TOKYO J. MATH.
VOL. 40, NO. 1, 2017

Characterization of Non-F-split del Pezzo Surfaces of Degree 2

Natsuo SAITO

Hiroshima City University

(Communicated by H. Kaji)

Abstract. We investigate a smooth del Pezzo surface of degree 2 which is not Frobenius split. We give a characterization of non-F-split del Pezzo surfaces of degree 2 which exist only if the characteristic of the ground field is 2 or 3. Moreover, we prove that the set of centers of the blow-ups on $\mathbf{P}^2$ which gives a non-F-split del Pezzo surface is projectively equivalent to the only complete 7-arc over $\mathbf{F}_9$ if the characteristic is 3.

0. Introduction

Let k be an algebraically closed field of characteristic p. In algebraic geometry, del Pezzo surfaces are one of the most important objects. It is well known that they are obtained as the blow-ups of $(9-d)$-points on $\mathbf{P}^2$, where d is the degree of the surface. The classical theory of del Pezzo surfaces almost works in positive characteristic, with some modifications in characteristic 2 ([2]).

If the degree is 3, each del Pezzo surface can be expressed as a cubic surface in $\mathbf{P}^3$. In [7], Homma gives some results which contain a characterization of the Fermat cubic surface in characteristic 2:

THEOREM 0.1 (Homma). *Let X be a smooth del Pezzo surface of degree 3 over k. Suppose that X is obtained as the blow-up of $\mathbf{P}^2$ with centers $P_1, \dots, P_6$. Then the following conditions are equivalent*:

(i) $p = 2$ *and the anticanonical embedding of X is projectively equivalent to the Fermat cubic surface*;
(ii) $p > 0$ *and each smooth member of $|-K_X|$ is a supersingular elliptic curve*;
(iii) *Each point of $P_1, \dots, P_6$ is the concurrent point of all tangent lines of the conic passing through the remaining five points. The set $\{P_1, \dots, P_6\}$ is projectively equivalent to*

$$\{[1,0,0],[0,1,0],[0,0,1],[1,1,1],[1,\alpha,\alpha^2],[1,\alpha^2,\alpha]\},$$

which is the only complete 6-arc over $\mathbf{F}_4$, where $\alpha \in \mathbf{F}_4$ with $\alpha^2+\alpha+1=0$.

Received December 1, 2015; revised September 9, 2016
Mathematics Subject Classification: 14J26
Key words and phrases: del Pezzo surfaces, Frobenius splitting

Another important characterization of the Fermat cubic surface in characteristic 2 is non-F-splitting. Hara proved that a del Pezzo surface X can be non-F-split only if the characteristic of k and the degree of X are both small ([5]):

THEOREM 0.2 (Hara). *Let X be a smooth del Pezzo surface over k. Then X is F-split unless*

(1) $K_X^2 = 3$, $p = 2$;
(2) $K_X^2 = 2$, $p = 2, 3$;
(3) $K_X^2 = 1$, $p = 2, 3, 5$.

Especially, if X is a del Pezzo surface of degree 3, then X is not F-split if and only if X is the Fermat cubic surface in characteristic 2.

In this paper, we give various characterizations of non-F-split del Pezzo surfaces of degree 2. The main theorem is as follows:

THEOREM 0.3. *Let X be a smooth del Pezzo surface of degree 2 over an algebraically closed field k of characteristic $p > 0$. The anticanonical system $|-K_X|$ induces a double cover of $\mathbf{P}^2$, which we denote by $\pi : X \to \mathbf{P}^2$. Suppose that X is obtained as the blow-up of $\mathbf{P}^2$ with centers $P_1, \dots, P_7$. Then the following conditions are equivalent:*

(i) *$p = 2, 3$, and the branch locus C of the double cover $\pi : X \to \mathbf{P}^2$ is isomorphic to*
  (a) *the Fermat quartic curve in characteristic 3,*
  (b) *a double line in characteristic 2.*

(ii) *X is not F-split.*
(iii) *Each conic passing through 5 points in $\{P_1, \dots, P_7\}$ is tangent to the line passing through the remaining 2 points.*
(iv) *Each smooth member of $|-K_X|$ is a supersingular elliptic curve.*

In particular, when $p = 3$, the following condition is also equivalent to the above:

(v) *the set $\{P_1, \dots, P_7\}$ is projectively equivalent to*

$$\mathscr{P}_0 = \{[1, 0, 0], [0, 1, 0], [0, 0, 1], [1, 1, 1], [1, -1, \alpha], [1, -\alpha^3, \alpha^3], [1, -\alpha, -1]\},$$

which is the only complete 7-arc over $\mathbf{F}_9$, where $\alpha \in \mathbf{F}_9$ with $\alpha^2 - \alpha - 1 = 0$.

REMARK 0.4. For a normal projective variety X, we say that X is globally F-regular if there exists an integer $e \geq 1$ such that the natural map $\mathscr{O}_X \to F_*^e \mathscr{O}_X(D)$ splits as an $\mathscr{O}_X$-module for every effective divisor D on X. Global F-regularity is a stronger condition than F-splitting, but if a smooth Fano variety is F-split, then it is also globally F-regular by [11, Theorem 3.10] or [9, Theorem 3.9]. Thus Theorem 0.3 also gives equivalent conditions of non-globally F-regular smooth del Pezzo surfaces of degree 2.

1. Facts about Frobenius splittings

We work over an algebraically closed field k of characteristic $p > 0$. We recall some basic definitions and facts about Frobenius splittings.

DEFINITION 1.1. (1) Suppose R is a ring. We say R is F-split if the map $F : R \to F_* R$ splits as a map of R-modules.
(2) Suppose X is a scheme over k. We say X is Frobenius split (or F-split) if the map $\mathscr{O}_X \to F_*\mathscr{O}_X$ splits as a map of $\mathscr{O}_X$-modules.

The notion of Frobenius splitting is very significant for studying varieties in positive characteristic. To check whether a given ring is F-split or not, one of the very useful tools is the Fedder's criterion:

LEMMA 1.2 (Fedder's criterion). *Let I be an ideal in $k[x_1, \ldots, x_n]$ defined by $I = \langle f_1, \ldots, f_m\rangle$. Then $k[x_1, \ldots, x_n]/\langle f_1, \ldots, f_m\rangle$ is F-split at a point $\mathfrak{m} \in \operatorname{Spec} k[x_1, \ldots, x_n]$ if and only if $I^{[p]} : I \not\subset \mathfrak{m}^{[p]}$, where we denote $\langle f_1^p, \ldots, f_m^p\rangle$ by $I^{[p]}$.*

Lemma 1.2 states the criterion in the local settings. Thanks to the following theorem (cf. [11]), we can apply Fedder's criterion to a projective variety:

THEOREM 1.3. *Let X be any projective scheme over a perfect field. The followings are equivalent*:

(i) *X is Frobenius split*;
(ii) *the ring $S_{\mathscr{L}} = \oplus_{n\in\mathbf{N}} H^0(X, \mathscr{L}^n)$ is Frobenius split for all invertible sheaves $\mathscr{L}$*;
(iii) *the section ring $S_{\mathscr{L}} = \oplus_{n\in\mathbf{N}} H^0(X, \mathscr{L}^n)$ is Frobenius split for some ample invertible sheaf $\mathscr{L}$.*

By Grothendieck duality for the finite map $F : X \to X$, we have an isomorphism $\mathscr{H}om_{\mathscr{O}_X}(F_*\mathscr{O}_X, \mathscr{O}_X) \cong F_*\omega_X^{1-p}$ for a normal projective variety X. This implies that if X is Frobenius split, then there exists a section $s \in H^0(X, \omega_X^{1-p})$, called a splitting section, corresponding to the F-splitting. The following useful criterion for Frobenius splitting is proved by Mehta and Ramanathan ([8, Proposition 7]):

PROPOSITION 1.4. *Let X be a smooth projective variety of dimension n. X is Frobenius split if there is a point P in X and a global section $s \in H^0(X, \omega_X^{-1})$ with divisor of zeros*

$$(s)_0 = Y_1 + \cdots + Y_n + Z\,,$$

where $Y_1, \ldots, Y_n$ are prime divisors intersecting transversally at P, and Z is an effective divisor not containing P.

2. k-arcs over finite fields

We recall some definitions and facts from projective geometry over finite fields.

DEFINITION 2.1. Let $\mathscr{P}$ be a finite set of points of $\mathbf{P}^2$ over $\mathbf{F}_q$. We say $\mathscr{P}$ is *k-arc* if $\mathscr{P}$ consists of k points and no three of them are collinear. A k-arc $\mathscr{P}$ is called *complete* if $\mathscr{P}$ is not contained in any $(k+1)$-arc.

A group $G(\mathscr{P})$ acting on a k-arc $\mathscr{P}$ is called the *projective group for $\mathscr{P}$* if $G(\mathscr{P})$, which is a subgroup of the projective general linear group, fixes $\mathscr{P}$ as a set.

LEMMA 2.2. (1) *There exists a unique* 6*-arc over* $\mathbf{F}_4$, *which is complete. It is projectively equivalent to*

$$\{[1,0,0],[0,1,0],[0,0,1],[1,1,1],[1,\alpha,\alpha^2],[1,\alpha^2,\alpha]\},$$

where $\alpha \in \mathbf{F}_4$ *with* $\alpha^2+\alpha+1=0$.

(2) *There exist three* 7*-arcs over* $\mathbf{F}_9$ *up to projective equivalences. Only one of them is complete, which is projectively equivalent to*

$$\mathscr{P}_0 = \{[1,0,0],[0,1,0],[0,0,1],[1,1,1],[1,-1,\alpha],[1,-\alpha^3,\alpha^3],[1,-\alpha,-1]\},$$

where $\alpha \in \mathbf{F}_9$ *with* $\alpha^2-\alpha-1=0$. *Moreover*, $G(\mathscr{P}_0)$ *is isomorphic to* $\mathbf{Z}/3\mathbf{Z} \rtimes \mathbf{Z}/7\mathbf{Z}$.

PROOF. See Hirschfeld [6, Sections 14.3 and 14.7]. □

3. Aronhold sets

In this section, we assume that $p > 2$. Over the complex numbers, it is well known that a plane quartic curve has 28 bitangents. This also holds in positive characteristic if $p \neq 2$.

DEFINITION 3.1. Let C be a smooth plane quartic curve. A set $\mathscr{K} = \{\ell_1, \dots, \ell_7\}$ of 7 bitangents of C among 28 ones is called an *Aronhold set* if for each subtriple $\{\ell_i, \ell_j, \ell_k\} \subset \mathscr{K}$ there exists no conic passing through the 6 contact points of $\ell_i \cup \ell_j \cup \ell_k$ with C.

Note that two of the contact points may be infinitely near. Aronhold sets can also be defined by using the theory of theta characteristics. There are 288 Aronhold sets for each smooth plane quartic curve. For more details, see [4].

For a del Pezzo surface of degree 2, there is a one-to-one correspondence between each Aronhold set of a quartic curve which is a branch curve of the double cover induced by anticanonical divisor, and a set of 7 points, the centers of the blow-up, on the dual projective plane:

PROPOSITION 3.2. *Let* $\sigma : X \to \mathbf{P}^2$ *is the blow-up of* $\mathbf{P}^2$ *with centers* $\{P_1, \dots, P_7\}$, *and* C *a branch quartic curve of the double cover* $\pi : X \to \mathbf{P}^2$ *induced by* $|-K_X|$. *Then* 7 *bitangents which are the images of* (-1)*-curves* $\sigma^{-1}(P_i)$ *form an Aronhold set of* C. *Conversely, for each Aronhold set* $\mathscr{K}$ *of* C, *there exists a set* $\mathscr{P} = \{P_1, \dots, P_7\}$ *of* 7 *points on* $\mathbf{P}^2$ *such that each* (-1)*-curve* $\sigma^{-1}(P_i)$ *corresponds to a member in* $\mathscr{K}$.

PROOF. See [4, Propositions 6.3.10, 6.3.11]. □

4. Proof of Main Theorem

Now we prove Theorem 0.3. Note that X can be expressed as a hypersurface of degree 4 in the weighted projective space $\mathbf{P}(1,1,1,2)$. Thus the defining equation of X can be written as $f(x_0,x_1,x_2,y) = y^2 + f_2(x_0,x_1,x_2)y + f_4(x_0,x_1,x_2) = 0$, where x_i's $(i = 0,1,2)$ are coordinates of weight 1 and y is of weight 2, and $f_2(x_0,x_1,x_2)$ and $f_4(x_0,x_1,x_2)$ are homogeneous polynomials of degrees 2 and 4. We may assume that $f_2(x_0,x_1,x_2) = 0$ if $p \neq 2$.

PROOF. (i) $\Rightarrow$ (ii): This is a direct consequence of Lemma 1.2 and Theorem 1.3. Suppose $p = 3$. If the condition (i) holds, then the equation of X is $y^2 = x_0^4 + x_1^4 + x_2^4$. We can easily check that it is not F-split by Lemma 1.2. In characteristic 2, we can see that the branch divisor of $\pi : X \to \mathbf{P}^2$ is a double line if and only if $f_2(x_0,x_1,x_2)$ does not have the term x_0x_1 after suitable coordinate change. Hence the conditions (i) and (ii) are equivalent in characteristic 2.

(ii) $\Leftrightarrow$ (iv): This also follows from Lemma 1.2 and Theorem 1.3. For each smooth member E of $|-K_X|$, there exists a line ℓ in $\mathbf{P}^2$ such that $E = \pi^*\ell$. Let $E_{\alpha,\beta}$ be an elliptic curve which is obtained by the inverse image of a line defined by $x_2 = \alpha x_0 + \beta x_1$ where $\alpha, \beta \in k$, and $f_{E_{\alpha,\beta}}$ the equation of $E_{\alpha,\beta}$ in $\mathbf{P}(1,1,2)$.

If $E_{\alpha,\beta}$ is an ordinary elliptic curve, then Theorem 1.3 implies that the ring $k[x_0,x_1,x_2,y]/(x_2 - \alpha x_0 - \beta x_1, f(x_0,x_1,x_2,y))$ is F-split. Since this ring is a complete intersection, it follows from Lemma 1.2 that $(x_2 - \alpha x_0 - \beta x_1)^{p-1} f(x_0,x_1,x_2,y)^{p-1} \notin (x_0,x_1,x_2,y)^{[p]}$. In particular, we have $f(x_0,x_1,x_2,y)^{p-1} \notin (x_0,x_1,x_2,y)^{[p]}$, which implies that X is F-split.

On the other hand, suppose that X is F-split. If $p \geqq 3$, then it is easy to see that $f(x_0,x_1,x_2,y)$ has the term $x_0^2x_1^2$, after the change of coordinates if necessary. Since we can write $f_{E_{\alpha,\beta}} = y^2 - f_4(x_0,x_1,\alpha x_0 + \beta x_1)$, $(f_{E_{\alpha,\beta}})^{p-1}$ has the term $(y^2)^{\frac{p-1}{2}}(x_0^2x_1^2)^{\frac{p-1}{2}} = y^{p-1}x_0^{p-1}x_1^{p-1}$, which means that $E_{\alpha,\beta}$ is an ordinary elliptic curve. In the same manner, we can see that the assertion also holds in characteristic 2.

(ii) $\Rightarrow$ (iii): Let l be a line in $\mathbf{P}^2$. We have $\pi^*l \in |-K_X|$, and there exist 28 lines l in $\mathbf{P}^2$ such that $\pi^*l = \Gamma_1 + \Gamma_2$, where Γ_1 and Γ_2 are (-1)-curves. Moreover, among these 28 pairs of (-1)-curves, 21 pairs are the proper inverse transforms of a conic passing through 5 points in $\{P_1,\ldots,P_7\}$ and the line passing through remaining 2 points. Hence if there exists a pair of a conic and a line passing through the centers of blow-ups such that they intersect transversally, then Proposition 1.4 implies X is F-split. In fact, we see that the conditions (ii) and (iii) are equivalent from [1, Theorem 0.1] if $p = 2$.

(iii) $\Rightarrow$ (i): We may assume $p \neq 2$. A smooth quartic curve $C \subset \mathbf{P}^2$ is the Fermat quartic curve in characteristic 3 if and only if the number of hyperflexes of C is 28. Moreover, if C is not the Fermat quartic in characteristic 3, the sum of the weight of Weierstrass points is equal to 24 ([12]). This implies the number of hyperflexes is at most 12, since the weight of each hyperflex is 2. Hence the claim (i) holds since C has at least 21 hyperflexes if the condition (iii) holds.

From now on, we assume $p = 3$.

(v) $\Rightarrow$ (iii): We can check that the set of 7 points described in (v) satisfies the condition (iii) by direct calculation.

(i) $\Rightarrow$ (v): Let $C \subset \mathbf{P}^2$ be the Fermat quartic curve in characteristic 3. Consider a set of 7 points

$$\mathscr{P}_0' = \{[0, \alpha, 1], [0, 1 - \alpha, 1], [-\alpha, 0, 1], [\alpha - 1, 0, 1], [\alpha, 1, 0], [1 - \alpha, 1, 0], [1, 1, 1]\}$$

on C, where $\alpha \in \mathbf{F}_9$ with $\alpha^2 - \alpha - 1 = 0$. By direct calculation, we can check that there exists no conic passing through any three points in $\mathscr{P}_0'$ such that it is tangent to the tangent lines of C at those points. Hence $\mathscr{P}_0'$ is an Aronhold set for C. Moreover, since there exists a projective transformation $T : \mathscr{P}_0' \to \mathscr{P}_0$ given by

$$\begin{bmatrix} 1 & \alpha - 1 & \alpha \\ \alpha - 1 & \alpha & 1 \\ \alpha & 1 & \alpha - 1 \end{bmatrix} \in PGL(3, \mathbf{F}_9)\,,$$

$\mathscr{P}_0'$ is projectively equivalent to $\mathscr{P}_0$. Thus $\mathscr{P}_0'$ is the complete 7-arc.

It is known that the automorphism group of C is isomorphic to the projective unitary group $U_3(3)$ (see [10] or [3] for example). By Lemma 2.2, the projective group $G(\mathscr{P}_0')$ for $\mathscr{P}_0'$ is isomorphic to $\mathbf{Z}/3\mathbf{Z} \rtimes \mathbf{Z}/7\mathbf{Z}$, in which $\mathbf{Z}/3\mathbf{Z}$ is induced by changing cyclic coordinates, and $\mathbf{Z}/7\mathbf{Z}$ a cyclic transformation of the 7 points given by

$$\begin{bmatrix} 1 & \alpha + 1 & 1 - \alpha \\ \alpha + 1 & \alpha & 1 \\ 1 - \alpha & 1 & -1 \end{bmatrix} \in PGL(3, \mathbf{F}_9)\,.$$

It is easy to see that $G(\mathscr{P}_0')$ is a subgroup of $U_3(3)$. Thus the number of orbits of $\mathscr{P}_0'$ under the action of $U_3(3)$ on C is

$$\frac{|U_3(3)|}{|G(\mathscr{P}_0')|} = \frac{6048}{21} = 288\,,$$

which implies the set of Aronhold sets is transitive for the action of $U_3(3)$ since there are exactly 288 Aronhold sets for each smooth plane quartic curve. Hence by Proposition 3.2, each 7-arc which is obtained as the centers of the blow-ups is projectively equivalent to each other. $\square$

REMARK 4.1. Together with the Geiser involution, we see that an automorphism group of a non-F-split del Pezzo surface of degree 2 in characteristic 3 is isomorphic to

$U_3(3) \times \mathbf{Z}/2\mathbf{Z}$. For automorphism groups of del Pezzo surfaces in positive characteristic, see [3].

ACKNOWLEDGMENT. I am very grateful to Professor Nobuo Hara for raising some questions on a geometric characterization of configuration of points which are the centers of blow-ups to give non-F-split del Pezzo surfaces, and for helpful comments. I would also like to express my gratitudes to Professors Masayuki Hirokado, Hiroyuki Ito and Shunsuke Takagi for valuable conversations and helpful comments. I would also like to thank the referee for many useful comments and thoughtful suggestions.

The author was partially supported by Grants-in-Aid for Scientific Research 24540046, Japan Society for the Promotion of Science.

References

[1] P. CRAGNOLINI and P. A. OLIVERIO, On the proof of Castelnuovo's rationality criterion in positive characteristic, J. Pure and Applied Algebra **68** (1990), 297–323.

[2] M. DEMAZURE, *Surfaces de del Pezzo II-V*, Lecture Notes in Mathematics 777, Springer, 1980, 23–69.

[3] I. DOLGACHEV, Finite subgroups of the plane Cremona group, Algebraic Geometry in East Asia-Seoul 2008, Adv. St. in Pure Math. **60** (2010), 1–49.

[4] I. DOLGACHEV, *Classical Algebraic Geometry: A Modern View*, Cambridge University Press, 2012.

[5] N. HARA, A characterization of rational singularities in terms of injectivity of Frobenius maps, Amer. J. Math. **120** (1998), 981–996.

[6] J. W. P. HIRSCHEFELD, *Projective geometries over finite fields*, Oxford Science Publications, 1998.

[7] M. HOMMA, A combinatorial characterization of the Fermat cubic surface in characteristic 2, Geometriae Dedicata **64** (1997), 311–318.

[8] V. B. MEHTA and A. RAMANATHAN, Frobenius Splitting and Cohomology Vanishing for Schubert Varieties, Annals of Mathematics **122** (1985), 27–40.

[9] K. SCHWEDE and K. E. SMITH, Globally F-regular and log Fano varieties, Adv. Math. **224** (2010), 863–894.

[10] T. SHIODA, Arithmetic and Geometry of Fermat Curves, Proc. Algebraic Geometry Seminar, Singapore, World Scientific, 1987, 95–102.

[11] K. E. SMITH, Globally F-regular varieties: applications to vanishing theorems for quotients of Fano varieties, Michigan Math. J. **48** (2000), 553–572.

[12] K.-O. STÖHR and J. F. VOLOCH, Weierstrass points and curves over finite fields, Proc. London Math. Soc. **52** (1986), 1–19.

Present Address:
GRADUATE SCHOOL OF INFORMATION SCIENCES,
HIROSHIMA CITY UNIVERSITY,
OZUKA-HIGASHI, ASAMINAMI-KU, HIROSHIMA 731–3194, JAPAN.
e-mail: natsuo@math.info.hiroshima-cu.ac.jp