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The Doeblin-Lenstra Conjecture for a Complex Continued Fraction Algorithm

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Abstract. We prove the analogue of the Doeblin-Lenstra conjecture for a certain complex continued fraction algorithm introduced by Julius Hurwitz [5] in 1895 and further investigated by Shigeru Tanaka [12] in 1985 using ergodic theoretical means.

1. Introduction and Statement of the Main Results

In 1888, Adolf Hurwitz [4] investigated the continued fraction expansion of complex numbers with partial quotients defined by the nearest Gaussian integer. Building on this important work of his famous younger brother, Julius Hurwitz wrote a doctoral thesis [5] on this topic (published as [6]). For the interesting story behind this thesis and the late mathematical career of Julius we refer to Oswald & Steuding [8]. Where Adolf Hurwitz chose the set of Gaussian integers $\mathbf{Z}[i]$ for the partial quotients Julius took the ideal $(1+i)\mathbf{Z}[i]$. The same complex continued fraction expansion was studied by Shigeru Tanaka [12] by ergodic means in slightly different (modern) notation (and without knowing of Julius Hurwitz' contribution). In this note we shall continue their line of investigation and consider the approximation coefficients; in particular, we prove the complex analogue of the Doeblin–Lenstra conjecture on the existence of a limiting distribution function. Moreover, we discuss the restriction of Julius Hurwitz' resp. Tanaka's complex continued fraction expansion to the real case which is not unrelated to previous work by Fritz Schweiger [10, 11] on real continued fractions with even partial quotients.

According to Tanaka we define a transformation T by $T(0) = 0$ and

$$T(z) := \frac{1}{z} - \left[\frac{1}{z}\right]_T$$

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otherwise, where the bracket $[z]_T$ is given by

$$[z]_T := \left\lfloor \lambda + \frac{1}{2} \right\rfloor \alpha + \left\lfloor \mu + \frac{1}{2} \right\rfloor \overline{\alpha} \quad \text{for } z = \lambda\alpha + \mu\overline{\alpha}$$

with $\alpha := 1 + i$ (and $\overline{\alpha} = 1 - i$), and $\lfloor x \rfloor$ with a real number x denotes the largest rational integer less than or equal to x. It is not difficult to see that T maps the square $X := \{z = \lambda\alpha + \mu\overline{\alpha} \ : \ -\frac{1}{2} \leq \lambda, \mu < \frac{1}{2}\}$ onto X and each $z \in X$ has a unique complex continued fraction expansion

$$z = a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3 + T^3 z} = [a_0, a_1, a_2, a_3 + T^3 z] \quad \text{with } a_n := \left[\frac{1}{T^n z} \right]_T$$

as long as $T^n z \neq 0$. Here we write Tz in place of $T(z)$ and the nth transformation T^n is defined recursively by $T^0 = \mathrm{id}$, $T^1 = T$, and $T^n = T^{n-1} \circ T$. Already Julius Hurwitz proved that *this continued fraction expansion for $z \in X$ terminates with a partial quotient divisible by $1 + i$ if, and only if, either the numerator or the denominator is divisible by $1 + i$*, i.e. if $z = \frac{u}{v}$ in lowest terms, then either $u \equiv 0$ and $v \equiv 1 \bmod \alpha$ or $u \equiv 1$ and $v \equiv 0 \bmod \alpha$ ([5] , pp. 27; [6], p.246). Notice that $\mathbf{Z}[i]/(\alpha) \simeq \{0, 1\}$ where (α) denotes the ideal generated by α (that is $(\alpha) = (1 + i)\mathbf{Z}[i] = \{m + ni \ : \ m \equiv n \bmod 2\}$). For all other complex z the expansion is infinite and unique. The case of finite expansions is not our concern, since we are mainly interested in diophantine and ergodic properties.

Given $z = [a_0, a_1, \ldots] \in X$, we define sequences of Gaussian integers p_n and q_n by

$$\begin{aligned} p_{-1} = \alpha\,, \quad p_0 = 0\,, \quad &\text{and} \quad p_{n+1} = a_{n+1} p_n + p_{n-1} \quad \text{for } n \geq 1\,, \\ q_{-1} = 0, \ q_0 = \alpha\,, \quad &\text{and} \quad q_{n+1} = a_{n+1} q_n + q_{n-1} \quad \text{for } n \geq 1\,. \end{aligned}$$

It follows from the representation

$$z = \frac{p_n + p_{n-1} T^n z}{q_n + q_{n-1} T^n z}$$

that the so-called convergents $\frac{p_n}{q_n}$ converge to z as $n \to \infty$. The above result of Julius Hurwitz about terminating complex continued fractions follows easily by induction from the equivalences

$$\frac{p_n}{\alpha} \in (\alpha) \quad \Leftrightarrow \quad 2|n \quad \Leftrightarrow \quad \frac{q_n}{\alpha} \notin (\alpha)\,.$$

One can show that the absolute values of the denominators q_n increase strictly with n. Here we want to examine the approximation coefficients

$$\theta_n := \theta_n(z) := |q_n|^2 \left| z - \frac{p_n}{q_n} \right| .$$

Our main result proves the almost sure existence of a limiting distribution function for this quantity:

THEOREM 1. *For $x > 0$ define*

$$\ell(x) := \lim_{N\to\infty} \frac{1}{N} \sharp\{n \le N \,:\, \theta_n(z) \le x\} \,.$$

Then, for almost all z, the distribution function $\ell(x)$ exists and is given by

$$\ell(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{n\le N} \chi_{A_x}(\mathbf{T}^n(z,0)) = \mu(A_x) = \frac{1}{G} \iint_{A_x} \frac{d\lambda(u,v)}{|1+uv|^4} \,, \tag{1}$$

where $\mathbf{T}$ is defined by (3) below, λ is the Lebesgue measure, χ_{A_x} is the characteristic function of the set

$$A_x := \left\{ (u,v) \in X \times Y \,:\, \left| \frac{1}{u} + v \right| \ge \frac{1}{x} \right\} ,$$

and the normalizing constant

$$G := \iint_{X\times Y} \frac{d\lambda(u,v)}{|1+uv|^4} = 4\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \,. \tag{2}$$

equals 4π times the Catalan constant.

Interestingly, Catalan's constant appears in various results in the ergodic theory of continued fractions, e.g., Nakada [7]. Our main theorem is a complex analogue of the Doeblin-Lenstra conjecture for the ordinary regular continued fraction expansion of real numbers, conjectured by Hendrik W. Lenstra jr. and, implicitly, by Wolfgang Doeblin [3], and first proved by Wieb Bosma, Hendrik Jager & Freek Wiedijk [1] in 1983. However, the following statement shows a significant difference from the real case:

THEOREM 2. *For almost all z, the limiting function $x \mapsto \ell(x)$ is strictly increasing for $x > 2$.*

Already in Julius Hurwitz' thesis [5] one can find the statement that the sequence of $\theta_n(z)$ has always a finite limit point (p. 34), in general, however, the sequence of the $\theta_n(z)$ appears to have divergent subsequences. This is rather different from the real case where $\ell(x)$ is constant for $x \ge 1$ as follows from the classical estimate $|z - \frac{p_n}{q_n}| \le \frac{1}{q_n^2}$ giving $\theta_n(z) \le 1$. This is in perfect agreement with studies of Schweiger [10, 11] who considered the continued fraction expansion with even partial quotients for real numbers $z > 0$ and showed

$$\lim_{N\to\infty} \frac{1}{N} \sharp\{n \le N \,:\, \theta_n(z) \le x\} = 0$$

for all positive x. Consequently, there exist z for which $\theta_n(z)$ diverges to infinity. Schweiger's result follows from the non-finiteness of the underlying invariant measure. In fact, restricting Julius Hurwitz' (Tanaka's) complex continued fraction expansion to real numbers one ends up with exactly Schweiger's continued fractions with even partial quotients. This follows essentially from the observation that $(\alpha) \cap \mathbf{Z} = 2\mathbf{Z}$. The corresponding invariant measure has density given by $(1 + uv)^{-2}$ for $(u, v) \in [-1, 1]$ (resp. $(u, v) \in [0, 1] \times [-1, 1]$ in Schweiger's articles), and indeed

$$\iint_{[-1,1]^2} \frac{d\lambda(u, v)}{(1 + uv)^2}$$

diverges. The case of those real continued fractions with even partial quotients seems to be rather different from the general complex situation with respect to diophantine properties. In a recent paper [9], Ian Short & Mairi Walker give a beautiful geometric representation of real continued fractions with even partial quotients in the context of the Farey tree and the hyperbolic plane.

2. Proof of Theorem 1

Our approach follows closely the ergodic proof of Bosma et al. [1]. Therefore, we recall some ergodic properties from Tanaka [12]. Let $Y = \{w \in \mathbf{C} : |w| \leq 1\}$ denote the closed unit disk. Following Tanaka, we define the dual transformation $S : Y \to Y$ of T by $S0 = 0$ and

$$Sw = \frac{1}{w} - \left[\frac{1}{w}\right]_S \quad \text{for } w \neq 0$$

with $[w]_S = a$ if $w \in a + V(a)$, where

$$V(a) := \begin{cases} Y, & \text{if } a = 0, \\ V_j, & \text{if } a \in J_j. \end{cases}$$

Here $V_1 := \{w \in Y : |w + \alpha| \geq 1\}$, $V_{j+1} := -i \times V_j$ for $j = 1, 2, 3, 5, 6, 7$, as well as $V_5 := V_1 \cap V_2$ (such that $\mathbf{C} = \bigcup_{a \in (\alpha)} (a + V(a))$), and $J_1 = \{n\alpha : n > 0\}$, $J_{v+1} := -i \times J_v$ for $v = 1, 2, 3, 5, 6, 7$, as well as $J_5 = \{n\alpha + m\overline{\alpha} : m > 0\}$. Setting

$$b_n = b_n(w) = \left[\frac{1}{S^{n-1}w}\right]_S,$$

which lies in (α), one obtains an expansion of $w \in Y$ as continued fraction, namely

$$w = [b_1, b_2, \ldots, b_n + S^n w]$$

with convergents $v_n := [b_1, b_2, \ldots, b_n]$. Tanaka showed that a sequence $a_1, \ldots, a_n$ of numbers in (α) is T-admissible if, and only if, the inverse sequence of partial quotients $a_n, \ldots, a_1$ is S-admissible. Furthermore, he proved that the map $\mathbf{T} : X \times Y \to X \times Y$ defined by

$$\mathbf{T}(z, w) = \left(Tz, \frac{1}{a_1(z) + w}\right), \tag{3}$$

where $a_1 = \left[\frac{1}{z}\right]_T$ is a natural extension; in particular he showed that T and S are ergodic. Moreover, the function $h : X \times Y \to \mathbf{R}$ given by

$$h(u, v) = \frac{1}{|1 + uv|^4} \tag{4}$$

is the density function of a finite absolutely continuous $\mathbf{T}$-invariant measure μ. Now we are in the position to begin with the proof of Theorem 1.

It follows from the dual transformation that

$$v_n = \frac{q_{n-1}}{q_n} = [0; a_n, a_{n-1}, \ldots, a_2, a_1] ,$$

and thus we obtain the equivalence

$$\theta_n(z) \le x \quad \Leftrightarrow \quad \left| \frac{T^n z}{1 + v_n T^n z} \right| \le x . \tag{5}$$

By (3), we find

$$\mathbf{T}^n(z, w) = \left(T^n z, [0; a_n, a_{n-1}, \ldots, a_2, a_1 + w]\right) ,$$

and, in particular, $\mathbf{T}^n(z, 0) = (T^n z, v_n)$. In view of (5) this shows that $\theta_n(z) \le x$ if, and only if,

$$\mathbf{T}^n(z, 0) \in A_x = \left\{ (u, v) \in X \times Y : \left| \frac{u}{1 + uv} \right| \le x \right\} ,$$

equivalently

$$\mathbf{T}^n(z, 0) \in A_x = \left\{ (u, v) \in X \times Y : \left| \frac{1}{u} + v \right| \ge \frac{1}{x} \right\} .$$

Comparing the quantities $\mathbf{T}^n(z, w) = (T^n z, [0; a_n, a_{n-1}, \ldots, a_1 + w])$ and $\mathbf{T}^n(z, 0) = (T^n z, [0; a_n, a_{n-1}, \ldots, a_1])$, it follows that for every $\varepsilon > 0$ there exists $n_0(\varepsilon)$ such that, for all $n \ge n_0(\varepsilon)$ and all $w \in Y$, we have

$$\mathbf{T}^n(z, w) \in A_{x+\varepsilon} \Rightarrow \mathbf{T}^n(z, 0) \in A_x ,$$

as well as

$$\mathbf{T}^n(z, 0) \in A_x \Rightarrow \mathbf{T}^n(z, w) \in A_{x-\varepsilon} .$$

We define $A_x^N := \{n \le N : \mathbf{T}^n(z, w) \in A_x\}$. Since $\mathbf{T}$ is ergodic, the following limits exist and the inequalities in between hold: for all $\varepsilon > 0$,

$$\lim_{N\to\infty} \frac{1}{N} \sharp A_{x+\varepsilon}^N \le \liminf_{N\to\infty} \frac{1}{N} \sharp A_x^N \le \limsup_{N\to\infty} \frac{1}{N} \sharp A_x^N \le \lim_{N\to\infty} \frac{1}{N} \sharp A_{x-\varepsilon}^N .$$

Since the underlying dynamical system $(X \times Y, \Sigma, \mu, \mathbf{T})$ with the corresponding invariant probability measure μ (with density given by (4)) and σ-algebra Σ is ergodic, application of Birkhoff's pointwise ergodic theorem yields (1).

In order to compute the normalizing factor G, we shall use the following derivation of the geometric series expansion

$$\frac{1}{(1-w)^2} = \sum_{m\geq 1} m w^{m-1} ,$$

valid for $|w| < 1$. To prevent difficulties that could arise from singularities of h on the boundary of $X \times Y$ we define, for $0 < \rho < 1$,

$$X_\rho := \left\{ z = x + iy \,:\, -\frac{\rho}{2} \leq x, y \leq \frac{\rho}{2} \right\} \quad \text{and} \quad Y_\rho := \{ w \in \mathbf{C} \,:\, |w| \leq \rho \} .$$

In view of (4) and $|1+uv|^2 = (1+uv)(1+\overline{uv})$ we find

$$\begin{aligned} G_\rho &:= \iint_{X_\rho \times Y_\rho} \frac{d\lambda(u,v)}{|1+uv|^4} \\ &= \sum_{m,n\geq 1} mn(-1)^{m+n} \int_{X_\rho} u^{m-1}\overline{u}^{n-1} d\lambda(u) \int_{Y_\rho} v^{m-1}\overline{v}^{n-1} d\lambda(v) . \end{aligned}$$

We compute

$$\begin{aligned} \int_{Y_\rho} v^{m-1}\overline{v}^{n-1} d\lambda(v) &= \int_{Y_\rho} |v|^{2(n-1)} v^{m-n} d\lambda(v) \\ &= \int_0^{2\pi} \int_0^\rho r^{m+n-1} e^{i\varphi(m-n)} dr d\varphi = \begin{cases} \pi \frac{\rho^{2n}}{n} , & \text{if } m = n, \\ 0, & \text{otherwise} . \end{cases} \end{aligned}$$

Applying the derivation of the geometric series expansion once again, leads to

$$\begin{aligned} G_\rho &= \pi \sum_{n\geq 1} n\rho^{2n} \int_{X_\rho} |u|^{2(n-1)} d\lambda(u) \\ &= \pi \int_{X_\rho} \sum_{n\geq 1} n\rho^{2n} |u|^{2(n-1)} d\lambda(u) = \pi \int_{X_\rho} \frac{d\lambda(u)}{\rho^{-2} - |u|^2} . \end{aligned}$$

Now with $\rho \to 1-$ we get $G_\rho \to G_1 = G$ by Lebesgue's theorem on monotone convergence. Next, we use the symmetry of the fundamental domain and split it along the axes into four parts of equal size. We consider the part located in the first quadrant $\triangle := \{a + ib \,:\, 0 \leq a, b \leq 1; b - 1 < a\}$ and deduce

$$G = 4\pi \iint_\triangle \frac{dadb}{1 - |a+ib|^2} = 4\pi \int_0^1 \left(\int_0^{1-a} \frac{db}{1-(a^2+b^2)} \right) da ,$$

where

$$\int_0^{1-a} \frac{db}{1-a^2-b^2} = \int_0^{1-a} \left(\frac{1}{\sqrt{1-a^2}-b} + \frac{1}{\sqrt{1-a^2}+b} \right) db \frac{1}{2\sqrt{1-a^2}}$$
$$= \Big[-\log|b-\sqrt{1-a^2}| + \log(b+\sqrt{1-a^2}) \Big]_{b=0}^{1-a} \cdot \frac{1}{2\sqrt{1-a^2}}$$
$$= \frac{1}{2\sqrt{1-a^2}} \log \frac{\sqrt{1-a^2}+1-a}{\sqrt{1-a^2}-1+a} .$$

Altogether this gives

$$G = 2\pi \int_0^1 \log \frac{\sqrt{1-a^2}+1-a}{\sqrt{1-a^2}-1+a} \frac{da}{\sqrt{1-a^2}}$$
$$= 2\pi \Big[A - B + i \sum_{k=1}^{\infty} \frac{(-\exp(i \arcsin(a)))^k}{k^2} - i \sum_{k=1}^{\infty} \frac{(\exp(i \arcsin(a)))^k}{k^2} \Big]_{a=0}^{1}$$

with

$$A = \arcsin(a) \log(1 - \exp(i \arcsin(a))) ,$$
$$B = \arcsin(a) \log(1 + \exp(i \arcsin(a))) .$$

Hence, we obtain (2). This computation is inspired by Clausen [2].

In comparison to the real case for the ordinary regular continued fraction expansion [1] the analogous expressions for the complex situation are by far more complicated. Using Tanaka's results [12] one may compute $\ell(x)$ in the latter case numerically, however, it seems difficult to find an explicit expression for the limiting distribution function. Another instance of this difficulty is given by Tanaka [12] himself and his non-explicit representation of the entropy of T and S.

3. Proof of Theorem 2

Since h is continuous and positive, it suffices to show that

$$A_{x_1} \subsetneqq A_{x_2} \quad \text{for } 2 < x_1 < x_2 .$$

Of course, if $x_1 < x_2$, then every $(u, v) \in X \times Y$ satisfying $|\frac{1}{u} + v| \geq \frac{1}{x_1}$ satisfies $|\frac{1}{u} + v| \geq \frac{1}{x_2}$ as well; this proves $A_{x_1} \subset A_{x_2}$. That this is a strict inclusion follows from the fact that the set

$$B := \left\{ (u, v) \in X \times Y \ : \ \frac{1}{x_2} < \left| \frac{1}{u} + v \right| < \frac{1}{x_1} \right\} \subset A_{x_2} \setminus A_{x_1}$$

is non-empty and open. The latter follows from the strict inequalities whereas the non-emptiness can be seen as follows: for real $u \in X$ near -1 write $u = \varepsilon - 1$, say. Then,

$-\frac{1}{u} = 1 + \varepsilon'$ for some small positive real number ε' and any sufficiently large disk centered at $-\frac{1}{u}$ will have a non-empty intersection with Y.

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