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Trajectory-harps and Horns Applied to the Study of the Ideal Boundary of a Hadamard Kähler Manifold

Qingsong SHI and Toshiaki ADACHI

Nagoya Institute of Technology

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Abstract. A trajectory-harps is made of a trajectory for a Kähler magnetic field $\mathbf{B}_\kappa$ and an associated variation of geodesics, and a trajectory-horn is made of a geodesic and an associated variation of trajectories. On a Hadamard Kähler manifold M we study thickness and string-angles of trajectory-harps, and study tube-lengths and tube-angles of trajectory-horns. As an application of these we show that two distinct points on the compactification of M with geometric ideal boundary can be joined by a trajectory for $\mathbf{B}_\kappa$ if the strength $|\kappa|$ is less than the upper bound of sectional curvatures of M.

1. Introduction

On a Kähler manifold M we have a canonical closed 2-form $\mathbf{B}_J$ induced by the complex structure J, which is called the Kähler form. We say constant multiples of this form to be *Kähler magnetic fields* (cf. [1]). Generally, a closed 2-form on a Riemannian manifold is said to be a magnetic field because it can be regarded as a generalization of static magnetic fields on a Euclidean 3-space (see [8, 13]). We call a smooth curve γ on M parameterized by its arclength a *trajectory* for a Kähler magnetic field $\mathbf{B}_\kappa = \kappa\mathbf{B}_J$ if it satisfies the differential equation $\nabla_{\dot\gamma}\dot\gamma = \kappa J\dot\gamma$. When $\kappa = 0$, the magnetic field $\mathbf{B}_0 = 0$ is the trivial magnetic field and its trajectories are geodesics. Hence we may say that trajectories for Kähler magnetic fields are perturbations of geodesics. In this sense the authors consider that properties of trajectories for Kähler magnetic fields show features of the underlying Kähler manifold just like geodesics do (cf. [9, 12]).

In order to study behavior of trajectories the second author defined trajectory-harps in [5]. A trajectory-harp is made of a trajectory and geodesics joining each point of this trajectory and its origin. Since a trajectory-harp gives a variation of geodesics, by applying Rauch's comparison theorem, he gave an estimate of its string-lengths, the length of geodesic segments joining two points of the trajectory. In this paper, we study more on trajectory-harps

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on a Hadamard Kähler manifold. A Hadamard manifold is a simply connected complete Riemannian manifold of nonpositive sectional curvature. On this manifold, if the strength $|\kappa|$ of a Kähler magnetic field $\mathbf{B}_\kappa$ is not greater than the square root of the infimum of the absolute values of sectional curvatures, every trajectory for $\mathbf{B}_\kappa$ is unbounded. For a trajectory-harp made by such a trajectory, we give an estimate of the distance between this trajectory and each string of this harp. Also, we define trajectory-horns in this paper. A trajectory-horn is made of a geodesic and trajectories joining each point of this geodesic and its origin. We give estimates of arclength of each trajectory-segment joining two points of this geodesic and of the angle between two trajectory-segments at the origin.

As an application we study asymptotic behavior of trajectories. For a Hadamard manifold M we can define its ideal boundary ∂M as the set of asymptotic classes of geodesic half-lines. Since the ideal boundary inherits properties of the outside of some compact subset of the manifold by definition, the geometry of the ideal boundary shows some properties of the manifold itself (see [6, 7, 11] for example). In this sense we are interested in whether trajectories show some properties of the underlying Kähler manifold in connection with its ideal boundary. In this paper we show that trajectories have the same properties as for geodesics when the strength of a Kähler magnetic field is less than the absolute value of the upper bound of sectional curvatures of the underlying manifold.

2. Trajectory-harps

Let (M, J) be a Hadamard Kähler manifold with complex structure J. We denote by $\mathbf{B}_J$ its Kähler form. We consider Kähler magnetic fields, which are closed 2-forms given as constant multiples of the Kähler form. Since M is complete, every trajectory for Kähler magnetic fields is defined on the whole line $\mathbf{R}$. A trajectory-harp in this paper consists of a trajectory half-line and geodesics. Given a trajectory half-line $\gamma : [0, \infty) \to M$ for a Kähler magnetic field $\mathbf{B}_\kappa = \kappa \mathbf{B}_J$ ($\kappa \in \mathbf{R}$) on M, which is the restriction of a trajectory to the interval $[0, \infty)$, we define a variation $\alpha_\gamma : [0, \infty) \times \mathbf{R} \to M$ of geodesics by the following conditions:

i) $\alpha_\gamma(t, 0) = \gamma(0)$;
ii) when $t = 0$, the curve $s \mapsto \alpha_\gamma(0, s)$ is the geodesic of initial vector $\dot\gamma(0)$;
iii) when $t > 0$, the curve $s \mapsto \alpha_\gamma(t, s)$ is the geodesic of unit speed joining $\gamma(0)$ and $\gamma(t)$.

We call this the *trajectory-harp* associated with γ. We denote by $\ell_\gamma(t)$ the distance $d\big(\gamma(0), \gamma(t)\big)$ between $\gamma(0)$ and $\gamma(t)$, and set $\delta_\gamma(t) = \big\langle \dot\gamma(t), \frac{\partial \alpha_\gamma}{\partial s}\big(t, \ell_\gamma(t)\big)\big\rangle$. We call $\ell_\gamma(t)$ and $\delta_\gamma(t)$ the *string-length* and the *string-cosine* of this trajectory-harp at t, respectively. These satisfy $\frac{d}{dt}\ell_\gamma(t) = \delta_\gamma(t)$. In [4] the second author gave a comparison theorem on trajectory-harps. For a negative c and a constant κ with $|\kappa| \le \sqrt{|c|}$, we define a function $\ell_\kappa(\cdot; c) : [0, \infty) \to [0, \infty)$ by the following relation:

$$\sqrt{|c|-\kappa^2}\sinh \tfrac{1}{2}\sqrt{|c|}\,\ell_\kappa(t; c) = \sqrt{|c|}\sinh \tfrac{1}{2}\sqrt{|c|-\kappa^2}\,t, \quad \text{when } |\kappa| < \sqrt{|c|}, \tag{2.1}$$

$$2\sinh\tfrac{1}{2}\sqrt{|c|}\,\ell_\kappa(t;c) = \sqrt{|c|}\,t\,, \qquad \text{when } \kappa = \pm\sqrt{|c|}\,. \tag{2.2}$$

Also, we define a function $\delta_\kappa(\cdot;c) : [0,\infty) \to (0,1]$ by

$$\delta_\kappa(t;c) = \begin{cases} \dfrac{\sqrt{|c|-\kappa^2}\,\cosh\bigl(\sqrt{|c|-\kappa^2}\,t/2\bigr)}{\sqrt{|c|\cosh^2\bigl(\sqrt{|c|-\kappa^2}\,t/2\bigr)-\kappa^2}}\,, & \text{when } |\kappa| < \sqrt{|c|}\,, \\ \dfrac{2}{\sqrt{|c|t^2+4}}\,, & \text{when } \kappa = \pm\sqrt{|c|}\,. \end{cases}$$

Since one can easily check that $\frac{d}{dt}\ell_\kappa(t;c) = \delta_\kappa(t;c) > 0$, we see that $\ell_\kappa(\cdot;c)$ is monotone increasing. Denoting by $\tau_\kappa(\cdot;c)$ the inverse function of $\ell_\kappa(\cdot;c)$, we have the following.

PROPOSITION 1 ([4]). *Suppose sectional curvatures of a Hadamard manifold M satisfy* $\mathrm{Riem}^M \le c < 0$ *for some constant c. For each trajectory-harp α_γ for a Kähler magnetic field $\mathbf{B}_\kappa$ with $|\kappa| \le \sqrt{|c|}$, its string length and string cosine satisfy $\ell_\gamma(t) \ge \ell_\kappa(t;c)$ and $\delta_\gamma(t) \ge \delta_\kappa\bigl(\tau_\kappa\bigl(\ell_\gamma(t);c\bigr);c\bigr)$ for $t \ge 0$. In particular, every trajectory half-line for $\mathbf{B}_\kappa$ with $|\kappa| \le \sqrt{|c|}$ is unbounded.*

For a trajectory-harp α_γ, we denote the geodesic segment $\alpha_\gamma(t,\cdot) : [0,\ell_\gamma(t)] \to M$ by σ_γ^t and call it the *harp-string* at $\gamma(t)$. First, we estimate angles between two harp-strings at the origin and show that every trajectory-harp has a limit string.

THEOREM 1. *Let M be a Hadamard Kähler manifold of sectional curvature* $\mathrm{Riem}^M \le c < 0$. *If $|\kappa| \le \sqrt{|c|}$, then for every trajectory half-line γ for $\mathbf{B}_\kappa$ its trajectory-harp α_γ has a limit* $\lim_{t\to\infty} \frac{\partial\alpha_\gamma}{\partial s}(t,0) \in U_{\gamma(0)}M$ *of initial vectors of harp-strings in the unit tangent space.*

PROOF. We set $Z_t(s) = \frac{\partial\alpha_\gamma}{\partial s}(t,s)$, which is a Jacobi field along the geodesic $s \mapsto \alpha_\gamma(t,s)$. By Rauch's comparison theorem on Jacobi fields, if $\mathrm{Riem}^M \le c < 0$ we have $\|Z_t(s)\| \ge \|\nabla_{\frac{\partial\alpha_\gamma}{\partial s}} Z_t(0)\| \times \bigl(1/\sqrt{|c|}\bigr)\sinh\sqrt{|c|}\,t$. On the other hand, as $\alpha_\gamma\bigl(t;\ell_\gamma(t)\bigr) = \gamma(t)$, we have $\dot\gamma(t) = Z_t\bigl(\ell_\gamma(t)\bigr) + \delta_\gamma(t)\frac{\partial\alpha_\gamma}{\partial s}\bigl(t;\ell_\gamma(t)\bigr)$, which shows that $\|Z_t\bigl(\ell_\gamma(t)\bigr)\|^2 = 1-\delta_\gamma(t)^2$. Thus we have

$$\|\nabla_{\frac{\partial\alpha_\gamma}{\partial s}} Z_t(0)\| \le \frac{\sqrt{|c|}\,\|Z_t\bigl(\ell_\gamma(t)\bigr)\|}{\sinh\sqrt{|c|}\,\ell_\gamma(t)} < \frac{\sqrt{|c|}}{\sinh\sqrt{|c|}\,\ell_\gamma(t)} \le \frac{\sqrt{|c|}}{\sinh\sqrt{|c|}\,\ell_\kappa(t;c)}\,.$$

When $|\kappa| < \sqrt{|c|}$, by the relation (2.1), we have

$$\begin{aligned}
&\frac{1}{\sqrt{|c|}} \sinh \sqrt{|c|}\, \ell_\kappa(t; c) \\
&\quad = \frac{2}{\sqrt{|c|-\kappa^2}} \sinh \frac{1}{2}\sqrt{|c|-\kappa^2}\, t \sqrt{\frac{|c|}{|c|-\kappa^2} \sinh^2 \frac{1}{2}\sqrt{|c|-\kappa^2}\, t + 1} \\
&\quad > \frac{1}{\sqrt{|c|-\kappa^2}} \sinh \sqrt{|c|-\kappa^2}\, t\,,
\end{aligned}$$

we therefore obtain

$$\int_{t_1}^{t_2} \left\| \nabla_{\frac{\partial \alpha_\gamma}{\partial t}} \frac{\partial \alpha_\gamma}{\partial s}(t, 0) \right\| dt = \int_{t_1}^{t_2} \left\| \nabla_{\frac{\partial \alpha_\gamma}{\partial s}} Z_t(0) \right\| dt \le \int_{t_1}^{t_2} \frac{\sqrt{|c|-\kappa^2}}{\sinh \sqrt{|c|-\kappa^2}\, t}\, dt$$

for all $t_2 > t_1 > 0$. When $\kappa = \pm\sqrt{|c|}$, by (2.2) we have $\sinh \sqrt{|c|}\, \ell_\kappa(t; c) = \sqrt{|c|}\, t \sqrt{(|c|/4)t^2 + 1} > |c| t^2/2$. We hence obtain

$$\int_{t_1}^{t_2} \left\| \nabla_{\frac{\partial \alpha_\gamma}{\partial t}} \frac{\partial \alpha_\gamma}{\partial s}(t, 0) \right\| dt = \int_{t_1}^{t_2} \left\| \nabla_{\frac{\partial \alpha_\gamma}{\partial s}} Z_t(0) \right\| dt < \int_{t_1}^{t_2} \frac{2}{\sqrt{|c|}\, t^2}\, dt\,.$$

Since we have

$$\angle\left(\frac{\partial \alpha_\gamma}{\partial s}(t_1, 0), \frac{\partial \alpha_\gamma}{\partial s}(t_2, 0) \right) \le \int_{t_1}^{t_2} \left\| \nabla_{\frac{\partial \alpha_\gamma}{\partial t}} \frac{\partial \alpha_\gamma}{\partial s}(t, 0) \right\| dt\,,$$

and have $\int_1^\infty (\sinh \sqrt{|c|-\kappa^2}\, t)^{-1}\, dt < \infty$ and $\int_1^\infty t^{-2}\, dt < \infty$, we find that the limit $\lim_{t\to\infty} \frac{\partial \alpha_\gamma}{\partial s}(t, 0) \in U_{\gamma(0)}M$ exists. □

We shall call the geodesic half-line σ_γ with initial vector $\lim_{t\to\infty} \frac{\partial \alpha_\gamma}{\partial s}(t, 0)$ the *limit harp-string* of a trajectory harp α_γ.

REMARK 1. As we have

$$\frac{2}{\sqrt{|c|-\kappa^2}} \sinh \frac{1}{2}\sqrt{|c|-\kappa^2}\, t > t \quad \text{and} \quad \frac{|c|}{|c|-\kappa^2} \sinh^2 \frac{1}{2}\sqrt{|c|-\kappa^2}\, t + 1 > \frac{|c|}{4} t^2\,,$$

the proof of Theorem 1 shows that under the same assumption for every trajectory harp α_γ for $\mathbf{B}_\kappa$ we have $\angle\big(\frac{\partial \alpha_\gamma}{\partial s}(t_1, 0), \frac{\partial \alpha_\gamma}{\partial s}(t_2, 0)\big) \le \big(2/\sqrt{|c|}\big) \int_{t_1}^{t_2} t^{-2}\, dt$ for all $t_2 > t_1 > 0$. Hence we have

$$\angle\left(\frac{\partial \alpha_\gamma}{\partial s}(t_1, 0), \frac{\partial \sigma_\gamma}{\partial s}(0) \right) \le \frac{2}{\sqrt{|c|}} \int_{t_1}^{\infty} t^{-2}\, dt$$

for each $t_1 > 0$.

Next we study "thickness" of a trajectory-harp (cf. [5]). Given a trajectory-harp α_γ associated with a trajectory γ for $\mathbf{B}_\kappa$, we set $\Gamma_\gamma^t = \{\alpha_\gamma(\tau, s) \mid 0 \le \tau \le t,\ 0 \le s \le \ell_\gamma(t)\}$ and $\Gamma_\gamma = \bigcup_{t>0} \Gamma_\gamma^t$. We call Γ_γ the body of α_γ. For a smooth curve γ, we denote by $U(\gamma; r)$ the tube $\{p \in M \mid d(p, \gamma) \le r\}$ of radius r around it.

THEOREM 2. *Let M be a Hadamard Kähler manifold whose sectional curvatures satisfy* $\mathrm{Riem}^M \le c < 0$, *and let κ be a real number with $|\kappa| < \sqrt{|c|}$. We put $\rho(\kappa; c) = |\kappa|\pi \big/ \big(2\sqrt{|c|(|c| - \kappa^2)}\big)$. Then for each trajectory half-line γ for $\mathbf{B}_\kappa$, the following properties hold.*

(1) *At each $t > 0$, the harp-sector Γ_γ^t at t is contained in the tube $U\big(\sigma_\gamma^s, \rho(\kappa; c)\big)$ around the string σ_γ^t.*

(2) *The body Γ_γ is contained both in the tube $U\big(\gamma, \rho(\kappa; c)\big)$ around γ and in the tube $U\big(\sigma_\gamma, \rho(\kappa; c)\big)$ around the limit harp-string σ_γ.*

PROOF. Since $\ell_\gamma' = \delta_\gamma$, we see by Proposition 1 that ℓ_γ is monotone increasing. We denote by τ_γ the inverse function of ℓ_γ. In order to show the assertion we are enough to estimate the length of a curve $t \mapsto \alpha_\gamma(t, \ell)$ ($\tau_\gamma(\ell) \le t < \infty$) for an arbitrary positive ℓ.

We take a trajectory $\hat\gamma$ for $\mathbf{B}_\kappa$ on a complex hyperbolic space $\mathbf{C}H^1(c)$ of constant holomorphic sectional curvature c, and take the trajectory-harp $\hat\alpha_{\hat\gamma} : [0, \infty) \times \mathbf{R} \to \mathbf{C}H^1(c)$ associated with $\hat\gamma$. We then find that its string-length is given by $\ell_\kappa(t; c)$ and its string-cosine is given by $\delta_\kappa(t; c)$. We put $Z_t(s) = \frac{\partial\alpha_\gamma}{\partial t}(t, s)$ and $\widehat{Z}_t(s) = \frac{\partial\hat\alpha_{\hat\gamma}}{\partial t}(t, s)$, which are Jacobi fields along geodesics $s \mapsto \alpha_\gamma(t, s)$ and $s \mapsto \hat\alpha_{\hat\gamma}(t, s)$, respectively. By Proposition 1 we have

$$\big\|Z_t\big(\ell_\gamma(t)\big)\big\|^2 = 1 - \delta_\gamma(t)^2 \le 1 - \delta_\kappa\big(\tau_\kappa\big(\ell_\gamma(t); c\big); c\big)^2 = \big\|\widehat{Z}_{\tau_\kappa(\ell_\gamma(t); c)}\big(\ell_\gamma(t)\big)\big\|^2 .$$

Since $s \mapsto \|Z_t(s)\|/\|\widehat{Z}_{\tau_\kappa(\ell_\gamma(t);c)}(s)\|$ is monotone increasing by Rauch's comparison theorem, we find $\|Z_t(s)\| \le \|\widehat{Z}_{\tau_\kappa(\ell_\gamma(t);c)}(s)\|$ for $0 \le s \le \ell_\gamma(t)$. If we put $u = \tau_\kappa\big(\ell_\gamma(t); c\big)$, we have $\frac{du}{dt} = \delta_\gamma(t) \big/ \delta_\kappa\big(\tau_\kappa\big(\ell_\gamma(t); c\big); c\big) \ge 1$, because $\tau_\kappa(\cdot; c)$ is the inverse function of $\ell_\kappa(\cdot; c)$. Thus, by taking a positive r so that $\ell_\gamma(r) = \ell$, we obtain

$$\begin{aligned}
(\text{the length of the curve } t \mapsto \alpha_\gamma(t, \ell)) &= \int_r^\infty \|Z_t(\ell)\|\, dt \\
&\le \int_r^\infty \|\widehat{Z}_{\tau_\kappa(\ell_\gamma(t);c)}(\ell)\|\, dt \le \int_{\tau_\kappa(\ell;c)}^\infty \|\widehat{Z}_u(\ell)\|\, du .
\end{aligned}$$

As the Jacobi field $\widehat{Z}_u$ on $\mathbf{C}H^1$ satisfies

$$\|\widehat{Z}_u(s)\| = \Big\|\nabla_{\frac{\partial\hat\alpha_{\hat\gamma}}{\partial s}} \widehat{Z}_u(0)\Big\| \times \frac{1}{\sqrt{|c|}} \sinh\sqrt{|c|}\, s = \frac{1}{\sqrt{|c|}} \Big\|\nabla_{\frac{\partial\hat\alpha_{\hat\gamma}}{\partial u}} \frac{\partial\hat\alpha_{\hat\gamma}}{\partial s}(u, 0)\Big\| \sinh\sqrt{|c|}\, s ,$$

we have

$$
\begin{aligned}
&(\text{the length of the curve } t \mapsto \alpha_\gamma(t,\ell)) \\
&\le \frac{1}{\sqrt{|c|}} \sinh \sqrt{|c|}\,\ell \int_{\tau_\kappa(\ell;c)}^{\infty} \left\| \nabla_{\frac{\partial\hat\alpha_{\hat\gamma}}{\partial u}} \frac{\partial \hat\alpha_{\hat\gamma}}{\partial s}(u,0) \right\| du \\
&= \frac{1}{\sqrt{|c|}} \sinh \sqrt{|c|}\,\ell \lim_{u\to\infty} \angle\left(\frac{\partial\hat\alpha_{\hat\gamma}}{\partial s}(u,0), \frac{\partial\hat\alpha_{\hat\gamma}}{\partial s}\big(\tau_\kappa(\ell;c),0\big)\right).
\end{aligned}
$$

Here, by use of the inequality $2\theta/\pi \le \sin\theta$ for $0 \le \theta \le \pi/2$ and the addition theorem in trigonometry, and by (2.1), we have

$$
\begin{aligned}
&\lim_{u\to\infty} \angle\left(\frac{\partial\hat\alpha_{\hat\gamma}}{\partial s}(u,0), \frac{\partial\hat\alpha_{\hat\gamma}}{\partial s}\big(\tau_\kappa(\ell;c),0\big)\right) = \lim_{u\to\infty}\left\{\cos^{-1}\delta_\kappa(u;c) - \cos^{-1}\delta_\kappa\big(\tau_\kappa(\ell;c);c\big)\right\} \\
&\le \frac{\pi}{2}\sin\left\{\cos^{-1}\sqrt{(|c|-\kappa^2)/|c|} - \cos^{-1}\delta_\kappa\big(\tau_\kappa(\ell;c);c\big)\right\} \\
&= \frac{|\kappa|\pi\sqrt{|c|-\kappa^2}}{2|c|\cosh\frac12\sqrt{|c|}\,\ell}\left\{\sqrt{\sinh^2\frac12\sqrt{|c|}\,\ell + \frac{|c|}{|c|-\kappa^2}} - \sinh\frac12\sqrt{|c|}\,\ell\right\} \\
&= \frac{|\kappa|\pi}{2\sqrt{|c|-\kappa^2}\ \cosh\frac12\sqrt{|c|}\,\ell} \times \frac{1}{\sqrt{\sinh^2\frac12\sqrt{|c|}\,\ell + \frac{|c|}{|c|-\kappa^2}} + \sinh\frac12\sqrt{|c|}\,\ell} \\
&\le \frac{|\kappa|\pi}{2\sqrt{|c|-\kappa^2}} \times \frac{1}{\sinh\sqrt{|c|}\,\ell}\,.
\end{aligned}
$$

Therefore we obtain

$$
(\text{the length of the curve } t \mapsto \alpha_\gamma(t,\ell)) \le |\kappa|\pi/\big(2\sqrt{|c|(|c|-\kappa^2)}\,\big),
$$

and get the conclusion. □

REMARK 2. In view of the above proof, under the same condition as in Theorem 2, for every trajectory-harp α_γ we have $d\big(\sigma_\gamma^{t_1}(s), \sigma_\gamma^{t_2}(s)\big) \le |\kappa|\pi/\big(2\sqrt{|c|(|c|-\kappa^2)}\,\big)$ for $0 < t_1 < t_2$ and $0 < s \le \ell_\gamma(t_1)$. Trivially, this guarantees $d\big(\sigma_\gamma^{t}(s), \sigma_\gamma(s)\big) \le |\kappa|\pi/\big(2\sqrt{|c|(|c|-\kappa^2)}\,\big)$ for $t > 0$ and $0 < s \le \ell_\gamma(t)$.

3. Trajectory-horns

In order to study the behavior of trajectories we need also to study a family of trajectories associated with a given geodesic. A smooth map $\beta : \mathbf{R} \times (-\varepsilon, \varepsilon) \to M$ on a Kähler manifold M is said to be a variation of trajectories for a Kähler magnetic field $\mathbf{B}_\kappa$ if for each $s \in (-\varepsilon,\varepsilon)$ the curve $t \to \beta(t,s)$ is a trajectory for $\mathbf{B}_\kappa$.

Let M be a Hadamard Kähler manifold whose sectional curvatures satisfy $\mathrm{Riem}^M \le c < 0$ with some constant c. It was shown in [5] that when $|\kappa| \le \sqrt{|c|}$ for given distinct two points $p, q \in M$ there exists a unique trajectory for $\mathbf{B}_\kappa$ which goes from p to q. Given a geodesic half-line $\sigma : [0, \infty) \to M$ of unit speed we define a variation $\beta_\sigma^\kappa : [0, \infty) \times \mathbf{R} \to M$ of trajectories for $\mathbf{B}_\kappa$ with $|\kappa| \le \sqrt{|c|}$ by the following condition:

i) $\beta_\sigma^\kappa(s, 0) = \sigma(0)$;
ii) when $s = 0$, the curve $t \mapsto \beta_\sigma^\kappa(0, t)$ is the trajectory for $\mathbf{B}_\kappa$ with initial vector $\dot\sigma(0)$;
iii) when $s > 0$, the curve $t \mapsto \beta_\sigma^\kappa(s, t)$ is the trajectory for $\mathbf{B}_\kappa$ joining $\sigma(0)$ and $\sigma(s)$.

We call this the *trajectory-horn* for $\mathbf{B}_\kappa$ associated with σ. We denote by $r_{\sigma,\kappa}(s)$ the arc-length of the trajectory segment $\beta_\sigma^\kappa(s, \cdot)$ from $\sigma(0)$ to $\sigma(s)$, and call it the $\mathbf{B}_\kappa$-*tube-length* at s. Trivially we have $r_{\sigma,\kappa}(s) \ge s$. We set $\varepsilon_{\sigma,\kappa}(s) = \left\langle \dot\sigma(s), \frac{\partial \beta_\sigma^\kappa}{\partial t}\big(s, r_{\sigma,\kappa}(s)\big)\right\rangle$ and call it the $\mathbf{B}_\kappa$-*tube-cosine* at $\sigma(s)$. If we denote by γ_s the trajectory $t \mapsto \beta_\sigma^\kappa(s, t)$, we see that $\varepsilon_{\sigma,\kappa}\big(\ell_{\gamma_s}(t)\big) = \delta_{\gamma_s}(t)$. For a negative c and a constant κ with $|\kappa| \le \sqrt{|c|}$, we define a function $\varepsilon_\kappa(s; c) : [0, \infty) \to (0, 1]$ by

$$\varepsilon_\kappa(s; c) = \sqrt{1 - \frac{\kappa^2}{|c|} \tanh^2 \frac{\sqrt{|c|}}{2} s}\,.$$

We note that $\varepsilon_\kappa(s; c) = \delta_\kappa\big(\tau_\kappa(s; c); c\big)$ holds. Thus, as a consequence of Proposition 1, we have the following.

PROPOSITION 2. *Let σ be a geodesic on a Hadamard Kähler manifold M whose sectional curvatures satisfy* $\mathrm{Riem}^M \le c < 0$ *for some constant c. We take the trajectory-horn β_σ^κ for $\mathbf{B}_\kappa$ with $|\kappa| \le \sqrt{|c|}$ which is associated with σ. We then have the following*:

(1) *Its $\mathbf{B}_\kappa$-tube-length satisfies $s \le r_{\sigma,\kappa}(s) \le \tau_\kappa(s; c)$*;
(2) *Its $\mathbf{B}_\kappa$-tube-cosine satisfies $\varepsilon_{\sigma,\kappa}(s) \ge \varepsilon_\kappa(s; c)$ for $s \ge 0$.*

PROOF. By Proposition 1 we find that $t \mapsto \ell_{\gamma_s}(t)$ is monotone increasing. We denote by τ_{γ_s} the inverse function of ℓ_{γ_s}.

(1) By Proposition 1, we have

$$\ell_\kappa\big(\tau_\kappa(s; c); c\big) = s = \ell_{\gamma_s}\big(r_{\sigma,\kappa}(s)\big) \ge \ell_\kappa\big(r_{\sigma,\kappa}(s); c\big).$$

As $\ell_\kappa(\cdot, c)$ is monotone increasing, we get the first assertion.

(2) By definition of $\varepsilon_{\sigma,\kappa}$ we have

$$\varepsilon_{\sigma,\kappa}(s) = \delta_{\gamma_s}\big(\tau_{\gamma_s}(s)\big) \ge \delta_\kappa\big(\tau_\kappa(\ell_{\gamma_s}(\tau_{\gamma_s}(s)); c); c\big) = \delta_\kappa\big(\tau_\kappa(s; c); c\big) = \varepsilon_\kappa(s; c)$$

and get the conclusion. □

We here study embouchure angles of trajectory horns. A vector field Y along a trajectory

γ for $\mathbf{B}_\kappa$ is said to be a normal magnetic Jacobi field if it satisfies

$$\begin{cases}\nabla_{\dot\gamma}\nabla_{\dot\gamma}Y-\kappa J\big(\nabla_{\dot\gamma}Y\big)+R(Y,\dot\gamma)\dot\gamma=0\,,\\ \langle\nabla_{\dot\gamma}Y,\dot\gamma\rangle=0\,.\end{cases}$$

Variations of trajectories induce normal magnetic Jacobi fields, and for each normal magnetic Jacobi field there exists a variation of trajectories which induces this field (see [2]). For a normal magnetic Jacobi field Y along a trajectory γ, we denote by $Y^\sharp$ its component orthogonal to $\dot\gamma$. We have the following result which corresponds to Rauch's comparison theorem.

PROPOSITION 3 ([3]). *Let $\mathbf{B}_\kappa$ be a Kähler magnetic field with $|\kappa|\le\sqrt{|c|}$ on a Kähler manifold M whose sectional curvatures satisfy* $\mathrm{Riem}^M\le c<0$ *for some constant c. Then, every normal magnetic Jacobi field Y with $Y(0)=0$ along a trajectory γ for $\mathbf{B}_\kappa$ satisfies*

$$\begin{cases}\|Y^\sharp(t)\|\ge\|\nabla_{\dot\gamma}Y^\sharp(0)\|\times\dfrac{1}{\sqrt{|c|-\kappa^2}}\sinh\sqrt{|c|-\kappa^2}\,t\,, & \textit{when } |\kappa|<\sqrt{|c|},\\ \|Y^\sharp(t)\|\ge\|\nabla_{\dot\gamma}Y^\sharp(0)\|\,t\,, & \textit{when } \kappa=\pm\sqrt{|c|}\,.\end{cases}$$

For a trajectory-horn β_σ^κ, we denote the trajectory segment $\beta_\sigma^\kappa(s,\cdot):[0,r_{\sigma,\kappa}(s)]\to M$ by $\gamma_{\sigma,\kappa}^s$ and call it the *horn-tube* at $\sigma(s)$. We estimate angles between two horn-tubes at the origin and show that every trajectory-horn has a limit tube.

THEOREM 3. *Let M be a Hadamard Kähler manifold whose sectional curvatures satisfy* $\mathrm{Riem}^M\le c<0$ *for some constant c. If $|\kappa|<\sqrt{|c|}$, then for every geodesic half-line σ of unit speed its trajectory-horn β_σ^κ for $\mathbf{B}_\kappa$ satisfies*

$$\angle\left(\frac{\partial\beta_\sigma^\kappa}{\partial t}(s_1,0),\frac{\partial\beta_\sigma^\kappa}{\partial t}(s_2,0)\right)\le\int_{s_1}^{s_2}\frac{\sqrt{|c|-\kappa^2}}{\sinh\sqrt{|c|-\kappa^2}\,s}ds$$

for all $s_2>s_1>0$. In particular, it has a limit $\lim_{s\to\infty}\frac{\partial\beta_\sigma^\kappa}{\partial t}(s,0)\in U_{\beta(0,0)}M$ *of initial vectors of horn-tubes.*

PROOF. As we have $\sigma(s)=\beta\big(s,r_{\sigma,\kappa}(s)\big)$ we see $\dot\sigma(s)=Y_s\big(r_{\sigma,\kappa}(s)\big)+r'_{\sigma,\kappa}(s)\dot\gamma_s\big(r_{\sigma,\kappa}(s)\big)$, where γ_s denotes the trajectory $t\mapsto\beta(s,t)$ and $Y_s(t)=\frac{\partial\beta_\sigma^\kappa}{\partial s}(s,t)$. This shows that $Y_s^\sharp\big(r_{\sigma,\kappa}(s)\big)=\dot\sigma(s)-\varepsilon_{\sigma,\kappa}(s)\dot\gamma_s\big(r_{\sigma,\kappa}(s)\big)$. We hence obtain $\|Y_s^\sharp\big(r_{\sigma,\kappa}(s)\big)\|^2=1-\varepsilon_{\sigma,\kappa}(s)^2$. Since $\beta_\sigma^\kappa(s,0)=\beta_\sigma^\kappa(0,0)$ we have $Y_s(0)=0$. This leads us to $\nabla_{\frac{\partial\beta_\sigma^\kappa}{\partial t}}Y_s(0)=\nabla_{\frac{\partial\beta_\sigma^\kappa}{\partial t}}Y_s^\sharp(0)$. Therefore, by use of Proposition 3, we obtain

$$\left\|\nabla_{\frac{\partial\beta_\sigma^\kappa}{\partial t}}Y_s(0)\right\|\le\frac{\sqrt{|c|-\kappa^2}\,\|Y_s^\sharp\big(r_{\sigma,\kappa}(s)\big)\|}{\sinh\sqrt{|c|-\kappa^2}\,r_{\sigma,\kappa}(s)}\le\frac{\sqrt{|c|-\kappa^2}}{\sinh\sqrt{|c|-\kappa^2}\,s}.$$

Since we have

$$\angle\left(\frac{\partial\beta_\sigma^\kappa}{\partial t}(s_1,0),\ \frac{\partial\beta_\sigma^\kappa}{\partial t}(s_2,0)\right) \le \int_{s_1}^{s_2}\left\|\nabla_{\frac{\partial\beta_\sigma^\kappa}{\partial s}} Y_s(0)\right\| ds\,,$$

we get the estimate.

By the estimate we have $\angle\big(\frac{\partial\beta_\sigma^\kappa}{\partial t}(s_1,0), \frac{\partial\beta_\sigma^\kappa}{\partial t}(s_2,0)\big) \le 4\exp\big(-\sqrt{|c|-\kappa^2}s_1/2\big)$ for $s_2 > s_1 \ge 1/\sqrt{|c|-\kappa^2}$. As $U_{\beta_\sigma^\kappa(0,0)}M$ is compact we get the conclusion. □

We shall call the trajectory half-line γ_σ^κ with initial vector $\lim_{s\to\infty}\frac{\partial\beta_\sigma^\kappa}{\partial t}(s,0)$ the limit horn-tube of a trajectory-horn β_σ^κ.

We have a similar estimate on embouchure angles of trajectory-horns for $\mathbf{B}_{\pm\sqrt{|c|}}$ on a Hadamard manifold M satisfying $\mathrm{Riem}^M \le c < 0$. But our estimate on tube-lengths from below is too rough to get some properties on trajectory-horns for $\mathbf{B}_{\pm\sqrt{|c|}}$.

Given a trajectory-horn β_σ^κ for $\mathbf{B}_\kappa$ associated with a geodesic half-line σ of unit speed, we set $\varSigma_{\sigma,\kappa} = \big\{\beta_\sigma^\kappa(s,t)\ \big|\ s\ge 0,\ 0\le t\le r_{\sigma,\kappa}(s)\big\}$ and call it the body of β_σ^κ. By Theorem 2 we have the following.

PROPOSITION 4. *Let M be a Hadamard Kähler manifold whose sectional curvatures satisfy* $\mathrm{Riem}^M \le c < 0$, *and let κ be a real number with $|\kappa| < \sqrt{|c|}$. For each geodesic half-line σ of unit speed, the body $\varSigma_{\sigma,\kappa}$ of a trajectory-horn for $\mathbf{B}_\kappa$ on M is contained in the tube $U\big(\gamma_\sigma^\kappa;\rho(\kappa;c)\big)$ around the limit horn-tube γ_σ^κ, where $\rho(\kappa;c) = |\kappa|\pi/\big(2\sqrt{|c|(|c|-\kappa^2)}\big)$.*

PROOF. For each trajectory $t\mapsto \beta_\sigma^\kappa(s,t)$, the geodesic σ can be regarded as a string of the trajectory-harp associated with this trajectory, which we denote by γ_s. By Remark 2, we have $d\big(\gamma_s(t),\sigma\big(\ell_{\gamma_s}(t)\big)\big) \le |\kappa|\pi/\big(2\sqrt{|c|(|c|-\kappa^2)}\big)$. As $\gamma_\sigma^\kappa(t) = \lim_{s\to\infty}\gamma_s(t)$, we have $d\big(\gamma_\sigma^\kappa(t),\sigma\big(\ell_{\gamma_\sigma^\kappa}(t)\big)\big) \le |\kappa|\pi/\big(2\sqrt{|c|(|c|-\kappa^2)}\big)$, and get the conclusion. □

4. Asymptotic behavior of trajectory half-lines

For a Hadamard manifold M, its (geometric) ideal boundary ∂M is defined as the set of all asymptotic classes of geodesic half-lines of unit speed. Here, two geodesic half-lines $\sigma_1,\sigma_2 : [0,\infty)\to M$ of unit speed on M is said to be asymptotic to each other if the distance function $t\mapsto d\big(\sigma_1(t),\sigma_2(t)\big)$ is uniformly bounded. For a geodesic half-line σ of unit speed we denote by $\sigma(\infty)$ its asymptotic class, and call it its point at infinity. When a geodesic half-line σ of unit speed is contained in an asymptotic class $z\in\partial M$ (i.e. $\sigma(\infty)=z$), we say that this σ joins $\sigma(0)$ and z. On the union $M\cup\partial M$ a topology which is called the cone topology is introduced. We briefly recall its definition. For a point $p\in M$ and two points $x_1,x_2\in M\cup\partial M$, we take geodesic segments or geodesic half-lines joining p and x_i $(i=1,2)$. We denote them by σ_1,σ_2, where we set their parameters as $\gamma_i(0)=p$. We set $\angle_p(x_1,x_2) = \angle\big(\dot\gamma_1(0),\dot\gamma_2(0)\big)$. For $z\in\partial M$, we take an arbitrary point $p\in M$ and positive numbers R, ε, set $V_p(z; R, \varepsilon) = \{x \in M \cup \partial M \mid \angle_p(x, z) < \varepsilon,\ d(x, p) > R\}$, and consider the family of such sets as the fundamental neighborhood system at z. On M we use its original topology. As M is nonpositively curved, the function $t \mapsto d\big(\sigma_1(t), \sigma_2(t)\big)$ for two geodesic half-lines of unit speed is a convex function. Therefore we see that every exponential map $\exp_p : T_pM \to M$ induces a bijection $\partial\exp_p : U_pM \to \partial M$ (see [10]). With the cone topology this induced map is a homeomorphism.

In this section we study limit points of trajectory half-lines. Since trajectories for Kähler magnetic fields can be regarded as perturbations of geodesics, it is natural to consider that they have similar properties as of geodesics. The authors are interested in the relationship the maximum of strengths of Kähler magnetic fields whose trajectories have similar properties as of geodesics and sectional curvatures of the underlying Kähler manifold.

Let M be a Hadamard Kähler manifold whose sectional curvatures satisfy $\mathrm{Riem}^M \le c < 0$. We take a Kähler magnetic field $\mathbf{B}_\kappa$ of strength $|\kappa| \le \sqrt{|c|}$. For each trajectory half-line γ for $\mathbf{B}_\kappa$, Proposition 1 guarantees $\lim_{t\to\infty} \ell_\gamma(t) = \infty$, hence Theorem 1 shows that γ has its point at infinity $\gamma(\infty) := \lim_{t\to\infty}\gamma(t) \in \partial M$ and that it coincides with the point $\sigma_\gamma(\infty)$ at infinity of the limit harp-string σ_γ of the trajectory-harp associated with γ. Our goal in this section is the following.

THEOREM 4. *Let M be a Hadamard Kähler manifold of sectional curvature* $\mathrm{Riem}^M \le c < 0$. *We take a Kähler magnetic field $\mathbf{B}_\kappa$ on M with $|\kappa| \le \sqrt{|c|}$.*

(1) *For arbitrary points $p \in M$ and $z \in \partial M$, there exists a trajectory γ satisfying $\gamma(0) = p$ and $\lim_{t\to\infty}\gamma(t) = z$. Moreover when $|\kappa| < \sqrt{|c|}$, such a trajectory is uniquely determined.*
(2) *When $|\kappa| < \sqrt{|c|}$, for arbitrary distinct points $z,\ w \in \partial M$, there exists a trajectory γ satisfying $\lim_{t\to-\infty}\gamma(t) = z$ and $\lim_{t\to\infty}\gamma(t) = w$.*

For a Kähler magnetic field $\mathbf{B}_\kappa$ on a Kähler manifold M, we define the *magnetic exponential map* $\mathbf{B}_\kappa\exp_p : T_pM \to M$ at a point $p \in M$ by

$$\mathbf{B}_\kappa\exp_p(w) = \begin{cases} \gamma_{w/\|w\|}(w), & \text{if } w \neq 0_p\,, \\ p, & \text{if } w = 0_p\,. \end{cases}$$

Here, for a unit tangent vector $u \in U_pM$ we denote by γ_u the trajectory for $\mathbf{B}_\kappa$ of initial vector u. It is clear that when $\kappa = 0$ it is the ordinary exponential map $\exp_p$ at p. As we mentioned before, if M is a Hadamard Kähler manifold whose sectional curvatures satisfy $\mathrm{Riem}^M \le c < 0$ and $|\kappa| \le \sqrt{|c|}$, it is known that the magnetic exponential map $\mathbf{B}_\kappa\exp_p : T_pM \to M$ is bijective (see [4]). Since every trajectory half-line has its point at infinity, we see that the magnetic exponential map $\mathbf{B}_\kappa\exp_p$ at p induces a map $\partial\mathbf{B}_\kappa\exp_p : U_pM \ni v \mapsto \gamma_v(\infty) \in \partial M$. Our first assertion in Theorem 4 is equivalent to the assertion that this map is surjective when $|\kappa| \le \sqrt{|c|}$ and is bijective when $|\kappa| < \sqrt{|c|}$. We shall study induced maps step by step.

First, we study the image of the induced map $\partial\mathbf{B}_\kappa\exp_p$.

PROPOSITION 5. *Let M be a Hadamard Kähler manifold whose sectional curvatures satisfy* $\mathrm{Riem}^M \le c < 0$. *If* $|\kappa| \le \sqrt{|c|}$, *the induced map* $\partial\mathbf{B}_\kappa \exp_p : U_pM \to \partial M$ *is surjective.*

PROOF. We take an arbitrary $z \in \partial M$ and choose a geodesic ray $\sigma : [0,\infty) \to M$ satisfying $\sigma(0) = p$ and $\sigma(\infty) = z$. First we study the case $|\kappa| < \sqrt{|c|}$. We consider the trajectory-horn β_σ^κ for $\mathbf{B}_\kappa$ associated with σ. By Theorem 3 we have its limit horn-tube γ_σ^κ. For this trajectory for $\mathbf{B}_\kappa$ we take the associated trajectory-harp. Then it has limit harp-string $\sigma_{\gamma_\sigma^\kappa}$ by Theorem 1. By Remark 2 and by the proof of Proposition 4 we find that $d\big(\sigma(t), \sigma_{\gamma_\sigma^\kappa}(t)\big) \le |\kappa|\pi/\sqrt{|c|(|c|-\kappa^2)}$. Hence we find $\sigma = \sigma_{\gamma_\sigma^\kappa}$ and $\gamma_\sigma^\kappa(\infty) = \sigma(\infty) = z$. Thus we obtain that $\partial\mathbf{B}_\kappa \exp_p$ is surjective when $|\kappa| < \sqrt{|c|}$.

Next we study the case $\kappa = \pm\sqrt{|c|}$. We take a sequence $\{\kappa_j\}_{j=1}^\infty$ satisfying $\lim_{j\to\infty} \kappa_j = \kappa$ and $|\kappa_j| < \sqrt{|c|}$. Considering the trajectory-horn $\beta_\sigma^{\kappa_j}$ for $\mathbf{B}_{\kappa_j}$ associated with σ we take its limit horn-tube $\gamma_j := \gamma_\sigma^{\kappa_j}$. As we see above, the limit harp-string σ_{γ_j} of the trajectory-harp α_{γ_j} associated with γ_j coincides with σ. Therefore by Remark 1 we have $\angle\big(\frac{\partial \alpha_{\gamma_j}}{\partial s}(T,0), \frac{\partial\sigma}{\partial s}(0)\big) \le \big(2/\sqrt{|c|}\big)\int_T^\infty t^{-2}\,dt$ for every $T > 0$.

Since $U_{\sigma(0)}M$ is compact we have a convergent subsequence $\{\dot\gamma_{j_i}(0)\}_{i=1}^\infty$. We denote by γ_∞ the trajectory for $\mathbf{B}_\kappa$ with initial vector $\lim_{i\to\infty}\dot\gamma_{j_i}(0)$. We shall show $\gamma_\infty(\infty) = z$. We take the trajectory-harp α_{γ_∞} associated with γ_∞. By perturbation theory we see $\lim_{i\to\infty}\gamma_{j_i}(T) = \gamma_\infty(T)$ for each T. We therefore have $\lim_{i\to\infty}\frac{\partial\alpha_{\gamma_{j_i}}}{\partial s}(T,0) = \frac{\partial \alpha_{\gamma_\infty}}{\partial s}(T,0)$, and hence obtain

$$\angle\left(\frac{\partial\alpha_{\gamma_\infty}}{\partial s}(T,0), \frac{\partial\sigma}{\partial s}(0)\right) \le \frac{2}{\sqrt{|c|}}\int_T^\infty t^{-2}\,dt\,.$$

As $\lim_{t\to\infty}\ell_{\gamma_\infty}(t) = \infty$ by Proposition 1, this estimate shows that $\gamma_\infty(\infty) = \sigma(\infty) = z$. Thus we get the conclusion also in this case. □

When a Hadamard Kähler manifold M satisfies $\mathrm{Riem}^M \le c < 0$, for a constant κ with $|\kappa| \le \sqrt{|c|}$, we can define a map $\Phi_p^\kappa : U_pM \to U_pM$ by $v \mapsto \dot\sigma_{\gamma_v}(0)$, where γ_v denotes the trajectory for $\mathbf{B}_\kappa$ with initial vector v and σ_{γ_v} the limit harp-string of the trajectory-harp associated with γ_v. This map satisfies $\gamma_v(\infty) = \sigma_{\Phi_p^\kappa(v)}(\infty)$. On the other hand, when $|\kappa| < \sqrt{|c|}$, by the proof of Proposition 5, we can define a map $\Psi_p^\kappa : U_pM \to U_pM$ by $v \mapsto \dot\gamma_{\sigma_v}^\kappa(0)$, where σ_v denotes the geodesic with $\dot\sigma_v(0) = v$ and $\gamma_{\sigma_v}^\kappa$ the limit horn-tube of the trajectory-horn for $\mathbf{B}_\kappa$ associated with σ_v. This map satisfies $\sigma_v(\infty) = \gamma_{\Psi_p^\kappa(v)}(\infty)$. We shall use these maps to study the induced map $\partial\mathbf{B}_\kappa\exp_p$.

Next, we study the injective property of the induced map $\partial\mathbf{B}_\kappa\exp_p$.

PROPOSITION 6. *Let M be a Hadamard Kähler manifold whose sectional curvatures satisfy* $\mathrm{Riem}^M \le c < 0$. *If* $|\kappa| < \sqrt{|c|}$, *the induced map* $\partial\mathbf{B}_\kappa\exp_p : U_pM \to \partial M$ *is injective.*

PROOF. In order to show the assertion we are enough to show that the map $\Phi_p^\kappa : U_pM \to U_pM$ is injective. As usual, for a unit tangent vector $v \in UM$ we denote by σ_v the geodesic with $\dot\sigma_v(0) = v$ and by γ_v the trajectory for $\mathbf{B}_\kappa$ with $\dot\gamma_v(0) = v$.

We consider the composition $\Phi_p^\kappa \circ \Psi_p^\kappa : U_pM \to U_pM$. For $v \in U_pM$ we put $w = \Psi_p^\kappa(v)$, $u = \Phi_p^\kappa \circ \Psi_p^\kappa(v)$. We then have $\sigma_v(\infty) = \gamma_w(\infty) = \sigma_u(\infty)$. As $\partial\exp_p : U_pM \to \partial M$ is bijective, we find that $v = u$, which means that $\Phi_p^\kappa \circ \Psi_p^\kappa$ is the identity. Thus, to show that Φ_p^κ is injective we only need to show that Ψ_p^κ is surjective.

Given $v \in U_pM$ and an arbitrary positive t, we consider the trajectory-horn $\beta_t : [0,\infty) \times \mathbf{R} \to M$ for $\mathbf{B}_\kappa$ associated with the geodesic σ_t which joins $p = \gamma_v(0)$ and $\gamma_v(t)$. Hence $u \mapsto \beta_t(s,u)$ is the trajectory for $\mathbf{B}_\kappa$ joining $p = \sigma_t(0)$ and $\sigma_t(s)$. We denote by $r_t(s)$ the tube-length of β_t at s and set $w_t^s = \frac{\partial\beta_t}{\partial u}(s,0) \in U_pM$. By Proposition 1 we see $r_t(s) \le \tau_\kappa(s;c)$. Thus we have a subsequence $\{t_j\}_{j=1}^\infty$ depending on s which satisfies that both $\{w_{t_j}^s\}_{j=1}^\infty$ $\big(\subset U_pM\big)$ and $\{r_{t_j}(s)\}_{j=1}^\infty$ $\big(\subset \mathbf{R}\big)$ converge. We set $w_\infty^s = \lim_{j\to\infty} w_{t_j}^s$ and $r_\infty(s) = \lim_{j\to\infty} w_{t_j}(s)$. On the other hand, by Theorem 1, we find that $\lim_{t\to\infty}\dot\sigma_t(0) = \Phi_p^\kappa(v)$, hence find that $\lim_{t\to\infty}\sigma_t(s) = \sigma_{\Phi_p^\kappa(v)}(s)$ for each s. As $\sigma_t(s) = \beta_t\big(s, r_t(s)\big)$ we obtain that

$$\sigma_{\Phi_p^\kappa(v)}(s) = \lim_{j\to\infty} \gamma_{w_{t_j}^s}\big(r_{t_j}(s)\big) = \gamma_{w_\infty^s}\big(r_\infty(s)\big)\,.$$

This shows that each $\gamma_{w_\infty^s}$ is a tube of the trajectory-horn associated with $\sigma_{\Phi_p^\kappa(v)}$. By Proposition 3 we have

$$\angle(w_t^s, v) \le \int_s^{\ell_{\gamma_v}(t)} \frac{\sqrt{|c|-\kappa^2}}{\sinh\sqrt{|c|-\kappa^2}\,\xi}\,d\xi$$

for $s \le \ell_{\gamma_v}(t)$, hence obtain

$$\angle(w_\infty^s, v) \le \int_s^{\infty} \frac{\sqrt{|c|-\kappa^2}}{\sinh\sqrt{|c|-\kappa^2}\,\xi}\,d\xi < \infty\,.$$

Thus we find that $\lim_{s\to\infty} w_\infty^s = v$ and get $\Psi_p^\kappa\big(\Phi_p^\kappa(v)\big) = v$. This shows that Ψ_p^κ is surjective or more precisely shows that $\Psi_p^\kappa \circ \Phi_p^\kappa$ is the identity. We therefore get the conclusion. □

By Propositions 5, 6, we get the first assertion of Theorem 4.

REMARK 3. (1) We take an arbitrary geodesic half-line σ of unit speed emanating from $p \in M$. The condition $\Phi_p^\kappa \circ \Psi_p^\kappa = Id$ means that for the limit horn-tube γ_σ^κ of the trajectory-horn for $\mathbf{B}_\kappa$ associated with σ the limit harp-string $\sigma_{\gamma_\sigma^\kappa}$ of its trajectory-harp is σ.

(2) We take an arbitrary trajectory half-line γ for $\mathbf{B}_\kappa$ which is emanating from $p \in M$. The condition $\Psi_p^\kappa \circ \Phi_p^\kappa = Id$ means that for the limit harp-string σ_γ of the trajectory-harp associated with γ the limit horn-tube $\gamma_{\sigma_\gamma}^\kappa$ of its trajectory-horn is γ.

Finally we show the second assertion of Theorem 4. Given two distinct points $z, w \in \partial M$ in the ideal boundary of a Hadamard Kähler manifold M with $\mathrm{Riem}^M \le c < 0$, we take a geodesic σ satisfying $z = \lim_{t\to-\infty}\sigma(t)$ and $w = \lim_{t\to\infty}\sigma(t)$. Given a Kähler magnetic field $\mathbf{B}_\kappa$ with $|\kappa| < \sqrt{|c|}$, for each positive s we take a trajectory γ_s for $\mathbf{B}_\kappa$ joining $\sigma(-s)$ and $\sigma(s)$. We take the parameter of γ_s so that $\gamma_s(0) = \sigma(-s)$ and $\gamma_s(t_s) = \sigma(s)$ with some positive t_s. As a restriction of σ is a harp-string of the trajectory-harp α_{γ_s} associated with γ_s for each s, Remark 2 guarantees the following:

1) If we take positive r_s satisfying $s = \ell_{\gamma_s}(r_s)$, we have $d\big(\sigma(0), \gamma_s(r_s)\big) < |\kappa|\pi/\big(2\sqrt{|c|(|c|-\kappa^2)}\big)$;
2) For $0 \le t \le t_s$ we have $d\big(\gamma_s(t), \sigma\big) \le |\kappa|\pi/\big(2\sqrt{|c|(|c|-\kappa^2)}\big)$.

We take the geodesic ball B of radius $|\kappa|\pi/\sqrt{|c|(|c|-\kappa^2)}$ centered at $\sigma(0)$. As $\gamma_s(r_s) \in B$, we can choose a monotone increasing sequence $\{s_j\}_{j=1}^{\infty}$ so that it satisfies $\lim_{j\to\infty} s_j = \infty$ and that $\big\{\dot\gamma_{s_j}(r_{s_j})\big\}_j \subset UM|_B$ converges. We denote by γ_∞ the trajectory whose initial is $\lim_{j\to\infty}\dot\gamma_{s_j}(r_{s_j})$. By perturbation theory of differential equations we see that $\mathbf{B}_\kappa\exp_p$ is smooth with respect to p. Therefore, we find $d\big(\gamma_\infty(t), \sigma\big)$ is not greater than $|\kappa|\pi/\big(2\sqrt{|c|(|c|-\kappa^2)}\big)$ for each t. This shows that $\lim_{t\to-\infty}\gamma_\infty(t) = z$ and $\lim_{t\to\infty}\gamma_\infty(t) = w$. This completes the proof of the second assertion of Theorem 4.

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Present Addresses:
QINGSONG SHI
DIVISION OF MATHEMATICS AND MATHEMATICAL SCIENCE,
GRADUATE SCHOOL OF ENGINEERING,
NAGOYA INSTITUTE OF TECHNOLOGY,
GOKISO, NAGOYA 466–8555, JAPAN.
e-mail: sqs120012@yahoo.co.jp

TOSHIAKI ADACHI
DEPARTMENT OF MATHEMATICS,
NAGOYA INSTITUTE OF TECHNOLOGY,
NAGOYA 466–8555, JAPAN.
e-mail: adachi@nitech.ac.jp