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Ravels Arising from Montesinos Tangles

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Abstract. A ravel is a spatial graph which is non-planar but contains no non-trivial knots or links. We characterize when a Montesinos tangle can become a ravel as the result of vertex closure with and without replacing some number of crossings by vertices.

1. Introduction

One of the earliest results in spatial graph theory was the discovery by Suzuki [11] in 1970 of an embedding of an abstractly planar graph which had the property that it was non-planar but every subgraph of the embedding was planar. Note that a graph is said to be *abstractly planar* if it can be embedded in $\mathbb{R}^2$, and a particular embedding of a graph in $\mathbb{R}^3$ is said to be *planar* if there is an ambient isotopy of it into $\mathbb{R}^2 \subseteq \mathbb{R}^3$. Two years after Suzuki's result, Kinoshita [5] found an embedding of a θ_3 graph which had this property. Many results about such embeddings have been obtained since then, though several different terms are used to refer to them. In particular, we have the following definition.

DEFINITION 1.1. An embedding G of an abstractly planar graph in $\mathbb{R}^3$ is said to be *almost unknotted* (equivalently *almost trivial*, *minimally knotted*, or *Brunnian*) if G is non-planar but $G - \{e\}$ is planar for any edge e of G.

One of the most significant results in the study of almost unknotted graphs is the result obtained by Kawauchi [4] and Wu [14] that every abstractly planar graph without valence one vertices has an almost unknotted embedding.

We are now interested in a larger class of embedded graphs defined below.

DEFINITION 1.2. An embedding G of an abstractly planar graph in $\mathbb{R}^3$ is said to be a *ravel* if G is non-planar but contains no non-trivial knots or links.

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Any almost unknotted graph G is a ravel, unless G is topologically a non-trivial knot or a Brunnian link. However, the converse is not true. For example, starting with an almost unknotted embedding of a graph G, add an additional edge e' parallel to an existing edge e to get a new embedded graph G'. Since G was almost unknotted, G' will contain no non-trivial knots or links. However, the removal of the edge e' will not make G' planar, and hence G' is a ravel that is not almost unknotted.

The term *ravel* was originally coined as a way to describe hypothetical molecular structures whose complexity results from "an entanglement of edges around a vertex that contains no knots or links" [1]. The first molecular ravel to be identified was a metal-ligand complex synthesized by Feng Li et al. in 2011 [6]. In order to formalize the notion of entanglement about a vertex, we require the "entanglement" to be properly embedded in a ball. If we bring the endpoints of the edges in the boundary sphere together into a single vertex, we obtain a spatial graph which is known as the *vertex closure* $V(T)$ of the entanglement T. In Figure 1.1, we illustrate an entanglement whose vertex closure is a ravel.

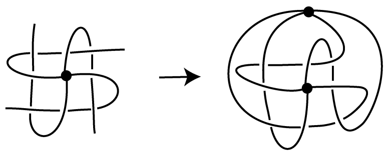

FIGURE 1.1. The entanglement on the left becomes a ravel when the endpoints are brought together on the right.

In this paper, we characterize when a Montesinos tangle can become a ravel as the result of vertex closure with and without replacing some number of crossings by vertices. In particular, our main results are the following.

THEOREM 3.3. *Let $T = T_1 + \cdots + T_n$ be a Montesinos tangle such that n is minimal and not both T_1 and T_n are trivial vertical tangles. If T is rational, then the vertex closure $V(T)$ is planar. If T is not rational and some rational subtangle T_i has ∞-parity, then $V(T)$ contains a non-trivial knot or link. Otherwise, $V(T)$ is a ravel.*

THEOREM 4.7. *Let $T = T_1 + \cdots + T_n$ be a projection of a Montesinos tangle in standard form, and let T' be obtained from T by replacing at least one crossing by a vertex. Then the vertex closure $V(T')$ is a ravel if and only if T' is an exceptional vertex insertion.*

While we postpone defining an exceptional vertex insertion until Section 4, Theorem 4.7 has the following more easily stated corollary.

COROLLARY 4.8. *Let $T = T_1 + \cdots + T_n$ be a projection of a Montesinos tangle in standard form, and let T' be obtained from T by replacing at least one crossing by a vertex. If $V(T')$ is a ravel, then precisely one T_i has ∞-parity.*

2. Background

For completeness we include some well-known definitions and results about knots, links, and tangles.

DEFINITION 2.1. A 2-string tangle T in a ball B is said to be *rational* if there is an ambient isotopy of B setwise fixing ∂B that takes T to a trivial tangle.

DEFINITION 2.2. The *sum* and *product* of tangles R and S are shown in Figure 2.1.

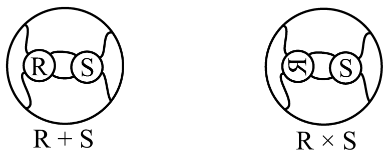

FIGURE 2.1. The sum and product of tangles R and S.

DEFINITION 2.3. A tangle is said to be *Montesinos* if it can be written as the sum of finitely many rational tangles. A tangle is said to be *algebraic* or *arborescent* if it can be written in terms of sums and products of finitely many rational tangles.

Note that Montesinos tangles and algebraic tangles are not necessarily 2-string tangles since they may contain simple closed curves in addition to the two strings.

DEFINITION 2.4. Let T be a 2-string tangle. The knot or link obtained by joining the NE and SE points together and the NW and SW points together is called the *denominator closure of* T, and denoted by $D(T)$. The knot or link obtained by joining the NW and NE points together and the SW and SE points together is called the *numerator closure of* T, and denoted by $N(T)$.

DEFINITION 2.5. A 2-string tangle is said to have ∞-*parity* if the NW and SW boundary points are on the same strand, and 0-*parity* if the NW and NE boundary points are on the same strand.

DEFINITION 2.6. Any tangle obtained from a trivial horizontal tangle by twisting together the NE and SE ends is said to be a *horizontal* tangle.

We will use the following results repeatedly in our proofs.

WOLCOTT'S THEOREM ([13]). Let T be a rational tangle. Then $D(T)$ is the unknot if and only if T is a horizontal tangle; and $D(T)$ is an unlink if and only if T is a trivial vertical tangle.

SCHUBERT'S THEOREM ([9]). Let L_1 and L_2 be knots or links. Then $L_1 \# L_2$ is trivial if and only if both L_1 and L_2 are trivial.

THISTLETHWAITE'S THEOREM ([12]). A reduced alternating projection of a link has the minimum number of crossings.

3. Vertex closure of rational and Montesinos tangles

DEFINITION 3.1. Let T be a 1-string or 2-string tangle (possibly with additional closed components) in a ball B. The embedded graph $V(T)$ obtained by bringing the endpoints of the string(s) together into a single vertex w in ∂B is said to be the *vertex closure of* T and w is said to be the *closing vertex*.

We begin with the following observation about when the vertex closure of a tangle is planar.

LEMMA 3.2. *Let T be a 2-string tangle with tangle ball B. Then the vertex closure $V(T)$ of T is planar if and only if T is rational.*

PROOF. It follows from the definition of rational that T is rational if and only if it can be made planar by moving the endpoints of the strands of T around in ∂B. However, moving the endpoints of the strands around in ∂B corresponds to moving the edges of $V(T)$ about the closing vertex. So T is rational if and only if $V(T)$ can be made planar by moving the edges of $V(T)$ about the vertex. □

The following theorem characterizes when the vertex closure of a Montesinos tangle is a ravel.

THEOREM 3.3. *Let $T = T_1 + \cdots + T_n$ be a Montesinos tangle such that n is minimal and not both T_1 and T_n are trivial vertical tangles. If T is rational, then the vertex closure $V(T)$ is planar. If T is not rational and some rational subtangle T_i has ∞-parity, then $V(T)$ contains a non-trivial knot or link. Otherwise, $V(T)$ is a ravel.*

PROOF. We know by Lemma 3.2 that if T is rational, then $V(T)$ is planar. So we assume that $n > 1$. Since n is minimal, none of the T_i is horizontal. Without loss of generality we can assume that T_n is not a trivial vertical tangle.

First suppose that at least one of the rational subtangles has ∞-parity. Let T_i be the rightmost such tangle in the sum $T = T_1 + \cdots + T_n$. Thus both ends of the NE-SE strand of T_i can be extended to the right until they are joined together at the closing vertex w, giving us a loop L_1 (illustrated in grey on the left in Figure 3.1), though we may have $i = n$.

Suppose that $i < n$. Then the loop L_1 is a connected sum of the denominator closure $D(T_n)$ together with a (possibly trivial) knot to the left. Since T_n does not have ∞-parity, $D(T_n)$ has a single component. Now since T_n is not horizontal, by Wolcott's Theorem $D(T_n)$ is a non-trivial knot, and hence by Schubert's Theorem, L_1 is a non-trivial knot contained in $V(T)$.

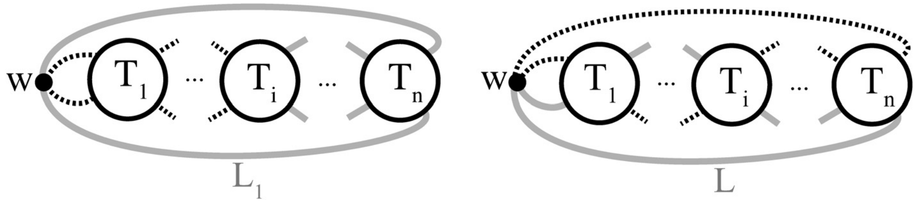

FIGURE 3.1. On the left T_i has ∞-parity and on the right no subtangle has ∞-parity.

Next suppose that $i = n$ and T_n is the only T_j with ∞-parity. Then we can extend both ends of the NW-SW strand of T_n to the left until they join together at the closing vertex w. We denote this loop by L_2. Since $n > 1$, L_2 is the connected sum of $D(T_1)$ and another (possibly trivial) knot. Now since T_1 is not horizontal, by Wolcott's Theorem and Schubert's Theorem L_2 is a non-trivial knot in $V(T)$.

Now suppose that some rational subtangle in addition to T_n has ∞-parity. Let T_i be the tangle with ∞-parity that is closest to T_n. Then both ends of the NE-SE strand of T_n can be extended rightward to w to obtain a loop L_1; and both ends of the NW-SW strand of T_n can be extended leftward until they are joined together in T_i, to obtain a loop L_2. Then the link $L = L_1 \cup L_2$ is the connected sum of $D(T_n)$ with some possibly trivial knot. Now since T_n is not a trivial vertical tangle, by Wolcott's Theorem and Schubert's Theorem, L is a non-trivial link in $V(T)$.

Finally, suppose that no T_i has ∞-parity. Then $V(T)$ is an embedding of the wedge of two circles and hence $V(T)$ cannot contain a two component link. Let L denote the vertex closure of a single strand of T. Since no T_i has ∞-parity, L passes through each T_i exactly once, as illustrated by the grey arcs on the right side of Figure 3.1. Now since each T_i is rational, by Lemma 3.2, each individual $V(T_i)$ is planar. It follows that the vertex closure of each of the single strands $T_i \cap L$ is unknotted. Now the loop L is the connected sum of the loops $V(T_1 \cap L), \ldots, V(T_n \cap L)$, each of which is unknotted. Hence L is a trivial knot. Thus $V(T)$ contains no non-trivial knots or links. However, since T is not rational, we know by Lemma 3.2 that $V(T)$ is non-planar. Hence in this case $V(T)$ is a ravel. □

The tangle in Figure 3.2 illustrates why Theorem 3.3 has the hypothesis that not both T_1 and T_n are trivial vertical tangles. In this case, $V(T_1 + T_2)$ is planar even though $T_1 + T_2$ is a non-rational Montesinos tangle.

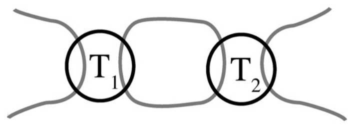

FIGURE 3.2. A Montesinos tangle where both T_1 and T_2 are trivial vertical tangles.

COROLLARY 3.4. *Let T be a non-rational algebraic tangle written as the sum and product of rational tangles $T_1, \dots, T_n$ where either $n > 2$ or not both T_1 and T_2 are trivial vertical tangles. If each strand of T passes through each T_i exactly once, then $V(T)$ is a ravel.*

PROOF. Observe that by our hypotheses, $V(T)$ must be a wedge of two circles. Thus the argument is analogous to the last case in the proof of Theorem 3.3. □

The algebraic tangle T in Figure 3.3 illustrates that the converse of Corollary 3.4 does not hold. In particular, the grey strand does not pass through T_1 or T_2. Observe that the vertex closure $V(T)$ contains no non-trivial knots or links. However, since T is non-rational, it follows from Lemma 3.2 that $V(T)$ is non-planar. Thus $V(T)$ is a ravel.

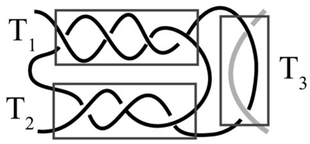

FIGURE 3.3. A counterexample to the converse of Corollary 3.4.

4. Vertex closure with crossing replacement

We are now interested in whether we can obtain a ravel from a projection of a Montesinos tangle by replacing some number of crossings by vertices and taking the vertex closure. In this case, we need to specify what types of projections we are considering.

DEFINITION 4.1. A projection of a rational tangle T is said to be in *alternating 3-braid form* if it is alternating and has the form of Figure 4.1, where each box A^i consists of some number of horizontal twists, and this number is non-zero for all $i > 1$.

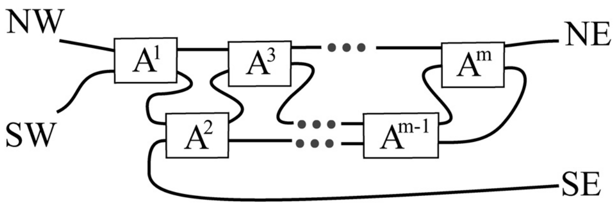

FIGURE 4.1. The 3-braid form of a rational tangle.

It follows from Schubert [10] and more recently Kauffman and Lambropoulou [3] that every rational tangle has a projection in alternating 3-braid form.

DEFINITION 4.2. A projection of a Montesinos tangle T is said to be in *standard form* if it is expressed as $T = T_1 + \cdots + T_n$, where each T_i is a non-trivial rational tangle in alternating 3-braid form and n is minimal.

Note that every Montesinos tangle with no trivial vertical tangle as a summand has a projection in standard form, though this projection may not be alternating.

DEFINITION 4.3. Let T be a projection of a knot, link, or tangle. The embedded graph obtained from T by replacing some number of crossings by vertices of valence 4 is denoted by T' and referred to as an *insertion of vertices* into T.

Note that if T is a tangle then T' is not technically an embedded graph, because its endpoints are not vertices. However, for convenience we will abuse notation and refer to T' as an embedded graph. Now let $T = T_1 + \cdots + T_n$ be a projection of a Montesinos tangle in standard form. Then T_i' denotes the subgraph of T' obtained from T_i by vertex insertion, and $(A_i^j)'$ denotes the subgraph obtained from the jth box of twists A_i^j in T_i by vertex insertion. If there are no vertices in T_i' or in $(A_i^j)'$, then we write $T_i' = T_i$ or $(A_i^j)' = A_i^j$, respectively.

The following result shows that a ravel cannot occur in the special case where a single crossing is replaced by a vertex and $V(T')$ is a θ_4 graph (i.e., the graph consists of two vertices and four edges between them).

THEOREM 4.4 (Farkas, Flapan, Sullivan [2]). *Let $T = T_1 + \cdots + T_n$ be a projection of a Montesinos tangle in standard form with $n > 1$, and let T' be obtained from T by replacing a single crossing by a vertex such that the vertex closure $V(T')$ is a θ_4 graph. Then $V(T')$ contains a non-trivial knot and hence is not a ravel.*

To see the necessity of the hypothesis that $V(T')$ is a θ_4 graph consider the Montesinos tangle in standard form illustrated on the left in Figure 4.2. By replacing a crossing in T_2 with a vertex and taking the vertex closure as illustrated on the right, we obtain a ravel which is not a θ_4 graph.

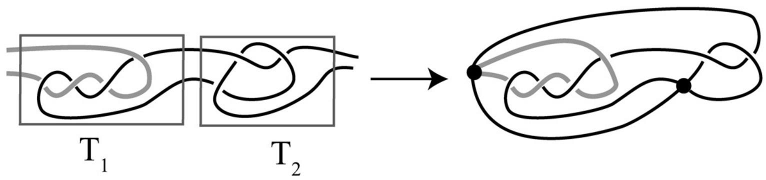

FIGURE 4.2. A Montesinos tangle which becomes a ravel by inserting one vertex and taking the vertex closure.

We will now consider the case where we replace any number of crossings of a Montesinos tangle in standard form by vertices. We begin with some technical definitions.

DEFINITION 4.5. Let T be a projection of a rational tangle in alternating 3-braid form with boxes $A^1, \ldots, A^m$ as illustrated in Figure 4.1. A vertex or crossing x of T' is said to be to the *right of* a vertex or crossing y if either x and y are in the same box $(A^i)'$ and y is to the right of x in $(A^i)'$, or x is in the box $(A^i)'$ and y is in the box $(A^j)'$ and $i > j$.

Observe that given a rational tangle T in 3-braid form, a subtangle R of T containing consecutive boxes $A^j, \ldots, A^m$ (illustrated in Figure 4.3) is itself a rational tangle in 3-braid form.

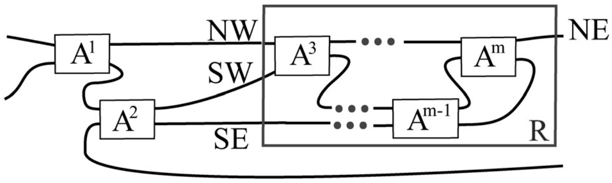

FIGURE 4.3. The subtangle R is itself a rational tangle in 3-braid form.

DEFINITION 4.6. Let $T = T_1 + \cdots + T_n$ be a projection of a Montesinos tangle in standard form, and let T' be obtained by replacing some non-zero number of crossings by vertices. Then T' is said to be an *exceptional vertex insertion* if all of the following conditions hold.

(1) There exists precisely one T_j with ∞-parity, and T'_j has no vertices.
(2) For all $k \neq j$, T'_k contains exactly one vertex v_k, and v_k is in $(A^2_k)'$ or possibly in $(A^3_k)'$ if A^2_k has a single crossing.
(3) For all $k \neq j$, the subtangle $R_k \subseteq T_k$ containing the boxes of T_k to the right of the vertex v_k has at least two crossings.
(4) For all $k \neq j$, T'_k has a loop containing v_k.

We see as follows that the insertion of vertices illustrated for the tangle in Figure 4.2 is exceptional.

(1) T_1 is the only T_i with ∞-parity, and T'_1 has no vertices.
(2) T'_2 contains exactly one vertex, and it is in the second box of T'_2. (Note that the first box of T'_2 has zero crossings).
(3) The subtangle $R_2 \subseteq T_2$ has two crossings.
(4) T'_2 has a loop containing its vertex.

Figure 4.4 illustrates a generalization of the exceptional vertex insertion in Figure 4.2. Here T_3 is any tangle with ∞-parity; for each $k \neq 3$, T'_k contains exactly one vertex and it replaces the only crossing in A^2_k; and R_k is any rational tangle containing at least two crossings such that T'_k has a loop containing v_k. As in Figure 4.2, we obtain a ravel by taking the vertex closure of this exceptional vertex insertion.

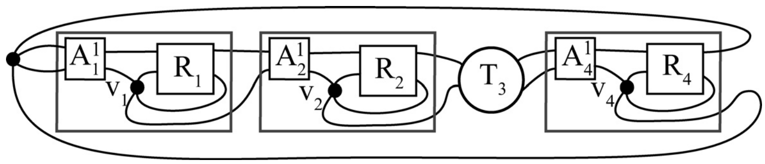

FIGURE 4.4. A ravel obtained by an exceptional vertex insertion together with vertex closure.

The remainder of the paper is devoted to proving the following theorem.

THEOREM 4.7. *Let $T = T_1 + \cdots + T_n$ be a projection of a Montesinos tangle in standard form, and let T' be obtained from T by replacing at least one crossing by a vertex. Then the vertex closure $V(T')$ is a ravel if and only if T' is an exceptional vertex insertion.*

Observe that requirement (4) of an exceptional vertex insertion implies that if T' is an exceptional vertex insertion, then $V(T')$ cannot be a θ_4 graph. Thus Theorem 4.7 is a generalization of Theorem 4.4. Theorem 4.7 immediately implies the following more simply stated corollary.

COROLLARY 4.8. *Let $T = T_1 + \cdots + T_n$ be a projection of a Montesinos tangle in standard form, and let T' be obtained from T by replacing at least one crossing by a vertex. If $V(T')$ is a ravel, then precisely one T_i has ∞-parity.*

Observe that if $T = T_1 + \cdots + T_n$ is a projection of a non-rational Montesinos tangle in standard form, then no T_i is horizontal since otherwise n would not be minimal. Also, by the definition of standard form, no T_i is a trivial tangle. Finally, because every vertex in $V(T')$ has valence 4, no arc in T' is forced to terminate at a vertex. This means that any arc in a T'_i can be extended to go from one of the points NE, SE, NW, SW of T'_i to another. We will make use of these facts together with the following simplifying assumptions that allow us to remove unnecessary crossings in any T'_i.

Simplifying Assumptions

(1) If there are vertices in some box $(A_i^j)'$ of T'_i, then we can untwist about them to remove all of the crossings of $(A_i^j)'$. Thus we assume that there are no crossings in any box containing a vertex.
(2) If there is a single crossing to the right of the rightmost vertex of some T'_i, then the crossing can be removed by untwisting about the vertex as illustrated in Figure 4.5. Thus we assume that there are either zero crossings or at least two crossings to the right of the rightmost vertex in any T'_i.

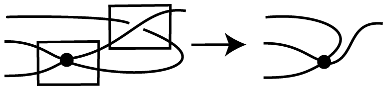

FIGURE 4.5. A single crossing to the right of the rightmost vertex of T_i' can be removed.

The rest of the paper is organized as follows. In Section 5, we prove two lemmas that we will use to prove the forward direction of Theorem 4.7. We then prove the forward direction in Section 6, and prove the backward direction in Section 7.

5. Lemmas for the Forward Direction

LEMMA 5.1. *Let T be a projection of a non-trivial rational tangle in 3-braid form. Suppose that T' has at least one vertex and there are no crossings to the right of its rightmost vertex v^R. Then for any pair of distinct points p_1 and p_2 in T', there is a simple path between p_1 and p_2 in T'.*

PROOF. Observe from Figure 4.1 that if a path P starts at the NE point of T and goes leftward, it will go through the rightmost box A^m precisely once. In particular, once a path exits from A^m, it cannot return to A^m. Thus both strands of T must go through A^m.

Recall that for all $i \neq 1$, the box A^i contains a non-zero number of crossings. Since there are no crossings to the right of v^R, this means that v^R occurs in the rightmost box $(A^m)'$ of T' as illustrated in Figure 5.1. Thus both strands of T are involved in the crossing that becomes v^R.

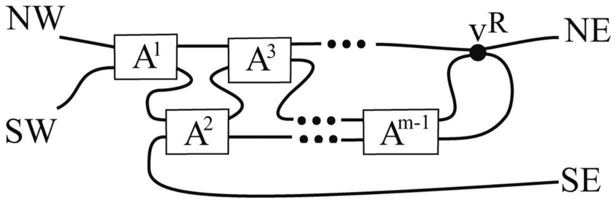

FIGURE 5.1. Both strands of T are part of the crossing that becomes v^R.

Now consider distinct points p_1 and p_2 in T'. Since both strands of T are part of the crossing that becomes v^R, there are paths P_1 and P_2 in T' joining both p_1 and p_2 to v^R. Thus $P = P_1 \cup P_2$ is a path in T' between p_1 and p_2. By removing any loops in P we obtain a simple path joining p_1 and p_2. □

It follows from Lemma 5.1 that if there are no crossings to the right of the rightmost vertex of T', then there is a simple path in T' between any pair of the NW, SW, NE, SE points of T'. We use Lemma 5.1 to prove our next lemma.

LEMMA 5.2. *Let $T = T_1 + \cdots + T_n$ be a projection of a Montesinos tangle written in standard form, and let T' be obtained from T by replacing at least one crossing by a vertex. Suppose that every T_i' containing a vertex has at most one crossing to the right of its rightmost vertex v_i^R. Then $V(T')$ is not a ravel.*

PROOF. We assume that $V(T')$ does not contain any non-trivial knots or links, and we will prove that $V(T')$ can be isotoped into the plane.

By Simplifying Assumption (2), we can assume that no T_i' has any crossings to the right of its rightmost vertex v_i^R. Thus any T_i' that contains a vertex must have v_i^R in its rightmost box $(A_i^m)'$. Now we see in Figure 5.2 that we can remove all of the crossings between v_i^R and the next vertex to its left in T_i', or all of the crossings in T_i' if v_i^R is the only vertex in T_i'. Thus we assume there are no such crossings.

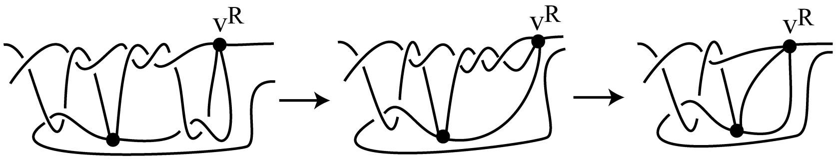

FIGURE 5.2. We remove all of the crossings between v_i^R and the next vertex to the left in T_i'.

We now sequentially prove the following list of claims showing that we can remove all of the crossings of $V(T')$ to obtain a planar embedding.

(1) Every T_i' contains a vertex.
(2) $V(T')$ can be simplified so that there is at most one crossing between any pair of adjacent vertices in each T_i'.
(3) All crossings to the left of the leftmost vertex in each T_i' can be removed.
(4) All of the crossings of $V(T')$ can be removed.

CLAIM 1. *Every T_i' contains a vertex.*

First we consider the case where T_1' is the only T_i' containing a vertex. Then there are no crossings to the right of its rightmost vertex, and hence by Lemma 5.1 there is a simple path L_1 in T_1' between its NE and SE points. Now we extend the ends of L_1 to the right until either they meet in some T_i with ∞-parity or at the closing vertex w. This gives us a simple closed curve L. Since there are no crossings in T_1 to the right of its rightmost vertex, L is the connected sum of $D(T_2)$ together with a possibly trivial knot to its right. Now, since T is in standard form, T_2 is not horizontal and not a trivial vertical tangle. Thus by Wolcott's Theorem, $D(T_2)$ is a non-trivial knot or link. It now follows from Schubert's Theorem that L is a non-trivial knot or link. As this is contrary to our assumption, this case does not occur.

Thus we now assume that for some i, T'_{i+1} contains a vertex and T'_i does not. Let v^R_{i+1} be the rightmost vertex of T'_{i+1}. Since there are no crossings in T'_{i+1} to the right of v^R_{i+1}, we can apply Lemma 5.1 to obtain a simple path L_1 in T'_{i+1} between its NW and SW points. We will now argue that there is also a simple path between the NW and SW points of T_i whose interior is to the left of T_i.

Suppose no T'_k to the left of T'_i contains a vertex or has ∞-parity. Then there are disjoint simple paths P_i and Q_i going leftwards from the NW and SW points of T_i to the closing vertex w. In this case, $L_2 = P_i \cup Q_i$ is a simple path between the NW and SW points of T_i whose interior is to the left of T_i.

Thus we assume that either some T'_k to the left of T'_i contains a vertex or some T'_k to the left of T'_i has ∞-parity and contains no vertices. Let T'_k be the closest such subgraph to the left of T_i. If T'_k contains a vertex, then there are no crossings to the right of its rightmost vertex, and hence by Lemma 5.1 there is a simple path in T'_k between its NE and SE points. If $T_k = T'_k$ has ∞-parity, then the NE-SE strand is a simple path in T_k. Thus in either case T'_k contains a simple path between its NE and SE points. By combining this path in T'_k with the strands of all T_j with $k < j < i$ and the arcs between these T_j from T'_k to T'_i, we obtain a simple path L_2 between the SW and NW points of T_i whose interior is to the left of T_i. Figure 5.3 illustrates the paths L_1 and L_2 as dotted arcs.

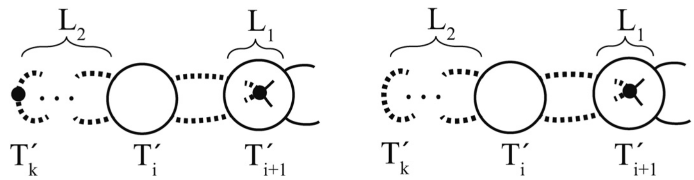

FIGURE 5.3. On the left, T'_k contains a vertex; and on the right, T'_k has ∞-parity.

Now in any of the above cases, let L denote the arc L_1 (in T'_{i+1}) together with the arc L_2 (to the left of T'_{i+1}), as well as the strands of T_i, and the arcs joining T'_i and T'_{i+1}. If T_i has ∞-parity, then L has two components, and otherwise L has a single component. In either case, observe that L is the connected sum of some (possibly trivial) knot that lies to the left of T_i, together with the denominator closure $D(T_i)$, and some (possibly trivial) knot that lies to the right of T_i. Now, as in the case at the beginning of the claim, L is a non-trivial knot or link. As this is contrary to our assumption, this proves Claim 1.

Hence from now on we assume that for every i, T_i' contains a vertex, and hence v_i^R is in the rightmost box of T_i'. Thus it follows from Lemma 5.1 that every pair of distinct points in any T_i' is joined by a simple path in T_i'.

CLAIM 2. *$V(T')$ can be simplified so that there is at most one crossing between any pair of adjacent vertices in each T_i'.*

Suppose that some T_i' has two or more crossings between a pair of adjacent vertices v_1 and v_2 contained in boxes $(A_i^j)'$ and $(A_i^k)'$, respectively. Note that by Simplifying Assumption (1), we can assume that there are no crossings in any box containing a vertex. Thus the boxes $(A_i^j)'$ and $(A_i^k)'$ must be distinct. Hence without loss of generality $j < k$. Thus v_2 is to the right of v_1. Also, as we saw at the beginning of the proof, we can assume that there are no crossings between v_i^R and the next vertex to its left. Hence v_2 cannot be the rightmost vertex of T_i'.

Now let B be a ball containing v_1 and v_2 together with the portion of T_i' between v_1 and v_2 as illustrated in Figure 5.4; and let a and b be the points of $\partial B \cap T_i'$ which are separated from v_1 and v_2 by crossings, as indicated in the figure. Note that since v_1 and v_2 are adjacent vertices, there are no other vertices in B. Thus either $T_i' \cap B$ contains two edges which go between v_1 and v_2 creating a simple closed curve K and a disjoint arc A going from a to b (as illustrated on the left in Figure 5.4), or $T_i' \cap B$ contains a single arc A joining a and b which goes through both v_1 and v_2 (as illustrated on the right in Figure 5.4).

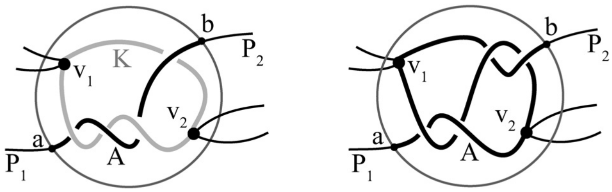

FIGURE 5.4. The ball B contains the portion of T_i' between v_1 and v_2.

Now since v_1 is contained in the box $(A_i^j)'$, there is a path P_1 going leftward from a to the NW, SW, or SE point of T_i' which does not pass through any box $(A_i^t)'$ with $t \geq j$. Also, there is a path P_2 going rightward from b to the rightmost vertex v_i^R, and then to the NE point of T_i' which does not pass through any $(A_i^s)'$ with $s \leq k$. In particular, neither P_1 nor P_2 contains v_1 or v_2.

We will now define a simple closed curve J that contains $P_1 \cup P_2 \cup A$. We first do this in the case where P_1 goes from a to the SE point of T_i' as illustrated by the dotted arc in Figure 5.5. In this case, if $i \neq n$, we extend P_1 and P_2 rightward to the SW and NW points of T_{i+1}' respectively. Then by Claim 1 and Lemma 5.1 applied to T_{i+1}', we can join P_1 and P_2

by a simple path in T'_{i+1}. If $i = n$, then we can extend P_1 and P_2 rightward until they meet at w. Thus either way we can join P_1 and P_2 by a simple path. This gives us a simple path P from a to b whose interior is disjoint from B. Now let J denote the simple closed curve $P \cup A$. Then J meets ∂B only in the points a and b.

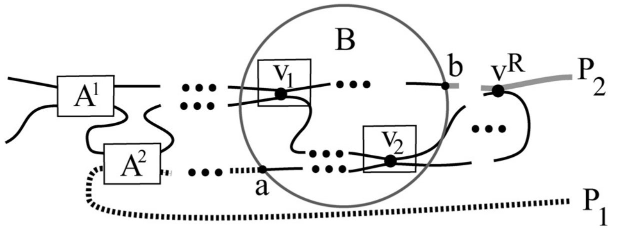

FIGURE 5.5. There is a simple path P from a to b consisting of P_1, P_2, and an arc in T'_{i+1}.

Next suppose that P_1 does not go to the SE point of T'_i. Since P_1 does not go to the NE point of T'_i, without loss of generality we can assume P_1 goes from a to the SW point of T'_i as illustrated by the dotted arc in Figure 5.6. By Claim 1 and Lemma 5.1, we can now extend P_1 and P_2 leftward and rightward respectively until they meet at the closing vertex w giving us a simple path P from a to b whose interior is disjoint from B. Now let J denote the simple closed curve $P \cup A$. Again J meets ∂B only in the points a and b.

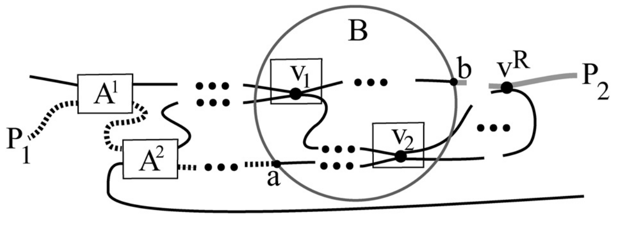

FIGURE 5.6. We extend P_1 leftward and P_2 rightward until they meet at the closing vertex w.

In either of the above cases, if the arc A contains v_1 and v_2, we let $L = J$, and otherwise we let $L = J \cup K$. Now since T'_i has at least two crossings between v_1 and v_2, the vertex closure $V(L \cap B)$ contains at least two crossings and is reduced and alternating. Thus it follows from Thistlethwaite's Theorem and Schubert's Theorem that L is a non-trivial knot or link in $V(T')$. As this is contrary to our assumption, we have proven Claim 2.

Hence from now on we assume that there is at most one crossing between any pair of adjacent vertices in each T'_i.

CLAIM 3. *All crossings to the left of the leftmost vertex in each T'_i can be removed.*

If T is rational, then we can remove all of the crossings to the left of the leftmost vertex v^L by untwisting about the closing vertex w from left to right, as illustrated in Figure 5.7. Thus for the rest of the proof of this claim we assume that T is not rational, and hence $n > 1$.

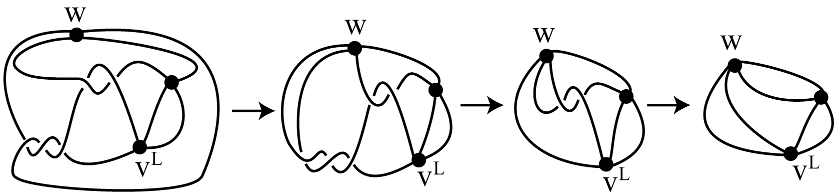

FIGURE 5.7. If T is rational, we can remove all of the crossings to the left of v^L.

Let T_i' be the leftmost subgraph of T' that contains at least one crossing to the left of its leftmost vertex v_i^L. By Simplifying Assumption (1), there are no crossings in the box with v_i^L. Thus $(A_i^1)'$ cannot contain any vertices, and hence $A_i^1 = (A_i^1)'$. Suppose that A_i^1 contains a crossing. Let c denote the leftmost crossing of A_i^1; let R denote a ball whose intersection with $V(T')$ is $(T_1 + \cdots + T_{i-1})'$; and let F denote the part of T_i' to the right of c (see the left side of Figure 5.8).

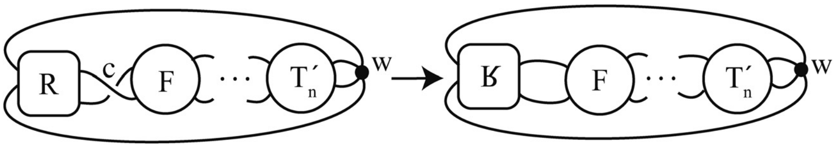

FIGURE 5.8. Removing a crossing of A_i'.

We flip R over to remove the crossing c. This adds a crossing to the left of R which can be removed by untwisting the strands around the closing vertex w. Thus we get the illustration on the right of Figure 5.8.

We repeat this operation until we have removed all of the crossings in A_i^1. This proves the claim in the case where v_i^L is in $(A_i^1)'$ or $(A_i^2)'$. Thus we assume for the sake of contradiction that v_i^L is not in $(A_i^1)'$ or $(A_i^2)'$.

Next suppose there is only one crossing in T_i' to the left of v_i^L. Since every box $(A_i^k)'$ with $k > 1$ must either contain a crossing or a vertex, this means that $(A_i^2)'$ contains one crossing and v_i^L is in $(A_i^3)'$. Hence $(A_i^2)'$ has no vertices and $(A_i^3)'$ has no crossings. Thus we have the illustration on the left of Figure 5.9, where the ball R contains the subgraphs $T_1', \ldots,$

T'_{i-1} and the ball F contains the boxes $(A_i^j)'$ with $j > 3$. We can now remove the crossing in A_i^2 by flipping both R and F and untwisting the strands around w as illustrated on the right side of Figure 5.9. Thus we assume there are at least two crossings in T'_i to the left of v_i^L.

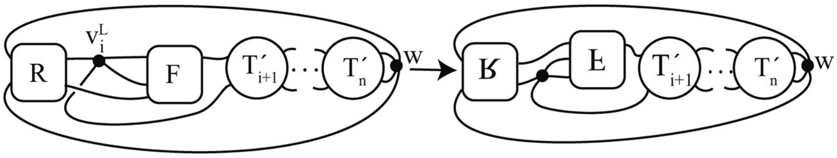

FIGURE 5.9. Removing a single crossing in A_i^2 when v_i^L is in $(A_i^3)'$.

Now let B be a ball containing v_i^L together with the part of T'_i that is to the left of v_i^L as illustrated in Figure 5.10. Since T'_i contains the vertex v_i^R in its rightmost box, the arc marked x can be extended rightward by a simple path to the NE point of T'_i. Now the NW and SW points of T'_i are joined together either by an arc through v_{i-1}^R or through w in the case where $i = 1$, and the NE and SE points of T'_i are joined together either by an arc through T'_{i+1} or through w if $i = n$. These arcs, together with the bold black and grey arcs in Figure 5.10, give us one or two simple closed curves which we denote by L. Since $V(L \cap B)$ is reduced and alternating and has at least two crossings, it follows from Thistlethwaite's Theorem and Schubert's Theorem that L is a non-trivial knot or link. As this contradicts our assumption, this case cannot arise.

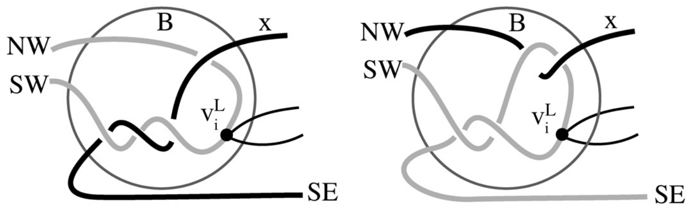

FIGURE 5.10. We can extend the bold black and grey arcs to get one or two simple closed curves.

Thus we have proven Claim 3. Hence from now on we assume that there are no crossings to the left of the leftmost vertex in every T'_i.

It follows from our hypotheses together with Claims 1–3 that the only crossings remaining in $V(T')$ are isolated crossings between two adjacent vertices within a single T'_i.

Let x denote the leftmost crossing in $V(T')$. Then x is between a pair of vertices v_a and v_b in some T'_i. Since there is at most one crossing between any pair of adjacent vertices v_a

and v_b are in boxes $(A_i^j)'$ and $(A_i^{j+2})'$ respectively, and the crossing x is in the box $(A_i^{j+1})' = A_i^{j+1}$. Let G denote a ball around all of the boxes $(A_i^k)'$ with $k < j$ in T_i', and let F denote a ball around all of the boxes $(A_i^k)'$with $k > j+2$ in T_i' (see Figure 5.11).

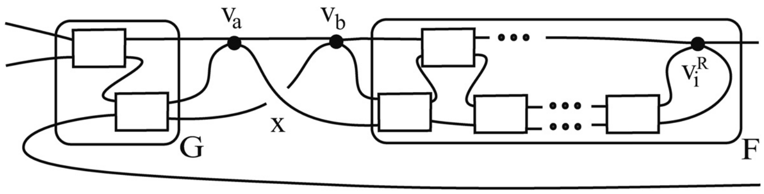

FIGURE 5.11. T_i' has a single crossing between v_a and v_b.

Let B denote a ball containing all of the T_s' such that $s < i$. Then $V(T')$ is the embedded graph illustrated on the top of Figure 5.12. Note that B and G contain no crossings since x is the leftmost crossing in $V(T')$. We now flip F over to remove the crossing x from $V(T')$ as illustrated on the bottom of Figure 5.12. We repeat the above argument to sequentially remove all of the remaining crossings in the projection of $V(T')$.

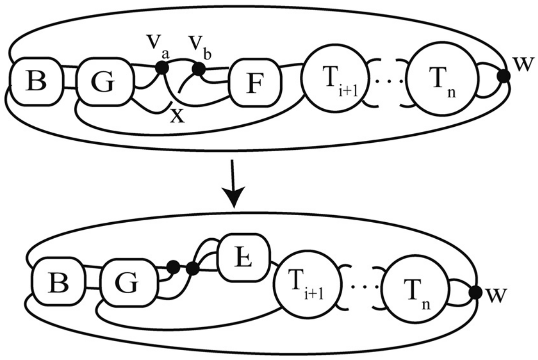

FIGURE 5.12. We can flip F over to remove the crossing between v_a and v_b.

This gives us a planar embedding of $V(T')$. Thus $V(T')$ is not a ravel. □

6. Proof of the Forward Direction of Theorem 4.7

PROPOSITION 6.1. *Let $T = T_1 + \cdots + T_n$ be a projection of a Montesinos tangle in standard form, and let T' be obtained from T by replacing at least one crossing by a vertex. Suppose that the vertex closure $V(T')$ is a ravel. Then T' is an exceptional vertex insertion.*

PROOF. Given any k such that T'_k contains a vertex, let v_k^R denote the rightmost vertex of T'_k. Then T'_k has the form illustrated in Figure 6.1, where v_k^R may be in the top or the bottom row, and R_k is a ball containing all of the boxes of T'_k that are to the right of v_k^R.

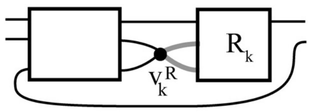

FIGURE 6.1. v_k^R is the rightmost vertex of T'_k, and R_k contains all of the crossings of T'_k that are to the right of v_k^R.

We now prove that T' is an exceptional vertex insertion by sequentially proving the following list of claims.

(1) Some T_j has ∞-parity, and T'_j has no vertices.
(2) For all $k \neq j$, T_k does not have ∞-parity, T'_k has at least one vertex, and R_k has at least two crossings.
(3) For all $k \neq j$, T'_k contains exactly one vertex v_k.
(4) For all $k \neq j$, the vertex v_k is in $(A_k^2)'$ or possibly in $(A_k^3)'$ if A_k^2 has a single crossing.
(5) For all $k \neq j$, the tangle T'_k has a loop containing v_k.

We begin by proving Observation 1 which will be used in the proof of Claim 1.

OBSERVATION 1. If for some k, there are at least two crossings in R_k, then there is no path in T'_k between its NE and SE point. Hence T'_k contains paths from its NE and SE points to its NW and SW points.

To prove Observation 1, suppose that there are at least two crossings in R_k and there is a path in T'_k between its NE and SE point. Then we can extend this path rightward until its ends meet either at a vertex in some T'_j, in some T_j with ∞-parity, or at the closing vertex w. This gives us a simple closed curve L_1. If L_1 contains v_k^R, then all of the crossings of R_k are in L_1. In this case, let $L = L_1$. Otherwise, as can be seen in Figure 6.1, there is a grey simple closed curve $L_2 \subseteq T'_k$ containing v_k^R such that all of the crossings in R_k are contained in $L_1 \cup L_2$. In this case, we let $L = L_1 \cup L_2$.

Now let B_k denote the tangle ball for T_k. Then L is the union of $L \cap B_k$ together with an arc outside of B_k. Since $L \cap B_k$ contains at least two crossings and is reduced and alternating, by Thistlethwaite's Theorem L is a non-trivial knot or link. However, this contradicts the hypothesis that $V(T')$ is a ravel. Thus we have proven the observation.

CLAIM 1. *Some T_j has ∞-parity, and T'_j has no vertices.*

Since $V(T')$ is a ravel, we know by Lemma 5.2 that some T'_k containing a vertex has at least two crossings in R_k. Now by Observation 1, there is an arc P_k in T'_k from its NE point to its SW or NW point. Suppose the endpoints of P_k can be extended rightward and leftward to w, so that we obtain a simple closed curve L_1. If L_1 contains v_k^R, let $L = L_1$. Otherwise, there is another simple closed curve $L_2 \subseteq T'_k$ containing v_k^R such that all of the crossings in R_k are contained in $L_1 \cup L_2$. In this case, we let $L = L_1 \cup L_2$. Now as in the proof of Observation 1, this implies that L is a non-trivial knot or link contradicting the hypothesis that $V(T')$ is a ravel. Thus P_k cannot be extended so that it passes through every T'_j.

Thus there must exist some j such that T_j has ∞-parity. Now suppose that T'_j has at least one vertex. If there are less than two crossings to the right of the rightmost vertex v_j^R, then by Simplifying Assumption (2), we can assume there are no crossings in T'_j to the right of v_j^R. Hence by Lemma 5.1, we could again extend P_k through T_j. On the other hand if there are at least two crossings to the right of v_j^R, then by Observation 1, there are paths from the NE and SE points of T'_j to the NW and SW points of T'_j. Thus we could again extend P_k through T'_j. Hence T'_j cannot have any vertices. Thus we have proven Claim 1.

We now prove Observation 2, which will be used in the proof of Claim 2.

OBSERVATION 2. A single strand of T_j cannot be extended to a simple closed curve in T'.

Suppose that some strand of T_j can be extended to a simple closed curve L_1 in T'. We now extend the ends of the other strand of T_j until they meet at or before w. Since T_j has ∞-parity, this gives us a simple closed curve L_2 which is disjoint from L_1. Then $L = L_1 \cup L_2$ is the connected sum of $D(T_j)$ and two (possibly trivial) knots. Since T_j has ∞-parity, $D(T_j)$ is a link; and because T_j is non-trivial, by Wolcott's Theorem $D(T_j)$ is non-trivial. Hence L is also a non-trivial link. As this contradicts the hypothesis that $V(T')$ is a ravel, Observation 2 follows.

CLAIM 2. *For all $k \neq j$, T_k does not have ∞-parity, T'_k has at least one vertex, and R_k has at least two crossings.*

Suppose that there is some $k \neq j$ such that T_k has ∞-parity. Without loss of generality, $k > j$. Then we can extend one of the strands of T_j to the right so that the ends meet either in T_k or before. As this violates Observation 2, no T_k with $k \neq j$ can have ∞-parity.

Suppose that some T'_k with $k \neq j$ has no vertices. Without loss of generality, $k > j$. Since T_j has ∞-parity, we can extend the western endpoints of T_k leftward until they meet at or before T_j, and we can extend the eastern endpoints of T_k rightward until they meet at or before w. Let L be the simple closed curve obtained as the union of T_k with these rightward and leftward extensions. Now L is the connected sum of $D(T_k)$ with (possibly trivial) knots on the right and left. Recall that since T is in standard form, T_k is not horizontal. Thus by

Wolcott's Theorem L is a non-trivial knot. As this is contrary to our hypothesis, T'_k must have at least one vertex.

Finally, suppose that some T'_k has at most one crossing in R_k. Then by Simplifying Assumption (2), T'_k has no crossings to the right of v_k^R. Hence by Lemma 5.1, there is a simple path in T'_k between its NW and SW point. Thus again we can extend one of the strands of T_j to a simple closed curve in T'. As this again violates Observation 2, R_k must have at least two crossings.

CLAIM 3. *For all $k \neq j$, T'_k contains exactly one vertex.*

Suppose that some T'_k contains at least two vertices. Without loss of generality $k > j$. Let v_k^L be the leftmost vertex in T'_k. Then we can illustrate T'_k by Figure 6.2, where all of the crossings of T'_k are in the balls Q_k, S_k, and R_k, and any other vertices of T'_k are contained in S_k. Note that in spite of the way we have illustrated them, v_k^L and v_k^R can each be in either the top or the bottom row.

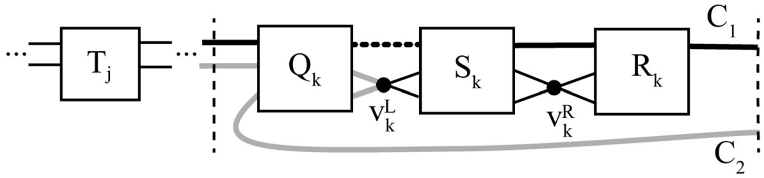

FIGURE 6.2. L_1 intersects T'_k in disjoint arcs C_1 and C_2.

Now we extend the ends of the NE-SE strand of T_j rightward until they meet. This must occur at the closing vertex w or else it would violate Observation 2. After removing any loops, we obtain a simple closed curve L_1 which intersects T'_k in a pair of disjoint arcs C_1 and C_2 each going between an eastern and western point of T'_k. Without loss of generality, we assume the endpoints of C_1 are the NE and NW points of T'_k and the endpoints of C_2 are the SE and SW points of T'_k as illustrated in Figure 6.2.

Observe that two arcs of C_2 enter Q_k from the left. Since Q_k has no vertices and does not contain the rightmost box of T_k, an arc that enters on the left must leave on the right. Thus two arcs of C_2 must exit Q_k on the right. Now C_1 also exits Q_k on the left, and hence must enter Q_k on the right. Since C_1 and C_2 are disjoint, this means that the two arcs entering v_k^L from the left must belong to C_2 and the dotted black arc in Figure 6.2 belongs to C_1. Furthermore, since C_2 does not contain any loops, C_2 cannot continue rightward beyond v_k^L. In particular, v_k^R cannot be in C_2.

Next suppose that the arc of C_1 from R_k to S_k passes through v_k^R. Then T'_k is illustrated in Figure 6.3, where the grey dotted arcs entering S_k on the right are connected in some way to the grey dotted arcs exiting S_k on the left. In this case, there is a path in T_k' going from its NE endpoint passing through both v_k^L and v_k^R and exiting T_k' from its SE endpoint. However, by Claim 2 we know that R_k contains at least two crossings, and hence by Observation 1 no such path can exist. Thus the arc of C_1 from R_k to S_k cannot pass through v_k^R as it does in Figure 6.3.

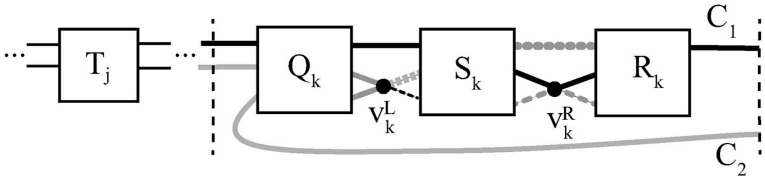

FIGURE 6.3. L_1 intersects T_k' in disjoint arcs C_1 and C_2.

Hence either C_1 goes through v_k^R and then reenters R_k, or T_k' contains a simple closed curve L_2 that goes through v_k^R and is disjoint from C_1. In the first case, since R_k is alternating and contains at least two crossings, L_1 is a non-trivial knot. As this is contrary to our assumption, the second case must occur. However, R_k is itself a rational tangle in alternating 3-braid form, and by Simplifying Assumption 1 there are no crossings in the same box as v_k^R. Thus there must be at least two crossings between L_1 and L_2. But this implies that $L_1 \cup L_2$ is a non-trivial link. As this is again contrary to our hypothesis, T_k' must contain exactly one vertex.

CLAIM 4. *For all $k \neq j$, the vertex v_k is in $(A_k^2)'$ or possibly in $(A_k^3)'$ if A_k^2 has a single crossing.*

If some T_k' has its vertex in the first box, as illustrated in Figure 6.4, then there would be a path in T_k' between its NE and SE points, which would violate Observation 1.

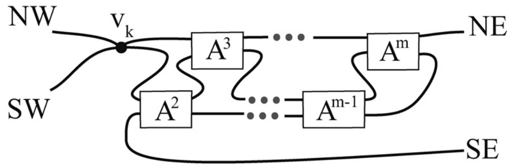

FIGURE 6.4. If v_k is in A^1, then there are paths in T_k' joining the NW and SW points and joining the NE and SE points.

Now suppose that some T_k' has its vertex v_k in a box $(A_k^p)'$ where either $p > 3$ or $p = 3$ and A_k^2 has more than one crossing. Let W_k be the tangle consisting of v_k and the part of T_k'

to the left of v_k, as illustrated in Figure 6.5 (though v_k could be in a box to the right of A_3^k). Note that W_k includes the black arcs to the left of v_k but not the grey arcs to the right of v_k. Then W_k is a rational tangle; and since there are at least two crossings in $A_k^2, \dots, A_k^{p-1}$, the tangle W_k is neither a horizontal tangle nor a trivial vertical tangle.

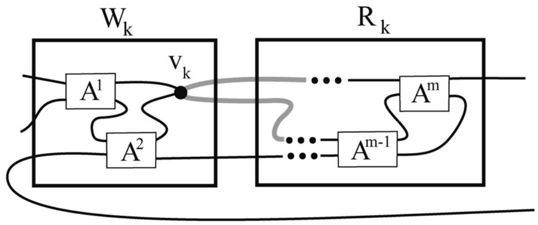

FIGURE 6.5. W_k is a rational tangle which is neither a horizontal tangle nor a trivial vertical tangle.

Now there is a path that goes from the NW and SW points of W_k leftward until its ends meet in T_j, at w, or at some other vertex. Also there is a path that goes rightward from the NE and SE points of W_k until the ends meet in T_j, at w, or at some other vertex. Note that since T_j has ∞-parity, at most one of these paths contains w. The union of the two strands of W_k together with these leftward and rightward paths is the connected sum of $D(W_k)$ with two (possibly trivial) knots. Since W_k is neither horizontal nor a trivial vertical tangle, this connected sum is a non-trivial knot or link. Thus the vertex v_k must either be in $(A_k^2)'$ or possibly in $(A_k^3)'$ if A_k^2 has only one crossing.

CLAIM 5. *For all $k \neq j$, T_k' has a loop containing v_k.*

It follows from Claim 4 that T_k' has one of the forms illustrated in Figure 6.6.

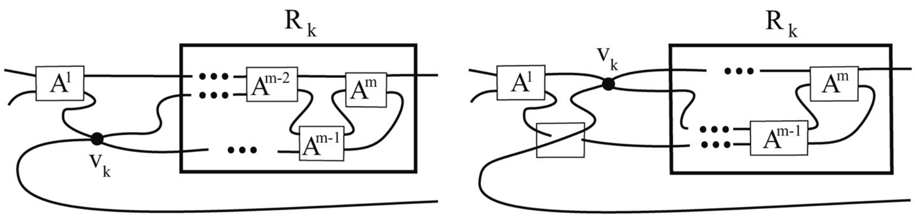

FIGURE 6.6. T_k' has one of these forms.

First we consider the illustration on the left in Figure 6.6. In this case, if R_k has 0-parity, then the strands going into v_k from the right are connected together in R_k. Hence they are part of a loop in T'_k. On the other hand, if R_k does not have 0-parity, then there is a path from the NE point of R_k to v_k. We can then extend this path to get a path in T'_k from its NE point to its SE point. As this violates Observation 1, this cannot occur.

Next we consider the illustration on the right in Figure 6.6. Now if R_k has ∞-parity, then the strands going into v_k from the right are connected together in R_k. Hence they are part of a loop in T'_k. But if R_k does not have ∞-parity, then there is a path from the NE point of R_k to v_k. Again we can extend this path to get a path in T'_k from its NE point to its SE point violating Observation 1. Thus in either case, T'_k has a loop containing v_k.

Now it follows from Claims 1 through 5 that T' is an exceptional vertex insertion. □

7. The Proof of the Backward Direction of Theorem 4.7

In order to prove the backward direction of Theorem 4.7, we make use of the following definition and theorem due to Sawollek [8].

DEFINITION 7.1. Let G be a 4-valent graph embedded in $\mathbb{R}^3$. The set of *associated links* $S(G)$ consists of all knots and links that can be obtained from G by replacing a neighborhood of each vertex of G by a rational tangle.

SAWOLLEK'S THEOREM ([8]). Let G be a 4-valent graph embedded in $\mathbb{R}^3$. The set of associated links $S(G)$ is an isotopy invariant of G.

PROPOSITION 7.2. *Let $T = T_1 + \cdots + T_n$ be a projection of a Montesinos tangle in standard form, and suppose that T' is obtained from T by an exceptional vertex insertion. Then the vertex closure $V(T')$ is a ravel.*

PROOF. By the definition of an exceptional vertex insertion, there is a single T_j with ∞-parity, and T'_j has no vertices. Without loss of generality we assume that $1 < j \leq n$. Also, for all $k \neq j$, T'_k has a single vertex v_k which is either in $(A^2_k)'$ or possibly in $(A^3_k)'$ if A^2_k has only one crossing. Furthermore, R_k (the subtangle of T_k consisting of the boxes to the right of v_k) is a rational tangle with at least two crossings and T'_k has a loop containing v_k. Now for each k such that v_k is in $(A^3_k)'$, we move v_k to $(A^2_k)'$ by flipping R_k as illustrated in Figure 7.1.

Next, for each sequential $k > 1$ such that $k \neq j$, we flip the part of the projection of $V(T')$ to the left of A^1_k repeatedly to move the crossings of A^1_k to A^1_1. Then we remove all of the accumulated crossings from A^1_1 by twisting the strands around w. We illustrate this in Figure 7.2, where A^1_1 begins with zero crossings and A^1_2 begins with three crossings. In the second picture we have moved the three crossings of A^1_2 to A^1_1, and in the third picture we have removed all of the crossings from A^1_1. This gives us a projection of $V(T')$ such that for each $k \neq j$, the vertex v_k is in $(A^2_k)'$ and all of the crossings of T'_k are in R_k.

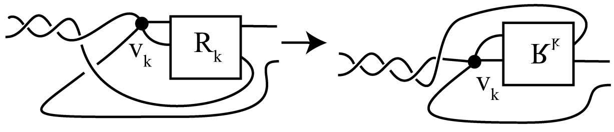

FIGURE 7.1. When v_k is in A_k^3, we flip R_k to move v_k to $(A_k^2)'$.

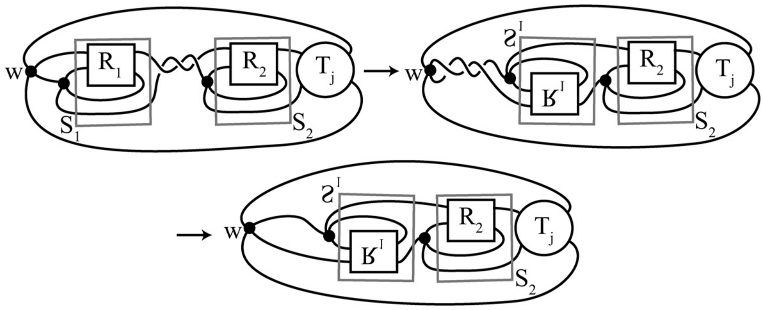

FIGURE 7.2. We can remove the crossings from the first box of each T_k' with $k \neq j$.

After doing the moves illustrated in Figure 7.1 and Figure 7.2, there are four different ways that the edges can go in and out of each R_k, which we illustrate in Figure 7.3.

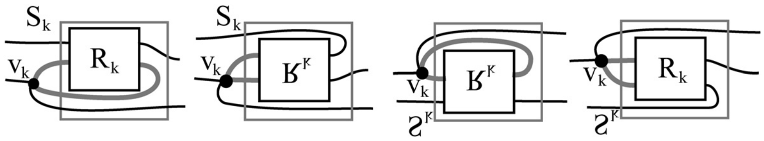

FIGURE 7.3. The possibilities for how the strands enter and exit each R_k, with the loop c_k indicated in grey.

The leftmost illustration occurs if v_k was originally in the second box so we did not have to flip R_k as in Figure 7.1, and in moving the crossings in the first boxes of the T_i' to the left as in Figure 7.2 we flipped S_k zero or an even number of times. The second illustration in Figure 7.3 occurs if v_k was originally in the third box so we flipped R_k as in Figure 7.1, and in moving the crossings in the first boxes of the T_i' to the left as in Figure 7.2 we flipped S_k zero or an even number of times. The third illustration occurs if v_k was in the second box so we did not flip R_k as in Figure 7.1, but in moving the crossings in the first boxes of the T_i' to the left as in Figure 7.2 we flipped S_k an odd number of times. The rightmost illustration in Figure 7.3 occurs if v_k was in the third box so we flipped R_k as in Figure 7.1, and in moving the crossings in the first boxes of the T_i' to the left as in Figure 7.2 we flipped S_k an odd number of times.

Observe that regardless of which of the four illustrations occur, the only difference between the edges outside of S_k is that the "dangling edge" at the left of S_k (that is the one not going into v_k) may be above or below the vertex v_k.

In Figure 7.4 we define a labeling of the edges of $V(T')$, keeping in mind that T_j has ∞-parity and none of the T_k with $k \neq j$ have ∞-parity. In particular, we label the loop containing v_k by c_k and label the edges which are not loops consecutively as follows. Let a_1 be the edge from w to v_1, and let a_2 be the other edge with one endpoint at v_1. We label the rest of the edges whose vertices are to the left of T_j consecutively from one vertex to the next as $a_3, \ldots, a_j$. Then a_j will have one endpoint at w, and hence $a = a_1 \cup a_2 \cup \cdots \cup a_j$ will be a simple closed curve. Similarly, let b_n be the edge of $V(T')$ from w to the rightmost vertex v_n, and then consecutively label the edges whose endpoints are to the right of T_j as $b_{n-1}, \ldots,$ b_j. Then b_j will also have an endpoint at w. Thus $b = b_n \cup b_{n-1} \cup \cdots \cup b_j$ will also be a simple closed curve.

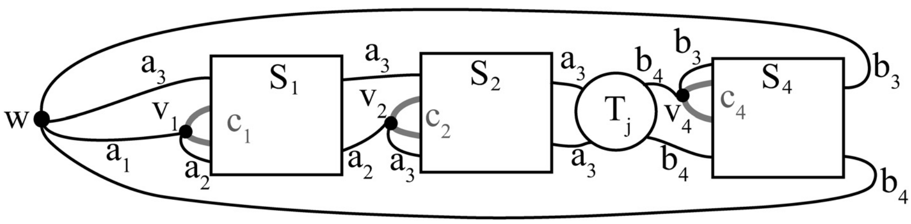

FIGURE 7.4. We label the edges of $V(T')$ in this way.

In Figure 7.4, for $k = 1$ and $k = 2$ we illustrate the dangling edge at the left of S_k above v_k, while for $k = 4$ we illustrate the dangling edge at the left of S_k below v_4. In fact, it makes no difference which of these illustrations occur.

Now observe from Figure 7.4 that there are no crossings between the projections of any pair of grey loops c_k and c_i with $i \neq k$, and hence no such pair can be linked. Also, observe from Figure 7.3 that each individual c_k is the numerator or denominator closure of a single strand of the rational tangle R_k, and so must be unknotted. In addition, the loops a and b are each connected sums of numerator or denominator closures of single strands of rational tangles. Hence a and b are also each unknotted. Finally, a has no crossings with any c_k with $k > j$ and meets every c_k with $k < j$ at the vertex v_k. Hence a cannot be linked with any c_k. Similarly, b cannot be linked with any c_k. It follows that $V(T')$ contains no non-trivial knots or links.

In order to show that $V(T')$ is non-planar we will show that the subgraph G obtained by deleting the loops c_k and vertices v_k for all $k > 1$ with $k \neq j$ is non-planar. The possibilities for S_k with c_k and v_k deleted are illustrated in Figure 7.5. Observe that since R_k is rational, after the deletion of c_k, the tangle R_k is left with a single unknotted strand. Thus in G, each S_k with $k > 1$ and $k \neq j$ is a trivial horizontal tangle.

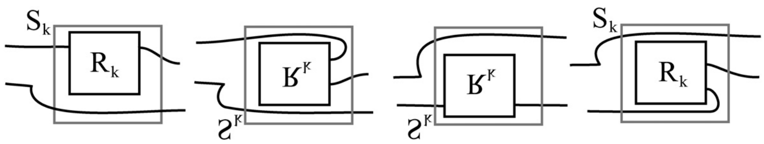

FIGURE 7.5. The forms of S_k after c_k and v_k have been deleted.

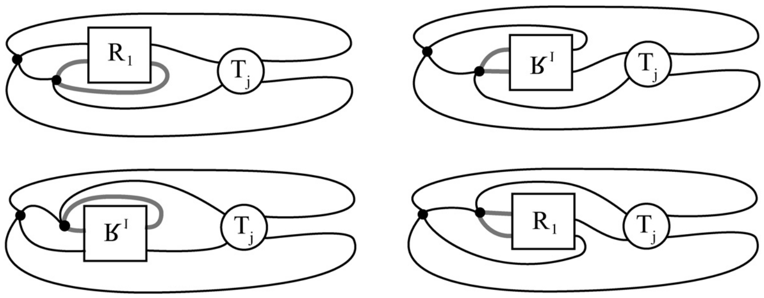

FIGURE 7.6. G has one of these forms.

It now follows that the spatial graph G has one of the forms illustrated in Figure 7.6. Since T_j has ∞-parity, regardless of which form G has, as an abstract graph G is isomorphic to the illustration on the left of Figure 7.7. We now let G_0 denote the planar embedding of G illustrated on the right side of Figure 7.7, and obtain the set of associated links $S(G_0)$ by replacing the two vertices of G_0 by rational tangles P and Q.

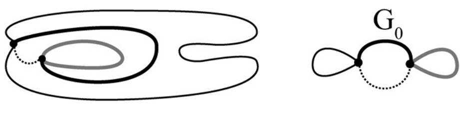

FIGURE 7.7. G as an abstract graph on the left, and a planar embedding G_0 on the right.

The denominator closure of a rational tangle is a 2-bridge knot or link. Thus all of the non-trivial, non-split links in $S(G_0)$ are either the connected sum of two 2-bridge knots or links or a single 2-bridge knot or link. Hence, by Sawollek's Theorem, to show that the spatial graph G is non-planar, it suffices to show that the set of associated links $S(G)$ contains some prime knot or link which is not 2-bridge.

In Figure 7.8, we replace the vertices w and v_1 of G by the rational tangles P and Q to get the elements of $S(G)$. Then in Figure 7.9, we group the rational tangles R_1 and Q together to create a single tangle U.

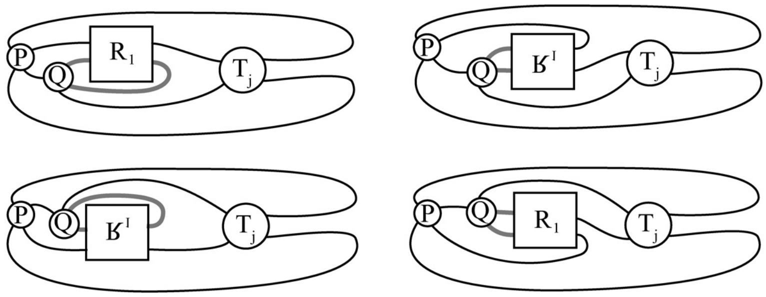

FIGURE 7.8. The elements of $S(G)$ have one of these forms.

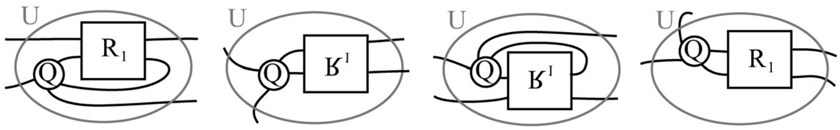

FIGURE 7.9. We group R_1 and Q together into a single tangle U.

Recall that R_1 is an alternating rational tangle with at least two crossings. Thus we can choose a rational tangle Q so that U is a non-trivial rational tangle which is not horizontal. Note that the choice of Q will depend on R_1 as well as on which form U has.

Now since T_j has ∞-parity, it cannot be horizontal; and by hypothesis T_j cannot be trivial. Thus for any rational tangle P, the knot or link $L = N(P + U + T_j)$ will be the numerator closure of a Montesinos tangle, where neither U nor T_j is horizontal or trivial.

It follows that the double branched cover $\Sigma(L)$ is a Seifert fibered space over S^2, and as long as P is not horizontal or trivial, $\Sigma(L)$ has three exceptional fibers. Now by the classification of Seifert manifolds [7], we can choose a rational tangle P such that $\Sigma(L)$ is irreducible, not $S^1 \times S^2$, and has infinite fundamental group. For such a P, we know that L will be a prime link which is not 2-bridge. Thus $S(G)$ contains a link which is not in $S(G_0)$. It follows that G is non-planar, and hence $V(T')$ must also be non-planar. Thus we have shown that $V(T')$ is a ravel. □

Propositions 6.1 and 7.2 together prove Theorem 4.7.

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