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A Remark on the Deformation Equivalence Classes of Hopf Surfaces

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Abstract. Let S be a compact complex surface. With the exception of some complex surfaces, it is known that there are only finitely many deformation types of complex surfaces with the same homotopy type as S. Let $\mathcal{H}(S)$ be the set of deformation equivalence classes of complex surfaces homotopy equivalent to S. We evaluate $\#\mathcal{H}(S)$ when S is a Hopf surface. As a corollary, we construct a sequence of Hopf surfaces (S_n) such that although $\#\mathcal{H}(S_n)$ is finite for all n, the sequence $(\#\mathcal{H}(S_n))$ is unbounded.

Introduction

In this paper, we study the number of deformation types of complex surfaces which are homotopy equivalent to a compact complex surface S. Recall that two compact complex surfaces S_1 and S_2 are deformation equivalent if there exist connected complex spaces χ and T, a smooth proper holomorphic map $\Phi : \chi \to T$, and two points $t_1, t_2 \in T$ such that $\Phi^{-1}(t_i)$ is biholomorphic to S_i, where $i = 1, 2$ [4, Ch.1 Def. 1.1]. We denote by $\mathcal{H}(S)$ the set of deformation equivalence classes of complex surfaces which are homotopy equivalent to S.

Let us review some results on the finiteness of the set $\mathcal{H}(S)$. Suppose that $b_1(S) \neq 1$, where $b_n(S)$ is the n-th Betti number of S. Further assume that S is not homotopy equivalent to an elliptic surface whose fundamental group $\pi_1(S)$ is finite-cyclic. Then, it is known that $\mathcal{H}(S)$ is finite, which is a result obtained by summarizing the works of several authors (see Appendix). However, we shall note that there exists a simply connected elliptic surface S, such that $\mathcal{H}(S)$ is infinite (see [3], [10]).

We now turn to compact complex surfaces S with $b_1(S) = 1$. In our previous paper [9], we have shown that if $b_2(S) = 0$, then the set $\mathcal{H}(S)$ is finite. If $b_2(S) = 1$ or $b_2(S) = 2$, then Teleman's results [12], [13] implies that $\mathcal{H}(S)$ is finite. Although the finiteness of $\mathcal{H}(S)$ is not known when $b_1(S) = 1$ and $b_2(S) > 2$, we can show that if the global spherical shell conjecture (see [13]) is true, then $\mathcal{H}(S)$ is finite.

One can then ask whether the cardinality of $\mathcal{H}(S)$ is bounded if S is not homotopy equivalent to an elliptic surface whose fundamental group is finite-cyclic. We will show that the

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above statement is false by constructing a sequence of Hopf surfaces (S_n) such that the sequence $(\#\mathcal{H}(S_n))$ is not bounded.

Now let us recall the definition and the basic properties of Hopf surfaces. A Hopf surface is a compact complex surface whose universal cover is biholomorphic to $\mathbb{C}^2 \setminus \{0\}$. Let S be a Hopf surface. By [8], we have $b_1(S) = 1$ and $b_2(S) = 0$, which implies that $\mathcal{H}(S)$ is finite. If $\pi_1(S)$ is non-abelian, then $\#\mathcal{H}(S) \leq 2$ (see [4, Ch.1 Lemma 7.20] and [4, Ch.2 Cor. 7.17]). Since we are interested in unbounded sequences, we may further assume that $\pi_1(S)$ is abelian. If $\pi_1(S)$ is infinite cyclic, then $\#\mathcal{H}(S) = 1$ (Dabrowski [2], Wehler [15]). Hence we evaluated the cardinality of $\mathcal{H}(S)$ when the order of the torsion subgroup of $\pi_1(S)$ is at least two:

THEOREM. *Let S be a Hopf surface whose fundamental group is abelian, and let n be the order of the torsion subgroup of $\pi_1(S)$. Assume that $n \geq 2$. Then*

$$\frac{\sqrt[3]{n}}{16} < \#\mathcal{H}(S) \leq \frac{n+1}{2}.$$

It is well known that for every $n \geq 2$, there always exists a Hopf surface which satisfies the assumption of our theorem above. Hence we obtain the following:

COROLLARY. *Let $(S_n)_{n\geq 2}$ be a sequence of Hopf surfaces such that $\pi_1(S_n)$ is abelian, and that the order of the torsion subgroup of $\pi_1(S_n)$ is equal to n. Then the sequence $(\#\mathcal{H}(S_n))_{n\geq 2}$ is not bounded.*

This paper is organized as follows. Let S be as in our theorem. We first review the general theory of Hopf surfaces. In particular, we show that $\mathcal{H}(S) = \mathcal{H}(S^1 \times L(n, q))$ for some integer q relatively prime to n, where $L(n, q)$ is the lens space. Using the results of [4], we construct a bijection between $\mathcal{H}(S)$ and the quotient set $D_{n,q}/\approx$, where $D_{n,q} = \{\pm k^2 q^{-1} \bmod n \mid (k, n) = 1\}$ and $\approx$ is an equivalence relation. Here $a \approx b$ if and only if $a = b$ or $a = b^{-1}q^2$. Our theorem is obtained by evaluating the cardinality of $D_{n,q}/\approx$, using elementary number theory.

1. Preliminaries

Let us begin by reviewing the general theory of Hopf surfaces. Recall that a Hopf surface is a primary Hopf surface if its fundamental group is isomorphic to $\mathbb{Z}$. Otherwise, it is called a secondary Hopf surface. Throughout this paper, a surface will always mean a compact complex manifold of dimension two. The following lemma allows us to restrict our attention to Hopf surfaces:

LEMMA 1. *Any surface homotopy equivalent to a Hopf surface is a Hopf surface.*

PROOF. Let S be a Hopf surface. Then $b_1(S) = 1$ and $b_2(S) = 0$. Let S' be a surface homotopy equivalent to S. If we denote by $\kappa(S')$ the Kodaira dimension of S', then $\kappa(S') = -\infty$, 0 or 1 since $b_1(S')$ is odd. By the Enriques-Kodaira classification, if $\kappa(S') \geq 0$, then S' is an elliptic surface whose universal cover is homotopy equivalent to $\mathbb{C}^2$. If $\kappa(S') = -\infty$, by

Bogomolov [1] and Teleman [14], S' is either a Hopf surface or an Inoue surface with $b_2 = 0$. However, the universal cover of an Inoue surface with $b_2 = 0$ is $\mathbb{H} \times \mathbb{C}$ [5]. Since the universal cover is invariant under homotopy equivalence, S' must be a Hopf surface. $\square$

According to [4, Ch.1 Lem. 7.20], if S is a Hopf surface whose fundamental group is abelian, then $\pi_1(S)$ is isomorphic to $\mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z})$ for some positive integer n. Thus, throughout the rest of the article, we let S be a Hopf surface with $\pi_1(S) \cong \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z})$, for some $n \geq 2$. We start by reviewing the deformation invariant of S introduced by Friedman and Morgan [4, Ch.1 Prop. 7.22] (see also [7]). Let $S = (\mathbb{C}^2 \setminus \{0\})/G$, where G is the deck transformation group of $\mathbb{C}^2 \setminus \{0\}$ with respect to S. Let τ be a generator of the torsion subgroup of G. By Hartogs' theorem, τ extends uniquely to an automorphism $\hat{\tau}$ over $\mathbb{C}^2$. Obviously, the origin is a fixed point of $\hat{\tau}$. Let $J_{\hat{\tau}}(0)$ be the Jacobian matrix of $\hat{\tau}$ at the origin, and λ_1, λ_2 be the eigenvalues of $J_{\hat{\tau}}(0)$. Then, λ_1 and λ_2 are both primitive roots of unity. Moreover, either $\lambda_1 = \lambda_2^q$ or $\lambda_2 = \lambda_1^q$ for some integer q relatively prime to n. Hence, we may associate the surface S with the set $\{q \mod n, q^{-1} \mod n\}$. Friedman and Morgan have shown that the set $\{q \mod n, q^{-1} \mod n\}$ is a deformation invariant of S. Further, they have shown the following:

PROPOSITION 1. *Let S and S' be secondary Hopf surfaces whose fundamental groups are isomorphic to $\mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z})$. Then S and S' are deformation equivalent if and only if they both have the same invariant* $\{q \mod n, q^{-1} \mod n\}$.

Now let $S(n, q)$ be a quotient of $\mathbb{C}^2 \setminus \{0\}$ by the subgroup of $\mathrm{Aut}(\mathbb{C}^2 \setminus \{0\})$ generated by the automorphisms

$$g : (w, z) \mapsto \left(\frac{1}{2}w, \frac{1}{2}z\right),$$

$$h : (w, z) \mapsto \left(\rho^q w, \rho z\right),$$

where $\rho = e^{2\pi\sqrt{-1}/n}$. The diffeomorphism

$$\mathbb{C}^2 \setminus \{0\} \to \mathbb{R} \times S^3 : z \mapsto \left(|z|, \frac{z}{|z|}\right)$$

induces a diffeomorphism $S(n, q) \to S^1 \times L(n, q)$, where $L(n, q)$ is the lens space. According to [7], any secondary Hopf surface S with $\pi_1(S) \cong \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z})$ is deformation equivalent to $S(n, q)$ for some integer q relatively prime to n. Thus we have shown the following: A secondary Hopf surface S with $\pi_1(S) \cong \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z})$, and an invariant $\{q \mod n, q^{-1} \mod n\}$ is homotopy equivalent to $S^1 \times L(n, q)$. Moreover, S deforms to $S(n, q)$. Conversely, if S is a surface homotopy equivalent to $S^1 \times L(n, q)$, then S is homotopy equivalent to $S(n, q)$ constructed above. By Lemma 1, S is a secondary Hopf surface. Therefore,

$$\mathcal{H}(S) = \mathcal{H}(S^1 \times L(n, q)).$$

We note that the fact that $S^1 \times L(n,q)$ is homotopy equivalent to $S(n,q)$ implies that the set $\mathcal{H}(S^1 \times L(n,q))$ is never empty.

2. Cardinality of the set $\mathcal{H}(S^1 \times L(n,q))$

We now study the number of deformation equivalence classes of surfaces homotopy equivalent to $S^1 \times L(n,q)$. Let us start by reviewing the homotopy type of $S^1 \times L(n,q)$. Any homotopy equivalence $S^1 \times L(n,q) \to S^1 \times L(n,q')$ lifts to a homotopy equivalence $\mathbb{R} \times L(n,q) \to \mathbb{R} \times L(n,q')$. Hence, $S^1 \times L(n,q)$ is homotopy equivalent to $S^1 \times L(n,q')$ if and only if $L(n,q)$ is homotopy equivalent to $L(n,q')$. Recall that $L(n,q)$ and $L(n,q')$ are homotopy equivalent if and only if there exists an integer k relatively prime to n such that $q' \equiv \pm k^2 q^{-1} \mod n$ [11]. Thus, if we let $D_{n,q} = \{\pm k^2 q^{-1} \mod n \mid k \in \mathbb{Z}, (k,n) = 1\}$, then

$$\mathcal{H}(S^1 \times L(n,q)) = \{\text{Deformation equivalence class of } S(n,q') \mid q' \mod n \in D_{n,q}\}.$$

We define an equivalence relation $\sim$ on $D_{n,q}$ by letting $a \sim b$ if and only if $a = b$ or $a = b^{-1}$. Since the surface $S(n,q)$ has an invariant $\{q \mod n, q^{-1} \mod n\}$, by Proposition 1 the map

$$\begin{array}{ccc} \mathcal{H}(S^1 \times L(n,q)) & \to & D_{n,q}/\sim \\ \Psi & & \Psi \\ \text{the deformation equivalence class of } S(n,q') & \mapsto & \text{the equivalence class of } q' \end{array}$$

is a bijection. Now, let $\approx$ be another equivalence relation on $D_{n,q}$, where $a \approx b$ if and only if $a = b$ or $a = b^{-1}q^2$. Then, multiplication by q induces a bijection between the quotient sets $D_{n,q}/\sim$ and $D_{n,q}/\approx$. Thus, we have constructed a one-to-one correspondence between the set $\mathcal{H}(S^1 \times L(n,q))$ and the set $D_{n,q}/\approx$.

Let $A_n = \{k^2 \mod n \mid k \in \mathbb{Z}, (k,n) = 1\}$, and $A'_n = \{-k^2 \mod n \mid k \in \mathbb{Z}, (k,n) = 1\}$. We will now express the cardinality of $D_{n,q}/\approx$ in terms of $\#A_n/\approx$.

LEMMA 2. *Let $n = 2^e p_1^{e_1} \dots p_m^{e_m}$ be the prime factorization of n. Then,*

$$A_n \cap A'_n = \begin{cases} A_n, & \text{if } e = 0 \text{ or } 1, \text{ and } p_1, \dots, p_m \equiv 1 \mod 4\,, \\ \emptyset, & \text{otherwise}\,. \end{cases}$$

PROOF. Assume that $A_n \cap A'_n$ is nonempty. Then there exist integers k and l such that $(k,n) = (l,n) = 1$ and $k^2 \equiv -l^2 \mod n$. By the Chinese remainder theorem, we obtain a system of congruences $k^2 \equiv -l^2 \mod 2^e$, and $k^2 \equiv -l^2 \mod p_i^{e_i}$, for all $i = 1, 2, \dots, m$. We first show that $p_i \equiv 1 \mod 4$. By [6, Prop. 4.2.3], $-l^2$ is a quadratic residue modulo $p_i^{e_i}$ if and only if $-l^2$ is a quadratic residue modulo p_i. It follows easily from the multiplicative property of the Legendre symbol that $p_i \equiv 1 \mod 4$. We also see that if $p_i \equiv 3 \mod 4$ for some i, then the set $A_n \cap A'_n$ is empty. Next we show that $e = 0, 1$. It is obvious that

$$A_{2^e} \cap A'_{2^e} = \begin{cases} \{1 \mod 2^e\}, & \text{if } e = 0, 1\,, \\ \emptyset, & \text{if } e = 2\,. \end{cases}$$

Suppose that $e \geq 3$. Then, an integer N is a quadratic residue mod 2^e if and only if $N \equiv 1 \mod 8$. Thus $-N \equiv 7 \mod 8$, hence $A_{2^e} \cap A'_{2^e} = \emptyset$. Therefore $e = 0, 1$. □

It is easy to see that if $\alpha \in A_n$, then the equivalence class of α by the relation $\approx$ is a subset of A_n. Thus we have:

COROLLARY 1. *Let $n = 2^e p_1^{e_1} \dots p_m^{e_m}$ be the prime factorization of n. Then,*

$$\#\mathcal{H}(S^1 \times L(n, q)) = \begin{cases} \#(A_n/\approx), & \text{if } e = 0, 1 \text{ and } p_1, \dots, p_m \equiv 1 \mod 4\,, \\ 2\#(A_n/\approx), & \text{otherwise}\,. \end{cases}$$

Let $C_{n,q} = \{k^2 \mod n \mid k \in \mathbb{Z}, (k, n) = 1, k^4 \equiv q^2 \mod n\}$. We view $C_{n,q}$ as the set $\{a \in A_n \mid a^2 \equiv q^2 \mod n\}$. By definition, the equivalence class of $\alpha \in A_n$ by $\approx$ contains two elements except when $\alpha \in C_{n,q}$. In this case, the equivalence class of α is a unit set. Hence we get:

PROPOSITION 2. *The cardinality of $A_n/\approx$ is equal to* $\dfrac{\#A_n + \#C_{n,q}}{2}$.

We now study the cardinality of the set A_n. View A_n as a group whose operation is multiplication. Let $n = 2^e p_1^{e_1} p_2^{e_2} \dots p_m^{e_m}$ be the prime factorization of n. By the Chinese remainder theorem, A_n is isomorphic to $A_{2^e} \times A_{p_1^{e_1}} \times \cdots \times A_{p_m^{e_m}}$. Denote by ϕ the Euler's totient function i.e. $\phi(n) = \#(\mathbb{Z}/n\mathbb{Z})^\times$, where $(\mathbb{Z}/n\mathbb{Z})^\times$ is the multiplicative group of integers modulo n. We first consider the case where n is odd. If p is an odd prime, then $\#A_p = \dfrac{\phi(p)}{2} = \dfrac{p-1}{2}$, since $\mathbb{Z}/p\mathbb{Z}$ is a field. Further, [6, Prop. 4.2.3] implies that if $n = p^m$ for some positive integer m, then $\#A_n = \dfrac{\phi(n)}{2}$. By the multiplicative property of the Euler's totient function, we obtain the following:

LEMMA 3. *Let $n = p_1^{e_1} p_2^{e_2} \dots p_m^{e_m}$, where $p_1, \dots, p_m$ is an odd prime. Then*

$$\#A_n = \frac{\phi(n)}{2^m}\,.$$

Next, we consider the case where n is a power of two.

LEMMA 4. *The cardinality of A_{2^e} is given as follows*:

$$\#A_{2^e} = \begin{cases} 1, & \text{if } e = 1, 2, 3\,, \\ 2^{e-3}, & \text{if } e \geq 4\,. \end{cases}$$

PROOF. The case $e = 1, 2, 3$ is obvious since $A_{2^e} = \{1 \mod 2^e\}$. Assume that $e \geq 4$. By [6, Prop. 4.2.4], every odd quadratic residue modulo 2^e is a square of four elements in $\mathbb{Z}_{2^e}^{\times}$. Thus, $\#A_{2^e} = \dfrac{\phi(n)}{4} = 2^{e-3}$. □

From Lemma 3 and Lemma 4, we get:

PROPOSITION 3. *Let $n = 2^e p_1^{e_1} p_2^{e_2} \dots p_m^{e_m}$ be the prime factorization of n. Then*

$$\#A_n = \frac{\phi(n)}{2^m f(e)},$$

where

$$f(e) = \begin{cases} 1, & \text{if } e = 0, 1, \\ 2, & \text{if } e = 2, \\ 4, & \text{if } e \geq 3. \end{cases}$$

We are now ready to evaluate the cardinality of the set $\#\mathcal{H}(S^1 \times L(n, q))$. Our proof relies on inequalities with Euler's totient function, which are valid if n is sufficiently large. For small n, we have the following lemma:

LEMMA 5. *If $2 \leq n \leq 254$, then*

$$\frac{\sqrt[3]{n}}{16} < \#\mathcal{H}(S^1 \times L(n, q)) \leq \frac{n+1}{2}.$$

REMARK. We have checked Lemma 5 computationally by counting the cardinality of $D_{n,q}/\approx$ for all pairs $(n, q) \in \mathbb{Z}^2$ which satisfy $2 \leq n \leq 254$ and $(n, q) = 1$. This method was the best we could do at present.

PROPOSITION 4. *The cardinality of the set $\mathcal{H}(S^1 \times L(n, q))$ is greater than $\dfrac{\sqrt[3]{n}}{16}$.*

PROOF. Let $n = 2^e p_1^{e_1} p_2^{e_2} \dots p_m^{e_m}$ be the prime factorization of n. We may assume that $3 \leq p_1 \leq \dots \leq p_m$. Then

$$\#\mathcal{H}(S^1 \times L(n, q)) \geq \#(A_n/\approx) \geq \frac{\#A_n}{2} = \frac{\phi(n)}{2^{m+1} f(e)}.$$

Now assume that $n \geq 42$. Then $\phi(n) \geq n^{2/3}$. Since $f(e)$ is at most four,

$$\begin{aligned} \frac{\phi(n)}{2^{m+1} f(e)} &\geq \frac{\sqrt[3]{p_1^{e_1}}}{2} \times \dots \times \frac{\sqrt[3]{p_m^{e_m}}}{2} \times \frac{\sqrt[3]{n}\sqrt[3]{2^{2e}}}{2 f(e)} \\ &\geq \frac{\sqrt[3]{p_1^{e_1}}}{2} \times \dots \times \frac{\sqrt[3]{p_m^{e_m}}}{2} \times \frac{\sqrt[3]{n}}{8}. \end{aligned}$$

Let $K = \dfrac{\sqrt[3]{p_1^{e_1}}}{2} \times \cdots \times \dfrac{\sqrt[3]{p_m^{e_m}}}{2}$. We show that $K > \dfrac{1}{2}$ for any positive integer m. If $m \geq 4$, then

$$K \geq \frac{\sqrt[3]{3}}{2} \times \frac{\sqrt[3]{5}}{2} \times \frac{\sqrt[3]{7}}{2} \times \frac{\sqrt[3]{p_4}}{2} \times \cdots \times \frac{\sqrt[3]{p_m}}{2}.$$

Since $p_4, p_5, \ldots, p_m \geq 11$, we have $\dfrac{\sqrt[3]{p_i}}{2} > 1$ for $i \geq 4$. Hence, $K > \dfrac{1}{2}$ if $m \geq 4$. The case $m = 1, 2, 3$ is obvious. Thus, we have $\#\mathcal{H}(S^1 \times L(n, q)) > \dfrac{\sqrt[3]{n}}{16}$. □

Finally, we shall prove that the set $\mathcal{H}(S^1 \times L(n, q))$ has at most $\dfrac{n+1}{2}$ elements. We first consider the case where n is prime.

LEMMA 6. *Let p be a prime number such that $p \equiv 1 \mod 4$. Then,*

$$\#\mathcal{H}(S^1 \times L(p, q)) = \begin{cases} \frac{p+3}{4}, & \text{if } q^{(p-1)/2} \equiv 1 \mod p\,, \\ \frac{p-1}{4}, & \text{otherwise}. \end{cases}$$

PROOF. Recall that $\#\mathcal{H}(S^1 \times L(p, q)) = \#(A_p/\approx)$. If $a \in C_{n,q}$, there exists an integer k relatively prime to n such that $k^4 \equiv a^2 \equiv q^2 \mod p$. Since $\mathbb{Z}_p$ is a field, $k^2 \equiv \pm q \mod p$. Using the multiplicative property of the Legendre symbol, we see that q is a quadratic residue modulo p if and only if $-q$ is a quadratic residue modulo p. Hence,

$$\#C_{n,q} = \begin{cases} 2, & \text{if } q^{(p-1)/2} \equiv 1 \mod p\,, \\ 0, & \text{otherwise}. \end{cases}$$

By Proposition 2, we are done. □

LEMMA 7. *Let p be a prime number such that $p \equiv 3 \mod 4$. Then,*

$$\#\mathcal{H}(S^1 \times L(p, q)) = \frac{p+1}{2}.$$

PROOF. In this case we have $\#\mathcal{H}(S^1 \times L(p, q)) = 2\#(A_p/\approx)$. By the multiplicative property of the Legendre symbol, we see that there are exactly two solutions of the equation $k^4 \equiv q^2 \mod p$. Thus, $\#C_{n,q} = 1$. The rest follows from Proposition 2. □

We now consider the case where n is a power of a prime p. First, let p be an odd prime and $n = p^m$ for some positive integer m. By [6, Prop. 4.2.3], we have $\#C_{p^m,q} = \#C_{p,q}$, hence $\#\mathcal{H}(S^1 \times L(n, q)) \leq \frac{n+1}{2}$. Now consider the case $n = 2^m$. If $m = 1$ or $m = 2$, then $\#C_n$ is at most one. If $m \geq 3$ then $\#C_n$ is at most two. Thus, $\#\mathcal{H}(S^1 \times L(2^m, q)) \leq \dfrac{2^m + 1}{2}$. Thus, we conclude that if p is a prime and $n = p^m$ for some positive integer m, then $\#\mathcal{H}(S^1 \times L(n, q))$ is at most $\frac{n+1}{2}$.

Finally, we consider the case when n is a composite number with at least two distinct prime factors. Then, $\phi(n) \leq n - \sqrt{n}$. Let $n = 2^e p_1^{e_1} \dots p_m^{e_m}$ be the prime factorization of n. By the Chinese remainder theorem, $\#C_{n,q}$ is at most 2^{m+2}. Hence,

$$\begin{aligned} \#\mathcal{H}(S^1 \times L(n,q)) \leq 2\,\#(A_n/\approx) &\leq \frac{\phi(n)}{2^m f(m)} + 2^{m+2} \\ &\leq \frac{n-\sqrt{n}}{2^m} + 2^{m+2}\,. \end{aligned}$$

Let $g(x) = \dfrac{n+1}{2} - \dfrac{n-\sqrt{n}}{x} - 4x$, where $x > 0$. We end our proof by showing that if $n \geq 255$, then $g(2^m) > 0$. By assumption, $g(2)$ and $g(\sqrt{n})$ are both positive. Since the function g is concave up on $x > 0$, by $2 \leq 2^m \leq \sqrt{n}$, we get $g(2^m) > 0$.

REMARK. The following conditions are equivalent (see [4, Ch.1 Prop. 7.24]).

(1) $L(n,q)$ is homeomorphic to $L(n,q')$.
(2) $S^1 \times L(n,q)$ is homeomorphic to $S^1 \times L(n,q')$.
(3) $L(n,q)$ is diffeomorphic to $L(n,q')$.
(4) $S^1 \times L(n,q)$ is diffeomorphic to $S^1 \times L(n,q')$.
(5) $q' \equiv \pm q^{\pm 1} \mod n$.

Thus, the number of deformation equivalence classes of surfaces which are diffeomorphic to $S^1 \times L(n,q)$ is at most two.

3. Appendix

The following claim is obtained by combining the works of several authors:

CLAIM. *Let S be a surface with $b_1(S) \neq 0$. Assume that S is not homotopy equivalent to an elliptic surface whose fundamental group is finite-cyclic. Then the set $\mathcal{H}(S)$ is finite.*

Since we could not find a reference for the claim above, we give a proof here for the convenience of the reader:

PROOF. Let A_i be the set of deformation equivalence classes of a surface S' such that

(i) S' is homotopy equivalent to S;
(ii) the Kodaira dimension $\kappa(S') = i$.

Here, $i = -\infty, 0, 1, 2$. Then, obviously $\mathcal{H}(S) = A_{-\infty} \cup A_0 \cup A_1 \cup A_2$. The sets A_0 and A_2 are finite by [4, Ch.1 Cor. 2.9]. If $S' \in A_{-\infty}$, then S' is algebraic since $b_1(S') \neq 1$ [4, Ch.1 p.20]. Hence $A_{-\infty}$ is finite by [4, Ch.1 Cor. 2.9]. Finally, if $S' \in A_1$, then S' is an elliptic surface. If the Euler number $\chi(S') > 0$, then A_1 is finite. This is because the deformation type of elliptic surfaces with positive Euler number is determined by the Euler number and the fundamental group as long as the fundamental group is not finite-cyclic ([4, Ch.2 p138]).

Now assume that $\chi(S') = 0$. Then the fundamental group of S' must be nonabelian. This is because an elliptic surface with Euler number zero and an abelian fundamental group is either a complex torus, a ruled surface over an elliptic curve, or a Hopf surface, which are all surfaces with Kodaira dimension not equal to one [4, Ch.2 Prop. 7.18]. Thus the cardinality of A_1 is at most two, since any elliptic surface deformation equivalent to S' is either S' or the conjugate complex manifold of S' [4, Ch.2 Cor. 7.17].

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