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Historic Behaviour for Random Expanding Maps on the Circle

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Abstract. F. Takens constructed a residual subset of the state space consisting of initial points with historic behaviour for expanding maps on the circle. We prove that this statistical property of expanding maps on the circle is preserved under small random perturbations. The proof is given by establishing a random Markov partition, which follows from a random version of Shub's Theorem on topological conjugacy with the folding maps.

1. Introduction

Let M be a compact smooth Riemannian manifold. For a dynamical system $f : M \to M$, the orbit issued from an initial point x in M is said to have historic behaviour when there exists a continuous function $\varphi : M \to \mathbf{R}$ such that the time average

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi\big(f^j(x)\big)$$

does not exist. Several statistical quantities are given as the time average of some observable φ, and thus it is a natural question in smooth dynamical systems theory whether the set of points with historic behaviour is not negligible in some sense; it is typically formulated as a positive measure set with respect to the normalised Lebesgue measure. Takens asked in [12] whether there are persistent classes of smooth dynamical systems such that the set of initial points with historic behaviour is of positive Lebesgue measure (*Takens' Last Problem*). Very recently, it was affirmatively answered by Kiriki and Soma [9] with diffeomorphisms having homoclinic tangencies. The reader is asked to see [11], [12] and [4] for the background of historic behaviour in this measure-theoretical sense; see also [6] and [10] for the (recent) contribution to Takens' Last Problem for one-dimensional mappings and flows.

As another measurement to investigate historic behaviour, Takens [12] considered whether the set of points with historic behaviour is a residual subset of the state space (i.e., the set of points with historic behaviour is not negligible in a topological sense). For the doubling map on the circle, he constructed a residual subset consisting of initial points with historic

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behaviour by using symbolic dynamics. Since the method of symbolic dynamics is applicable to any expanding maps on the circle, the proof can be literally translated to expanding maps on the circle (see also [3] for another investigation of historic behaviour for expanding maps, and [8] for historic behaviour of geometric Lorenz flows in topological sense). Our goal in this paper is to extend his result for historic behaviour (and its generalisation to expanding maps on the circle) to a random setting. For a random version of Takens' problem in measure-theoretical sense, we refer to the result by Araújo [1]; *in contrast to* our result in topological setting, he gave a negative answer to a random version of Takens' problem in measure-theoretical sense for general diffeomorphisms under some conditions on noise.

As in the unperturbed case, the key step in the proof is establishing a (random) Markov partition. This is given through proving a random version of Shub's Theorem on topological conjugacy of expanding maps with the folding maps.

1.1. Definitions and results. Let $\mathbf{S}^1$ be a circle given by $\mathbf{S}^1 = \mathbf{R}/\mathbf{Z}$. We endow $\mathbf{S}^1$ with a metric $d_{\mathbf{S}^1}(\cdot,\cdot)$, where $d_{\mathbf{S}^1}(x,y)$ is the infimum of $|\tilde{x}-\tilde{y}|$ over all representatives $\tilde{x},\tilde{y}$ of $x,y\in\mathbf{S}^1$, respectively. Let $\mathscr{C}^r(\mathbf{S}^1,\mathbf{S}^1)$ and Homeo$(\mathbf{S}^1,\mathbf{S}^1)$ be the spaces of all endomorphisms of class $\mathscr{C}^r$ and homeomorphisms on the circle $\mathbf{S}^1$, endowed with the usual $\mathscr{C}^r$ and $\mathscr{C}^0$ metrics $d_{\mathscr{C}^r}(\cdot,\cdot)$ and $d_{\mathscr{C}^0}(\cdot,\cdot)$, respectively, with $r>1$. (Given that $r=k+\gamma$ for some $k\in\mathbf{N}$, $k\geq 1$ and $0\leq\gamma\leq 1$, $f\in\mathscr{C}^r(\mathbf{S}^1,\mathbf{S}^1)$ denotes the k-th derivative of f is γ-Hölder.) Let $\pi_{\mathbf{S}^1}:\mathbf{R}\to\mathbf{S}^1$ be the canonical projection on $\mathbf{S}^1$, i.e., $\pi_{\mathbf{S}^1}(\tilde{x})$ is the equivalent class of $\tilde{x}\in\mathbf{R}$. We call $A(\subsetneq\mathbf{S}^1)$ a closed interval of $\mathbf{S}^1$ if $A=\pi_{\mathbf{S}^1}(\tilde{A})$ for some closed interval $\tilde{A}$ of $\mathbf{R}$. Open intervals or left-closed and right-open intervals of $\mathbf{S}^1$ are defined in a similar manner. We let $\mathcal{F}(\mathbf{S}^1)$ denote the space consisting of all nonempty left-closed and right-open intervals of $\mathbf{S}^1$, all point sets of $\mathbf{S}^1$ and the state space $\mathbf{S}^1$, endowed with the Hausdorff metric $d_H(\cdot,\cdot)$. We endow $\mathscr{C}^r(\mathbf{S}^1,\mathbf{S}^1)$, Homeo$(\mathbf{S}^1,\mathbf{S}^1)$ and $\mathcal{F}(\mathbf{S}^1)$ with the Borel σ-field.

Let Ω be a separable complete metric space endowed with the Borel σ-field $\mathcal{B}(\Omega)$ with a probability measure $\mathbf{P}$. Given a smooth map $f_0:\mathbf{S}^1\to\mathbf{S}^1$ of class $\mathscr{C}^r$, let $\{f_\varepsilon\}_{\varepsilon>0}$ be a family of continuous mappings defined on Ω with values in $\mathscr{C}^r(\mathbf{S}^1,\mathbf{S}^1)$ such that

$$\sup_{\omega\in\Omega} d_{\mathscr{C}^r}\big(f_\varepsilon(\omega),f_0\big)\to 0 \quad \text{as } \varepsilon\to 0\,. \tag{1}$$

For each $\varepsilon>0$, adopting the notation $f_\varepsilon(\omega,\cdot)=f_\varepsilon(\omega)$, the distance between $f_\varepsilon(\omega,x)$ and $f_\varepsilon(\omega',x)$ is bounded by $d_{\mathscr{C}^r}(f_\varepsilon(\omega),f_\varepsilon(\omega'))$ for each $x\in\mathbf{S}^1$ and each $\omega,\omega'\in\Omega$. Thus, it is straightforward to realise that $f_\varepsilon:\Omega\times\mathbf{S}^1\to\mathbf{S}^1$ is a continuous (in particular, measurable) mapping. When convenient, we will identify $f_0:\mathbf{S}^1\to\mathbf{S}^1$ with the constant map $\Omega\ni\omega\mapsto f_0$.

Let f_0 be an orientation-preserving expanding map on the circle, i.e., there exists a constant $\lambda_0>1$ such that $\inf_x \frac{d}{dx}f_0(x)\geq\lambda_0$. For the properties of expanding maps, the reader is referred to [7]. Let $k\geq 2$ be the degree of the covering map f_0. In view of (1) it then follows that $f_\varepsilon(\omega)$ is an orientation-preserving expanding map for each $\omega \in \Omega$ if ε is sufficiently small. In fact, if $\lambda_0 = \inf_x \frac{d}{dx} f_0(x)$ and we set $\lambda = (\lambda_0 + 1)/2$, then $\lambda > 1$ and we can find an $\varepsilon_0 > 0$ such that

$$\inf_{\omega} \inf_{x} \frac{\partial}{\partial x} f_\varepsilon(\omega, x) \geq \lambda, \quad 0 \leq \varepsilon < \varepsilon_0 . \tag{2}$$

Let $\delta_0 < \frac{1}{2}(1 - \frac{1}{k})$ be a positive number and η a positive number such that

$$\eta < \min\left\{\frac{1}{2}, (\lambda - 1)\delta_0\right\} . \tag{3}$$

We also assume ε_0 to be sufficiently small such that

$$\sup_{\omega \in \Omega} d_{\mathscr{C}^0}\big(f_\varepsilon(\omega), f_0\big) < \eta, \quad 0 \leq \varepsilon < \varepsilon_0 . \tag{4}$$

In particular, $f_\varepsilon(\omega)$ is a covering map of $\mathbf{S}^1$ with degree k for each $\omega \in \Omega$.

Since f_0 is a covering of $\mathbf{S}^1$ of degree $k \geq 2$, there exists a fixed point $p_0 \in \mathbf{S}^1$ for f_0. In view of (1) and (2) together with that f_0 is locally a diffeomorphism of the circle, we can find a closed interval $B = B_\varepsilon$ including p_0 such that if $0 \leq \varepsilon < \varepsilon_0$, then $f_\varepsilon(\omega) : B \to f_\varepsilon(\omega)(B)$ is a diffeomorphism such that $B \subset f_\varepsilon(\omega)(B)$ for each $\omega \in \Omega$ (by taking ε_0 sufficiently small if necessary). We assume that ε_0 is sufficiently small such that

$$|B| \leq \delta_0 \tag{5}$$

for all $0 \leq \varepsilon < \varepsilon_0$, where $|B|$ is the length of B with respect to $d_{\mathbf{S}^1}(\cdot, \cdot)$.

REMARK 1. When there is no ambiguity, the noise level ε will sometimes be omitted from the notation, in particular when the dependence on the noise parameter $\omega \in \Omega$ is already displayed. Throughout the rest of the paper we will also permit us to use ε_0 as a way to denote the upper bound of a range $0 \leq \varepsilon < \varepsilon_0$ for which (2), (4) and (5) hold, even if ε_0 may change between occurrences.

Let $f : \Omega \to \mathscr{C}^r(\mathbf{S}^1, \mathbf{S}^1)$ be a continuous mapping, and $\theta : \Omega \to \Omega$ a measure-preserving homeomorphism on $(\Omega, \mathbf{P})$ (see Remark 3 for this condition). For simplicity, we also assume θ to be ergodic. For each $n \geq 1$, let $f^{(n)}(\omega, x)$ be the fibre component in the n-th iteration of the skew product mapping

$$\Theta(\omega, x) = \big(\theta\omega, f(\omega, x)\big), \quad (\omega, x) \in \Omega \times \mathbf{S}^1 ,$$

where we simply write $\theta\omega$ for $\theta(\omega)$. Setting the notation $f_\omega = f(\omega, \cdot)$ and $f_\omega^{(n)} = f^{(n)}(\omega, \cdot)$, the explicit form of $f_\omega^{(n)}$ is

$$f_\omega^{(n)} = f_{\theta^{n-1}\omega} \circ f_{\theta^{n-2}\omega} \circ \cdots \circ f_\omega .$$

For convenience, we set $f_\omega^{(0)} = \mathrm{id}_{\mathbf{S}^1}$ for each $\omega \in \Omega$. We call $\{f_\omega^{(n)}(x)\}_{n\geq 0} = \{x, f_\omega(x), f_\omega^{(2)}(x), \ldots\}$ the *random orbit* of f issued from $(\omega, x) \in \Omega \times \mathbf{S}^1$. As in the treatment of the unperturbed case undertaken by Takens [12], we now define historic behaviour for random orbits.

DEFINITION 1. We say that a random orbit issued from (ω, x) has *historic behaviour* if the empirical measure

$$\frac{1}{n}\sum_{j=0}^{n-1} \delta_{f_\omega^{(j)}(x)}$$

does not converge to any probability measure on the circle in weak* sense, where δ_x is the Dirac measure at x.

Our goal is to prove the following theorem.

THEOREM 1. *Let* $0 \leq \varepsilon < \varepsilon_0$. *For* $\mathbf{P}$*-almost every* $\omega \in \Omega$, *we can find a residual subset* $\mathcal{R}^\omega$ *of* $\mathbf{S}^1$ *such that for any* $x \in \mathcal{R}^\omega$ *the random orbit of* f_ε *issued from* (ω, x) *has historic behaviour.*

REMARK 2. It is natural to ask if it might be possible to replace "$\mathbf{P}$-almost every" with "every" in Theorem 1 since the assumption on our perturbations is given for every $\omega \in \Omega$ as in (1). However, this strengthening of the conclusion in Theorem 1 seems to be impossible if the proof utilises the ergodicity of $(\theta, \mathbf{P})$ as in this paper, because the ergodicity of $(\theta, \mathbf{P})$ (together with Birkhoff's ergodic theorem) only ensures that time averages coincide with their corresponding space averages for $\mathbf{P}$-almost every $\omega \in \Omega$. See Remark 3 for the detail. Another direction to improve Theorem 1 is to weaken the assumption on our perturbations, see Remark 4 for the detailed discussion about the direction.

2. The proof

We start the proof by considering a random version of Dowker's Theorem. This theorem asserts that if we find a dense orbit with historic behaviour (of the unperturbed dynamics), then there exists a residual subset of the phase space such that the orbit of each point in the set has historic behaviour. For this purpose, we need to give stronger forms of the definitions of denseness and historic behaviour for random orbits. When there is no confusion, we employ the notation $f_\omega^{(n)} = f^{(n)}(\omega, \cdot)$ for $f = f_\varepsilon$ once $0 \leq \varepsilon < \varepsilon_0$ is given. For a continuous function $\varphi : \mathbf{S}^1 \to \mathbf{R}$, $\omega \in \Omega$, $x \in \mathbf{S}^1$ and $n \geq 1$, we define the truncated time average $B_n(\varphi; \omega, x)$ of the observable φ by

$$B_n(\varphi; \omega, x) = \frac{1}{n}\sum_{j=0}^{n-1} \varphi \circ f_\omega^{(j)}(x) .$$

Note that the orbit issued from (ω, x) has historic behaviour if $B_n(\varphi; \omega, x)$ does not converge for some continuous function $\varphi : \mathbf{S}^1 \to \mathbf{R}$.

DEFINITION 2. Let $f : \Omega \to \mathscr{C}^r(\mathbf{S}^1, \mathbf{S}^1)$ be a measurable mapping. Let X be a random variable on Ω with values in $\mathbf{S}^1$. We call

$$\left\{X(\omega),\, f_\omega(X(\omega)),\, f_\omega^{(2)}(X(\omega)), \ldots\right\}$$

the *future random orbit* of X at ω, and

$$\left\{X(\omega),\, f_{\theta^{-1}\omega}\big(X(\theta^{-1}\omega)\big),\, f^{(2)}_{\theta^{-2}\omega}\big(X(\theta^{-2}\omega)\big), \ldots\right\}$$

the *past random orbit* of X at ω (see [2] for this notation).

We say that X has *historic behaviour* at ω if there exist real numbers α, β and a continuous function $\varphi : \mathbf{S}^1 \to \mathbf{R}$ such that

$$\liminf_{n\to\infty} B_n\big(\varphi; \theta^{-\ell}\omega, X\big(\theta^{-\ell}\omega\big)\big) < \alpha < \beta < \limsup_{n\to\infty} B_n\big(\varphi; \theta^{-\ell}\omega, X\big(\theta^{-\ell}\omega\big)\big) \tag{6}$$

for each $\ell \geq 0$ (that is, the time average of φ along the future random orbit of X does not exist at each $\theta^{-\ell}\omega$).

PROPOSITION 1. *Let $f : \Omega \to \mathscr{C}^r(\mathbf{S}^1, \mathbf{S}^1)$ be a measurable mapping and $\omega \in \Omega$. Assume that there exists a measurable mapping $X : \Omega \to \mathbf{S}^1$ such that the past random orbit of X at ω is dense in $\mathbf{S}^1$ and X has historic behaviour at ω. Then, we can find a residual subset $\mathcal{R}^\omega$ of $\mathbf{S}^1$ such that for any $x \in \mathcal{R}^\omega$, the random orbit issued from (ω, x) has historic behaviour.*

PROOF. Let α, β and φ be constants and a continuous function given in (6) for a measurable mapping $X : \Omega \to \mathbf{S}^1$ and $\omega \in \Omega$, where X has historic behaviour at ω by hypothesis. We let $\mathcal{R}^\omega = s_1^\omega \cap s_2^\omega$, where

$$s_1^\omega = \bigcap_{N\geq 1} \bigcup_{n\geq N} \left\{x \in \mathbf{S}^1 : B_n(\varphi; \omega, x) < \alpha\right\},$$

$$s_2^\omega = \bigcap_{N\geq 1} \bigcup_{n\geq N} \left\{x \in \mathbf{S}^1 : B_n(\varphi; \omega, x) > \beta\right\}.$$

Then it is straightforward to see that for any $x \in \mathcal{R}^\omega$, we have

$$\liminf_{n\to\infty} B_n(\varphi; \omega, x) \leq \alpha < \beta \leq \limsup_{n\to\infty} B_n(\varphi; \omega, x).$$

In particular, the random orbit issued from (ω, x) has historic behaviour. Note also that $\mathcal{R}^\omega$ is a countable intersection of open sets.

The rest of the proof is devoted to showing that the past random orbit of X at ω is included in $\mathcal{R}^\omega$. Since the past random orbit of X at ω is dense in $\mathbf{S}^1$ by hypothesis, it leads to that $\mathcal{R}^\omega$ is dense in $\mathbf{S}^1$, and we complete the proof. Let $\ell \geq 1$. Due to the observation $f_\omega^{(i)} \circ f_{\theta^{-\ell}\omega}^{(\ell)} = f_{\theta^{-\ell}\omega}^{(i+\ell)}$ for any $i \geq 0$, we have

$$B_n\left(\varphi; \omega, f_{\theta^{-\ell}\omega}^{(\ell)}\big(X\big(\theta^{-\ell}\omega\big)\big)\right) = \frac{1}{n}\sum_{j=\ell}^{n+\ell-1} \varphi \circ f_{\theta^{-\ell}\omega}^{(j)}\big(X\big(\theta^{-\ell}\omega\big)\big).$$

Therefore, for each $n \geq \ell$

$$\begin{aligned} &B_n\left(\varphi; \omega, f_{\theta^{-\ell}\omega}^{(\ell)}\big(X\big(\theta^{-\ell}\omega\big)\big)\right) - B_n\big(\varphi; \theta^{-\ell}\omega, X\big(\theta^{-\ell}\omega\big)\big) \\ &\qquad = \frac{1}{n}\left(\sum_{j=n}^{n+\ell-1} \varphi \circ f_{\theta^{-\ell}\omega}^{(j)}\big(X\big(\theta^{-\ell}\omega\big)\big) - \sum_{j=0}^{\ell-1} \varphi \circ f_{\theta^{-\ell}\omega}^{(j)}\big(X\big(\theta^{-\ell}\omega\big)\big)\right). \end{aligned}$$

Hence we have

$$\left|B_n\left(\varphi; \omega, f_{\theta^{-\ell}\omega}^{(\ell)}\big(X\big(\theta^{-\ell}\omega\big)\big)\right) - B_n\big(\varphi; \theta^{-\ell}\omega, X\big(\theta^{-\ell}\omega\big)\big)\right| \leq \frac{2\ell\|\varphi\|_{\mathscr{C}^0}}{n}, \tag{7}$$

which goes to zero as n goes to infinity for any fixed $\ell \geq 1$.

Let $\{n_k\}_{k\geq 1}$ be a sequence such that $B_{n_k}(\varphi; \theta^{-\ell}\omega, X(\theta^{-\ell}\omega))$ converges to the infimum limit in (6) as k goes to infinity. Then, in view of (7) we have

$$B_{n_k}\left(\varphi; \omega, f_{\theta^{-\ell}\omega}^{(\ell)}\big(X\big(\theta^{-\ell}\omega\big)\big)\right) < \alpha \quad \text{if } k \text{ is sufficiently large.}$$

This implies that $f_{\theta^{-\ell}\omega}^{(\ell)}(X(\theta^{-\ell}\omega))$ is in s_1^ω. In a similar manner, we can show that $f_{\theta^{-\ell}\omega}^{(\ell)}(X(\theta^{-\ell}\omega))$ is in s_2^ω. Therefore, $f_{\theta^{-\ell}\omega}^{(\ell)}(X(\theta^{-\ell}\omega))$ is in $\mathcal{R}^\omega$, and the past random orbit of X at ω is included in $\mathcal{R}^\omega$. □

Due to Proposition 1, Theorem 1 is reduced to constructing *one* measurable mapping X such that for $\mathbf{P}$-almost every $\omega \in \Omega$, the past random orbit of X at ω is dense in $\mathbf{S}^1$ and X has historic behaviour at ω.

2.1. Shub's Theorem and Markov partition. In the next subsection, we establish the coding of *graphs* associated with a *random Markov partition* of f_ε, which is a key ingredient in our proof. For that purpose, we will need the following extension of Shub's Theorem on topological conjugacy between expanding mappings and the folding mapping with the same degree to our random setting.

Set $\mathcal{A} = \{0, 1, \ldots, k-1\}$. For $s = (s_0, s_1, \ldots, s_m, s_{m+1}, \ldots, s_n, \ldots, s_{N-1}) \in \mathcal{A}^N$ with integers $0 \leq m \leq n \leq N-1$, we denote $(s_m, s_{m+1}, \ldots, s_n)$ by $[s]_m^n$. Recall that in view of (4), the degree of $f_\varepsilon(\omega, \cdot) : \mathbf{S}^1 \to \mathbf{S}^1$ is k if $0 \leq \varepsilon < \varepsilon_0$ and $\omega \in \Omega$.

THEOREM 2. *Suppose that* $0 \leq \varepsilon < \varepsilon_0$. *Then we can find a continuous mapping* $h : \Omega \to \mathrm{Homeo}(\mathbf{S}^1, \mathbf{S}^1)$ (*in particular, measurable mapping*) *which satisfies that*

$$h(\theta\omega) \circ E_k = f_\varepsilon(\omega) \circ h(\omega) \tag{8}$$

for any $\omega \in \Omega$, *where* $E_k : \mathbf{S}^1 \to \mathbf{S}^1$ *is the* k*-folding mapping defined by* $E_k(x) = \pi_{\mathbf{S}^1}(k\tilde{x})$ *for each* $x \in \mathbf{S}^1$ *with a representative* $\tilde{x}$ *of* x.

Furthermore, for every $\omega, \omega' \in \Omega$ *we have*

$$d_{\mathscr{C}^0}\big(h(\omega), h(\omega')\big) \leq \delta_0 , \tag{9}$$

where δ_0 *is given in* (3).

PROOF. Throughout the proof, we fix $0 \leq \varepsilon < \varepsilon_0$ and omit it in notions if there is no confusion. Recall that B is a closed interval of $\mathbf{S}^1$ (given above (5)) such that $B \subset f_\varepsilon(\omega)(B)$ for each $\omega \in \Omega$. Then, one can find a lift $\tilde{f}_\varepsilon(\omega) : \mathbf{R} \to \mathbf{R}$ of the k-covering $f_\varepsilon(\omega) : \mathbf{S}^1 \to \mathbf{S}^1$ with $\omega \in \Omega$ and a closed interval $\tilde{B} \subset \mathbf{R}$ such that $\pi_{\mathbf{S}^1}(\tilde{B}) = B$ and $\tilde{B} \subset \tilde{f}_\varepsilon(\omega)(\tilde{B})$ for each $\omega \in \Omega$.

We shall first find a continuous mapping $p : \Omega \to \tilde{B}$, which is invariant under $\tilde{f}_\varepsilon$, i.e., $\tilde{f}_\varepsilon(\omega, p(\omega)) = p(\theta\omega)$ for every $\omega \in \Omega$. We denote for each $\omega \in \Omega$ the inverse mapping of the diffeomorphism $\tilde{f}_\varepsilon(\omega)$ on $\mathbf{R}$ by $F(\omega)$. Then, due to (2), we have

$$\sup_{\omega} \sup_{x} \left| \frac{d}{dx} \left[F(\omega)\right](x) \right| \leq \lambda^{-1} . \tag{10}$$

Let $\mathscr{C}^0(\Omega, \tilde{B})$ be the space of all continuous mappings from Ω to the closed interval $\tilde{B} \subset \mathbf{R}$, endowed with the usual $\mathscr{C}^0$ metric $d_{\mathscr{C}^0}(\cdot, \cdot)$ defined by $d_{\mathscr{C}^0}(\phi_1, \phi_2) = \sup_\omega |\phi_1(\omega) - \phi_2(\omega)|$ for $\phi_1, \phi_2 \in \mathscr{C}^0(\Omega, \tilde{B})$. For each $\varphi \in \mathscr{C}^0(\Omega, \tilde{B})$, we define a mapping $\mathcal{G}(\varphi) : \Omega \to \tilde{B}$ by

$$\mathcal{G}(\varphi)(\omega) = F(\omega)(\varphi(\theta\omega)), \quad \omega \in \Omega .$$

(Note that $\mathcal{G}(\varphi)(\omega)$ is indeed in $\tilde{B}$ since $\tilde{B} \subset \tilde{f}_\varepsilon(\omega)(\tilde{B})$ for each $\omega \in \Omega$.) It is straightforward to see that $(\omega, x) \mapsto F(\omega)(x)$ is a continuous mapping from $\Omega \times \tilde{B}$ to $\tilde{B}$ (see the argument below (1)). Thus, it follows from the continuity of θ and φ that $\mathcal{G}(\varphi) : \Omega \to \tilde{B}$ is also a continuous mapping. (The transformation $\mathcal{G} : \mathscr{C}^0(\Omega, \tilde{B}) \to \mathscr{C}^0(\Omega, \tilde{B})$ is called the *graph transformation* induced by F.) Furthermore, by virtue of (10) together with the mean value theorem, we have

$$\begin{aligned} d_{\mathscr{C}^0}\big(\mathcal{G}(\phi_1), \mathcal{G}(\phi_2)\big) &\leq \sup_{\omega\in\Omega} \big|F(\omega)\big(\phi_1(\theta\omega)\big) - F(\omega)\big(\phi_2(\theta\omega)\big)\big| \\ &\leq \lambda^{-1} \sup_{\omega\in\Omega} |\phi_1(\theta\omega) - \phi_2(\theta\omega)| = \lambda^{-1} d_{\mathscr{C}^0}(\phi_1, \phi_2) , \end{aligned}$$

for all $\phi_1, \phi_2 \in \mathscr{C}^0(\Omega, \tilde{B})$, i.e., $\mathcal{G}$ is a contraction mapping on the complete metric space $\mathscr{C}^0(\Omega, \tilde{B})$. Therefore, there exists a unique fixed point p of $\mathcal{G}$. By construction, $p : \Omega \to \tilde{B} \subset \mathbf{R}$ is an $\tilde{f}_\varepsilon$-invariant continuous mapping.

We next construct a sequence $\{h_n\}_{n\geq 0}$ of continuous mappings $h_n : \Omega \to \mathrm{Homeo}(\mathbf{S}^1, \mathbf{S}^1)$, which shall uniformly converge to the desired mapping h as n goes to infinity. Since $\tilde{f}_\varepsilon(\omega) : \mathbf{R} \to \mathbf{R}$ for $\omega \in \Omega$ is a lift of the orientation-preserving k-covering

$f_\varepsilon(\omega)$ of $\mathbf{S}^1$ and $\tilde{f}_\varepsilon(\omega)(p(\omega)) = p(\theta\omega)$, we have $\tilde{f}_\varepsilon(\omega)(p(\omega)+1) = p(\theta\omega)+k$ and $\tilde{f}_\varepsilon(\omega) : [p(\omega), p(\omega)+1] \to [p(\theta\omega), p(\theta\omega)+k]$ is a monotonically increasing homeomorphism for all ω. Now we define points $a^n_j(\omega)$ in $[p(\omega), p(\omega)+1)$ for $n \geq 0$, $0 \leq j \leq k^n - 1$ and $\omega \in \Omega$, inductively with respect to n. In the case $n = 0$, let $a^0_0(\omega) = p(\omega)$ for each $\omega \in \Omega$. For given integer $n \geq 0$, we assume that $a^n_j(\omega)$'s are well defined for each $\omega \in \Omega$ and $0 \leq j \leq k^n - 1$. Then we define $a^{n+1}_j(\omega)$ for $0 \leq j \leq k^{n+1} - 1$ of the form $j = \ell \cdot k^n + j_1$ with integers $0 \leq \ell \leq k-1$ and $0 \leq j_1 \leq k^n - 1$ by

$$a^{n+1}_j(\omega) = F(\omega)\big(a^n_{j_1}(\theta\omega) + \ell\big), \quad \omega \in \Omega\,. \tag{11}$$

Setting the notation $F_\ell(\omega) = F(\omega)(\cdot + \ell)$ and $\bar{s} = \sum_{i=0}^{n-1} s_i \cdot k^i$ for $s = (s_0, s_1, \dots, s_{n-1})$ in $\mathcal{A}^n$, the explicit form of $a^n_{\bar{s}}(\omega)$ with $s \in \mathcal{A}^n$ and $\omega \in \Omega$ is

$$a^n_{\bar{s}}(\omega) = F_{s_{n-1}}(\omega) \circ F_{s_{n-2}}(\theta\omega) \circ \cdots \circ F_{s_0}\big(\theta^{n-1}\omega\big)\big(p(\theta^n\omega)\big)\,. \tag{12}$$

We denote the composition of mappings in the right-hand side of (12) by $F^{(n)}_s(\omega)$, i.e., $a^n_{\bar{s}}(\omega) = F^{(n)}_s(\omega)(p(\theta^n\omega))$. Then it is straightforward to see that $F^{(n+m)}_s(\omega) = F^{(n)}_{[s]^{n+m-1}_m}(\omega) \circ F^{(m)}_{[s]^{m-1}_0}(\theta^n\omega)$ for each integer $m \geq 0$, $s \in \mathcal{A}^{n+m}$ and $\omega \in \Omega$. Therefore, for any $s \in \mathcal{A}^n$, integer $m \geq 0$ and $\omega \in \Omega$,

$$\begin{aligned} a^{n+m}_{k^m\cdot\bar{s}}(\omega) &= F^{(n)}_s(\omega) \circ F^{(m)}_{(00\dots0)}(\theta^n\omega)\big(p(\theta^{n+m}\omega)\big) \\ &= F^{(n)}_s(\omega)\big(p(\theta^n\omega)\big) = a^n_{\bar{s}}(\omega)\,. \end{aligned} \tag{13}$$

(Note that $k^m \cdot \bar{s} = \bar{t}$ with the concatenation t of two words $(00\dots0) \in \mathcal{A}^m$ and s, and that $F_0(\omega)(p(\theta\omega)) = p(\omega)$ for any $\omega \in \Omega$.) For convenience, we set $a^n_{k^n}(\omega) = p(\omega)+1$ for each $\omega \in \Omega$.

For the time being, we fix $n \geq 0$ and $\omega \in \Omega$. Then it follows from the monotonicity of $\tilde{f}_\varepsilon(\omega)$ in (11) that $a^n_0(\omega) < a^n_1(\omega) < \cdots < a^n_{k^n}(\omega)$. Thus we can define a continuous, monotonically increasing, piecewise linear mapping $\tilde{h}_n(\omega) : [0,1] \to [p(\omega), p(\omega)+1]$ such that

$$\tilde{h}_n(\omega)\left(\frac{j}{k^n}\right) = a^n_j(\omega), \quad 0 \leq j \leq k^n\,, \tag{14}$$

and that $\tilde{h}_n(\omega) : [j/k^n, (j+1)/k^n] \to [a^n_j(\omega), a^n_{j+1}(\omega)]$ is an affine mapping of slope $k^n \cdot (a^n_{j+1}(\omega) - a^n_j(\omega))$. We finally define $h_n(\omega) : \mathbf{S}^1 \to \mathbf{S}^1$ by

$$h_n(\omega)(x) = \pi_{\mathbf{S}^1} \circ \tilde{h}_n(\omega)(\tilde{x}), \quad x \in \mathbf{S}^1\,, \tag{15}$$

where $\tilde{x}$ is a representative of x in $[0,1]$. (This is well-defined since $\tilde{h}_n(\omega)(1) = \tilde{h}_n(\omega)(0) + 1$.)

Now we change $n \geq 0$ while ω is still fixed. For any nonnegative integers n, m and $s \in \mathcal{A}^n$, it follows from (13) and (14) that

$$\tilde{h}_{n+m}(\omega)\left(\frac{\bar{s}}{k^n}\right) = \tilde{h}_{n+m}(\omega)\left(\frac{k^m \cdot \bar{s}}{k^{n+m}}\right) = a^{n+m}_{k^m \cdot \bar{s}}(\omega) = a^n_{\bar{s}}(\omega) = \tilde{h}_n(\omega)\left(\frac{\bar{s}}{k^n}\right).$$

Hence, by the monotonicity of $\tilde{h}_n(\omega)$ and $\tilde{h}_{n+m}(\omega)$, the supremum norm of $h_n(\omega)(x) - h_{n+m}(\omega)(x)$ (over $x \in \mathbf{S}^1$) is bounded by

$$\max_{0 \leq j \leq k^n - 1} \left| a^n_{j+1}(\omega) - a^n_j(\omega) \right| . \tag{16}$$

On the other hand, since $[a^n_j(\omega), a^n_{j+1}(\omega)]$ is diffeomorphically mapped onto $[p(\theta^n\omega) + j, p(\theta^n\omega) + j + 1]$ by $\tilde{f}^{(n)}_\omega$, the length of each $[a^n_j(\omega), a^n_{j+1}(\omega)]$ does not exceed λ^{-n} (independently of ω) by virtue of (2), so that $\{\pi_{\mathbf{S}^1}(a^n_j(\omega)) \mid n \geq 0, 0 \leq j \leq k^n\}$ is dense in $\mathbf{S}^1$ and (16) uniformly converges to zero as n goes to infinity with respect to ω. Therefore, $\{h_n\}_{n\geq 0}$ is a Cauchy sequence of continuous mappings from Ω to Homeo($\mathbf{S}^1, \mathbf{S}^1$) and there exists the limit mapping $h = \lim_{n\to\infty} h_n$, which is also by construction a continuous mapping from Ω to Homeo($\mathbf{S}^1, \mathbf{S}^1$).

We see the conjugacy (8) between f_ε and E_k. Let $\omega \in \Omega$, $n \geq 0$ and $0 \leq j \leq k^{n+1} - 1$ of the form $j = \ell \cdot k^n + j_1$ with some integers $0 \leq \ell \leq k - 1$ and $0 \leq j_1 \leq k^n - 1$. Then on the one hand, in view of (11), (14) and (15), we have

$$f_\varepsilon(\omega) \circ h(\omega) \circ \pi_{\mathbf{S}^1}\left(\frac{j}{k^{n+1}}\right) = \pi_{\mathbf{S}^1} \circ \tilde{f}_\varepsilon(\omega)\big(a^{n+1}_j(\omega)\big) = \pi_{\mathbf{S}^1}\big(a^n_{j_1}(\theta\omega)\big).$$

On the other hand,

$$h(\theta\omega) \circ E_k \circ \pi_{\mathbf{S}^1}\left(\frac{j}{k^{n+1}}\right) = h(\theta\omega) \circ \pi_{\mathbf{S}^1}\left(\frac{j_1}{k^n}\right) = \pi_{\mathbf{S}^1}\big(a^n_{j_1}(\theta\omega)\big).$$

Since $\{\pi_{\mathbf{S}^1}(j/k^n) \mid n \geq 0, 0 \leq j \leq k^n - 1\}$ is dense in $\mathbf{S}^1$, we get (8).

In the end, we prove the estimate (9) of h by showing

$$|a^n_j(\omega) - a^n_j(\omega')| \leq \delta_0 \tag{17}$$

for all $n \geq 0$, $0 \leq j \leq k^n - 1$ and $\omega, \omega' \in \Omega$. We show (17) by induction with respect to $n \geq 0$. Due to (5), this inequality holds for $n = 0$. Suppose that (17) is true for given $n \geq 0$. Let $\omega, \omega' \in \Omega$ and $0 \leq j \leq k^{n+1} - 1$ of the form $j = \ell \cdot k^n + j_1$ with some integers $0 \leq \ell \leq k - 1$ and $0 \leq j_1 \leq k^n - 1$. Then, it follows from (11) together with the triangle inequality that $|a^{n+1}_j(\omega) - a^{n+1}_j(\omega')|$ is bounded by $S_1 + S_2$, where

$$S_1 = \left| F_\ell(\omega)\big(a^n_{j_1}(\theta\omega)\big) - F_\ell(\omega')\big(a^n_{j_1}(\theta\omega)\big) \right|,$$

$$S_2 = \left| F_\ell(\omega')\big(a^n_{j_1}(\theta\omega)\big) - F_\ell(\omega')\big(a^n_{j_1}(\theta\omega')\big) \right|.$$

To estimate S_1, we let $x = F_\ell(\omega)(a^n_{j_1}(\theta\omega))$ and $x' = F_\ell(\omega')(a^n_{j_1}(\theta\omega))$. Then, $\tilde{f}_\varepsilon(\omega')(x') = \tilde{f}_\varepsilon(\omega)(x)$, and in view of (4) we have

$$\left|\tilde{f}_\varepsilon(\omega')(x) - \tilde{f}_\varepsilon(\omega')(x')\right| = \left|\tilde{f}_\varepsilon(\omega')(x) - \tilde{f}_\varepsilon(\omega)(x)\right| \le \eta\,.$$

Hence, it follows from (2) and the mean value theorem that

$$S_1 = |x - x'| \le \lambda^{-1}\left|\tilde{f}_\varepsilon(\omega')(x) - \tilde{f}_\varepsilon(\omega')(x')\right| \le \lambda^{-1}\eta\,.$$

On the other hand, by virtue of (10) and the mean value theorem together with the hypothesis of induction, we get

$$S_2 \le \lambda^{-1}\left|\left(a^n_{j_1}(\theta\omega) + \ell\right) - \left(a^n_{j_1}(\theta\omega') + \ell\right)\right| \le \lambda^{-1}\delta_0\,.$$

These estimates together with the condition (3) on η and δ_0 implies that $|a^{n+1}_j(\omega) - a^{n+1}_j(\omega')| \le \delta_0$, which completes the proof of the claim (17). □

REMARK 3. It is desirable to see if any topological condition on perturbations (in our case, e.g., the continuity of θ) is removable. This boils down to whether the random conjugacy $h : \Omega \to \mathrm{Homeo}(\mathbf{S}^1, \mathbf{S}^1)$ is measurable without the topological condition. However it is unclear to us whether requiring our topological setup is due to a substantial obstacle or is an artifact of our construction.

We need the following proposition for a random Markov partition.

DEFINITION 3. Let $f : \Omega \to \mathscr{C}^r(\mathbf{S}^1, \mathbf{S}^1)$ be a continuous mapping and denote $f(\omega)$ by f_ω. We say that a finite collection $\{\mathcal{I}^{(\cdot)}_j\}_{j\in\mathcal{A}}$ of continuous mappings defined on Ω with values in $\mathcal{F}(\mathbf{S}^1)$ is a *Markov partition* of f if it satisfies the following conditions:

- $\mathcal{I}^\omega_j$ is a nonempty left-closed and right-open interval for each $j \in \mathcal{A}$ and $\omega \in \Omega$,
- $\bigsqcup_{j\in\mathcal{A}} \mathcal{I}^\omega_j = \mathbf{S}^1$ for every $\omega \in \Omega$,
- $f_\omega(\mathcal{I}^\omega_j) = \mathbf{S}^1$ and $f_\omega : \mathcal{I}^\omega_j \to \mathbf{S}^1$ is a $\mathscr{C}^r$ diffeomorphism for each $j \in \mathcal{A}$ and $\omega \in \Omega$,
- $(f_\omega)^{-1}(\mathcal{I}^{\theta\omega}_i) \cap \mathcal{I}^\omega_j$ is a nonempty left-closed and right-open interval for each $i, j \in \mathcal{A}$ and $\omega \in \Omega$.

PROPOSITION 2. *Suppose that* $0 \le \varepsilon < \varepsilon_0$. *Then there is a Markov partition* $\{\mathcal{I}^{(\cdot)}_j\}^{k-1}_{j=0}$ *of* f_ε *such that for each* $0 \le j \le k-1$, *we can find a nonempty open interval* J' *such that* $\mathcal{I}^\omega_j$ *does not intersect* J' *for every* $\omega \in \Omega$.

PROOF. Let $I_j = \pi_{\mathbf{S}^1}([j/k, (j+1)/k))$ for $0 \le j \le k-1$. Then $\{I_j\}^{k-1}_{j=0}$ is a Markov partition of the k-folding map E_k. I.e.,

- I_j is a left-closed and right-open interval for each $j \in \mathcal{A}$,

- $\bigsqcup_{j\in\mathcal{A}} I_j = \mathbf{S}^1$,
- $E_k(I_j) = \mathbf{S}^1$ and $E_k : I_j \to \mathbf{S}^1$ is a $\mathscr{C}^r$ diffeomorphism,
- $(E_k)^{-1}(I_i) \cap I_j$ is a left-closed and right-open interval for each $i, j \in \mathcal{A}$.

We fix $0 \le \varepsilon < \varepsilon_0$, and let $h : \Omega \to \mathrm{Homeo}(\mathbf{S}^1, \mathbf{S}^1)$ be the continuous mapping satisfying (8) in Theorem 2. Set $\mathcal{I}_j^\omega = h(\omega)(I_j)$ for each $\omega \in \Omega$ and $0 \le j \le k-1$. Then due to Theorem 2, it is straightforward to see that $\{\mathcal{I}_j^{(\cdot)}\}_{j=0}^{k-1}$ is a Markov partition of f_ε such that

$$\sup_{(\omega,\omega')\in\Omega^2} d_H\big(\mathcal{I}_j^\omega, \mathcal{I}_j^{\omega'}\big) \le \frac{1}{k} + 2\delta_0 < 1$$

for each $0 \le j \le k-1$, where the last inequality follows from the condition of δ_0 given above (3). This immediately implies the conclusion. □

2.2. Coding of graphs. Throughout this subsection, we fix $0 \le \varepsilon < \varepsilon_0$, and let $\{\mathcal{I}_j^{(\cdot)}\}_{j=0}^{k-1}$ be the Markov partition of f_ε constructed in the proof of Proposition 2. For a word $s = (s_0, s_1, s_2, \dots s_{n-1}) \in \mathcal{A}^n$ and $\omega \in \Omega$, let $\mathcal{I}_s^\omega$ be a subset of $\mathbf{S}^1$ defined by

$$\mathcal{I}_s^\omega = \bigcap_{j=0}^{n-1} \left(f_\omega^{(j)}\right)^{-1}\left(\mathcal{I}_{s_j}^{\theta^j\omega}\right). \tag{18}$$

LEMMA 1. *For each $n \ge 1$ and $s \in \mathcal{A}^n$, $\mathcal{I}_s^\omega$ is a nonempty left-closed and right-open interval of $\mathbf{S}^1$ for every $\omega \in \Omega$. Moreover, the mapping $\omega \mapsto \mathcal{I}_s^\omega$ from Ω to $\mathcal{F}(\mathbf{S}^1)$ is continuous (in particular, measurable).*

PROOF. Let $n \ge 1$, $s \in \mathcal{A}^n$ and $\omega \in \Omega$ be given. Recall that $\mathcal{I}_{s_j}^{\theta^j\omega} = h(\theta^j\omega)(I_{s_j})$ for each $0 \le j \le n-1$ by construction. Then, it follows from Theorem 2 that

$$\left(f_\omega^{(j)}\right)^{-1}\left(\mathcal{I}_{s_j}^{\theta^j\omega}\right) = h(\omega) \circ \left(E_k^j\right)^{-1}\left(I_{s_j}\right).$$

Therefore, due to that $h(\omega)$ is a homeomorphism on $\mathbf{S}^1$, we have

$$\mathcal{I}_s^\omega = h(\omega)\left(I_s\right), \quad \text{where } I_s = \bigcap_{j=0}^{n-1}\left(E_k^j\right)^{-1}\left(I_{s_j}\right). \tag{19}$$

We next prove that I_s is a nonempty left-closed and right-open interval (and so is $\mathcal{I}_s^\omega$ for each $\omega \in \Omega$, due to (19)) by induction. It is obviously true for $n = 1$. Assume that the claim is true for a positive integer n, i.e., the set I_s is a nonempty left-closed and right-open interval for every $s \in \mathcal{A}^n$. Then, for every $s = (s_0, s_1, \dots, s_n) \in \mathcal{A}^{n+1}$, we have

$$I_s = I_{s_0} \cap \left((E_k)^{-1}\left(\bigcap_{j=1}^{n}\left(E_k^{j-1}\right)^{-1}\left(I_{s_j}\right)\right)\right) \tag{20}$$

$$= I_{s_0} \cap \left((E_k)^{-1} \left(I_{\sigma(s)} \right) \right) ,$$

where $\sigma : \mathcal{A}^{n+1} \to \mathcal{A}^n$ is the one-sided shift defined by $\sigma(s) = (s_1, s_2, \dots, s_n)$ for $s = (s_0, s_1, \dots, s_n) \in \mathcal{A}^{n+1}$. Note that $E_k : I_{s_0} \to \mathbf{S}^1$ is a diffeomorphism, and $I_{\sigma(s)}$ is a nonempty left-closed and right-open interval by the hypothesis of induction. Hence, I_s is also a nonempty left-closed and right-open interval, and we complete the proof of the claim.

We fix $n \geq 1$ and $s \in \mathcal{A}^n$. We finally prove the continuity of the mapping $\omega \mapsto h(\omega)(I_s)$ from Ω to $\mathcal{F}(\mathbf{S}^1)$, which implies the continuity of $\omega \mapsto \mathcal{I}_s^\omega$ by (19). Fix $\omega \in \Omega$, and let $\kappa = \kappa(\omega) > 0$ be a real number such that $\kappa < 1 - |h(\omega)(I_s)|$. By the continuity of $h : \Omega \to \mathrm{Homeo}(\mathbf{S}^1, \mathbf{S}^1)$ at ω, if ω' is sufficiently close to ω, then

$$d_{\mathscr{C}^0}\big(h(\omega), h(\omega')\big) < \min\left\{ \frac{\kappa}{2}, \frac{1 - \kappa - |h(\omega)(I_s)|}{2} \right\} . \tag{21}$$

For each ω' satisfying (21), it is easy to see that $|h(\omega')(I_s)| < |h(\omega)(I_s)| + \kappa$ and

$$d_{\mathscr{C}^0}\big(h(\omega), h(\omega')\big) < \min\left\{ \frac{1 - |h(\omega)(I_s)|}{2}, \frac{1 - |h(\omega')(I_s)|}{2} \right\} . \tag{22}$$

Let $\tilde{I}_s$ be an interval of $\mathbf{R}$ such that $\pi_{\mathbf{S}^1}(\tilde{I}_s) = I_s$. Let $\tilde{h}(\omega)$ and $\tilde{h}(\omega')$ be the lifts on $\mathbf{R}$ of homeomorphisms $h(\omega)$ and $h(\omega')$ on $\mathbf{S}^1$, respectively, such that $d_{\mathscr{C}^0}(\tilde{h}(\omega), \tilde{h}(\omega')) < 1$. Note that $h(\omega)$ and $h(\omega')$ are orientation-preserving by construction in the proof of Theorem 2, and thus $\tilde{h}(\omega)$ and $\tilde{h}(\omega')$ are monotonically increasing. We let ζ and ζ' denote the midpoints of $\tilde{h}(\omega)(\tilde{I}_s)$ and $\tilde{h}(\omega')(\tilde{I}_s)$, respectively. Then, it is straightforward to see that (22) implies $\tilde{h}(\omega')(\tilde{I}_s) \subset [\zeta - 1/2, \zeta + 1/2)$ and $\tilde{h}(\omega)(\tilde{I}_s) \subset [\zeta' - 1/2, \zeta' + 1/2)$. Therefore, we have

$$d_H\big(h(\omega)(I_s), h(\omega')(I_s)\big) = \max_{x \in \partial I_s} d_{\mathbf{S}^1}\big(h(\omega)(x), h(\omega')(x)\big),$$

which is bounded by $d_{\mathscr{C}^0}(h(\omega), h(\omega'))$, where ∂I_s is the boundary of I_s. By using the continuity of $h : \Omega \to \mathrm{Homeo}(\mathbf{S}^1, \mathbf{S}^1)$ at ω again, we obtain the continuity of $h(\cdot)(I_s)$ at ω. Since the choice of $\omega \in \Omega$ is arbitrary, $\omega \mapsto h(\omega)(I_s)$ is a continuous mapping from Ω to $\mathcal{F}(\mathbf{S}^1)$. □

Let $\mathbf{N}_0$ be the set of all nonnegative integers $\{0, 1, \dots\}$. For a sequence $s = (s_0, s_1, \dots, s_m, s_{m+1}, \dots, s_n, \dots) \in \mathcal{A}^{\mathbf{N}_0}$ with integers $n \geq m \geq 0$, we denote $(s_m, s_{m+1}, \dots, s_n)$ by $[s]_m^n$. We define $\mathcal{I}_s^\omega \in \mathcal{F}(\mathbf{S}^1)$ by $\mathcal{I}_s^\omega = \bigcap_{n \geq 0} \mathcal{I}_{[s]_0^n}^\omega$. Then, since the inverse branches of f_ω are contractions due to (2), it follows from Lemma 1 that $\mathcal{I}_s^\omega$ is a point set. We denote the point by $X_s(\omega)$. Then, by the continuity in Lemma 1, the mapping $\omega \mapsto \{X_s(\omega)\}$ from Ω to $\mathcal{F}(\mathbf{S}^1)$ is continuous, so is the mapping $X_s : \Omega \to \mathbf{S}^1$.

The following lemmas on the graphs of X_s's are simple but substantial in the proof of Theorem 1. Let $\sigma : \mathcal{A}^{\mathbf{N}_0} \to \mathcal{A}^{\mathbf{N}_0}$ be the one-sided shift given by $\sigma(s) = (s_1, s_2, \dots)$ for $s = (s_0, s_1, \dots) \in \mathcal{A}^{\mathbf{N}_0}$.

LEMMA 2. *For any $s \in \mathcal{A}^{\mathbf{N}_0}$, we have*

$$f_\omega\big(X_s(\omega)\big) = X_{\sigma(s)}(\theta\omega), \quad \omega \in \Omega\,,$$

and we can find a nonempty open interval J' such that $X_s(\omega)$ does not intersect J' for every $\omega \in \Omega$.

PROOF. From (20), it follows that

$$\begin{aligned} f_\omega\left(\mathcal{I}^{\omega}_{[s]_0^n}\right) &= f_\omega\left(\mathcal{I}^{\omega}_{s_0}\right) \cap \mathcal{I}^{\theta\omega}_{\sigma([s]_0^n)} \\ &= \mathbf{S}^1 \cap \mathcal{I}^{\theta\omega}_{\sigma([s]_0^n)} = \mathcal{I}^{\theta\omega}_{\sigma([s]_0^n)}\,. \end{aligned}$$

Thus, by the virtue of the continuity of $f_\omega : \mathcal{F}(\mathbf{S}^1) \to \mathcal{F}(\mathbf{S}^1)$, we get $f_\omega(\mathcal{I}^\omega_s) = \mathcal{I}^{\theta\omega}_{\sigma(s)}$ for every $\omega \in \Omega$. Together with the construction of X_s and Proposition 2, this implies the conclusion. □

LEMMA 3. *Let x be a point in $\mathbf{S}^1$ and $s = s(x) = (s_0, s_1, \ldots) \in \mathcal{A}^{\mathbf{N}_0}$ the coding of x by E_k, i.e., $E_k^j(x) \in I_{s_j}$ for each $j \geq 0$. Then, $h(\omega)(x) = X_s(\omega)$ for every $\omega \in \Omega$ and $x \in \mathbf{S}^1$, where h is given in Theorem 2.*

PROOF. It follows from (8) and (18) that for each $n \geq 1$, we have

$$\begin{aligned} \mathcal{I}^{\omega}_{[s]_0^n} &= \bigcap_{j=0}^{n-1} \left(f_\omega^{(j)}\right)^{-1} \circ h(\theta^j\omega)\left(I_{s_j}\right) \\ &= \bigcap_{j=0}^{n-1} \left(h(\omega) \circ E_k^{-j}(I_{s_j})\right)\,. \end{aligned}$$

Since $h(\omega) : \mathbf{S}^1 \to \mathbf{S}^1$ is a homeomorphism, this implies

$$\mathcal{I}^\omega_s = h(\omega)\left(\bigcap_{j=0}^{\infty} E_k^{-j}(I_{s_j})\right)\,.$$

Recalling $\mathcal{I}^\omega_s = \{X_s(\omega)\}$, we immediately get the conclusion. □

2.3. Ergodicity and SRB property. As a final preparation before turning to the proof of Theorem 1, we may need to find a sequence s'' such that time averages along the future random orbit of $X_{s''}$ **P**-almost surely coincide with their corresponding integrals with respect to a probability measure that is equivalent to the normalised Lebesgue measure, and that the past random orbit of $X_{s''}$ is **P**-almost surely dense. We prepare language for this purpose.

Fix $0 \leq \varepsilon < \varepsilon_0$ and let $h = h_\varepsilon$ be the mapping given in Theorem 2. Then, $(\omega, x) \mapsto h(\omega)(x)$ is a continuous mapping from $\Omega \times \mathbf{S}^1$ to $\mathbf{S}^1$ due to the continuity of $h : \Omega \to \mathrm{Homeo}(\mathbf{S}^1, \mathbf{S}^1)$. Therefore for each continuous function $\varphi : \mathbf{S}^1 \to \mathbf{R}$, the function

$\Phi_h : \Omega \times \mathbf{S}^1 \mapsto \mathbf{R}$ given by $\Phi_h(\omega, x) = \varphi(h(\omega)(x))$ for $(\omega, x) \in \Omega \times \mathbf{S}^1$ is continuous, in particular measurable. Moreover, it is straightforward to see that $\|\Phi_h\|_{L^1_{m\times\mathbf{P}}} \leq \|\varphi\|_{\mathscr{C}^0}$, i.e., Φ_h is an integrable function with respect to the product measure $m \times \mathbf{P}$, where m is the normalised Lebesgue measure on $\mathbf{S}^1$. Thus it follows from Fubini's theorem that $\omega \mapsto \int \Phi_h(\omega, \cdot)dm$ is measurable. Furthermore, since φ is an arbitrary continuous function, it follows from Riesz representation theorem that we can find a probability measure $(h(\omega))_* m$ on $\mathbf{S}^1$ such that $\int \Phi_h(\omega, \cdot)dm = \int \varphi d[(h(\omega))_* m]$ for each $\omega \in \Omega$. (This probability measure is called the pushforward of m by $h(\omega)$.)

THEOREM 3. *For any continuous function $\varphi : \mathbf{S}^1 \to \mathbf{R}$, there exist a full measure set $A = A(\varphi)$ of $(\mathbf{S}^1, m)$ and a family of full measure sets $\{\Gamma_x\}_{x\in A} = \{\Gamma_x(\varphi)\}_{x\in A}$ of $(\Omega, \mathbf{P})$ such that for each $x \in A$ and $\omega \in \Gamma_x$, if we set $s'' = s''(x) \in \mathcal{A}^{\mathbf{N}_0}$ as the coding of x by E_k given in Lemma* 3, *then we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f_\omega^{(j)}\big(X_{s''}(\omega)\big) = \int \left(\int \varphi d\left[(h(\cdot))_* m\right] \right) d\mathbf{P}\,.$$

PROOF. For given $x \in \mathbf{S}^1$, let $s'' \in \mathcal{A}^{\mathbf{N}_0}$ be the coding of x by E_k. By Theorem 2 and Lemma 3, we have

$$\varphi \circ f_\omega^{(j)}\big(X_{s''}(\omega)\big) = \varphi\big(f_\omega^{(j)} \circ h(\omega)(x)\big) = \varphi\big(h(\theta^j\omega) \circ E_k^j(x)\big)$$

for all $j \geq 0$ and $\omega \in \Omega$. Thus, if we define a measurable mapping $\Theta_{E_k} : \Omega \times \mathbf{S}^1 \to \Omega \times \mathbf{S}^1$ of the direct-product form

$$\Theta_{E_k}(\omega, x) = \big(\theta\omega, E_k(x)\big), \quad (\omega, x) \in \Omega \times \mathbf{S}^1\,,$$

then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f_\omega^{(j)}\big(X_{s''}(\omega)\big) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \Phi_h \circ \Theta_{E_k}^j(\omega, x)\,, \tag{23}$$

where $\Phi_h : \Omega \times \mathbf{S}^1 \to \mathbf{R}$ is the integral function given above Theorem 3. Recall that $(\theta, \mathbf{P})$ is assumed to be ergodic, and that (E_k, m) is known to be weak mixing (see, e.g., Proposition 4.2.11 in [7] together with Remark (1) below Definition 1.5 in [14]). Then, it follows from Corollary 1 in §1, Chapter 10 of [5] that the direct product $(\Theta_{E_k}, \mathbf{P} \times m)$ of $(\theta, \mathbf{P})$ and (E_k, m) is an ergodic transformation. Therefore, applying Birkhoff's ergodic theorem to $(\Theta_{E_k}, \mathbf{P} \times m)$ with the integral function Φ_h, we can find a $(\mathbf{P} \times m)$-full measure set G such that the time average in (23) coincides with $\int \Phi_h dm d\mathbf{P}$ for all $(\omega, x) \in G$.

Let $1_G : \Omega \times \mathbf{S}^1 \to \mathbf{R}$ be the indicator function of G. Then, it follows from Fubini's theorem that there is a full measure set A_1 of $(\mathbf{S}^1, m)$ such that $1_G(x, \cdot) : \Omega \to \mathbf{R}$ is integrable for every $x \in A_1$. Let $\Gamma_x = \mathrm{supp}(1_G(x, \cdot))$ and $A = \{x \in A_1 \mid \mathbf{P}(\Gamma_x) = \int 1_G(x, \cdot)d\mathbf{P} =$

$1\}$. Then $m(A) = 1$ (otherwise, it contracts to that $\mathbf{P} \times m(G) = 1$), and the conclusion immediately follows from (23) and the observation above Theorem 3. $\square$

REMARK 4. Theorem 3 is exactly the reason why Theorem 1 is only stated for $\mathbf{P}$-almost every $\omega \in \Omega$ (not for all $\omega \in \Omega$). See that the statements of Proposition 1, Proposition 2, Lemma 2 and Theorem 4 are given for all $\omega \in \Omega$, while Theorem 3 is only stated for $\mathbf{P}$-almost every ω. This restriction in Theorem 3 comes from that the convergence of time averages along the future random orbit of $X_{s''}$ is ensured by the ergodicity of $(\Theta_{E_k}, \mathbf{P} \times m)$, ultimately, by the ergodicity of $(\theta, \mathbf{P})$.

Theorem 3 tells that time averages along the future random orbit of $X_{s''}$ (with some $s'' \in \mathcal{A}^{\mathbf{N}_0}$) $\mathbf{P}$-almost surely coincide with their corresponding integrals with respect to a probability measure that is equivalent to $\mathbf{P} \times m$. However, this may not imply in general that the *past* random orbit of $X_{s''}$ is $\mathbf{P}$-almost surely dense in $\mathbf{S}^1$, and we need the following theorem to apply Proposition 1.

THEOREM 4. *There exists a full measure set A of $(\mathbf{S}^1, m)$ such that for each $x \in A$, if we set $s'' = s''(x) \in \mathcal{A}^{\mathbf{N}_0}$ as the coding of x by E_k given in Lemma 3, then we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^{(j)}_{\theta^{-j}\omega}\big(X_{s''}(\theta^{-j}\omega)\big) = \int \varphi d\left[(h(\omega))_* m\right], \tag{24}$$

for any continuous function $\varphi : \mathbf{S}^1 \to \mathbf{R}$ and $\omega \in \Omega$.

Furthermore, for any $s'' = s''(x)$ with $x \in A$, the past random orbit of $X_{s''}$ is dense at every $\omega \in \Omega$.

PROOF. We first recall that the normalised Lebesgue measure m is the unique SRB probability measure of E_k. I.e., there exists a full measure set $A \subset \mathbf{S}^1$ such that for every $x \in A$ and continuous function $\psi : \mathbf{S}^1 \to \mathbf{R}$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi\big(E_k^j(x)\big) = \int \psi dm. \tag{25}$$

A standard reference for this property is [13].

Let $\varphi : \mathbf{S}^1 \to \mathbf{R}$ be a continuous function and $\omega \in \Omega$. In a similar manner to the proof of Theorem 3, we have

$$\varphi \circ f^{(j)}_{\theta^{-j}\omega}\big(X_{s''}(\theta^{-j}\omega)\big) = \varphi\big(h(\omega) \circ E_k^j(x)\big)$$

for all $j \geq 0$. Thus,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^{(j)}_{\theta^{-j}\omega}\big(X_{s''}(\theta^{-j}\omega)\big) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ h(\omega)\big(E_k^j(x)\big).$$

Therefore, applying (25) with $\psi = \varphi \circ h(\omega)$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^{(j)}_{\theta^{-j}\omega}\big(X_{s''}(\theta^{-j}\omega)\big) = \int \varphi \circ h(\omega) dm = \int \varphi d\left[(h(\omega))_* m\right] .$$

The later assertion is shown by contradiction. Fix $s'' = s''(x)$ with some $x \in A$ and assume that the past random orbit of $X_{s''}$ is not dense at some $\omega \in \Omega$. Then, there is a nonempty open interval $J \subset \mathbf{S}^1$ with which $\{f^{(j)}_{\theta^{-j}\omega}(X_{s''}(\theta^{-j}\omega))\}_{j\geq 0}$ does not intersect. Thus if we let φ be a continuous nonnegative function whose support is a nonempty set included in J, then the left-hand side of (24) is equal to zero, while the right-hand side is nonzero since $h(\omega) : \mathbf{S}^1 \to \mathbf{S}^1$ is a homeomorphism so that the support of $(h(\omega))_* m$ coincides with $\mathbf{S}^1$. This is a contradiction, and we complete the proof. □

Since the intersection of two full measure sets is also a full measure set, we immediately get the following theorem by Theorems 3 and 4.

THEOREM 5. *For any continuous function* $\varphi : \mathbf{S}^1 \to \mathbf{R}$, *there exist a full measure set* $A = A(\varphi)$ *of* $(\mathbf{S}^1, m)$ *and a family of full measure sets* $\{\Gamma_x\}_{x\in A} = \{\Gamma_x(\varphi)\}_{x\in A}$ *of* $(\Omega, \mathbf{P})$ *such that for each* $x \in A$ *and* $\omega \in \Gamma_x$, *if we set* $s'' = s''(x) \in \mathcal{A}^{\mathbf{N}_0}$ *as the coding of* x *by* E_k *given in Lemma* 3, *then we have that*

(1) $\lim_{n\to\infty} B_n(\varphi; \omega, X_{s''}(\omega)) = \int \left(\int \varphi d\left[(h(\cdot))_* m\right]\right) d\mathbf{P}$;

(2) *the past random orbit of* $X_{s''}$ *at* ω *is dense in* $\mathbf{S}^1$.

2.4. The end of the proof. We will construct a sequence $\bar{s} \in \mathcal{A}^{\mathbf{N}_0}$ such that $X_{\bar{s}}$ $\mathbf{P}$-almost surely satisfies the hypotheses in Proposition 1. As in the treatment undertaken by Takens [12], this sequence will be an appropriate combination of a periodic sequence s' and a sequence s'' generating a probability measure that is equivalent to the normalised Lebesgue measure.

Fix $0 \leq \varepsilon < \varepsilon_0$ and let $h = h_\varepsilon$ be the mapping given in Theorem 2. Let $s' \in \mathcal{A}^{\mathbf{N}_0}$ be a periodic sequence. For simplicity, we set $s' = (00\ldots)$. By virtue of Proposition 2 and Lemma 2, there exist nonempty open intervals $J \subset J' \subset \mathbf{S}^1$ both of which do not intersect $\mathcal{I}^\omega_0$ (in particular, $X_{s'}(\omega) = X_{(00\ldots)}(\omega)$) for every ω, where the inclusion $J \subset J'$ is strict. Let $\varphi_0 : \mathbf{S}^1 \to \mathbf{R}$ be a $\mathscr{C}^1$ function such that

- the support of φ_0 is included in J',
- $\varphi_0(x) = 1$ for all $x \in J$,
- $0 \leq \varphi_0(x) \leq 1$ for all $x \in \mathbf{S}^1$.

Then, for all $n \geq 1$ and $\omega \in \Omega$, we have

$$B_n\big(\varphi_0; \omega, X_{s'}(\omega)\big) = 0 . \tag{26}$$

Moreover, it follows from Theorem 5 that we can find a sequence $s'' \in \mathcal{A}^{\mathbf{N}_0}$ and a full measure set $\tilde{\Gamma} = \tilde{\Gamma}_{s''}(\varphi_0)$ of $(\Omega, \mathbf{P})$ such that $B_n(\varphi_0; \omega, X_{s''}(\omega))$ converges to $\int(\int \varphi_0 d[(h(\cdot))_* m])d\mathbf{P}$ as n goes to infinity for every $\omega \in \tilde{\Gamma}$. Let $\Gamma = \cap_{n\geq 0}\theta^{-n}(\tilde{\Gamma})$, then it is straightforward to see that $\theta(\Gamma) \subset \Gamma \subset \tilde{\Gamma}$ and $\mathbf{P}(\Gamma) = 1$.[1] For convenience, we define a nonnegative number $\rho_n(\omega)$ with $n \geq 1$ and $\omega \in \Omega$ by

$$\rho_n(\omega) = \left| B_n\big(\varphi_0; \omega, X_{s''}(\omega)\big) - \int \left(\int \varphi_0 d[(h(\cdot))_* m] \right) d\mathbf{P} \right| . \tag{27}$$

Then $\rho_n(\theta^N \omega)$ converges to zero as n goes to infinity for every $\omega \in \Gamma$ and $N \geq 0$.

Recall that for a sequence $s = (s_0, s_1, \dots, s_m, s_{m+1}, \dots, s_n, \dots) \in \mathcal{A}^{\mathbf{N}_0}$ with some integers $n \geq m \geq 0$, $(s_m, s_{m+1}, \dots, s_n)$ is denoted by $[s]_m^n$. For a positive number a, we write $[a]$ for the integer part of a.

LEMMA 4. *For any integer $m \geq 0$ and real number $\rho > 0$, we can find an integer $n \geq 2m+2$ such that, for all $\omega \in \Omega$ and $s, t \in \mathcal{A}^{\mathbf{N}_0}$ satisfying that $[s]_m^{n-1} = [t]_m^{n-1}$, we have*

$$\left| B_{[\frac{n}{2}]}\big(\varphi_0; \omega, X_s(\omega)\big) - B_{[\frac{n}{2}]}\big(\varphi_0; \omega, X_t(\omega)\big) \right| \leq \rho .$$

PROOF. We fix $m \geq 0$ and $\rho > 0$. Note that for any $n' \geq m+1$, $\omega \in \Omega$ and $x \in \mathbf{S}^1$, we have

$$B_{n'}(\varphi_0; \omega, x) = \frac{m}{n'} B_m(\varphi_0; \omega, x) + \frac{n'-m}{n'} B_{n'-m}\big(\varphi_0; \theta^m \omega, f_\omega^{(m)}(x)\big) \tag{28}$$

and

$$\left| \frac{m}{n'} B_m(\varphi_0; \omega, x) \right| \leq \frac{m}{n'} \|\varphi_0\|_{\mathscr{C}^0} \leq \frac{m}{n'} . \tag{29}$$

Hence, if we take $n \geq 2m+2$ sufficiently large, then $\left[\frac{n}{2}\right] \geq m+1$ and we get

$$\begin{aligned} &\left| B_{[\frac{n}{2}]}\big(\varphi_0; \omega, X_s(\omega)\big) - B_{[\frac{n}{2}]}\big(\varphi_0; \omega, X_t(\omega)\big) \right| \\ &\quad \leq \frac{\rho}{2} + \left| B_{[\frac{n}{2}]-m}\Big(\varphi_0; \theta^m \omega, f_\omega^{(m)}\big(X_s(\omega)\big)\Big) - B_{[\frac{n}{2}]-m}\Big(\varphi_0; \theta^m \omega, f_\omega^{(m)}\big(X_t(\omega)\big)\Big) \right| . \end{aligned} \tag{30}$$

Let $s, t \in \mathcal{A}^{\mathbf{N}_0}$ satisfying that $[s]_m^{n-1} = [t]_m^{n-1}$ with some $n \geq 2m+2$. Then by the argument in the proof of Lemma 2, $f_\omega^{(m+\ell)}(X_s(\omega)) = f_{\theta^m\omega}^{(\ell)}(X_{\sigma^m s}(\theta^m \omega))$ and $f_\omega^{(m+\ell)}(X_t(\omega)) = f_{\theta^m\omega}^{(\ell)}(X_{\sigma^m t}(\theta^m \omega))$, and both points are in the interval $f_{\theta^m\omega}^{(\ell)}\Big(\mathcal{I}_{[s]_m^{n-1}}^{\theta^m\omega}\Big) = \mathcal{I}_{\sigma^\ell [s]_m^{n-1}}^{\theta^{m+\ell}\omega}$ for each $0 \leq \ell \leq \left[\frac{n}{2}\right] - m - 1$. Moreover, when $0 \leq \ell \leq \left[\frac{n}{2}\right] - m - 1$, due to (2) and

[1] Assume that $\mathbf{P}(\theta^{-1}(\tilde{\Gamma}) \cap \tilde{\Gamma}) < 1$. Then $\mathbf{P}(\theta^{-1}\tilde{\Gamma} \cup \tilde{\Gamma}) = \mathbf{P}(\theta^{-1}(\tilde{\Gamma})) + \mathbf{P}(\tilde{\Gamma}) - \mathbf{P}(\theta^{-1}(\tilde{\Gamma}) \cap \tilde{\Gamma}) > 1$ since θ is measure-preserving and $\mathbf{P}(\tilde{\Gamma}) = 1$. This contradicts to that $\mathbf{P}$ is a probability measure, so that $\mathbf{P}(\theta^{-1}(\tilde{\Gamma}) \cap \tilde{\Gamma}) = 1$. Reiterating the argument, we get that $\cap_{0\leq n\leq N}\theta^{-n}(\tilde{\Gamma})$ is a full measure set for each $N \geq 0$, which immediately implies $\mathbf{P}(\Gamma) = 1$.

(18), the length of the interval $\mathcal{I}^{\theta^{m+\ell}\omega}_{\sigma^\ell[s]_m^{n-1}}$ is less than $\lambda^{-(n-m-\ell)} \leq \lambda^{-\frac{n}{2}}$. (Notice that the length of the word $\sigma^\ell[s]_m^{n-1}$ is $n-m-\ell$.) Thus, it follows from the mean value theorem that $\left|\varphi_0\left(f_\omega^{(m+\ell)}(X_s(\omega))\right) - \varphi_0\left(f_\omega^{(m+\ell)}(X_t(\omega))\right)\right|$ is bounded by $\lambda^{-\frac{n}{2}}\|\varphi_0\|_{\mathscr{C}^1}$ for every $0 \leq \ell \leq \left[\frac{n}{2}\right] - m - 1$, and we have

$$\left|B_{[\frac{n}{2}]-m}\left(\varphi_0; \theta^m\omega, f_\omega^{(m)}\big(X_s(\omega)\big)\right) - B_{[\frac{n}{2}]-m}\left(\varphi_0; \theta^m\omega, f_\omega^{(m)}\big(X_t(\omega)\big)\right)\right| \\ \leq \left(\left[\frac{n}{2}\right] - m\right)\lambda^{-\frac{n}{2}}\|\varphi_0\|_{\mathscr{C}^1}.$$

This should be less than $\frac{\rho}{2}$ by taking n sufficiently large, and the conclusion immediately follows from (30). □

Now we inductively set an increasing sequence $\{N_j\}_{j\geq 0}$ of nonnegative integers. Let $\{\tilde\rho_j\}_{j\geq 1}$ be a sequence of real numbers in $(0, 1]$ such that $\tilde\rho_j$ converges to zero as j goes to infinity. Let $N_0 = 0$. For a given integer N_{j-1} with odd integer $j \geq 1$, we take an integer $N_j \geq 6N_{j-1}/\tilde\rho_j + 2$ such that if $s \in \mathcal{A}^{\mathbf{N}_0}$ satisfies that $[s]_{N_{j-1}}^{N_j-1} = [s']_0^{N_j-N_{j-1}-1}$, then

$$\left|B_{[N_j/2]}\big(\varphi_0; \omega, X_s(\omega)\big) - B_{[N_j/2]-N_{j-1}}\big(\varphi_0; \theta^{N_{j-1}}\omega, X_{s'}(\theta^{N_{j-1}}\omega)\big)\right| \leq \tilde\rho_j, \tag{31}$$

for all $\omega \in \Omega$. (We can indeed find such N_j: For each $\omega \in \Omega$, $\tilde s \in \mathcal{A}^{N_{j-1}}$ and $s \in \mathcal{A}^{\mathbf{N}_0}$ satisfying that $[s]_{N_{j-1}}^{N_j-1} = [\tilde s s']_{N_{j-1}}^{N_j-1} (= [s']_0^{N_j-N_{j-1}-1})$, where $\tilde s s' = (\tilde s_0 \tilde s_1 \ldots \tilde s_{N_{j-1}-1} s'_0 s'_1 \ldots)$ is the concatenation of $\tilde s = (\tilde s_0 \tilde s_1 \ldots \tilde s_{N_{j-1}-1})$ and s', we have

$$\left|B_{[N_j/2]}\big(\varphi_0; \omega, X_s(\omega)\big) - B_{[N_j/2]}\big(\varphi_0; \omega, X_{\tilde s s'}(\omega)\big)\right| \leq \tilde\rho_j/3,$$

by applying Lemma 4 with $\rho = \tilde\rho_j/3$, $m = N_{j-1}$, $n = N_j$ and $t = \tilde s s'$, and with the notation $\xi_j = 1 - N_{j-1}/[N_j/2]$, we also have

$$\left|B_{[N_j/2]}\big(\varphi_0; \omega, X_{\tilde s s'}(\omega)\big) - \xi_j \cdot B_{[N_j/2]-N_{j-1}}\big(\varphi_0; \theta^{N_{j-1}}\omega, X_{s'}(\theta^{N_{j-1}}\omega)\big)\right| \leq \tilde\rho_j/3$$

with $|1 - \xi_j| \leq \tilde\rho_j/3$, by applying (28) and (29) with $n' = [N_j/2]$, $m = N_{j-1}$ and $x = X_{\tilde s s'}(\omega)$, together with Lemma 2 and the fact that $N_{j-1}/[N_j/2] \leq \tilde\rho_j/3$.) On the other hand, for a given integer N_{j-1} with even integer $j \geq 1$, we take an integer $N_j \geq 6N_{j-1}/\tilde\rho_j + 2$ such that if $s \in \mathcal{A}^{\mathbf{N}_0}$ satisfies that $[s]_{N_{j-1}}^{N_j-1} = [s'']_0^{N_j-N_{j-1}-1}$, then

$$\left|B_{[N_j/2]}\big(\varphi_0; \omega, X_s(\omega)\big) - B_{[N_j/2]-N_{j-1}}\big(\varphi_0; \theta^{N_{j-1}}\omega, X_{s''}(\theta^{N_{j-1}}\omega)\big)\right| \leq \tilde\rho_j, \tag{32}$$

for all $\omega \in \Omega$. Finally, we set a sequence $\bar{s} \in \mathcal{A}^{\mathbf{N}_0}$ by

$$[\bar{s}]_{N_{j-1}}^{N_j-1} = \begin{cases} [s']_0^{N_j-N_{j-1}-1} & (j : \text{odd}), \\ [s'']_0^{N_j-N_{j-1}-1} & (j : \text{even}). \end{cases}$$

It follows from (26) and (31) that for any $\omega \in \Omega$ and odd integer $j \geq 1$, we have

$$\left| B_{[N_j/2]}(\varphi_0; \omega, X_{\bar{s}}(\omega)) - 0 \right| \leq \tilde{\rho}_j .$$

Hence for any $\omega \in \Omega$,

$$\lim_{\ell \to \infty} B_{[N_{2\ell+1}/2]}\big(\varphi_0; \omega, X_{\bar{s}}(\omega)\big) = 0 .$$

Furthermore, it follows from (27) and (32) that, for any $\omega \in \Omega$ and even integer $j \geq 1$,

$$\left| B_{[N_j/2]}\big(\varphi_0; \omega, X_{\bar{s}}(\omega)\big) - \int \left(\int \varphi_0 d[(h(\cdot))_* m] \right) d\mathbf{P} \right| \leq \tilde{\rho}_j + \rho_{[N_j/2]-N_{j-1}}\big(\theta^{N_{j-1}}\omega\big) .$$

Hence, due to that $[N_j/2] - N_{j-1} \geq N_{j-1}$ and $\theta(\Gamma) \subset \Gamma$, for any $\omega \in \Gamma$ we have

$$\lim_{\ell \to \infty} B_{[N_{2\ell}/2]}\big(\varphi_0; \omega, X_{\bar{s}}(\omega)\big) = \int \left(\int \varphi_0 d[(h(\cdot))_* m] \right) d\mathbf{P} > 0 .$$

The last inequality holds because the support of $h(\omega)_* m$ coincides with $\mathbf{S}^1$ for each ω. Consequently, $X_{\bar{s}}$ has historic behaviour for $\mathbf{P}$-almost every $\omega \in \Omega$.

At the end, we show that the past random orbit of $X_{\bar{s}}$ is $\mathbf{P}$-almost surely dense. Let j be an even integer and fix $\omega \in \Gamma$. We denote $\sigma^{N_{j-1}}\bar{s}$ by $\bar{s}^j$. Then, for each $0 \leq \ell \leq N_{j-1}$,

$$f^{(N_{j-1}+\ell)}_{\theta^{-(N_{j-1}+\ell)}\omega}\Big(X_{\bar{s}}\big(\theta^{-(N_{j-1}+\ell)}\omega\big)\Big) = f^{(\ell)}_{\theta^{-\ell}\omega}\Big(X_{\bar{s}^j}\big(\theta^{-\ell}\omega\big)\Big)$$

by Lemma 2, and $\big[\bar{s}^j\big]_\ell^{N_j-N_{j-1}-1} = \big[s''\big]_\ell^{N_j-N_{j-1}-1}$. Thus, by using Lemma 2 again, one can see that both $f^{(\ell)}_{\theta^{-\ell}\omega}(X_{\bar{s}^j}(\theta^{-\ell}\omega))$ and $f^{(\ell)}_{\theta^{-\ell}\omega}(X_{s''}(\theta^{-\ell}\omega))$ are in $\mathcal{I}_t^\omega$ with a word $t = \big[s''\big]_\ell^{N_j-N_{j-1}-1}$ of length $N_j - N_{j-1} - \ell$. Therefore, noting that $N_j - N_{j-1} - \ell \geq N_j$ for each $0 \leq \ell \leq N_{j-1}$, together with (2) and (18), we get

$$\left| f^{(\ell)}_{\theta^{-\ell}\omega}\big(X_{\bar{s}^j}(\theta^{-\ell}\omega)\big) - f^{(\ell)}_{\theta^{-\ell}\omega}\big(X_{s''}(\theta^{-\ell}\omega)\big) \right| \leq C_\omega \lambda^{-(N_j-N_{j-1}-\ell)} \leq C_\omega \lambda^{-N_{j-1}} ,$$

where $C_\omega > 0$ is the maximum of $|\mathcal{I}_j^\omega|$ over $0 \leq j \leq k-1$. Since the past random orbit of $X_{s''}$ is $\mathbf{P}$-almost surely dense on $\mathbf{S}^1$ by Theorem 5, we immediately conclude that the past random orbit of $X_{\bar{s}}$ is $\mathbf{P}$-almost surely dense on $\mathbf{S}^1$. This completes the proof of Theorem 1 by Proposition 1.

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