(n)}(x,\alpha) := \mathcal{H}_{\mathbf{w}}^{(n)}(x,\alpha)$, $\{D_{i,v_i,\mathbf{w}}^{(n)}(x,x_1)\}_{1\le i\le r, 0\le j\le s_i+1, \mathbf{w}\in\{\mathbf{0},\mathbf{w}_{i,j}\}} \cup \{-Q_{\mathbf{w}}^{(n)}(x,x_1)\}_{\mathbf{w}\in\{\mathbf{0},\mathbf{w}_{i,j}\}}$, D_n, E_n, $f^{(p)}$, $g^{(p)}$ and $h^{(p)}$ satisfy Assumptions 8.2 and 8.3. Applying Theorem 8.6, we obtain the desired estimation in Theorem 1.5. □

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