A Geometric Model of Mixing Lyapunov Exponents Inside Homoclinic Classes in Dimension Three

Xiaolong LI

Huazhong University of Science and Technology

(Communicated by M. Kurihara)

Abstract. For C^1 diffeomorphisms of three dimensional closed manifolds, we provide a geometric model of mixing Lyapunov exponents inside a homoclinic class of a periodic saddle p with non-real eigenvalues. Suppose p has stable index two and the sum of the largest two Lyapunov exponents is greater than $\log(1 - \delta)$, then δ-weak contracting eigenvalues are obtained by an arbitrarily small C^1 perturbation. Using this result, we give a sufficient condition for stabilizing a homoclinic tangency within a given C^1 perturbation range.

1. Introduction

For diffeomorphisms of smooth closed manifolds, homoclinic tangencies and heterodimensional cycles are understood as two basic phenomena of bifurcations beyond uniform hyperbolic systems. They are defined as follows: Let Λ and Γ be transitive hyperbolic sets of a diffeomorphism f (throughout the paper, the *index* of a transitive hyperbolic set Γ, denoted by ind (Γ), is defined as the dimension of its stable subspace).

- f has a *cycle* associated to Λ and Γ if the stable manifold $W^s(\Lambda)$ of Λ intersects the unstable manifold $W^u(\Gamma)$ of Γ and the same holds for $W^u(\Lambda)$ and $W^s(\Gamma)$. The cycle is called *heterodimensional* if the indices of Λ and Γ are different. In particular, the cycle is said to be *co-index one* if ind $(\Lambda) =$ ind $(\Gamma) \pm 1$.
- f has a *homoclinic tangency* associated to Γ if there exist $x, y \in \Gamma$ such that $W^s(x)$ intersects $W^u(y)$ non-transversally.

Obviously, by definition, heterodimensional cycles only exist on manifolds of dimension at least three. Lots of interesting phenomena, for instance, super exponential growth of the number of periodic points [BDF], existence of infinitely many sinks or sources [N1], non-hyperbolic robust transitivity [BDPR] and entropy-expansiveness [LVY] are closely related to them. It is conjectured by Palis that these two are typical mechanisms beyond uniform

hyperbolicity, especially in the C^1 topology (see [B] for a brief introduction on this topic). Let M be a closed manifold of dimension greater than or equal to two.

C^1 **Palis Conjecture** C^1 diffeomorphisms of M exhibiting either a homoclinic tangency or a heterodimensional cycle are C^1 dense in the complement of the C^1 closure of hyperbolic systems.

In particular, we can consider heterodimensional cycles and homoclinic tangencies associated to hyperbolic periodic saddles. Note that both heterodimensional cycles and homoclinic tangencies associated to periodic points contain non-transversal intersections, which can be easily destroyed by small perturbations.

Towards the study of Palis Conjecture, if one wants to develop perturbations keeping these bifurcations surviving, he needs to consider the robust version of them. More precisely, if there is a neighbourhood $\mathcal{U}$ of f such that for every $g \in \mathcal{U}$, the hyperbolic continuation Γ_g of Γ for g exhibits homoclinic tangencies, then, we say that f has a *robust homoclinic tangency* associated to Γ. *Robust heterodimensional cycles* are defined in a similar way. Obviously, robust homoclinic tangencies and robust heterodimensional cycles must be associated to non-trivial hyperbolic sets. Concrete examples of them can be found in [A] and [AS], for instance. Then, a natural question arises immediately: Starting from a homoclinic tangency (resp. heterodimensional cycle) associated to a hyperbolic periodic saddle p (resp. hyperbolic periodic saddles p and q) of f, is there an arbitrarily small perturbation g of f, exhibiting robust homoclinic tangencies (resp. robust heterodimensional cycles)? This problem is called the *stabilization* of homoclinic tangencies (resp. heterodimensional cycles).

In the C^2 topology, Newhouse gave a positive answer to the stabilization of homoclinic tangencies [N2]. This result was extended to higher dimensions by Palis and Viana [PV]. See [BC2] for the stabilization of homoclinic tangencies in some other situation, which gives an alternative proof of a theorem in [PV] on the existence of the Newhouse domain in higher dimensions. In the C^1 topology, for heterodimensional cycles, the first result was obtained by Bonatti and Díaz by introducing a model of blender horseshoe, a kind of thick hyperbolic set. They proved that every co-index one heterodimensional cycle can be stabilized [BD1]. Later, this result was improved by Bonatti, Díaz and Kiriki in [BDK] (see Lemma 2.4). Compared with [BD1], the stabilization in [BDK] is stronger in the following sense: the hyperbolic sets Γ and Λ (to which the robust heterodimensional cycle of g is associated) contain the continuation p_g and q_g respectively. Moreover, examples (called *fragile cycles*) which cannot be stabilized in this sharp sense were constructed [BD2].

By these observations, we propose the following question: In the C^1 topology, is it possible to stabilize a homoclinic tangency? In fact, this question only makes sense when the dimension of M (denoted by $\dim M$) is larger or equal to three. Since according to [Mo], for surface diffeomorphisms, C^1 robust homoclinic tangencies do not exist. In higher dimensional case, Bonatti and Díaz built the so-called folding manifolds which exhibit robust tangent intersections in a natural setting [BD3]. Later, Bonatti, Díaz, Crovisier and Gourmelon gave

a sufficient condition for the existence of robust homoclinic tangencies for homoclinic classes with the absence of some particular dominated splittings [BCDG]. In that paper, as a key step, weak eigenvalues inside homoclinic classes are obtained by using Bochi and Bonatti's result [BB]. It is well known that the absence of some type of dominated splittings leads to the C^1-creation of homoclinic tangencies [W]. So the above situation of three-dimensional case is close to ours, but here we construct a three-dimensional geometric model providing weak eigenvalues directly by perturbing a homoclinic tangency. This method gives a more concrete procedure than [BCDG] to provide weak eigenvalues in this particular case, which even allows us a slightly weaker hypothesis on the Lyapunov exponents of periodic points.

Now, we state the main theorem of this paper. Let M be a compact smooth Riemannian manifold without boundary. In particular, write M^d if it is necessary to emphasize the dimension d of M. Denote by $\mathrm{Diff}^1(M)$ the space of C^1 diffeomorphisms of M endowed with the C^1 topology. Recall that any Riemannian metric $\|\cdot\|$ on M can induce a distance d on TM. We define the C^1 distance between two diffeomorphisms f and g of M as follows:

$$\mathrm{dist}_{C^1}(f, g) = \sup_{v \in TM, w \in TM} \left\{ d(Df(v), Dg(v)), d\left(Df^{-1}(w), Dg^{-1}(w)\right) \right\}.$$

For $f \in \mathrm{Diff}^1(M^3)$ and a hyperbolic periodic point p of f, let $\mathrm{orb}(p)$ denote the orbit of p and $\mathrm{ind}(p)$ denote the index of p, i.e. the dimension of its stable subspace. Let $\chi_1(p) \leq \chi_2(p) \leq \chi_3(p)$ be the Lyapunov exponents of p, counting with multiplicities. We write $\|Df^{\pm}(p)\| = \max\{\|Df^{\beta}(x)\| : \beta = \pm 1, \ x \in \mathrm{orb}(p)\}$, where $\|A\|$ denotes the operator norm of a linear map A.

THEOREM A. *For any $a > 1$, there exists $\delta_0(a) > 0$ with $\delta_0(a) \to 0$ as $a \to 1$, such that if $0 < \delta < \delta_0(a)$ and $f \in \mathrm{Diff}^1(M^3)$ exhibits a homoclinic tangency associated to a hyperbolic periodic saddle p having non-real contracting eigenvalues satisfying $\chi_2(p) + \chi_3(p) > \log(1 - \delta)$, then there exists g with $\mathrm{dist}_{C^1}(f, g) < a\delta\|Df^{\pm}(p)\|$, exhibiting a robust heterodimensional cycle and a robust homoclinic tangency.*

REMARK 1.1.

- When $\mathrm{ind}(p) = 1$, replacing f by its inverse, the symmetric version of this theorem is also valid.
- If $\chi_2(p) + \chi_3(p) > 0$, then $\mathrm{dist}_{C^1}(f, g)$ can be required arbitrarily small, which also follows from [BCDG, Theorem 1].

It is worth mentioning that the aforementioned [BCDG] dealt with the stabilization of homoclinic tangencies in case of $\dim M \geq 3$ (an earlier version is due to Shinohara [S]). Within a fixed perturbation range, Theorem A says that the stabilization of homoclinic tangencies, at least in the weak sense, can be realized if $\chi_2(p) + \chi_3(p) > \log(1 - \delta)$ compared with [BCDG] which requires $\chi_2(p) + \chi_3(p) > -\delta$.

As a corollary, the following result is available if one wants to stabilize a homoclinic tangency in the strong sense. See Definition 2.2 for the definition of dominated splittings and

their dimensions. $\delta_0(a) > 0$ is the number in the statement of Theorem A.

COROLLARY B. *For any $a > 1$, suppose $f \in \text{Diff}^1(M^3)$ exhibits a homoclinic tangency associated to a hyperbolic periodic point p such that*

- *$H(p_g)$ does not admit dominated splittings of dimension $\text{ind}(p_g)$ for all g in a neighbourhood $\mathcal{U}_f$ of f; and*
- *p has non-real contracting eigenvalues satisfying $\chi_2(p) + \chi_3(p) > \log(1 - \delta)$,*

where $0 < \delta < \delta_0(a)$ is sufficiently small, depending on $\mathcal{U}_f$. Then, there exists g with $\text{dist}_{C^1}(f, g) < a\delta\|Df^{\pm}(p)\|$, exhibiting a robust heterodimensional cycle and a robust homoclinic tangency associated to a hyperbolic set Γ containing p_g.

Actually, in the proof of Theorem A, the main part is devoted to the creation of weak contracting eigenvalues. Let us be more precise. Suppose $H(p)$ is a homoclinic class of some hyperbolic periodic saddle p of f (see Section 2 for relevant definitions), then the set of hyperbolic periodic saddles of f which are homoclinically related to p is a dense subset of $H(p)$, which is denoted by $\pitchfork(p)$. We say that $H(p)$ has *weak eigenvalues* associated to periodic points homoclinically related to p if for any $\epsilon > 0$, there exists $q \in \pitchfork(p)$ such that q has some contracting eigenvalue $\lambda^s(q)$ satisfying $|\lambda^s(q)| > (1 - \epsilon)^{\pi(q)}$ or q has some expanding eigenvalue $\lambda^u(q)$ satisfying $|\lambda^u(q)| < (1 + \epsilon)^{\pi(q)}$, where $\pi(q)$ is the period of q. Such an eigenvalue is called ϵ-*weak*. It is not hard to show that if $H(p)$ does not admit dominated splittings of dimension $\text{ind}(p)$, then we can obtain arbitrarily weak eigenvalues associated to periodic points homoclinically related to p_g by an arbitrarily small perturbation g of f. However, unless additional assumptions are given, in general, we cannot designate in advance that such a weak eigenvalue to be contracting or expanding. For example, when $\dim M = 3$ and $\text{ind}(p) = 2$, if we want to use folding manifolds and blender horseshoes to construct robust homoclinic tangencies by small perturbations, as a preliminary step, we should find a periodic point $q \in \pitchfork(p)$ with sufficiently weak *contracting* eigenvalues and then decrease $\text{ind}(q)$ by stretching Df over $TM|\text{orb}(q)$. Otherwise, if the weak eigenvalue of $q \in \pitchfork(p)$ we obtain is always expanding, we might thus get nothing but a sink after C^1small perturbations, which of course escape the continuation of the original homoclinic class.

By this observation, we see that designating the type of a weak eigenvalue is very important in some situation. Along this direction, Bochi and Bonatti developed a method which said, in rough terms, that under some hypothesis, one can mix two consecutive Lyapunov exponents of some periodic point such that both of them move continuously towards their midpoint [BB, Theorem 4.1 and Proposition 3.1]. As a result, under the same setting as above (i.e. $\dim M = 3$ and $\text{ind}(p) = 2$), if we want to get δ-weak contracting eigenvalues by using [BB], the assumption of $\chi_2 + \chi_3 > -\delta$ is necessary. For otherwise, along the parameter curve, χ_3 decreases to zero before χ_2 increases to $-\delta$. In fact, according to the so-called isotopic Franks' Lemma (Lemma 2.5), in order to guarantee the above perturbation does not make the periodic point go out of the continuation of the original homoclinic class, none of the Lyapunov exponents is permitted to pass through zero. [BCDG] borrowed [BB] to obtain δ-weak

contracting eigenvalues. In this paper, by directly perturbing a homoclinic tangency (Sections 3 and 4), we also give a sufficient condition for getting δ-weak contracting eigenvalues, which is slightly better than [BB] when p has non-real eigenvalues.

THEOREM C. *Let δ be a positive real number. Suppose p is a hyperbolic periodic saddle of $f \in \mathrm{Diff}^1(M^3)$ satisfying*:

- *p has non-real contracting eigenvalues satisfying $\chi_2(p) + \chi_3(p) > \log(1 - \delta)$; and*
- *f exhibits a homoclinic tangency associated to p.*

Then, there exist g arbitrarily C^1 close to f and a hyperbolic saddle q of g, homoclinically related to p_g, having δ-weak contracting eigenvalues.

REMARK 1.2.

- When $\mathrm{ind}\,(p) = 1$, replacing f by its inverse, we can give the symmetric version of this theorem.
- According to its proof, this theorem is still valid when $\delta = 0$ (which also follows from [BCDG]). In this case, δ-*weak* should be read as *arbitrarily weak*. That is, for any $\epsilon > 0$, there exist g arbitrarily close to f and $q \in \pitchfork(p_g)$ admitting ϵ-weak contracting eigenvalues.

When δ is positive, to get δ-weak contracting eigenvalues, our assumption on Lyapunov exponents is a little weaker than that of [BCDG] which comes from [BB]. Indeed, the mixing process of Lyapunov exponents in [BB] is obtained by induction on dimensions. In case of planar dynamics, in a periodic orbit $\mathrm{orb}\,(q)$, once there exists some $r \in \mathrm{orb}\,(q)$ with small angle θ between its two eigendirections, by composing a rotation in the tangent space at r with size less than θ, one can mix the Lyapunov exponents of $\mathrm{orb}\,(q)$ (see [BDP, Lemma 3.2] for instance). But in our perturbations, only rotating at a single point is not sufficient, we also need additional perturbations on tangent spaces over many points in $\mathrm{orb}\,(q)$ with relatively large angles. These points are so many that the number of them take a positive proportion in $\mathrm{orb}\,(q)$ especially when q has a large period. The additional perturbations at the many points should make some effect on the exponential growth of tangent vectors which assists the eigenvalues condition of p, causing the weaker assumption of inequality than [BCDG]'s. The selection of such periodic orbit heavily relies on the delicate constructions of the horseshoe model in Section 4.

As mentioned before, the proof of Theorem C occupies the central position of this paper. Independent of [BB], we adopt a different way which is somewhat geometric. Let E^u (resp. E^s) denote the unstable (resp. stable) subspace of a periodic point. Our proof involves looking at the interplay between $\angle(E^s, E^u)$ and contracting rate of vectors in E^s (shortly, E^s-rate). Roughly speaking, for a sequence of periodic saddles $q_n \in \pitchfork(p)$, if $\angle(E^s(q_n), E^u(q_n))$ decrease to zero more rapidly than $E^s(q_n)$-rate, then weak contracting eigenvalues can be created by C^1 small perturbations inside the homoclinic class. In fact, in order to apply the isotopic Franks' Lemma, we need to find a continuous path C_t ($t \in [0, 1]$) of matrices which

connects the derivative of the original first return map and a matrix with weak contracting eigenvalues. In general, finding such a path is not so difficult, while ensuring its hyperbolicity is much harder and more important. Our strategy is the following: Choose a path $Dg^n(q) \circ C_t$ without paying attention to its hyperbolicity for a while. Then, modify this path by adding another matrix, say D_t, which aims to recover the expanding eigenvector. As a consequence, the expanding eigenvalue survives all the time along the modified path, which indicates that the weak eigenvalue we obtain must be contracting. Let us remark that although it is necessary only in theoretic sense, the introduction of D_t is the main reason for assuming $\chi_2(p) + \chi_3(p) > \log(1 - \delta)$. In the foreseeable future, this assumption is difficult to be removed, see [B, Conjecture 8].

This paper is organized as follows. In Section 2, through a brief review of some basic facts and background of this topic, we summarize without proofs some basic properties as the set-up of notation and terminology. In Section 3, we provide a sufficient condition for getting weak contracting eigenvalues inside homoclinic classes by arbitrarily small perturbations. Theorem C will be proved in Section 4 by building a horseshoe model near a homoclinic tangency. Theorem A and Corollary B are proved in Section 5 where index change is shown by using weak eigenvalues.

2. Preliminaries

Let $f \in \text{Diff}^1(M)$ and p be a hyperbolic periodic point of f, denote by $\pi(p)$ the period of p. Suppose the eigenvalues of $Df^{\pi(p)}(p)$, counting with multiplicities, satisfy $|\lambda^1(p)| \leq \cdots \leq |\lambda^s(p)| < 1 < |\lambda^{s+1}(p)| \leq \cdots \leq |\lambda^d(p)|$ where $d = \dim M$ and $s = \text{ind}(p)$, then, $\lambda^s(p)$ is called the *central contracting eigenvalue*. In particular, if $|\lambda^{s-1}(p)| < |\lambda^s(p)|$, we say that $\lambda^s(p)$ has multiplicity one. For any $1 \leq i \leq s$, we say that $Df^{\pi(p)}(p)$ has i-strong stable direction if $|\lambda^i(p)| < |\lambda^{i+1}(p)|$. In this case, one can define the i-strong stable manifold of p, denoted by $W_i^{ss}(p)$, as the unique submanifold in the stable manifold $W^s(p)$ of p, which is tangent to the i-strong stable direction of $Df^{\pi(p)}(p)$. Similar definitions can also be given for unstable eigenvalues.

A set is *residual* in $\text{Diff}^1(M)$ if it can be written as a countable intersection of open and dense subsets of $\text{Diff}^1(M)$. In particular, residual sets of $\text{Diff}^1(M)$ are dense. Throughout the paper, we say that a property holds *generically* in $\text{Diff}^1(M)$ if it is satisfied by diffeomorphisms contained in a residual subset of $\text{Diff}^1(M)$.

Generically in $\text{Diff}^1(M)$, homoclinic classes exhibit many good properties which are similar to the basic sets in the spectral decomposition theorem of Axiom A diffeomorphisms. For this reason, we will mainly focus on the dynamics of C^1 diffeomorphisms restricted to homoclinic classes. Recall that the *homoclinic class* of a hyperbolic periodic saddle p of f, denoted by $H(p)$, is defined as the closure of transversal intersections of the stable and unstable manifolds of p. We can equivalently define $H(p)$ as the closure of all hyperbolic periodic saddles q homoclinically related to p (i.e. the stable manifold manifold of p transversally

meets the unstable manifold of q and vice versa). Another sort of elementary dynamical pieces which are closely related to homoclinic classes are *chain recurrent classes*. In general, a homoclinic class is a proper subset of a chain recurrent class [BCGP]. However, it was shown by Bonatti and Crovisier that as long as periodic points are involved, C^1 generically, these two notions coincide.

LEMMA 2.1 ([BC1]). *Generically in* $\mathrm{Diff}^1(M)$, *every homoclinic class is a chain recurrent class*; *Equivalently*, *every chain recurrent class containing a periodic point p coincide with the homoclinic class of p.*

Recall that an ε-*pseudo-orbit* of f is a sequence $x_i \in M$ such that all the *jumps* $\mathrm{dist}(f(x_i), x_{i+1})$ are less than ε. A point $x \in M$ is called *chain recurrent* if for every $\epsilon > 0$, there exists ϵ-pseudo orbit starting and ending at x. The *chain recurrent class* of x, denoted by $C(x)$, is the collection of all points $y \in M$ such that there are pseudo orbits of arbitrarily small jumps from x to y and from y to x. The following fact is straightforward: Suppose f has a heterodimensional cycle associated to transitive hyperbolic sets Λ and Γ, then Λ and Γ are contained in the same chain recurrent class of f.

DEFINITION 2.2. Let $f \in \mathrm{Diff}^1(M)$ and let $\Lambda \subset M$ be a compact f-invariant subset. A continuous splitting $T_\Lambda M = E \oplus F$ of the tangent bundle over Λ is called *dominated* if it is Df-invariant and there exists $N \in \mathbb{N}$ such that for all $x \in \Lambda$, one has

$$\frac{\|Df^N(x)u\|}{\|Df^N(x)v\|} < \frac{1}{2},$$

where u and v are any unit vectors in E and F respectively. The *dimension* of this dominated splitting is defined as $\dim E$.

Nowadays, homoclinic tangencies are known to be closely related to the absence of some particular type of dominated splitting [W]. For the existence of robust tangencies, the following criterion is quite useful.

LEMMA 2.3 ([BD3, Theorem 1.2]). *Let* M *be a compact manifold with* $\dim M \geq 3$. *There is a residual subset* $\mathcal{R}$ *of* $\mathrm{Diff}^1(M)$ *such that, for every* $f \in \mathcal{R}$ *and every periodic saddle p of f such that*

- $H(p)$ *has a periodic saddle q with* $\mathrm{ind}(p) \neq \mathrm{ind}(q)$; *and*
- $H(p)$ *does not admit dominated splittings of dimension* $\mathrm{ind}(p)$,

the saddle p belongs to a transitive hyperbolic set having a C^1 *robust homoclinic tangency.*

As another kind of homoclinic bifurcation, heterodimensional cycles can be stabilized in most cases (see also [BD1] for an earlier result):

LEMMA 2.4 ([BDK, Theorem 1]). *Let f be a* C^1 *diffeomorphism with a co-index one heterodimensional cycle associated to periodic saddles p and q. Suppose that at least one of*

the homoclinic classes of these saddles is non-trivial. Then there exist an arbitrarily small perturbation g of f and hyperbolic sets $\Lambda \ni p_g$, $\Gamma \ni q_g$ such that g exhibits a robust heterodimensional cycle associated to Λ and Γ.

Now, let us introduce the basic tool which will be used in our perturbation. Usually, Franks' Lemma ([F, Lemma 1.1]) is well known as a simple but helpful result which allows us to realize linear perturbations of Df along a finite set of M by perturbing f itself in an arbitrarily small neighbourhood of that finite set. However, this result has an inherent disadvantage, especially when someone wants to perturb f along some periodic orbit with its homoclinic (or heteroclinic) relation with another periodic point being kept. In other words, unless additional assumptions are given (for instance, the homoclinic class is isolated), a periodic point might escape the continuation of the original homoclinic class (resp. chain recurrent classes). However, Gourmelon's result gave a sufficient condition for controlling the behaviour of stable/unstable manifolds. By applying this isotopic version of Franks' Lemma, we are allowed to give perturbations inside a homoclinic class.

LEMMA 2.5 (Isotopic Franks' Lemma [G1, G2]). *Given $f \in \mathrm{Diff}^1(M)$, let Q be a periodic point of f with period n. Consider $\epsilon > 0$ and $i, j \in \mathbb{N}$. Suppose*

$$(A_{l,t})_{l=0,\dots,n-1, t \in [0,1]}$$

is a one-parameter family of linear cocycle in $GL(\mathbb{R}, d)$ satisfying

- *$A_{l,0} = Df(f^l(Q))$ for $l = 0, \dots, n-1$;*
- *The radius of the curve, defined by*

$$\max_{\substack{l=0,\dots,n-1 \\ t \in [0,1]}} \left\{ \left\| A_{l,t} - A_{l,0} \right\|, \left\| A_{l,t}^{-1} - A_{l,0}^{-1} \right\| \right\},$$

  is less than ϵ;
- *For any $t \in [0,1]$, the product $\prod_{l=0}^{n-1} A_{l,t} = A_{n-1,t} \circ \cdots \circ A_{0,t}$ admits i-strong stable direction and j-strong unstable direction.*

Then, for any neighbourhood V of $\mathrm{orb}_f(Q)$ in M, there exists $g \in \mathrm{Diff}^1(M)$ such that

- *$\mathrm{dist}_{C^1}(g, f) < \epsilon$;*
- *$g = f$ on $\mathrm{orb}_f(Q)$ and on $M \setminus V$, in particular, $Q_g = Q$;*
- *$Dg(g^l(Q)) = A_{l,1}$ for $l = 0, \dots, n-1$;*
- *g preserves the local i-strong stable manifold of Q outside V and the local j-strong unstable manifold outside V.*

where the local i-strong stable manifold of Q outside V is the set of points contained in $W_i^{ss}(f^l(Q)) \cap (M \setminus V)$ whose positive iterations enter V without leaving it.

Since we will make a systematic application of this result, especially for preserving some particular homoclinic or heteroclinic relations, the following version of Lemma 2.5 is convenient.

LEMMA 2.6. *Under the hypothesis of Lemma 2.5, if we assume further that the i-strong stable manifold $W_i^{ss}(Q)$ of Q intersect the unstable manifold of another periodic point R of f, then the perturbed diffeomorphism g also satisfies $W_i^{ss}(Q_g) \cap W^u(R_g) \neq \emptyset$.*

Note that the existence of R_g is guaranteed since orb $_f(R)$ is outside the support of the perturbation. The proof of this lemma is similar in spirit to [S, Lemma 4.6], just noting that the statement there includes the transversality of the intersection, but in the proof, transversality is not used at all.

In the application of Lemma 2.6, we often give the perturbation of Df separately in its invariant subspaces, say, E and F with $E \oplus F = TM$. At this moment, we should be very careful because the angle between E and F might cause some trouble. When this angle is small, even if perturbations of $Df|_E$ and $Df|_F$ are both small, the total size of the perturbation probably becomes pretty large. For subspaces E and F of $\mathbb{R}^d$ with $E \cap F = \{0\}$, let Angle $(E, F) \in [0, \pi/2]$ denote the Euclidean angle between E and F, and define $\angle(E, F) \in [0, +\infty]$ as $\tan$ Angle (E, F). Obviously, when Angle (E, F) is very small, these two quantities become almost the same since $\lim_{\theta \to 0} \theta / \tan \theta = 1$. The following lemma will be frequently used when estimating the sizes of perturbations.

LEMMA 2.7 ([M, Lemma II.10]). *Let $\mathbb{R}^d = E \oplus F$, and $T : \mathbb{R}^d \to \mathbb{R}^d$ is a linear map having E and F as its invariant subspaces, then, the operator norm of T has an upper bound*:

$$\|T\| \leq \frac{1 + \angle(E, F)}{\angle(E, F)} \left(\|T|_E\| + \|T|_F\| \right).$$

3. Weak contracting eigenvalues

In this section, we give a sufficient condition for getting weak contracting eigenvalues inside homoclinic classes, which will be used in the proof of Theorem C. Recall that the index of a hyperbolic periodic point is defined as the dimension of its stable bundle and $\pitchfork(p)$ denotes the set of hyperbolic saddles which are homoclinically related to p.

PROPOSITION 1. *Let $f \in \mathrm{Diff}^1(M^3)$ and a hyperbolic periodic point p of f with index 2 be given. Suppose that there is a constant $\sigma > 0$ such that for any $\gamma > 0$ small, there exist $g \in \mathrm{Diff}^1(M)$ γ-close to f and coinciding with f outside a neighbourhood U of orb (p), a periodic point $q \in \pitchfork(p_g)$ of period n, and $\lambda \in (0, 1)$ satisfying the following*:

  (i) $\lambda^n < \gamma$;
  (ii) Angle $(Dg^n(q)\xi, \xi) > \sigma$;
  (iii) $\|Dg^n|_{E^s}(q)\| < \gamma\lambda^n$;
  (iv) $\lambda^n \angle(E^s(q), E^u(q)) < \gamma\|Dg^n(q)\xi\|$,

where ξ is the unit vector in the image of orthogonal projection of $E^u(q)$ into $E^s(q)$. Then, there exists a $c\gamma$-small C^1 perturbation h of f, coinciding with f outside U and admitting

$(1 - \lambda)$-*weak contracting eigenvalues associated to* $q \in \pitchfork(p_h)$, *where* $c > 0$ *is a constant that only depends on* f.

REMARK 3.0.1.

(1) If $\gamma < 1$, conditions (iii) and (iv) together imply that $\angle(E^s(q), E^u(q)) < \gamma$.
(2) By replacing f by f^{-1}, we obtain the symmetric version of Proposition 1 which provides arbitrarily weak expanding eigenvalues.
(3) The neighbourhood U of orb (p) in the statement will be specified in Section 4.
(4) We can require that the weak eigenvalue is real, central, and has multiplicity one. This is because we have the following:

LEMMA 3.0.2 ([GY, Lemma 2.3]). *For generic* f *in* $\mathrm{Diff}^1(M)$ *and any hyperbolic periodic point* p *of* f, *if* f *has a periodic point* $q \in \pitchfork(p)$ *having an* ϵ-*weak eigenvalue*, *then* f *has a periodic point* $p_1 \in \pitchfork(p)$ *with an* ϵ-*weak eigenvalue*, *whose eigenvalues are all real*.

Although it is stated as a property for C^1 generic diffeomorphisms, this lemma is actually a perturbation result. Moreover, by checking its proof, we see that if the weak eigenvalue in the hypothesis is contracting (resp. expanding), then after the perturbation, one gets also contracting (resp. expanding) weak eigenvalues. Once a real weak eigenvalue is obtained by Lemma 3.0.2, which is associated to some $q \in \pitchfork(p_g)$, then an additional arbitrarily small perturbation using the isotopic Franks' Lemma will help us to split the eigenvalues such that all of them have multiplicity one. This last perturbation still preserves the homoclinic relation because the matrix keeps its hyperbolicity in the process.

Now, let us turn to the proof of Proposition 1. It suffices to consider the case that p is a fixed point. The main idea is as follows: Firstly, we find a small perturbation g of f which induces weak contracting eigenvalues by modifying [M, Lemma II.9]. Secondly, we create a one-parameter family C_t ($0 \le t \le 1$) of matrices in $\mathrm{GL}(\mathbb{R}, 3)$ which connects the identity and the perturbation obtained in the previous step. But this isotopic perturbation cannot be used directly, since the hyperbolicity might be destroyed until t arrives at its endpoint. To avoid this happens, thirdly, we recover the unstable eigenvector by adding another isotopic perturbation D_t before C_t. In this step, we will make use of the own dynamics in the 2-dimensional subspace $E^s(q)$ to guarantee, that the additional perturbation D_t can be given separately in two invariant subspaces which have a relatively large angle. This will help us to control the size of the perturbation. Finally, we apply the isotopic Franks' Lemma to the new perturbation $Dg^n(q) \circ C_t \circ D_t$ of $Dg^n(q)$, obtaining the desired weak contracting eigenvalue.

3.1. A small perturbation. Given a hyperbolic fixed point of saddle type whose angle is very small. Following Mañe [M], we construct a perturbation which induces the desired eigenvalue.

LEMMA 3.1.1. *Let* $H \in \mathrm{GL}(\mathbb{R}, 3)$ *have stable index* 2. *Denote the stable (unstable) subspace of* H *by* E^s (*resp.* E^u) *and* $\angle(E^s, E^u)$ *by* θ. *If there is a constant* $\mu \in (0, 1)$

such that $\|H|_{E^s}\| < \frac{1}{2}\mu$. *Then, there exists a 6θ-perturbation H' of H which exhibits an eigenvector with eigenvalue μ.*

PROOF. We take an orthogonal coordinate chart $\{(E^s)^\perp, E^s\}$ of $\mathbb{R}^3$. Since E^s is H-invariant, we write $H = \begin{pmatrix} A & 0 \\ P & B \end{pmatrix}$ in this chart. Here

$$A = (Q \circ H)|_{(E^s)^\perp} \in \mathrm{GL}(\mathbb{R}, 1) = \mathbb{R} \setminus \{0\},$$

$$B = H|_{E^s} \in \mathrm{GL}(\mathbb{R}, 2),$$

where Q is the orthogonal projection of $\mathbb{R}^3$ into $(E^s)^\perp$. Clearly, $\|A^{-1}\| = |A|^{-1} < 1$ since $\dim(E^s)^\perp = 1$. Note that A is a 1×1 invertible matrix which can also be treated as a non-zero real number. For notational simplicity, in the following, the notation A can be read either as a 1×1 matrix or as a real number. This do not cause any confusion because for the scalar multiplication of a matrix, the real number and the matrix can change their position. For the same reason, a real number μ will be identified as the scalar matrix μI. Define a linear map $L : (E^s)^\perp \to E^s$ such that

$$E^u = \mathrm{graph}(L) = \left\{ v + Lv : v \in (E^s)^\perp \right\}.$$

Thus $\theta = \angle(E^s, E^u) = \|L\|^{-1}$. Since E^u is H-invariant, we obtain $LA = P + BL$. According to the assumption that $\|B\| \le \frac{1}{2}\mu < \frac{1}{2}$, we have

$$\|L\| \le \|PA^{-1}\| + \|BLA^{-1}\| \le \|PA^{-1}\| + \|B\| \cdot \|L\| \le \|PA^{-1}\| + \frac{1}{2}\|L\|,$$

which implies $\|PA^{-1}\|^{-1} \le 2\|L\|^{-1} = 2\theta$. Now, for $(x, y) \in \mathbb{R} \times \mathbb{R}^2$, consider the following linear equation:

$$\begin{pmatrix} A & 0 \\ P & B \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mu \begin{pmatrix} x \\ y \end{pmatrix} \tag{1}$$

where I is the identity matrix and the blocks of $\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}$ has the same sizes as those of H. The definition of C will be clear later. This equation is equivalent to

$$\begin{cases} x = -\left(I - \mu A^{-1}\right)^{-1} Cy & (2) \\ y = (\mu - B)^{-1} P (I - (I - \mu A^{-1})^{-1}) Cy. & (3) \end{cases}$$

The inverse does make sense, because $\mu - B = \mu(I - \mu^{-1}B)$ and by the assumption, $\|\mu^{-1}B\| \le \frac{1}{2}$, which implies that $(\mu - B)$ is invertible ([PdM, Lemma 2.4.2]). Notice that

$$I = (I - \mu A^{-1})(I - \mu A^{-1})^{-1} = (I - \mu A^{-1})^{-1} - \mu A^{-1}(I - \mu A^{-1})^{-1},$$

then (3) can be rewritten as

$$y = -\mu(\mu - B)^{-1}PA^{-1}(I - \mu A^{-1})^{-1}Cy.$$

To solve (1), we take $v \in (E^s)^{\perp}$ satisfying $\|v\| = \|PA^{-1}\|^{-1}$ and $\|PA^{-1}v\| = 1$. Let

$$y = \mu(\mu - B)^{-1}PA^{-1}v \in E^s.$$

By taking norms of the equality $\mu PA^{-1}v = (\mu - B)y$, we have

$$\mu = \|\mu PA^{-1}v\| = \|(\mu - B)y\| \leq \frac{3}{2}\mu\|y\|,$$

which gives $\|y\| \geq \frac{2}{3}$. Let $w = -(I - \mu A^{-1})v$, we conclude that $\|w\| \leq 2\|v\|$. Define C as a linear map from E^s to $(E^s)^{\perp}$ satisfying $Cy = w$ and $\|C\| = \frac{\|w\|}{\|y\|}$. By the previous estimations,

$$\|C\| \leq \frac{2\|v\|}{2/3} = 3\|v\| = 3\|PA^{-1}\|^{-1} \leq \frac{6}{\|L\|} = 6\theta.$$

It is easy to verify that for C defined as above,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} v \\ \mu(\mu - B)^{-1}PA^{-1}v \end{pmatrix}$$

is exactly the solution of (1), hence $H\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}$ is the desired perturbation of H. The proof is complete now.

3.2. Recovering the unstable subspace. As we explained in the beginning of Section 3, for $g \in \mathrm{Diff}^1(M^3)$ which satisfies the assumption of Proposition 1 for sufficiently small γ, since the hypothesis of Lemma 3.1.1 is satisfied (take $\mu = \lambda^n$ for some $\lambda \in (0,1)$ which is sufficiently close to 1), we want to consider the isotopic perturbation $(A_{l,t})_{\substack{l=0,\ldots,n-1 \\ t \in [0,1]}}$ of Dg on $\mathrm{orb}_g(q)$ which is defined as follows: (C is the matrix obtained in the proof of Lemma 3.1.1)

- $A_{0,t} = Dg(q) \circ C_t$, where $C_t = \begin{pmatrix} I & tC \\ 0 & I \end{pmatrix}$;
- $A_{l,t} = Dg(g^l(q))$ for $l = 1, \ldots, n-1$.

As a consequence of Lemma 3.1.1, $\prod_{l=0}^{n-1} A_{l,1}$ admits an eigenvector with eigenvalue λ^n as we desired, hence we can apply Lemma 2.6 to this one-parameter family. However, in general, when t moves from 0 to 1, the composition $\prod_{l=0}^{n-1} A_{l,t}$ (although it begins as a hyperbolic matrix $\prod_{l=0}^{n-1} A_{l,0} = Dg^n(q)$) might lose its hyperbolicity before t arrives at 1, which will

destroy the established plan. To overcome this obstacle, our strategy is to introduce another perturbation D_t which is used for ensuring the hyperbolicity of the perturbed derivatives by recovering its expanding eigenvector for every $t \in [0, 1]$. Using the same notations as in the proof of Lemma 3.1.1, in particular, $H = \begin{pmatrix} A & 0 \\ P & B \end{pmatrix}$ is a 3×3 hyperbolic matrix of index 2 whose stable and unstable subspace are E^s and E^u. Here, $A = (Q \circ H)|_{(E^s)^\perp} \in \mathrm{GL}(\mathbb{R}, 1)$, $B = H|_{E^s} \in \mathrm{GL}(\mathbb{R}, 2)$ where Q is the orthogonal projection of $\mathbb{R}^3$ into $(E^s)^\perp$, $L : (E^s)^\perp \to E^s$ is a linear map with $\mathrm{graph}(L) = E^u$. For every $t \in [0, 1]$, $C_t = \begin{pmatrix} I & tC \\ 0 & I \end{pmatrix}$ is a perturbation of identity such that $H \circ C_1$ exhibits an eigenvector (x, y) with eigenvalue μ. We will show the existence of a recovering matrix D_t which is sufficiently close to identity and satisfies the following conditions:

(D1)   For every $t \in [0, 1]$,

$$D_t \begin{pmatrix} v \\ Lv \end{pmatrix} = \begin{pmatrix} v - tCLv \\ Lv \end{pmatrix} ;$$

(D2)
$$D_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} .$$

First, let us observe a geometric fact. Denote by G the 2-dimensional plane spanned by $(v, 0)$ and $(0, Lv)$. Recall that $\xi \in E^s$ is the unit vector in the image of the orthogonal projection of E^u into E^s.

LEMMA 3.2.1.   *If* $\mathrm{Angle}\,(H\xi, \xi) > 0$, *then* $(x, y) \notin G$.

PROOF.   Suppose by contradiction that there are $b_1, b_2 \in \mathbb{R}$ satisfying

$$\begin{pmatrix} v \\ \mu(\mu - B)^{-1} PA^{-1}v \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = b_1 \begin{pmatrix} v \\ 0 \end{pmatrix} + b_2 \begin{pmatrix} 0 \\ Lv \end{pmatrix} .$$

As a consequence,

$$\mu(\mu - B)^{-1} PA^{-1}v = b_2 Lv .$$

Combining $Lv = PA^{-1}v + BLA^{-1}v$, and noticing that A is actually a real number, we obtain

$$\mu(\mu - B)^{-1}(Lv - BLA^{-1}v) = b_2 Lv$$
$$\mu Lv - \mu A^{-1} BLv = b_2(\mu - B)Lv ,$$
$$(1 - b_2)\mu Lv = (\mu A^{-1} - b_2)BLv .$$

But according to the assumption, Lv and BLv are linearly independent, we conclude

$$1 = b_2 = \mu A^{-1} < 1 ,$$

which is absurd. The lemma is proved.

Let $\theta = \angle(E^s, E^u)$ as before. Now, we are ready to construct D_t.

LEMMA 3.2.2.   *Suppose there is a $\sigma > 0$ such that for any $\mu > 0$ small enough and any $\epsilon > 0$ small enough, the following conditions hold*:

(a) Angle $(H\xi, \xi) > \sigma$;
(b) $\|H|_{E^s}\| < \mu\epsilon$;
(c) $\dfrac{600\mu\theta}{\|H\xi\|} < \epsilon$.

Then, there exists a one-parameter family of matrices $D_t \in GL(\mathbb{R}, 3)$ $(0 \le t \le 1)$ satisfying (D1) and (D2) such that D_t is ϵ-close to identity for all $t \in [0, 1]$.

PROOF.   According to Lemma 3.2.1, we will take $F := \mathrm{span}\{(x, y)\}$ and G as two invariant subspaces of D_t and give the definition of D_t $(0 \le t \le 1)$ in F and G separately.

- Define $D_t|_F$ as the identity map;
- Define $D_t|_G$ as a rotation of the form (under some 2-dimensional standard orthogonal coordinate chart of G)

$$D_t|_G = \rho_t \begin{pmatrix} \cos\omega_t & -\sin\omega_t \\ \sin\omega_t & \cos\omega_t \end{pmatrix}$$

such that

$$D_t \begin{pmatrix} v \\ Lv \end{pmatrix} = \begin{pmatrix} v - tCLv \\ Lv \end{pmatrix},$$

where

$$\rho_t = \left\| \begin{pmatrix} v - tCLv \\ Lv \end{pmatrix} \right\| \Big/ \left\| \begin{pmatrix} v \\ Lv \end{pmatrix} \right\| \quad \text{and}$$

$$\omega_t = \mathrm{Angle}\left(\begin{pmatrix} v \\ Lv \end{pmatrix}, \begin{pmatrix} v - tCLv \\ Lv \end{pmatrix} \right).$$

Obviously, (D1) and (D2) follow directly from this definition, it remains to estimate the distance between D_t and the identity map. In fact, since $\|C\| \le 6\theta$ by the proof of Lemma 3.1.1,

$$\omega_t \le \omega_1 \le \frac{\|v\| + \|CLv\|}{\|Lv\|} \le 2\theta + \|C\| \le 12\theta.$$

Moreover,

$$\rho_1 - 1 \le \frac{\sqrt{\|v - CLv\|^2 + \|Lv\|^2}}{\|Lv\|} - 1$$

$$= \frac{\|v - CLv\|^2}{\|Lv\|(\sqrt{\|v - CLv\|^2 + \|Lv\|^2} + \|Lv\|)}$$

$$\leq \frac{(\|v\| + 6\|v\|)^2}{\|Lv\|(\sqrt{\|Lv\|^2} + \|Lv\|)} = \frac{49}{2}\left(\frac{\|v\|}{\|Lv\|}\right)^2 \leq 49\theta^2,$$

hence

$$\rho_t - 1 \leq \rho_1 - 1 \leq \theta$$

whenever θ is sufficiently small. As a conclusion,

$$\|(D_t - \mathrm{id})|_G\| \leq \|(D_1 - \mathrm{id})|_G\| = \left\|\rho_1 \begin{pmatrix} \cos\omega_1 & -\sin\omega_1 \\ \sin\omega_1 & \cos\omega_1 \end{pmatrix} - \rho_1 \cdot \rho_1^{-1}\right\|$$

$$\leq \rho_1 \left(\left\|\begin{pmatrix} \cos\omega_1 & -\sin\omega_1 \\ \sin\omega_1 & \cos\omega_1 \end{pmatrix} - \mathrm{id}\right\| + \|\mathrm{id} - \rho_1^{-1}\|\right)$$

$$\leq 2\rho_1\omega_1 + (\rho_1 - 1) \leq 24\theta(1 + \theta) + \theta < 50\theta.$$

Let $\beta = \angle(F, G)$, we will estimate this angle in the triangle spanned by (v, Lv) and (x, y). Let

$$\vec{n} := \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} v \\ Lv \end{pmatrix} = \begin{pmatrix} v \\ \mu(\mu - B)^{-1}PA^{-1}v \end{pmatrix} - \begin{pmatrix} v \\ Lv \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \mu(\mu - B)^{-1}PA^{-1}v - Lv \end{pmatrix}$$

which is parallel to $E^s(q)$. Moreover, notice that

$$\mu(\mu - B)^{-1}PA^{-1}v - Lv = \mu(\mu - B)^{-1}(Lv - BLA^{-1}v) - (\mu - B)^{-1}(\mu - B)Lv$$

$$= (\mu - B)^{-1}(\mu Lv - \mu A^{-1}BLv - \mu Lv + BLv)$$

$$= \mu^{-1}(I - \mu^{-1}B)^{-1}(I - \mu A^{-1})BLv,$$

combining the assumption (b) and the fact that μ is sufficiently small, we can require that $\mu^{-1}B$ and μA^{-1} are sufficiently close to zero in advance. Since σ is independent of μ and ϵ, the assumption (a) gives

$$\mathrm{Angle}\,(\vec{n}, G) \geq \frac{1}{2}\,\mathrm{Angle}\,(BLv, Lv) \geq \frac{\sigma}{2}.$$

Moreover, by assumption (b),

$$\|\vec{n}\| = \|\mu^{-1}(I - \mu^{-1}B)^{-1}(I - \mu A^{-1})BLv\| \in \left[\frac{\|BLv\|}{4\mu}, \frac{4\|BLv\|}{\mu}\right].$$

Intuitively, we see that $\vec{n}$ has stood in G (see Figure 1). Again, since σ is uniform for every $\epsilon > 0$ small, we are allowed to estimate β using $\dfrac{\|\vec{n}\|}{\|Lv\|}$, up to a constant multiple, which
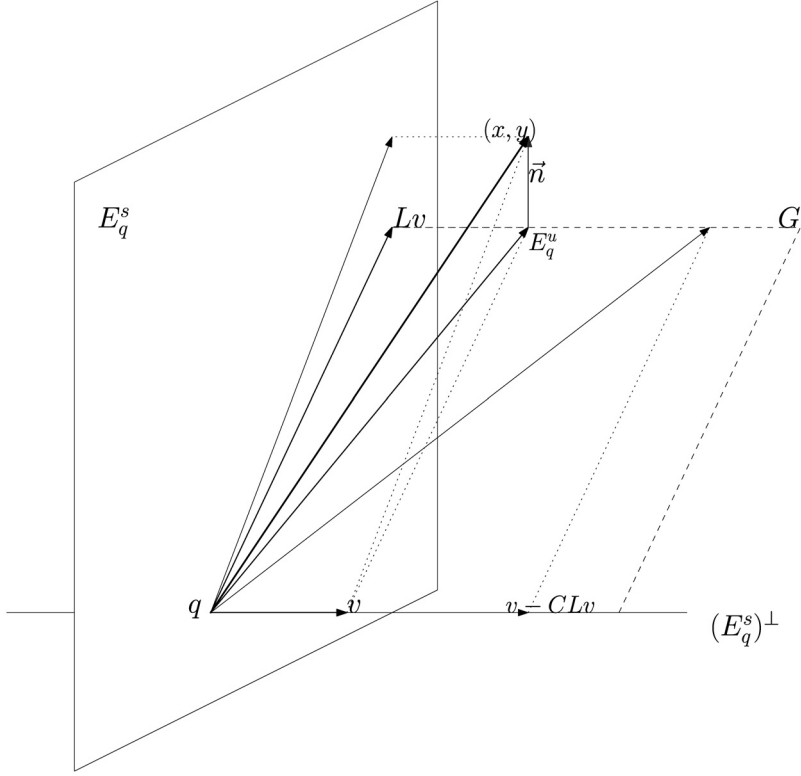
FIGURE 1.   Perturbations of Dg at $T_q M$

can be assumed equal to 1 for simplicity. Thus, using (b) again, we have

$$\frac{\|BLv\|}{4\mu\|Lv\|} \le \beta \le \frac{4\|BLv\|}{\mu\|Lv\|} \le \frac{4\|B\|}{\mu} \le 2\,.$$

Thus, by Lemma 2.7,

$$\|D_t - \mathrm{id}\| \le \|D_1 - \mathrm{id}\| \le \frac{1+\beta}{\beta}\big(\|(D_1 - \mathrm{id})|_F\| + \|(D_1 - \mathrm{id})|_G\|\big)$$

$$< \frac{1+\beta}{\beta}\big(0 + 50\theta\big) \le \frac{150\theta}{\beta} \le 600\mu\theta\frac{\|Lv\|}{\|BLv\|} \le \epsilon\,,$$

where the last inequality comes from (c). Recall that ξ is the unit vector in Lv direction.

Now, we complete showing the existence of the parameter curve D_t $(0 \le t \le 1)$ which is close to identity and satisfies (D1) and (D2). Using this D_t, let us finish the proof of Proposition 1.

3.3. Proof of Proposition 1. Under the hypothesis of Proposition 1 with a fixed point p, for any fixed $\gamma > 0$ and a neighbourhood U of p, we are going to construct a $c\gamma$-perturbation h of f having a periodic point homoclinically related to p_h with $(1 - \lambda)$-weak contracting eigenvalues. First, by assumption, we are allowed to select $g \in \mathrm{Diff}^{\,1}(M)$ coinciding with f outside a neighbourhood U of p, a saddle $q \in \pitchfork(p_g)$ of period n and λ satisfying the following conditions:

(K0)
$$\lambda^n < \gamma \,;$$

(K1)
$$\mathrm{dist}_{C^1}(g, f) < \gamma \,;$$

(K2)
$$\mathrm{Angle}\,(Dg^n(q)\xi, \xi) \geq \sigma \,,$$

(K3)
$$\|Dg^n|_{E^s}(q)\| < \gamma\lambda^n \,;$$

(K4)
$$\max\left\{6\theta_q, \frac{600\lambda^n\theta_q}{\|Dg^n(q)\xi\|}\right\} < \gamma \,,$$

where $\theta_q = \angle(E^s(q), E^u(q))$.

By letting $\mu = \lambda^n$, one easily verifies that the assumption of Lemmas 3.1.1 and 3.2.2 are satisfied by $H = Dg^n(q)$. Thus, we obtain one-parameter families of matrices C_t and D_t ($0 \leq t \leq 1$) respectively. Consider the following parameter curves $(A_{l,t})_{\substack{l=0,\ldots,n-1 \\ t\in[0,1]}}$:

- $A_{0,t} = Dg(q) \circ C_t \circ D_t$;
- $A_{l,t} = Dg(g^l(q))$ $(l = 1, \ldots, n-1)$.

Clearly, $A_{l,0} = Dg(g^l(q))$ for $l = 0, \ldots, n-1$. To estimate the radius of the path, since

$$\|C_1^{\pm 1} - \mathrm{id}\| \leq \|C\| \leq 6\theta_q \leq \gamma \leq 1$$

gives $\|C_1\| \leq 2$, as a consequence,

$$\max_{\substack{l=0,\ldots,n-1 \\ t\in[0,1]}} \left\{\|A_{l,t} - A_{l,0}\|\right\}$$

$$= \max_{t\in[0,1]} \|A_{0,t} - A_{0,0}\|$$

$$= \max_{t\in[0,1]} \|Dg(q) \circ C_t \circ D_t - Dg(q)\| \leq \|Dg(q)\| \cdot \max_{t\in[0,1]} \|C_t \circ D_t - \mathrm{id}\|$$

$$\leq D(U) \max_{t\in[0,1]} \|C_t\| \big(\|D_t - \mathrm{id}\| + \|\mathrm{id} - C_t^{-1}\|\big) < 2D(U)(\gamma + \gamma) = 4D(U)\gamma \,,$$

where

$$D(U) := \sup\{\|Df(x)\| + \|Df^{-1}(x)\| : x \in U\} + 1 \,.$$

In fact, since M is compact, $D(U)$ is a finite number. Moreover, for any g sufficiently C^1 close to f coinciding with f outside U, we have

$$\sup\{\|Dg(x)\| + \|Dg^{-1}(x)\| : x \in U\} \leq D(U).$$

Similar estimation shows that

$$\max_{\substack{l=0,\dots,n-1 \\ t\in[0,1]}} \left\{\|A_{l,t}^{-1} - A_{l,0}^{-1}\|\right\} < 4D(U)\gamma.$$

Hence, we conclude that the radius of the path is less than $4D(U)\gamma$.

Immediately, we have two cases: *either*

(I) $\prod_{l=0}^{n-1} A_{l,t}$ keeps its hyperbolicity during all the time when t varies from 0 to 1; *or*

(II) $\prod_{l=0}^{n-1} A_{l,t}$ loses its hyperbolicity for the first time at some $t_0 \in (0, 1]$.

If case (I) occurs, applying Lemma 2.6 to orb$_g(q)$ and $(A_{l,t})_{\substack{l=0,\dots,n-1 \\ t\in[0,1]}}$, we get a $4D(U)\gamma$-perturbation h of g (as a consequence, dist$_{C^1}(h, f) \leq$ dist$_{C^1}(h, g)$ + dist$_{C^1}(g, f) <$ $(4D(U) + 1)\gamma$ by (K1)), satisfying

- $h^l(q) = g^l(q)$ for $l = 0, \dots, n - 1$;
- q is homoclinically related to p_h;
- $Dh(h^l(q)) = A_{l,1}$ for $l = 0, \dots, n - 1$.

Since we also have, by (D2),

$$Dh^n(q)\begin{pmatrix} x \\ y \end{pmatrix} = Dg^n(q) \circ C_1 \circ D_1 \begin{pmatrix} x \\ y \end{pmatrix} = Dg^n(q) \circ C_1 \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} A & 0 \\ P & B \end{pmatrix}\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda^n \begin{pmatrix} x \\ y \end{pmatrix},$$

$Dh^n(q)$ admits a contracting eigenvalue λ^n. In other words, we have found a $(4D(U) + 1)\gamma$-perturbation h of f, having $(1 - \lambda)$-weak contracting eigenvalues associated to $q \in \pitchfork(p_h)$.

On the other hand, if case (II) occurs, $\prod_{l=0}^{n-1} A_{l,t}$ is hyperbolic for all $t \in [0, t_0)$. Then we cut the path $(A_{l,t})$ just before t arrives at t_0 such that its end point admits an eigenvalue as weak as we need (in particular, weaker than $1 - \lambda$). Applying Lemma 2.6 to the tail-cut curve, we also obtain a $(4D(U) + 1)\gamma$-perturbation h of f, having $(1 - \lambda)$-weak eigenvalues associated to $q \in \pitchfork(p_h)$. By our construction, this weak eigenvalue must be contracting. In fact, recalling (D1), for every $t \in [0, 1]$, we have

$$\left(\prod_{l=0}^{n-1} A_{l,t}\right)\begin{pmatrix} v \\ Lv \end{pmatrix}$$

$$= Dg^n(q) \circ C_t \circ D_t \begin{pmatrix} v \\ Lv \end{pmatrix} = Dg^n(q) \circ C_t \begin{pmatrix} v - tCLv \\ Lv \end{pmatrix}$$

$$= Dg^n(q) \begin{pmatrix} I & tC \\ 0 & I \end{pmatrix} \begin{pmatrix} v - tCLv \\ Lv \end{pmatrix} = Dg^n(q) \begin{pmatrix} v \\ Lv \end{pmatrix} = \lambda^u \begin{pmatrix} v \\ Lv \end{pmatrix}$$

where λ^u is the expanding eigenvalue of $Dg^n(q)$ associated to $E^u(q)$. In other words, (v, Lv) is an expanding eigenvector of $\prod_{l=0}^{n-1} A_{l,t}$ for all $t \in [0, 1]$, which indicates, when t increases from 0 to 1, that $\prod_{l=0}^{n-1} A_{l,t}$ loses its hyperbolicity for the first time by the absolute value of one of its contracting eigenvalues passes through 1 from the left to the right. Therefore, the weak eigenvalue obtained in case (II) should be contracting. Now, the proof of Proposition 1 is complete.

4. A horseshoe model: Proof of Theorem C

Theorem C is a straightforward consequence of Proposition 1 and the following proposition. Recall that $\chi_1(p) \leq \chi_2(p) \leq \chi_3(p)$ are the Lyapunov exponents of p.

PROPOSITION 2.   *For any $\delta > 0$, suppose p is a hyperbolic periodic saddle of $f \in$* Diff$^1(M^3)$ *satisfying*:

- *p has non-real contracting eigenvalues and $\chi_2(p) + \chi_3(p) > \log(1 - \delta)$; and*
- *f exhibits a homoclinic tangency associated to p.*

Then, there is a constant $\sigma > 0$ such that for any $\gamma > 0$ and any neighbourhood U of orb(p), *there exist $g \in$* Diff$^1(M)$ *γ-close to f coinciding with f outside U, a periodic point $q \in \pitchfork(p_g)$ of period n, and $\lambda \in (0, 1)$ satisfying* (i)–(iv) *as in the hypothesis of Proposition 1. Moreover,*

- *If $\chi_2(p) + \chi_3(p) \geq 0$, we have $\lambda \to 1-$ when $\gamma \to 0$;*
- *If $0 > \chi_2(p) + \chi_3(p) > \log(1 - \delta)$, we have $\lambda = 1 - \delta$ for any $\gamma > 0$ small enough.*

To prove this proposition, we can always assume that p is a fixed point of f. If not, it is enough to consider $f^{\pi(p)}$ instead of f.

Let us give the outline of the proof. First, by choosing an orthogonal chart in a sufficiently small neighbourhood U_p of p in M, up to an arbitrarily small perturbation, the dynamics of f in this neighbourhood can be identified with that of the linear map $Df(p)$ (Section 4.1). Fix a homoclinic point x of p in U_p. Then, for any $\epsilon > 0$, we are going to construct another periodic point q of some ϵ-perturbation g of f. Accordingly, by shrinking the original neighbourhood U_p into a proper subset of it, we find that some iteration of g exhibits a topological horseshoe Λ inside that subset (Section 4.2). By constructing a cone field, we prove in Section 4.3 that Λ is in fact a hyperbolic horseshoe whose dynamics is conjugate to a full shift of two symbols. This part is relatively standard. Readers can refer [PT, Section 2.3] for a two dimensional model although we are dealing with a three dimensional one. Next, we select a particular periodic point q in Λ (Section 4.4) and give appropriate 2ϵ-perturbations of g (still denoted by g) along its orbit at which E^s and E^u have large angles (Section 4.5). Thus, for any $\epsilon > 0$

fixed, we obtain a 3ϵ-perturbation g of f and a periodic point $q \in \pitchfork(p_g)$. Finally, we can check that the conclusion of Proposition 2 is satisfied by this perturbation (Section 4.6).

4.1. Some Preparations.   Under the assumption of Proposition 2, by an arbitrarily small C^1 perturbation of f (the readers can refer the proof of [F, Lemma 1.1]), there exists a local coordinate chart $\phi : U_p \to \mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R}$ defined in a small neighbourhood $U_p \subset M$ of p satisfying the following:

(A1) $\phi(U_p) = \{\mathbf{z} \in \mathbb{C} : |\mathbf{z}| < 1\} \times (-1, 1) =: B^s \times B^u =: B$;

(A2) $\phi(p) = (\mathbf{0}, 0) \in B$;

(A3) $B^s \subset \phi(W^s_{loc}(p))$ and $B^u \subset \phi(W^u_{loc}(p))$;

(A4) There exists $x = (\mathbf{z}, 0) \in B$ at which $\phi(W^s(p))$ intersects $\phi(W^u(p))$ non-transversally;

(A5) $\phi f \phi^{-1}$ defined on B acts as the linear transformation $Df(p)$, that is, for any $(\mathbf{z}, y) \in B$, we have $\phi f \phi^{-1}(\mathbf{z}, y) = (\mu^s \mathbf{z}, \mu^u y)$ where μ^s and μ^u denote the eigenvalues of p corresponding to $\chi_2(p)$ and $\chi_3(p)$ respectively;

(A6) There exists $T \in \mathbb{N}$ such that $\phi^{-1}(x) \notin f^{T-1}(U_p)$ but $\phi^{-1}(x) \in f^T(U_p)$. By replacing x if necessary, we can also assume $\phi^{-1}(x) \notin f^j(U_p)$ for $j = 1, \ldots, T-1$. We assume f^T which is defined in a small neighbourhood of $f^{-T}(\phi^{-1}(x))$ is affine and call it *transition map* ([BDPR, Lemma 3.2]).

(From now on, points and vectors in B^s will be identified with complex numbers.) It is worth pointing out that (A5) can be obtained by an arbitrarily small perturbation as long as U_p is taken small enough. For this U_p, define

$$D(\phi, U_p) := \sup\{\|D(\phi f \phi^{-1})(x)\| + \|D(\phi f^{-1}\phi^{-1})(x)\| : x \in \phi(U_p)\} + 1 < +\infty.$$

Since ϕ is a rescaling map, there is $C_1 > 0$ such that

$$D(U_p) := \sup\{\|Df(w)\| + \|Df^{-1}(w)\| : w \in U_p\} + 1 \le C_1 D(\phi, U_p).$$

Moreover, there is $C_2 > 0$ such that for any g sufficiently C^1 close to f, we have

$$\sup\{\|Dg(w)\| + \|Dg^{-1}(w)\| : w \in M\} \le C_2 D(U_p) \le C_1 C_2 D(\phi, U_p).$$

In the following, for notational simplicity, $T_x W^u(p)$ should be read as the tangent space of $W^u(p)$ at $\phi^{-1}(x)$ and the action of $\phi f \phi^{-1}$ in this coordinate chart will be written directly as f. We write $D = C_1 C_2 D(\phi, U_p)$ and also after identifying $\phi f \phi^{-1}$ with f in this coordinate chart, we still regard D as an upper bound of $\sup\{\|Dg(w)\| + \|Dg^{-1}(w)\| : w \in M\}$ for any g near f. This upper bound will be frequently used in this section.

Before constructing the horseshoe, as preparations, let us fix some important constants.

LEMMA 4.1.1.   *If $\chi_2(p) + \chi_3(p) > \log(1-\delta)$, then, for any $\epsilon > 0$, there are constants $\lambda \in (0, 1)$ and $\kappa > 2$ such that the following inequalities hold*:

$$\left|\mu^u(\mu^s)^{\kappa-1}\right| < 1. \tag{4}$$

$$0 < \lambda^\kappa < \left|\mu^u(\mu^s)^{\kappa-1}\right| . \tag{5}$$

$$\left|\mu^s(\mu^u)^{\kappa-1}\right| > 1 . \tag{6}$$

$$\lambda^\kappa > \left(1 - \frac{\epsilon}{D}\right)^{\kappa-2}\left|\mu^u(\mu^s)^{\kappa-1}\right| . \tag{7}$$

Recall that $D \geq \sup\{\|Dg(x)\| + \|Dg^{-1}(x)\| : x \in M\}$ for every g near f.

PROOF. For convenience, two cases are considered in separated ways: (Note that the case of $\chi_2(p) + \chi_3(p) = 0$ can be dealt by using a limit process. That is, by selecting a sequence $\delta_k \to 0+$, consider a sequence $f_k \to f$ with $0 > \chi_2(p_k) + \chi_3(p_k) > \log(1 - \delta_k)$. Then, it suffices to replace f by f_k for sufficiently large k in the following argument).

CASE (I): $\chi_2(p) + \chi_3(p) > 0$. For $\epsilon > 0$ fixed before, choose $\lambda = \lambda(\epsilon) \in (0, 1)$ sufficiently close to 1, such that

$$\frac{\log|\mu^s| - \log|\mu^u| + 2\log\left(1 - \frac{\epsilon}{D}\right)}{\log|\mu^s| + \log\left(1 - \frac{\epsilon}{D}\right) - \log\lambda} < 1 - \frac{\log|\mu^u|}{\log|\mu^s|} .$$

This is possible because the above inequality is equivalent to

$$\log|\mu^s|\left(\log|\mu^s| - \log|\mu^u| + 2\log\left(1 - \frac{\epsilon}{D}\right)\right)$$
$$< (\log|\mu^s| - \log|\mu^u|)\left(\log|\mu^s| + \log\left(1 - \frac{\epsilon}{D}\right) - \log\lambda\right) ,$$

$$2\log|\mu^s|\log\left(1 - \frac{\epsilon}{D}\right) < (\log|\mu^s| - \log|\mu^u|)\left(\log\left(1 - \frac{\epsilon}{D}\right) - \log\lambda\right) ,$$

$$\log|\mu^s\mu^u|\log\left(1 - \frac{\epsilon}{D}\right) < \log\lambda\log\left|\frac{\mu^u}{\mu^s}\right| .$$

By the assumption $|\mu^s\mu^u| > 1$, it suffices to choose $\lambda \in (0, 1)$ sufficiently close to 1. Note that $\lambda(\epsilon) \to 1-$ when $\epsilon \to 0$. Next, select $\kappa \in \mathbb{R}$ such that

$$1 - \frac{\log|\mu^u|}{\log|\mu^s|} < \kappa < \frac{\log|\mu^s| - \log|\mu^u|}{\log|\mu^s| - \log\lambda} .$$

This definition does make sense because it is easy to see

$$1 - \frac{\log|\mu^u|}{\log|\mu^s|} < \frac{\log|\mu^s| - \log|\mu^u|}{\log|\mu^s| - \log\lambda} \iff |\mu^s| < \lambda < 1 .$$

The choice of κ provides us two inequalities (4) and (5).

Moreover, since $|\mu^s\mu^u| > 1$ gives $-\frac{\log|\mu^u|}{\log|\mu^s|} > 1$, we know $\kappa > 2$. Thus

$$\left|\mu^s(\mu^u)^{\kappa-1}\right| > |\mu^s\mu^u| > 1 ,$$

which is (6). Besides, since

$$\kappa > 1 - \frac{\log|\mu^u|}{\log|\mu^s|} > \frac{\log|\mu^s| - \log|\mu^u| + 2\log(1 - \frac{\epsilon}{D})}{\log|\mu^s| + \log(1 - \frac{\epsilon}{D}) - \log\lambda},$$

we also obtain (7).

CASE (II): $0 > \chi_2(p) + \chi_3(p) > \log(1 - \delta)$. For $\epsilon > 0$ fixed before, let $\lambda = 1 - \delta$. Note that in contrast with the previous case, here, λ does not depend on ϵ. Thus, $0 > \chi_2(p) + \chi_3(p) > \log(1 - \delta)$ can be rewritten as $1 > |\mu^s\mu^u| > \lambda$. Moreover, we can always assume $|\mu^s| < \lambda$, for otherwise, p itself has δ-weak contacting eigenvalues (recall that p is a fixed point) and there is nothing to prove. By the choice of λ, for $\epsilon > 0$ small enough, we have

$$1 - \frac{\log|\mu^s|}{\log|\mu^u|} < \frac{\log|\mu^s| - \log|\mu^u| + 2\log(1 - \frac{\epsilon}{D})}{\log|\mu^s| + \log(1 - \frac{\epsilon}{D}) - \log\lambda}.$$

This is because, for $\epsilon > 0$ small enough, the above inequality is equivalent to

$$\log|\mu^s\mu^u|\log\left|\frac{\mu^u}{\mu^s}\right| > \log\lambda\log\left|\frac{\mu^u}{\mu^s}\right| + \log|\mu^s\mu^u|\log\left(1 - \frac{\epsilon}{D}\right),$$

but $|\mu^s\mu^u| > \lambda$, thus it suffice to shrink ϵ if necessary. Next, select $\kappa \in \mathbb{R}$ satisfying

$$\frac{\log|\mu^s| - \log|\mu^u| + 2\log(1 - \frac{\epsilon}{D})}{\log|\mu^s| + \log(1 - \frac{\epsilon}{D}) - \log\lambda} < \kappa < \frac{\log|\mu^u| - \log|\mu^s|}{\log\lambda - \log|\mu^s|}.$$

κ is well-defined since

$$\frac{\log|\mu^s| - \log|\mu^u| + 2\log(1 - \frac{\epsilon}{D})}{\log|\mu^s| + \log(1 - \frac{\epsilon}{D}) - \log\lambda} < \frac{\log|\mu^u| - \log|\mu^s|}{\log\lambda - \log|\mu^s|} \iff |\mu^s\mu^u| > \lambda^2.$$

The choice of κ provides inequalities (5) and (7).

Moreover,

$$\kappa > \frac{\log|\mu^s| - \log|\mu^u| + 2\log(1 - \frac{\epsilon}{D})}{\log|\mu^s| + \log(1 - \frac{\epsilon}{D}) - \log\lambda} > 1 - \frac{\log|\mu^s|}{\log|\mu^u|}$$

gives (6).

Since $|\mu^s\mu^u| < 1$ as we assumed in this case, $\kappa > 1 - \frac{\log|\mu^s|}{\log|\mu^u|} > 2$, which implies (4).

Therefore, we still have a constant $\kappa > 2$ satisfying completely the same inequalities (4)–(7) as in Case (I). The proof of Lemma 4.1.1 is complete.

It should be pointed out that as long as these inequalities are obtained, the following constructions are fit for both Case (I) and Case (II). The only difference is, $\lambda(\epsilon) \to 1 - (\epsilon \to 0)$ when $|\mu^s\mu^u| > 1$ while $\lambda = 1 - \delta$ does not depend on ϵ when $1 > |\mu^s\mu^u| > 1 - \delta$.

From now on, let us fix an $\epsilon > 0$ sufficiently small.

4.2.   Construction of the topological horseshoe.   For any $\theta \in (0, \epsilon)$, by an ϵ-small perturbation of f near $f^{-1}(x)$, we get g such that the above facts (A1)–(A6) still hold for g except (A4), which becomes

(A4') $x = (\mathbf{z}, 0) \in B$ is a transversal homoclinic point of $p_g = p$ with

$$\text{Angle} \, (T_x W^s(p), T_x W^u(p)) = \theta.$$

In fact, this perturbation can be obtained as follows. Fix a small neighbourhood V of x. Consider $\{e_1, e_2, e_3\}$ as the coordinate basis of $T_x M$ where

$$E^s(x) = T_x W^s(p) = \text{span}\{e_1, e_2\};$$

$$E^u(x) = T_x W^u(p) = \text{span}\{e_2\}.$$

Moreover, by changing the length of e_3, we can require that $\|e_3\| = \theta \|e_2\|$. Let $L : T_x M \to T_x M$ be a linear map satisfying

- $Le_2 = e_3$;
- $L = 0$ in $\text{span}\{e_1, e_3\}$;
- $\|L\| = \dfrac{\|e_3\|}{\|e_2\|} = \theta.$

By applying the Franks' Lemma to f at x, we obtain g with $\text{dist}_{C^1}(f, g) \leq \epsilon$ such that

$$Dg(x) = (\text{id} + L) \circ Df(x).$$

It can be immediately checked that (A4') holds for g. Moreover, $g = f$ outside V.

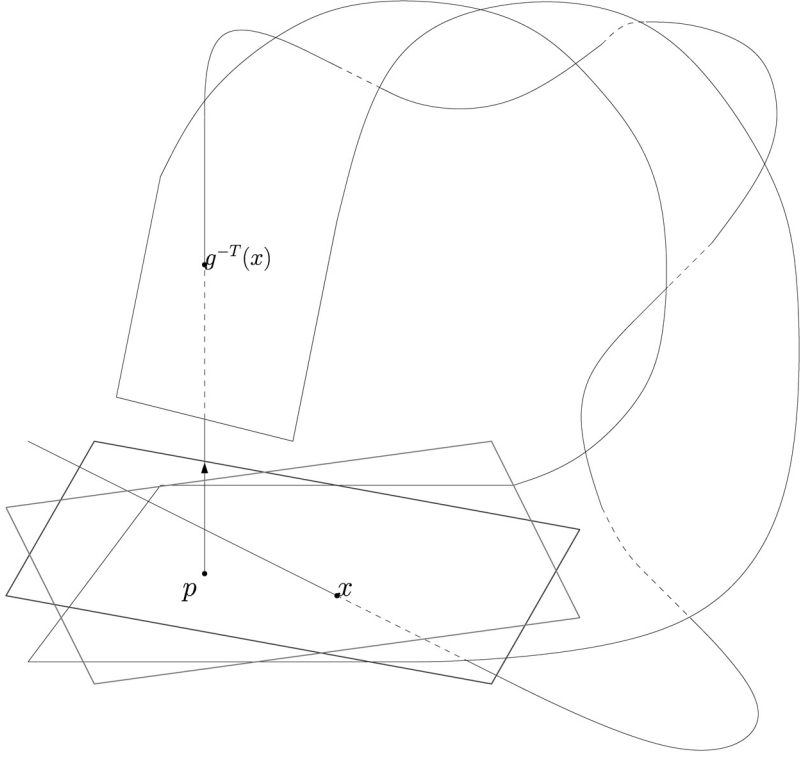Now, we are ready to construct the horseshoe. Define $N = N(\theta) \in \mathbb{R}$ by

$$\left| \frac{\mu^u}{\mu^s} \right|^N \theta = 1.$$

Clearly, $N \to \infty$ when $\theta \to 0$. For some iteration of g, we are going to build a horseshoe inside B, but notice that the above perturbation from f to g only affects the angle between $T_x W^s(p)$ and $T_x W^u(p)$ which is in general not sufficient to guarantee that $g^T(B)$ passes through B from its top to the bottom. Therefore, we need to cut B to get some subset with smaller height. To be more precise, define

$$B_H^\theta = B^s \times \left(\frac{1}{|\mu^u|^{[\kappa N]}} B^u \right) = \{\mathbf{z} \in \mathbb{C} : |\mathbf{z}| < 1\} \times \left(-\frac{1}{|\mu^u|^{[\kappa N]}}, \frac{1}{|\mu^u|^{[\kappa N]}} \right).$$

Here and subsequently, for $s \in \mathbb{R}$, let $[s]$ denote the smallest integer that is larger than s. (Note that this notation is different from the usual one.) When $\theta \to 0$, the height of B_H^θ, denoted by $h(B_H^\theta)$, decreases to zero, but by inequality (6), we have

$$\lim_{\theta \to 0} \frac{\theta}{h(B_H^\theta)} = \lim_{N \to \infty} \left(\left| \frac{\mu^s}{\mu^u} \right|^N |\mu^u|^{[\kappa N]} \right) \geq \lim_{N \to \infty} \left| \mu^s (\mu^u)^{\kappa-1} \right|^N = \infty,$$

FIGURE 2.   A perturbation g of f

which shows that $h(B_H^\theta)$ decreases more rapidly than θ. Therefore, by the local lineariza-
tion property, we can always assume that the connected component of $g^T(\{\mathbf{0}\} \times B^u) \cap B_H^\theta$
containing x is a 1-dimensional curve C^1-close to $T_x W^u(p)$ whose boundary is contained
in $B^s \times \partial \left(\dfrac{1}{|\mu^u|^{[\kappa N]}} B^u \right)$. Briefly, we say that $g^T(\{\mathbf{0}\} \times B^u)$ passes through B_H^θ along B^u-
direction. By continuity, we have:

FACT 4.2.1.   *If $D^s \subset B^s$ is a disk centered at $\mathbf{0} \in B^s$ whose radius is sufficiently
small, then $g^T(D^s \times B^u)$ passes through B_H^θ along B^u-direction. That is, for each $\mathbf{y} \in D^s$,
the leaf $g^T(\{\mathbf{y}\} \times B^u)$ passes through B_H^θ along B^u-direction.*

Symmetrically, let

$$B_V^\theta = \left(|\mu^s|^{[\kappa N]} B^s \right) \times B^u = \left\{ \mathbf{z} \in \mathbb{C} : |\mathbf{z}| < |\mu^s|^{[\kappa N]} \right\} \times (-1, 1).$$

When $\theta \to 0$, the width of B_V^θ, denoted by $v(B_V^\theta)$, decreases to zero. Moreover, since T

is independent of θ, there exists a constant $c_T \geq 1$ which only depends on T and a fixed neighbourhood of f, such that

$$\text{Angle}\,(T_{g^{-T}(x)}W^s(p), T_{g^{-T}(x)}W^u(p)) =: \Theta \in [c_T^{-1}\theta, c_T\theta]\,. \tag{8}$$

As above, by inequality (4), we have

$$\lim_{\theta \to 0} \frac{\Theta}{v(B_V^\theta)} \geq \lim_{\theta \to 0} \frac{c_T^{-1}\theta}{v(B_V^\theta)} = \lim_{N \to \infty} \left(\frac{c_T^{-1}}{|\mu^s|^{[\kappa N]}} \left| \frac{\mu^s}{\mu^u} \right|^N \right) \geq \lim_{N \to \infty} c_T^{-1} \left(\frac{1}{|\mu^u(\mu^s)^{\kappa-1}|} \right)^N = \infty\,,$$

which shows that $v(B_V^\theta)$ decreases to zero faster than $\text{Angle}\,(T_{g^{-T}(x)}W^s(p), T_{g^{-T}(x)}W^u(p))$. Hence by an arbitrarily small perturbation, we can assume the connected component of $g^{-T}(B^s \times \{0\}) \cap B_V^\theta$ containing $g^{-T}(x)$ is a 2-dimensional disk whose boundary is contained in $\partial(|\mu^s|^{[\kappa N]}B^s) \times B^u$. Briefly, we say that $g^{-T}(B^s \times \{0\})$ passes through B_V^θ along B^s-direction.

If $\theta > 0$ is very small, the above construction shows the existence of a topological horseshoe of $g^{[\kappa N]+T}$ inside B_H^θ. In fact,

$$\Lambda_H^\theta := \bigcap_{n=-\infty}^{+\infty} g^{([\kappa N]+T)n}(B_H^\theta)$$

is a $g^{[\kappa N]+T}$-invariant subset, on which $g^{[\kappa N]+T}$ is semi-conjugate to a full shift of two symbols (see [KY] for instance).

4.3. Hyperbolicity of the horseshoe. By using the cone field criterion (see [BS, Section 5.4] for instance), in this subsection, we will prove that Λ_H^θ is actually a hyperbolic horseshoe.

To see this, notice that $g^{[\kappa N]+T}(B_H^\theta) = g^T(B_V^\theta)$, thus, when θ is small, according to Fact 4.2.1, there are two connected components of $g^{[\kappa N]+T}(B_H^\theta) \cap B_H^\theta$, containing p and x respectively, both of them pass through B_H^θ along B^u-direction. As a result, $g^{-([\kappa N]+T)}(B_H^\theta) \cap B_H^\theta$ has another two components which pass through B_H^θ along B^s-direction. Therefore, $B_H^\theta \cap g^{[\kappa N]+T}(B_H^\theta) \cap g^{-([\kappa N]+T)}(B_H^\theta)$ has four components, three of which contain p, x, $g^{-([\kappa N]+T)}(x)$, respectively. We denote these three components by $\text{comp}(p)$, $\text{comp}(x)$, $\text{comp}(g^{-([\kappa N]+T)}(x))$ and the other one by $\text{comp}(\star)$. Let us define an unstable cone field on Λ_H^θ as follows:

(UC1) For every $w \in \text{comp}(p) \cup \text{comp}(g^{-([\kappa N]+T)}(x))$, let $C_w^u(1) = \{(\mathbf{z}, y) \in T_w M : |\mathbf{z}| \leq |y|\}$;

(UC2) For every $w \in \text{comp}(x) \cup \text{comp}(\star)$, let $C_w^u\left(\frac{4}{3\theta}\right) = \left\{(\mathbf{z}, y) \in T_w M : |\mathbf{z}| \leq \frac{4}{3\theta}|y|\right\}$.

In order to define a stable cone field, for convenience, we consider B_V^θ instead of B_H^θ. Similarly as before, since $g^{-([\kappa N]+T)}(B_V^\theta) = g^{-T}(B_H^\theta)$, when θ is small, there are two connected

components of $g^{-([\kappa N]+T)}(B_V^\theta) \cap B_V^\theta$, both of them pass through B_V^θ along B^s-direction. As a result, $g^{[\kappa N]+T}(B_V^\theta) \cap B_V^\theta$ has another two components which pass through B_V^θ along B^u-direction. We denote the four connected components of $B_V^\theta \cap g^{[\kappa N]+T}(B_V^\theta) \cap g^{-([\kappa N]+T)}(B_V^\theta)$ by $\mathrm{Comp}(p)$, $\mathrm{Comp}(g^{-T}(x))$, $\mathrm{Comp}(g^{[\kappa N]}(x))$ and $\mathrm{Comp}(\star)$, where the first three contains p, $g^{-T}(x)$ and $g^{[\kappa N]}(x)$, respectively. Define a stable cone field on Λ_V^θ as the following:

(SC1) For every $w \in \mathrm{Comp}(p) \cup \mathrm{Comp}(g^{[\kappa N]}(x))$, let $C_w^s(1) = \{(\mathbf{z}, y) \in T_w M : |y| \leq |\mathbf{z}|\}$;

(SC2) For every $w \in \mathrm{Comp}(g^{-T}(x)) \cup \mathrm{Comp}(\star)$, let $C_w^s\left(\frac{4}{3\Theta}\right) = \left\{(\mathbf{z}, y) \in T_w M : |y| \leq \frac{4}{3\Theta}|\mathbf{z}|\right\}$.

Here, we gave the definition of unstable cone field on B_H^θ while stable cone field on B_V^θ. Indeed, it does not matter because $g^{[\kappa N]}(B_H^\theta) = B_V^\theta$, which implies

$$\Lambda_V^\theta = \bigcap_{n=-\infty}^{+\infty} g^{([\kappa N]+T)n}(B_V^\theta) = \bigcap_{n=-\infty}^{+\infty} g^{([\kappa N]+T)n+[\kappa N]}(B_H^\theta)$$

$$= \bigcap_{n=-\infty}^{+\infty} g^{([\kappa N]+T)n-T}(B_H^\theta) = g^{-T}\left(\bigcap_{n=-\infty}^{+\infty} g^{([\kappa N]+T)n}(B_H^\theta)\right) = g^{-T}(\Lambda_H^\theta).$$

Thus Λ_V^θ and Λ_H^θ only differ by some fixed number of iterations of g. If we can show that C_w^s is a stable cone field on Λ_V^θ, by defining $E^s(w) = \bigcap_{n=0}^{\infty} Dg^{-([\kappa N]+T)n}(C_{g^{([\kappa N]+T)n}(w)}^s)$ for every $w \in \Lambda_V^\theta$, it follows that $g^T(E^s(g^{-T}(w)))$ is the stable bundle for every $w \in \Lambda_H^\theta$. In other words, to prove the hyperbolicity of Λ_H^θ (or Λ_V^θ), it suffices to show that C_w^u is an unstable cone field on Λ_H^θ and C_w^s is a stable cone field on Λ_V^θ.

LEMMA 4.3.1 (Uniform invariance).   *For every $\theta > 0$ sufficiently small,*

(1) $Dg^{[\kappa N]+T}(w)(C_w^u(\cdot)) \subset C_{g^{[\kappa N]+T}(w)}^u(\frac{6}{7}\cdot)$ *for every* $w \in \Lambda_H^\theta$;

(2) $Dg^{-([\kappa N]+T)}(w)(C_w^s(\cdot)) \subset C_{g^{-([\kappa N]+T)}(w)}^s(\frac{6}{7}\cdot)$ *for every* $w \in \Lambda_V^\theta$.

PROOF.   For convenience, we define the H-slope (V-slope) of a vector $v = (\mathbf{z}, y) \in B^s \times B^u$ by $|\mathbf{z}|/|y|$ (resp. $|y|/|\mathbf{z}|$). Thus the definition of unstable (stable) cone field can be easily rewritten using the notion of H-slope (resp. V-slope).

(1a) For any $w \in \Lambda_H^\theta \cap (\mathrm{comp}(p) \cup \mathrm{comp}(x))$, we have

$$g^{[\kappa N]+T}(w) \in \mathrm{comp}(p) \cup \mathrm{comp}(g^{-([\kappa N]+T)}(x)).$$

Take any vector $(\mathbf{z}, y) \in C_w^u(*)$, where $* = 1$ or $\frac{4}{3\theta}$, depending on the component that w belongs to. Then $Dg^{[\kappa N]+T}(w)(\mathbf{z}, y) = \left((\mu^s)^{[\kappa N]+T}\mathbf{z}, (\mu^u)^{[\kappa N]+T}y\right)$ whose H-slope is

$$\frac{|\mathbf{z}|}{|y|}\left|\frac{\mu^s}{\mu^u}\right|^{[\kappa N]+T} \leq \max\left\{1, \frac{4}{3\theta}\right\}\left|\frac{\mu^s}{\mu^u}\right|^{\kappa N+T} = \frac{4}{3}\left|\frac{\mu^s}{\mu^u}\right|^T \theta^{\kappa-1} < \frac{6}{7}$$

whenever θ is sufficiently small. Recall that $\kappa > 2$.

(1b) For any $w \in \Lambda_H^\theta \cap (\text{comp}(g^{-([\kappa N]+T)}(x)) \cup \text{comp}(\star))$, we have

$$g^{[\kappa N]+T}(w) \in \text{comp}(x) \cup \text{comp}(\star) \,.$$

Take any vector $(\mathbf{z}, y) \in C_w^u(*)$, where $* = 1$ or $\frac{4}{w\theta}$, depending on the component that w belongs to. then $Dg^{[\kappa N]}(w)(\mathbf{z}, y) = \left((\mu^s)^{[\kappa N]}\mathbf{z}, (\mu^u)^{[\kappa N]}y\right)$ whose H-slope is

$$\frac{|\mathbf{z}|}{|y|}\left|\frac{\mu^s}{\mu^u}\right|^{[\kappa N]} \le \max\left\{1, \frac{4}{3\theta}\right\}\left|\frac{\mu^s}{\mu^u}\right|^{\kappa N} = \frac{4}{3}\theta^{\kappa-1} \,.$$

Thus,

$$\text{Angle}\left(Dg^{[\kappa N]}(w)(\mathbf{z}, y), \{\mathbf{0}\} \times B^u\right) \le \arctan\left(\frac{4}{3}\theta^{\kappa-1}\right) , \qquad (9)$$

which is a higher order infinitesimal of θ. Since $g^T(\{\mathbf{0}\} \times B^u)$ intersects $B^s \times \{0\}$ at x with angle θ (see (A4') of Section 4.2), we obtain

$$\text{Angle}\left(Dg^{[\kappa N]+T}(w)(\mathbf{z}, y), B^s \times \{0\}\right) \in \left[\theta - c_T \arctan\left(\frac{4}{3}\theta^{\kappa-1}\right), \theta + c_T \arctan\left(\frac{4}{3}\theta^{\kappa-1}\right)\right] ,$$

recall (8) that $c_T > 1$ is a constant which only depends on T and a fixed neighbourhood of f.

The H-slope of $Dg^{[\kappa N]+T}(w)(\mathbf{z}, y)$ is smaller than

$$\cot\left(\theta - c_T \arctan\left(\frac{4}{3}\theta^{\kappa-1}\right)\right) < \frac{8}{7\theta} = \frac{6}{7}\cdot\frac{4}{3\theta}$$

whenever $\theta > 0$ is sufficiently small. The last inequality holds because

$$\lim_{\theta\to 0}\frac{7\theta}{8}\cot\left(\theta - c_T \arctan\left(\frac{4}{3}\theta^{\kappa-1}\right)\right) = \frac{7}{8} < 1 \,.$$

Now, (1a) and (1b) together imply (1).

(2a) For any $w \in \Lambda_V^\theta \cap (\text{Comp}(p) \cup \text{Comp}(g^{-T}(x)))$, we have

$$g^{-([\kappa N]+T)}(w) \in \text{Comp}(p) \cup \text{Comp}(g^{[\kappa N]}(x)) \,.$$

Take any vector $(\mathbf{z}, y) \in C_w^s(*)$, where $* = 1$ or $\frac{4}{w\Theta}$, depending on the component that w belongs to. Then $Dg^{-([\kappa N]+T)}(w)(\mathbf{z}, y) = \left((\mu^s)^{-([\kappa N]+T)}\mathbf{z}, (\mu^u)^{-([\kappa N]+T)}y\right)$ whose V-slope is (recall that $c_T^{-1}\theta \le \Theta \le c_T\theta$)

$$\frac{|y|}{|\mathbf{z}|}\left|\frac{\mu^u}{\mu^s}\right|^{-([\kappa N]+T)} \le \max\left\{1, \frac{4}{3\Theta}\right\}\left|\frac{\mu^s}{\mu^u}\right|^{\kappa N + T} \le \frac{4}{3}\left|\frac{\mu^s}{\mu^u}\right|^T c_T^\kappa \Theta^{\kappa-1} < \frac{6}{7}$$

whenever θ (hence Θ) is sufficiently small.

(2b) For any $w \in \Lambda_V^\theta \cap (\text{Comp}(g^{[\kappa N]}(x)) \cup \text{Comp}(\star))$, we have

$$g^{-([\kappa N]+T)}(w) \in \text{Comp}(g^{-T}(x)) \cup \text{Comp}(\star).$$

Take any vector $(\mathbf{z}, y) \in C_w^s(*)$ where $* = 1$ or $\frac{4}{3\Theta}$, depending on the component that w belongs to. Then $Dg^{-[\kappa N]}(w)(\mathbf{z}, y) = \left((\mu^s)^{-[\kappa N]}\mathbf{z}, (\mu^u)^{-[\kappa N]}y\right)$ whose V-slope is

$$\frac{|y|}{|\mathbf{z}|}\left|\frac{\mu^u}{\mu^s}\right|^{-[\kappa N]} \leq \max\left\{1, \frac{4}{3\Theta}\right\}\left|\frac{\mu^s}{\mu^u}\right|^{\kappa N} \leq \frac{4}{3}c_T^\kappa\Theta^{\kappa-1}.$$

Thus,

$$\text{Angle}\left(Dg^{-[\kappa N]}(w)(\mathbf{z}, y), B^s \times \{0\}\right) \leq \arctan\left(\frac{4}{3}c_T^\kappa\Theta^{\kappa-1}\right),$$

which is a higher order infinitesimal of Θ. Since $g^{-T}(B^s \times \{0\})$ intersects $\{0\} \times B^u$ at $f^{-T}(x)$ with angle Θ (recall (8)), we get

$$\text{Angle}\left(Dg^{-([\kappa N]+T)}(w)(\mathbf{z}, y), \{0\} \times B^u\right)$$
$$\in \left[\Theta - c_T\arctan\left(\frac{4}{3}c_T^\kappa\Theta^{\kappa-1}\right), \Theta + c_T\arctan\left(\frac{4}{3}c_T^\kappa\Theta^{\kappa-1}\right)\right].$$

The V-slope of $Dg^{-([\kappa N]+T)}(w)(\mathbf{z}, y)$ is smaller than

$$\cot\left(\Theta - c_T\arctan\left(\frac{4}{3}c_T^\kappa\Theta^{\kappa-1}\right)\right) < \frac{8}{7\Theta} = \frac{6}{7}\cdot\frac{4}{3\Theta}$$

whenever θ (hence Θ) is sufficiently small. The last inequality holds because

$$\lim_{\Theta\to 0}\frac{7\Theta}{8}\cot\left(\Theta - c_T\arctan\left(\frac{4}{3}c_T^\kappa\Theta^{\kappa-1}\right)\right) = \frac{7}{8} < 1.$$

Now, (2a) and (2b) together imply (2). the proof of Lemma 4.3.1 is complete.

REMARK 4.3.2.   In the previous proof, we used the definition of unstable cone field given by (UC). By this definition, it is relatively easy to check its invariance just by finding the bounds of H-slope and V-slope as we did before. However, in the following, more accurate estimations on the expanding rates of vectors in the cone field are needed. Thus, it is convenient to replace the previous cone field by a thinner one which is "slanting" with respect to the coordinate chart. Precisely, in the proof of (1b) above, since $c_T\arctan(\frac{4}{3}\theta^{\kappa-1})$ is a higher order infinitesimal of θ, as a result, for every $w \in \Lambda_H^\theta \cap \left(\text{comp}(g^{-([\kappa N]+T)}(x)) \cup \text{comp}(\star)\right)$, we see that $Dg^{[\kappa N]+T}(C_w^u)$ is contained in $\tilde{C}_{w'}^u := \{v \in T_{w'}M : \text{Angle}(v, T_xW^u(p)) \leq \frac{\theta}{4}\}$ where $w' = g^{[\kappa N]+T}(w) \in \Lambda_H^\theta \cap (\text{comp}(x) \cup \text{comp}(\star))$. Thus we can modify the definition of the unstable cone field as follows:

- For every $w \in \text{comp}(p) \cup \text{comp}(g^{-([\kappa N]+T)}(x))$, let $\tilde{C}_w^u = \{v \in T_wM : \text{Angle}(v, \{0\} \times B^u) \leq \frac{\pi}{4}\}$;

- For every $w \in \mathrm{comp}(x) \cup \mathrm{comp}(\star)$, let $\tilde{C}_w^u = \{v \in T_w M : \mathrm{Angle}\,(v, T_x W^u(p)) \leq \frac{\theta}{4}\}$.

Similarly, the definition of the stable cone field can be replaced by:

- $\tilde{C}_w^s = \{v \in T_w M : \mathrm{Angle}\,(v, B^s \times \{0\}) \leq \frac{\pi}{4}\}$ for every $w \in \mathrm{Comp}(p) \cup \mathrm{Comp}(g^{[\kappa N]}(x))$;
- $\tilde{C}_w^s = \{v \in T_w M : \mathrm{Angle}\,(v, T_{g^{-T}(x)} W^s(p)) \leq \frac{\Theta}{4}\}$ for every $w \in \mathrm{Comp}(g^{-T}(x)) \cup \mathrm{Comp}(\star)$.

From now on, to simplify the notation, we write $\tilde{C}_w^u$ and $\tilde{C}_w^s$ again by C_w^u and C_w^s. Obviously, Lemma 4.3.1 still holds for these newly defined C_w^u and C_w^s.

LEMMA 4.3.3 (Uniform expansion). *For every $\theta > 0$ sufficiently small,*

(1) $\|Dg^{[\kappa N]+T}(w)v\| \geq 2\|v\|$ *for all $w \in \Lambda_H^\theta$ and all $v \in C_w^u$;*
(2) $\|Dg^{-([\kappa N]+T)}(w)v\| \geq 2\|v\|$ *for all $w \in \Lambda_V^\theta$ and all $v \in C_w^s$.*

PROOF. By the definition of unstable cone field in the previous remark, vectors in C_w^u ($w \in \mathrm{comp}(p) \cup \mathrm{comp}(g^{-([\kappa N]+T)}(x))$) expands much more than vectors in C_w^u ($w \in \mathrm{comp}(x) \cup \mathrm{comp}(\star)$). Hence we only need to verify the expansion rate of the latter one. Take any $w \in \mathrm{comp}(x) \cup \mathrm{comp}(\star)$ and any unit vector $v = (\mathbf{z}, y) \in C_w^u$. Obviously, whenever θ is small, $|\mathbf{z}| > \frac{1}{2}$ and $\frac{|y|}{|\mathbf{z}|} \geq \tan(\frac{3}{4}\theta) \geq \frac{\theta}{2}$. We have $Dg^{[\kappa N]+T}(w)v = Dg^T(g^{[\kappa N]}(w))((\mu^s)^{[\kappa N]}\mathbf{z}, (\mu^u)^{[\kappa N]}y)$ whose norm is larger than

$$\frac{1}{D^T} \left\| ((\mu^s)^{[\kappa N]}\mathbf{z}, (\mu^u)^{[\kappa N]}y) \right\| \geq \frac{1}{D^T}|(\mu^u)^{[\kappa N]}y| = \frac{|\mathbf{z}|}{D^T}\left|\frac{y}{\mathbf{z}}\right||\mu^u|^{[\kappa N]}$$
$$\geq \frac{1}{4D^T}|\mu^s(\mu^u)^{\kappa-1}|^N > 2\,.$$

The last inequality holds because $4D^T$ is a constant which is independent of θ while (6) tell us $|\mu^s(\mu^u)^{\kappa-1}|^N$ goes to infinity as θ tends to 0. Recall that g is an ϵ-perturbation of f hence $D^{-1} \leq \|Dg(x)\| \leq D$ for all $x \in M$.

Symmetrically, by the definition of stable cone field in Remark 4.3.2, under negative iterations, vectors in C_w^s ($w \in \mathrm{Comp}(p) \cup \mathrm{Comp}(g^{[\kappa N]}(x))$) expand much more than vectors in C_w^s ($w \in \mathrm{Comp}(g^{-T}(x)) \cup \mathrm{Comp}(\star)$). Thus, it suffices to verify the latter one. Take any $w \in \mathrm{Comp}(g^{-T}(x)) \cup \mathrm{Comp}(\star)$ and any unit vector $v = (\mathbf{z}, y) \in C_w^s$, we have $|\mathbf{z}| \geq \tan(\frac{3}{4}\Theta) \geq \frac{\Theta}{2}$. Thus,

$$Dg^{-([\kappa N]+T)}(w)v = Dg^{-T}\big(g^{-[\kappa N]}(w)\big)\big((\mu^s)^{-[\kappa N]}\mathbf{z}, (\mu^u)^{-[\kappa N]}y\big)$$

whose norm is larger than

$$\frac{1}{D^T}\big|(\mu^s)^{-[\kappa N]}\mathbf{z}\big| \geq \frac{\Theta}{2D^T|\mu^s|^{\kappa N}} > \frac{1}{2c_T D^T|\mu^u(\mu^s)^{\kappa-1}|^N} > 2 \qquad (10)$$

whenever θ is small. The last inequality follows from (4). The proof of Lemma 4.3.3 is complete now.

To summarize, for any $\theta \in (0, \epsilon)$ sufficiently small, by the cone field criterion of uniform hyperbolicity (see [BS, Section 5.4] for instance), Lemmas 4.3.1 and 4.3.3 together imply that Λ_H^θ (also Λ_V^θ) is a uniformly hyperbolic horseshoe of $g^{[\kappa N]+T}$. In fact, for every $w \in \Lambda_H^\theta$,

$$E^u(w) = \bigcap_{n=0}^{\infty} g^{([\kappa N]+T)n} (C^u_{g^{-([\kappa N]+T)n}(w)}) \quad \text{and}$$

$$E^s(w) = g^T \left(\bigcap_{n=0}^{\infty} g^{-([\kappa N]+T)n} (C^s_{g^{([\kappa N]+T)n-T}(w)}) \right)$$

are the expanding and contracting bundles of Λ_H^θ. As a consequence, there are exactly two fixed points of $g^{[\kappa N]+T}$ in Λ_H^θ: one is p (which is also a fixed point of g) and the other one is denoted by $q = q(\theta)$. Recall that g also depends on θ, where we have omitted the symbol θ for simplicity. By construction, q is homoclinically related to p_g.

4.4. The dynamics of the orbit of q.   In this section, we will investigate the dynamics of orb (q), showing that the number of points in orb (q) whose angles are uniformly bounded away from zero divided by the period of q tends to a positive number (this proportion does not depend on θ) as the period goes to infinity. This property allows us to give further perturbations in Section 4.5. First, let us simplify the notation.

REMARK 4.4.1.   Lemmas 4.3.1 and 4.3.3 are proved under the hypothesis that (recall (8))

$$\text{Angle}\,(T_{g^{-T}(x)} W^s(p), T_{g^{-T}(x)} W^u(p)) = \Theta \in [c_T^{-1}\theta, c_T\theta]\,.$$

But from the calculations in Section 4.3, we see that actually, c_T does not bring any essential effect, just appears as a constant which is independent of ϵ and θ. For instance, when proving the uniform expansion of $Dg^{[\kappa N]+T}$ in the the stable cone, in (10), c_T appears as a constant that depends only T and a fixed neighbourhood of f. Hence, by shrinking θ if necessary, for N large enough, the above expansion rate is always larger than 2. Thus, in what follows, let us set $c_T = 1$ for simplicity. In other words, take $\Theta = \theta$ directly. The readers can verify step by step as in the proofs of Lemmas 4.3.1 and 4.3.3 that such a simplification involves no loss of generality.

LEMMA 4.4.2.   *The stable and unstable direction of q satisfy the following*:

(1) Angle $(E^u(q), T_x W^u(p)) \leq C\theta^{\kappa-1}$, *where $C > 0$ is a constant independent of θ. In particular*, Angle $(E^u(q), B^s \times \{0\}) \in (\frac{3}{4}\theta, \frac{5}{4}\theta)$;

(2) Angle $(E^s(q), B^s \times \{0\}) \in (\frac{4}{5}\theta^{\kappa-1}, \frac{4}{3}\theta^{\kappa-1})$;

(3) Angle $(E^s(q), E^u(q)) < 2\theta$.
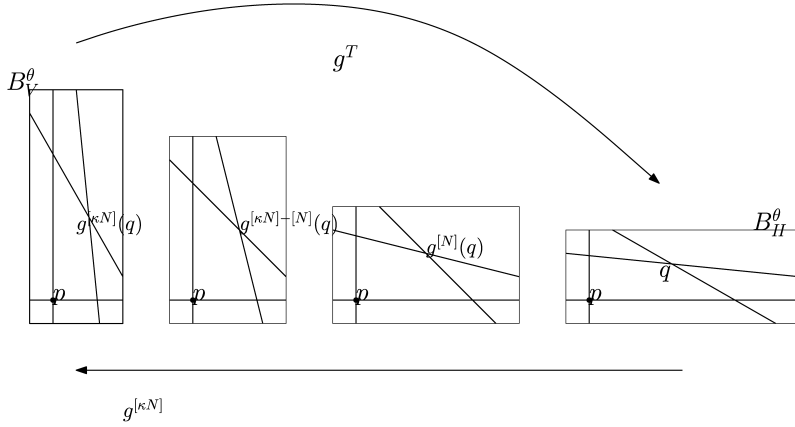
FIGURE 3. A conceptional picture of orb (q)

PROOF. Obviously, (3) follows directly from (1) and (2) because $\theta^{\kappa-1}$ is a higher order infinitesimal of θ (notice that $\kappa > 2$). We only prove the first two items.

(1) By (9) of Lemma 4.3.1, Angle $(Dg^{[\kappa N]}E^u(q), T_p W^u(p)) \leq \arctan\left(\frac{4}{3}\theta^{\kappa-1}\right)$ which is a higher order infinitesimal of θ. Thus, by noticing that $E^u(q) = Dg^{[\kappa N]+T}E^u(q)$, under the action of g^T which is assumed to be an affine map, we have Angle $(E^u(q), T_x W^u(p)) \leq C\theta^{\kappa-1}$.

(2) Take any two dimensional plane in $C^s_{g^{-T}(q)}$, let $v = (\mathbf{z}, y)$ be a vector contained in that plane with the largest V-slope. Thus, by Remark 4.4.1 and the definition of the unstable cone field in Remark 4.3.2, there exists $b \in \left[\frac{3}{4}, \frac{5}{4}\right]$ such that $\frac{|\mathbf{z}|}{|y|} = \tan(b\theta)$. Then, $Dg^{-[\kappa N]}v = \left((\mu^s)^{-[\kappa N]}\mathbf{z}, (\mu^u)^{-[\kappa N]}y\right)$ whose V-slope is $\frac{|y|}{|\mathbf{z}|}\left|\frac{\mu^u}{\mu^s}\right|^{-[\kappa N]}$ which is close to $\frac{1}{b}\theta^{\kappa-1} \in \left[\frac{4}{5}\theta^{\kappa-1}, \frac{4}{3}\theta^{\kappa-1}\right]$ when θ is small. Since $E^s(q)$ is contained in $Dg^{-[\kappa N]}C^s_{g^{-T}(q)}$, we obtain the desired conclusion and finish the proof of Lemma 4.4.2.

Let us divide the iteration from q to $g^{[\kappa N]}(q)$ into three parts: $[\kappa N] = [N] + ([\kappa N] - 2[N]) + [N]$ (see Figure 3 for a conceptional picture), recalling that $\kappa > 2$. The following lemma tells us that, in the middle part, E^s and E^u exhibit large angles.

LEMMA 4.4.3. *If $\theta > 0$ is small enough,*

$$\angle(E^s(g^i(q)), E^u(g^i(q))) > \frac{1}{2}$$

for $i = [N], \ldots, [\kappa N] - [N]$.

PROOF. By Lemma 4.4.2 (1) and (2), estimating in the similar way as in the proof of

Lemma 4.4.2, for any $a \in [0, \kappa]$,

$$\inf\{\text{V-slope}(u) : u \in C^u_{g^{[aN]}(q)}\} \geq \frac{3}{4}\theta \left|\frac{\mu^u}{\mu^s}\right|^{aN} = \frac{3}{4}\theta^{1-a},$$

$$\sup\{\text{V-slope}(u) : u \in C^s_{g^{[aN]}(q)}\} \leq \frac{4}{3\theta} \left|\frac{\mu^s}{\mu^u}\right|^{(\kappa-a)N} = \frac{4}{3}\theta^{\kappa-1-a}.$$

Thus, letting $\sigma_a = \text{Angle}\,(E^s(g^{[aN]}(q)), E^u(g^{[aN]}(q)))$,

$$\angle(E^s(g^{[aN]}(q)), E^u(g^{[aN]}(q))) = \tan\sigma_a \geq \tan\left(\arctan\left(\frac{3}{4}\theta^{1-a}\right) - \arctan\left(\frac{4}{3}\theta^{\kappa-1-a}\right)\right)$$

$$= \frac{\frac{3}{4}\theta^{1-a} - \frac{4}{3}\theta^{\kappa-1-a}}{1 + \theta^{\kappa-2a}} =: \Gamma_a. \tag{11}$$

As a lower bound of $\tan\sigma_a$, let us see how Γ_a varies with a.

CLAIM. $\Gamma_a \geq \Gamma_1 = \Gamma_{\kappa-1}\quad \text{for } a \in [1, \kappa - 1].$

In fact, letting $a = 1$ and $a = \kappa - 1$ in (11), we see that

$$\Gamma_1 = \Gamma_{\kappa-1} = \frac{\frac{3}{4} - \frac{4}{3}\theta^{\kappa-2}}{1 + \theta^{\kappa-2}}.$$

On the other hand, $\Gamma_a \geq \Gamma_1$ if and only if

$$\frac{\frac{3}{4} - \frac{4}{3}\theta^{\kappa-2}}{\theta^{a-1} + \theta^{\kappa-1-a}} = \frac{\frac{3}{4}\theta^{1-a} - \frac{4}{3}\theta^{\kappa-1-a}}{1 + \theta^{\kappa-2a}} \geq \frac{\frac{3}{4} - \frac{4}{3}\theta^{\kappa-2}}{1 + \theta^{\kappa-2}},$$

That is,

$$1 + \theta^{\kappa-2} \geq \theta^{a-1} + \theta^{\kappa-1-a}.$$

By analyzing the derivatives, when a increases from 1 to $\kappa - 1$, the right hand term first decreases, and then increases after reaching its minimum at $a = \kappa/2$, which implies that

$$\max_{a \in [1, \kappa-1]} \left\{\theta^{a-1} + \theta^{\kappa-1-a}\right\} = 1 + \theta^{\kappa-2}.$$

Thus the claim is proved. As a result, when θ is sufficiently small,

$$\tan\sigma_a \geq \tan\sigma_1 \geq \frac{1}{2}$$

which completes the proof of Lemma 4.4.3.

REMARK 4.4.4.   Since the angle in Lemma 4.4.3 is larger than a constant which does not depend on θ, when giving perturbations of the derivatives at these points keeping E^s and E^u invariant, we are allowed to estimate the size of the perturbation by considering the sum of its norms in E^s and E^u without paying attention to their angle (recall Lemma 2.7).
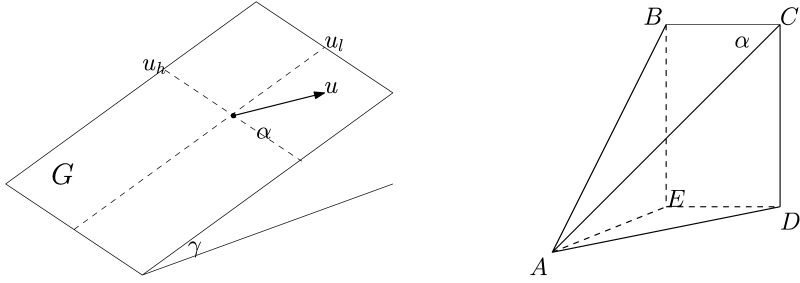
FIGURE 4.   Horizontal direction and slope direction

4.5.   Additional perturbations along the orbit of q.   In the coordinate chart $B^s \times B^u$, we set $B^s \times \{0\}$ as the horizontal plane and $\{0\} \times B^u$ as the vertical axis. Given any 2-dimensional plane G which is not parallel to $B^s \times \{0\}$, there are two uniquely defined unit vectors in G, denoted by u_h and u_l, where u_h is called the *horizontal direction*, which is parallel to $B^s \times \{0\}$ and u_l is called the *slope direction*, which is perpendicular to u_h (see Figure 4). Obviously, u_l has the largest angle with $B^s \times \{0\}$ among all vectors in G. It is very easy to check that under the iterations of g, the horizontal (slope) direction is still sent to the horizontal (slope) direction.

LEMMA 4.5.1.   *Suppose G is a two-dimensional plane with* Angle $(G, B^s \times \{0\}) = \gamma$. *Let u be any vector in G with* Angle $(u, u_h) = \alpha$. *Then*

$$\text{Angle } (u, B^s \times \{0\}) = \arcsin(\sin \gamma \sin \alpha).$$

PROOF.   Indeed, set $G = \text{span}\{AB, AC\}$, $B^s \times \{0\} = \text{span}\{AD, AE\}$, $u_h = BC$, $u_l = AB$, $u = AC$ and $|AB| = 1$ as in Figure 4. Then, by assumption, $|CD| = |BE| = \sin \gamma$ and $|AC| = \dfrac{1}{\sin \alpha}$. Hence in $\triangle ACD$,

$$\sin \text{Angle } (u, B^s \times \{0\}) = \sin \text{Angle } (AC, AD) = \frac{|CD|}{|AC|} = \sin \gamma \sin \alpha,$$

which gives the conclusion immediately.

LEMMA 4.5.2.   *Let $u_h(q)$ be the horizontal direction of $E^s(q)$ and $\xi(q)$ be the unit vector in the direction of the orthogonal projection of $E^u(q)$ into $E^s(q)$. Then, there exists $\theta > 0$ arbitrarily small such that*

$$\text{Angle } (u_h(q), \xi(q)) < \frac{\epsilon}{32cD},$$

where $c = \left| \dfrac{\mu^u}{\mu^s} \right|$ and $D \geq \sup\{\|Dg(x)\| + \|Dg^{-1}(x)\| : x \in M\}$ for any g near f. Recall that $\epsilon > 0$ is sufficiently small which was fixed at the end of Section 4.1.

PROOF. i) According to Lemma 4.4.2, when θ is small enough, both Angle $(E^s(q), B^s \times \{0\})$ and Angle $(E^u(q), T_x W^u(p))$ are higher order infinitesimals of θ (briefly, we write $o(\theta)$). Thus, if we denote the orthogonal projection of $T_x W^u(p)$ into $B^s \times \{0\}$ by ζ, we get Angle $(\xi(q), \zeta) = o(\theta)$. Note that by our construction in the beginning of Section 4.2, ζ does not depend on θ.

ii) On the other hand, when θ decreases to zero monotonically and continuously in $\mathbb{R}$, $[\kappa N] = [\kappa N(\theta)]$ increases to infinity monotonically and continuously in $\mathbb{N}$, where the topology of $\mathbb{N}$ is defined by the subspace topology of $\mathbb{R}$. This implies that when $[\kappa N]$ increases by one, $u_h(q)$ changes its direction in $E^s(q)$ by angle $\phi + o(\theta)$ where ϕ is the argument of the non-real contracting eigenvalue of p (By an arbitrarily small perturbation if necessary, we can always assume that ϕ irrational). Indeed, when θ is small, since $E^s(q)$ is $o(\theta)$-close to $g^{-[\kappa N]} T_{g^{-T}(x)} W^s(p)$, their horizontal directions are also $o(\theta)$-close. But the horizontal direction of $g^{-[\kappa N]} T_{g^{-T}(x)} W^s(p)$ moves by angle ϕ as $[\kappa N]$ increases by 1. Therefore, $u_h(q)$ moves in $E^s(q)$ by angle $\phi + o(\theta)$ when $[\kappa N]$ increases by 1.

Summarizing the above two aspects i) and ii), it follows that

$$\text{Angle }(u_h(q), \xi(q)) = o(\theta) + [\kappa N](\phi + o(\theta)) \pmod{2\pi}$$

when $\theta \to 0$. As a result, for $\epsilon > 0$ which has been fixed at the very beginning, by shrinking θ if necessary, we are allowed to take θ such that Angle $(u_h(q), \xi(q)) < \dfrac{\epsilon}{32cD}$ as desired. We need some explanation for the choice of θ. In fact, recall that $N \in \mathbb{R}$ is defined by (see Section 4.2)

$$\left| \frac{\mu^u}{\mu^s} \right|^N \theta = 1 \,.$$

Thus,

$$0 \le o(\theta)[\kappa N] \le o(\theta)(\kappa N + 1) = o(\theta)(\kappa A \log \theta^{-1} + 1) \,,$$

where $A = (\log |\frac{\mu^u}{\mu^s}|)^{-1} > 0$. Since

$$\lim_{\theta \to 0} o(\theta) \kappa A \log \theta^{-1} = 0 \,,$$

we obtain

$$\text{Angle }(u_h(q), \xi(q)) = o(\theta) + [\kappa N](\phi + o(\theta)) \pmod{2\pi}$$

$$= o(\theta) + [\kappa N]\phi \pmod{2\pi} \,.$$

As a result, for $\epsilon > 0$ which has been fixed at the very beginning, by shrinking θ if necessary, we are allowed to choose $[\kappa N]$ as some large integer such that

$$0 < [\kappa N]\phi \pmod{2\pi} < \frac{\epsilon}{32cD} \,.$$

This is because when $[\kappa N]$ increases by one, $[\kappa N]\phi \pmod{2\pi}$ moves on the unit circle as an irrational rotation, but for irrational rotations, every orbit is dense on the unit circle. This completes the proof of Lemma 4.5.2.

Let us point out that we can replace $\theta \in (0, \epsilon)$ if necessary such that the conclusions of Lemmas 4.3.1, 4.3.3, 4.4.3 and 4.5.2 are all satisfied.

The following lemma shows that by the action of $g^{[N]}$, The angle in Lemma 4.5.2 does not change a lot. More precisely, it keeps to be the same order as ϵ.

LEMMA 4.5.3.   Angle $(Dg^{[N]}(q)\xi(q), u_h(g^{[N]}(q))) \leq \dfrac{\epsilon}{2D}$.

PROOF.   Write $\xi(q) = (\mathbf{z}, y) \in E^s(q)$. By item (2) of Lemma 4.4.2,

$$\text{Angle}\,(E^s(q), B^s \times \{0\}) \leq 2\theta^{\kappa-1}.$$

Applying Lemma 4.5.1, we get

$$\text{Angle}\,(\xi(q), B^s \times \{0\}) \leq \arcsin\left(\sin(2\theta^{\kappa-1})\sin\left(\frac{\epsilon}{32cD}\right)\right).$$

Thus,

$$\left|\frac{y}{\mathbf{z}}\right| = \tan \text{Angle}\,(\xi(q), B^s \times \{0\}) \leq \frac{\epsilon\theta^{\kappa-1}}{8cD}.$$

Then $Dg^{[N]}(q)\xi(q) = ((\mu^s)^{[N]}\mathbf{z}, (\mu^u)^{[N]}y)$ whose V-slope is

$$\left|\frac{(\mu^u)^{[N]}y}{(\mu^s)^{[N]}\mathbf{z}}\right| \leq c\left|\frac{(\mu^u)^N y}{(\mu^s)^N \mathbf{z}}\right| = \frac{c}{\theta}\left|\frac{y}{\mathbf{z}}\right| \leq \frac{\epsilon\theta^{\kappa-2}}{8D}.$$

On the other hand, by Lemma 4.4.2 (2), we have Angle $(E^s(q), B^s \times \{0\}) \geq \frac{1}{2}\theta^{\kappa-1}$. Thus

$$\text{Angle}\,(E^s(g^{[N]}(q)), B^s \times \{0\}) \geq \left|\frac{\mu^u}{\mu^s}\right|^N \frac{\theta^{\kappa-1}}{2} = \frac{\theta^{\kappa-2}}{2}.$$

Applying Lemma 4.5.1 again, we obtain

$$\frac{\epsilon\theta^{\kappa-2}}{8D} \geq \tan \text{Angle}\,(Dg^{[N]}(q)\xi(q), B^s \times \{0\})$$

$$\geq \tan \circ \arcsin\left(\sin\left(\frac{\theta^{\kappa-2}}{2}\right)\sin \text{Angle}\,\left(Dg^{[N]}(q)\xi(q), u_h(g^{[N]}(q))\right)\right)$$

$$\geq \frac{\theta^{\kappa-2}}{4}\,\text{Angle}\,(Dg^{[N]}(q)\xi(q), u_h(g^{[N]}(q))).$$

That is, Angle $(Dg^{[N]}(q)\xi(q), u_h(g^{[N]}(q))) \leq \dfrac{\epsilon}{2D}$ as desired. Now, Lemma 4.5.3 is proved.

Recall that in Section 4.4, we divided the iterations of g from q to $g^{[\kappa N]}(q)$ into three parts. In the following, we will give perturbations of Dg in the central part of $\mathrm{orb}_g(q)$. All of them take place in E^s while there is no perturbation in E^u. We point out that in this process, the angle between E^s and E^u need not to be considered, see Remark 4.4.4.

STEP 1.   Write $\omega = \mathrm{Angle}\,(Dg^{[N]}(q)\xi(q), u_h(g^{[N]}(q)))$. Under some standard orthogonal coordinate chart of $E^s(g^{[N]}(q))$, the isotopic perturbation $\begin{pmatrix} \cos t\omega & -\sin t\omega \\ \sin t\omega & \cos t\omega \end{pmatrix} \circ$ $Dg|_{E^s}(g^{[N]}(q))$ of $Dg|_{E^s}(g^{[N]}(q))$ sends $Dg^{[N]}(q)\xi(q)$ into $u_h(g^{[N]}(q))$, where $t \in [0, 1]$. It can be easily verified as we did in the proof of Lemma 4.3.3 that the corresponding path of the first return map keeps to be hyperbolic for all $t \in [0, 1]$. Thus, by the estimation in the previous lemma, there exists an ϵ-perturbation G_1 of g, satisfying $DG_1^{[N]}(q)\xi(q) = u_h(G_1^{[N]}(q))$ and $q \in \pitchfork(p_{G_1})$. That is, $DG_1^{[N]}\xi(q)$ is exactly in the horizontal direction of $E^s(G_1^{[N]}(q))$.

STEP 2.   In each of the following $([\kappa N]-2[N])$ iterations, we will contract DG_1 in the slope direction of E^s by a factor $(1-\frac{\epsilon}{D})$. More precisely, for $i = [N]+1, \ldots, [\kappa N]-[N]$, under the coordinate chart $\{u_h(G_1^i(q)), u_l(G_1^i(q))\}$ of $E^s(G_1^i(q))$, we use

$$\begin{pmatrix} 1 & 0 \\ 0 & 1-\frac{\epsilon}{D} \end{pmatrix} \circ DG_1|_{E^s}(G_1^i(q))$$

to replace $DG_1|_{E^s}(G_1^i(q))$, and leave $DG_1|_{E^u}(G_1^i(q))$ unchanged. By the isotopic Franks' Lemma, we get an ϵ-perturbation G_2 of G_1 such that from $G_2^{[N]}(q)$ to $G_2^{[\kappa N]-[N]}(q)$, the slope direction is contracted by $(1-\frac{\epsilon}{D})^{[\kappa N]-2[N]}$. Actually, the slope direction is mapped to the slope direction, thus the $(1-\frac{\epsilon}{D})$-contraction in each iteration can be accumulated. Clearly, q is still homoclinically related to p_{G_2}. Finally, since $DG_1^{[N]}(q)\xi(q)$ is exactly the horizontal direction of $E^s(G_1^{[N]}(q))$ and the above contraction in the slope direction does not affect the horizontal direction, we see that $DG_2^{[\kappa N]-[N]}\xi(q)$ is still in the horizontal direction of $E^s(G_2^{[\kappa N]-[N]}(q))$.

In what follows, we replace the notation G_2 by g again. Let us summarize all the perturbations we did since Section 4.2. Starting from f with a homoclinic tangency, for any $\epsilon > 0$ fixed in advance, first, by an ϵ perturbation, a hyperbolic horseshoe Λ_H^θ was constructed. Then, we selected $q \in \Lambda_H^\theta$. Finally, after the above two steps, we complete all the perturbations, obtaining g with $\mathrm{dist}_{C^1}(f, g) < \epsilon + 2\epsilon = 3\epsilon$.

4.6.   Proof of Proposition 2.   Under the hypothesis of Proposition 2 with a fixed point p, for any $\gamma > 0$ and any neighbourhood U_p of p, take $\epsilon \in (0, \gamma/3)$, we apply the previous construction for this ϵ, obtaining a γ perturbation g of f, such that $g = f$ outside U_p. To finish the proof, it remains to check conditions (i)–(iv) of Proposition 1.
(i) $\lambda^n < \gamma$.

Indeed, in the previous section, we see that θ is selected in Lemma 4.5.2 to get Angle $(\xi(q), u_h(q)) < \dfrac{\epsilon}{32cD}$. But in fact, for $\lambda = \lambda(\epsilon) \in (0, 1)$ fixed, this θ can be taken arbitrarily small in $(0, \epsilon)$, in particular, satisfying $\lambda^n < \gamma$. Recall that $n = \pi(q) = [\kappa N] + T$ where $N = N(\theta)$ can be arbitrarily large as long as θ is selected small.

(ii) Angle $(Dg^n(q)\xi, \xi) > \sigma$ where σ does not depend on γ.

In fact, using the previous notations, by Step 2 in Section 4.5, $Dg^{[\kappa N]-[N]}(q)\xi(q)$ is exactly in the horizontal direction of $E^s(g^{[\kappa N]-[N]}(q))$, hence $Dg^{[\kappa N]}(q)\xi(q)$ remains to stay in the horizontal direction of $E^s(g^{[\kappa N]}(q))$. Then, Dg^T sends the slope direction of $E^s(g^{[\kappa N]}(q))$ close to $\xi(q) \in E^s(q)$ (this is because before the perturbation, the tangent direction of $f^{-T}(x)$ is sent to the tangent direction of x but $\theta > 0$ is selected very small), and the closeness only depends on T and a fixed neighbourhood of f, we obtain

$$\text{Angle}\,(Dg^{\pi(q)}(q)\xi(q), \xi(q)) \geq \frac{\pi}{4}C_T > 0\,,$$

where $C_T > 0$ is a constant that only depends on T and a fixed neighbourhood of f. This estimation holds for every $\gamma > 0$ small.

(iii) $\dfrac{\|Dg^n|_{E^s}(q)\|}{\lambda^n} < \gamma$.

Note that the expansion rate of Dg^T is bounded while that of $Dg^{[\kappa N]}$ varies exponentially fast when θ tends to zero. It is easy to see that for $\theta > 0$ small enough, among all directions in $E^s(q)$, the slope direction have the weakest contracting rate. Let u_l denote the unit vector in the slope direction of $E^s(q)$. By formula (10) in the proof of Lemma 4.3.3, the minimal expansion rate of $Dg^{-n}|_{E^s}(q)$ is larger than $\dfrac{1}{2D^T|\mu^u(\mu^s)^{\kappa-1}|^N}$ (recall that we have taken c_T equal to 1). Hence, the maximal expansion rate of $Dg^n|_{E^s}(q)$ can be bounded from above as follows:

$$\|Dg^n(q)u_l\| \leq 2D^T|\mu^u(\mu^s)^{\kappa-1}|^N \left(1 - \frac{\epsilon}{D}\right)^{[\kappa N]-2[N]}\,,$$

where $(1 - \frac{\epsilon}{D})^{[\kappa N]-2[N]}$ comes from the additional contraction given by perturbations on the center $[\kappa N] - 2[N]$ times iterations in Step 2 of the last subsection. Thus,

$$\frac{\|Dg^n|_{E^s}(q)\|}{\lambda^n} \leq (2D)^T \left|\frac{\mu^u(\mu^s)^{\kappa-1}(1 - \frac{\epsilon}{D})^{\kappa-2}}{\lambda^\kappa}\right|^N < \gamma\,.$$

The last inequality holds because according to (7) in Lemma 4.1.1, we have

$$\left|\frac{\mu^u(\mu^s)^{\kappa-1}(1 - \frac{\epsilon}{D})^{\kappa-2}}{\lambda^\kappa}\right| < 1\,,$$

hence we can choose θ sufficiently small in $(0, \epsilon)$ (As a result, N becomes large enough) such

that $\left|\dfrac{\mu^u(\mu^s)^{\kappa-1}(1-\frac{\epsilon}{D})^{\kappa-2}}{\lambda^\kappa}\right|^N < \dfrac{\gamma}{(2D)^T}.$

(iv) $\dfrac{\lambda^n \angle(E^s(q), E^u(q))}{\|Dg^n(q)\xi\|} < \gamma.$

By Lemma 4.4.2 (3),

$$\angle(E^u(q), E^s(q)) = \tan \text{Angle}\,(E^u(q), E^s(q)) \leq 4\theta\,.$$

On the other hand, since $Dg^{[\kappa N]}|_{E^s}(q)$ contracts the most in the horizontal direction, we have

$$\|Dg^n(q)\xi\| \geq D^{-T}|\mu^s|^{[\kappa N]}\,.$$

Therefore,

$$\dfrac{\lambda^n \angle(E^s(q), E^u(q))}{\|Dg^n\xi\|} \leq 4D^T \dfrac{\lambda^{[\kappa N]}\theta}{|\mu^s|^{[\kappa N]}} \leq \dfrac{4D^T}{|\mu^s|}\left|\dfrac{\lambda^\kappa}{\mu^u(\mu^s)^{\kappa-1}}\right|^N < \gamma\,.$$

The last inequality is obtained in a similar way as above: we only need to use (5) and choose θ small enough in $(0, \epsilon)$ (hence N is very large) such that $\left|\dfrac{\lambda^\kappa}{\mu^u(\mu^s)^{\kappa-1}}\right|^N < \dfrac{|\mu^s|\gamma}{4D^T}.$

REMARK 4.6.1.   By investigating the previous proof carefully, it is easy to see that we can require orb (q) spends a large proportion of its iterations in a small neighbourhood of p, where the norms of its derivatives and the inverse are close to that of f at p. More precisely, using the previous notations, for any neighbourhood $U_p \subset M$ of p, by decreasing θ if necessary, $\dfrac{[\kappa N]}{[\kappa N]+T}$ can be taken close to one as much as we want. This fact will be useful in the proof of Theorem A.

5.   Index change: Proof of Theorem A and Corollary B

First, we cite several lemmas for the proof. Relative definitions can be found in Section 2.

LEMMA 5.1 ([BCDG, Proposition 7.1]).   *For every $D > 1$, $\epsilon > 0$, and $d \geq 2$, there exists a constant $k = k(D, \epsilon, d)$ with the following property. Consider $f \in \text{Diff}^1(M)$, $\dim M = d$, such that the norms of Df and Df^{-1} are bounded by D from above, p is a periodic point of f with index i which satisfies $2 \leq i \leq d$ such that $H(p)$ is non-trivial and has no k-dominated splitting of dimension $(i-1)$, then there exists an ϵ-perturbation g of f and $q(g) \in \text{⋔}(p_g)$ such that $q(g)$ has a central contracting eigenvalue with multiplicity one and $W^{ss}_{i-1}(q(g)) \cap W^u(q(g)) \setminus \{q(g)\} \neq \emptyset.$*

LEMMA 5.2 ([PPV, Proposition 4.3] or [BCDG, Claim 8.3]).   *Let $\delta > 0$, $f \in \text{Diff}^1(M)$ and p be a periodic point of f with $\text{ind}\,(p) = i \geq 2$. Suppose*

- *there exists $q_1 \in \pitchfork(p)$ satisfying $|\lambda^{cs}(q_1)| > (1 - \delta)^{\pi(q_1)}$;*
- *there exists $q_2 \in \pitchfork(p)$ satisfying $W^{ss}_{i-1}(q_2) \cap W^u(q_2) \setminus \{q_2\} \neq \emptyset$.*

Then, there is an arbitrarily small C^1 perturbation g of f which has a periodic point $q(g) \in \pitchfork(p_g)$ such that $q(g)$ inherits both of the above properties of q_1 and q_2. That is,

- *$|\lambda^{cs}(q(g))| > (1 - \delta)^{\pi(q(g))}$;*
- *$W^{ss}_{i-1}(q(g)) \cap W^u(q(g)) \setminus \{q(g)\} \neq \emptyset$.*

LEMMA 5.3 (Connecting Lemma ([H, Theorem A])). *Let a_f and b_f be a pair of saddles of $f \in \mathrm{Diff}^1(M)$ such that there are sequences of points y_n and of natural numbers k_n satisfying:*

- *$y_n \to y \in W^u_{loc}(a_f)$ $(n \to \infty)$, $y \neq a_f$;   and*
- *$f^{k_n}(y_n) \to z \in W^s_{loc}(b_f)$ $(n \to \infty)$, $z \neq b_f$.*

Then there is a diffeomorphism g arbitrarily C^1-close to f such that $W^u(a_g)$ and $W^s(b_g)$ have a non-empty intersection. This point of intersection can be taken arbitrarily close to y by changing g. In particular, $W^s_{loc}(b_f)$ and $W^s_{loc}(b_g)$ can be replaced by $W^{ss}_{loc}(b_f)$ and $W^{ss}_{loc}(b_g)$, respectively.

To prove Theorem A, firstly, under the assumption of non-existence of dominated splittings of dimension one, we construct a strong homoclinic intersection using Lemma 5.1 and transport this strong homoclinic intersection to periodic points with weak contracting eigenvalues using Lemma 5.2. Secondly, we apply the Connecting Lemma (Lemma 5.3) to create a strong heteroclinic intersection. Thirdly, we perturb the heteroclinic cycle to a heterodimensional cycle. Finally, by stabilizing this heterodimensional cycle, we obtain periodic points of different indices in $H(p_g)$, by which robust homoclinic tangencies follow immediately.

PROOF OF THEOREM A.   Given any $a > 1$, let us fix constants b and δ_0 such that

$$1 < b < a \text{ and } 0 < \delta_0 < 1 - \frac{b}{a}.$$

Obviously, $\delta_0 \to 0$ as $a \to 1$. For any $\delta \in (0, \delta_0)$, take $\epsilon > 0$ satisfying $3\epsilon < \delta(a - \frac{b}{1-\delta})\|Df^{\pm}(p)\|$. Fix neighbourhoods $\mathcal{U} \subset \mathrm{Diff}^1(M)$ of f and $U \subset M$ of $\mathrm{orb}(p)$ such that $\|Dg^{\beta}(x)\| \leq b\|Df^{\pm}(p)\|$ for all $g \in \mathcal{U}$ and $x \in U$ where $\beta = \pm 1$. By Theorem C, there exist an ϵ-perturbation $g_1 \in \mathcal{U}$ of f and $r \in \pitchfork(p_{g_1})$ admitting contracting eigenvalue $\lambda^2(r)$ satisfying $|\lambda^2(r)| > (1 - \delta)^{\pi(r)}$. With an additional arbitrarily small perturbation and replace r if necessary, we can assume that $\lambda^2(r)$ is central contracting, having multiplicity one (see Remark 3.0.1). Moreover, by Remark 4.6.1, the orbit of r spends a large proportion (close to one as much as we want) of time in U. By continuity, there is a neighbourhood $\mathcal{W}_1 \subset \mathcal{U}$ of g_1 such that

(F1) For every $g \in \mathcal{W}_1$, we have $r_g \in \pitchfork(p_g)$ and $|\lambda^2(r_g)| > (1 - \delta)^{\pi(r_g)}$.

Shrinking $\mathcal{W}_1$ if necessary, we can always assume dist $_{C^1}(f, g) < 2\epsilon$ whenever $g \in \mathcal{W}_1$. On the other hand, since p_{g_1} has non-real contracting eigenvalue, $H(p_{g_1})$ does not have dominated splitting of dimension one. Applying Lemma 5.1 to $H(p_{g_1})$, we get a perturbation $g_2 \in \mathcal{W}_1$ of g_1 which admits a strong homoclinic intersection associated to some $s(g_2) \in \pitchfork(p_{g_2})$, that is

(F2) $W_1^{ss}(s(g_2)) \cap W^u(s(g_2)) \setminus \{s(g_2)\} \neq \emptyset$.

Recall that $W_1^{ss}(s(g_2))$ is the 1-strong stable manifold of $s(g_2)$ (see Section 2 for the definition of i-strong stable manifolds). Since $g_2 \in \mathcal{W}_1$, combining facts (F1) and (F2) above, we conclude by Lemma 5.2 that there exist $g_3 \in \mathcal{W}_1$ arbitrarily close to g_2 and $q(g_3) \in \pitchfork(p_{g_3})$ satisfying

- $|\lambda^2(q(g_3))| > (1 - \delta)^{\pi(q(g_3))}$;
- $\exists x \in W_1^{ss}(q(g_3)) \cap W^u(q(g_3)) \setminus \{q(g_3)\}$.

Note that $q(g_3)$ can also be taken such that its orbit spends a large proportion in U. Since $q(g_3) \neq x \in H(p_{g_3})$ and $H(p_{g_3})$ is transitive, using the Connecting Lemma, we obtain an arbitrarily small perturbation $g_4 \in \mathcal{W}_1$ of g_3 satisfying $W_1^{ss}(q_{g_4}) \cap W^u(p_{g_4}) \neq \emptyset$. Moreover, since $q(g_3)$ is homoclinically related to p_{g_3}, by robustness of transversal intersections, $W^s(p_{g_4}) \cap W^u(q_{g_4})$ remains non-empty. Now, we apply Lemma 2.6 to orb (q_{g_4}). For $l = 0, \ldots, \pi(q_{g_4}) - 1$ and $t \in [0, 1]$, let

- $A_{l,t} = ((1 - t) + t\lambda^{-1}) \circ Dg_4(g_4^l(q_{g_4}))$ if $g_4^l(q_{g_4}) \in U$;
- $A_{l,t} = Dg_4(g_4^l(q_{g_4}))$ if $g_4^l(q_{g_4}) \notin U$,

where $\lambda \in (0, 1)$ is selected satisfying

$$\max \left\{ |\lambda^1(q_{g_4})|^{\frac{1}{\pi(q_{g_4})}}, (1 - \delta) \right\} < \lambda < |\lambda^2(q_{g_4})|^{\frac{1}{\pi(q_{g_4})}}.$$

Here, λ^1 denote the other contracting eigenvalue of q_{g_4}. By this definition and Remark 4.6.1, $(1 - \delta)^{-1}$-perturbations of Dg_4 in the tangent spaces over points of orb (q_{g_4}) inside U are sufficient for index change. Slightly different from before, this time we will pay attention to the behaviour of one dimensional strong stable manifold under the perturbation. By the choice of λ, one can easily verify that $\prod_{l=1}^{\pi(q_{g_4})-1} A_{l,t}$ keeps having a 1-dimensional strong stable direction for all $t \in [0, 1]$ and its endpoint $\prod_{l=1}^{\pi(q_{g_4})-1} A_{l,1}$ is a hyperbolic matrix with index one. Moreover, we have:

$$\max_{\substack{l=0,\ldots,\pi(q_{g_4})-1 \\ t \in [0,1]}} \left\{ \|A_{l,t} - A_{l,0}\| \right\} < b\|Df^{\pm}(p)\| \left(\frac{1}{\lambda} - 1 \right) < \frac{b\delta}{1 - \delta} \|Df^{\pm}(p)\|.$$

Similar estimation also works for the inverse. Thus, by Lemma 2.6 we get a perturbation g_5 of g_4 such that:

- $\mathrm{dist}_{C^1}(g_5, g_4) < \dfrac{b\delta}{1-\delta}\|Df^{\pm}(p)\|$;
- q_{g_5} is periodic with $\mathrm{ind}(q_{g_5}) = \mathrm{ind}(q_{g_4}) - 1 = 2 - 1 = 1$;
- $W^u(p_{g_5}) \cap W^s(q_{g_5}) \neq \emptyset$ and $W^u(q_{g_5}) \cap W^s(p_{g_5}) \neq \emptyset$.

In particular, g_5 has a co-index one heterodimensional cycle associated to p_{g_5} and q_{g_5}. Noticing that $H(p_{g_5})$ is non-trivial, by Lemma 2.4 there exists g arbitrarily close to g_5, admitting robust heterodimensional cycle associated to transitive hyperbolic sets $\Gamma_g \ni p_g$ and $\Lambda_g \ni q_g$. By robustness, we are allowed to select g in the residual set $\mathcal{R}$ of Lemma 2.3 and satisfying $\mathrm{dist}(g_5, g) < \epsilon$. Therefore, by Lemma 2.1, $H(p_g) = C(p_g)$ which contains periodic points of index one and two. Moreover, note that p_g has complex contracting eigenvalues, $H(p_g)$ does not have dominated splittings of dimension one. Now, by applying Lemma 2.3 to $H(q_g)$, it follows that g exhibits a robust homoclinic tangency. Finally,

$$\mathrm{dist}_{C^1}(g, f) \leq \mathrm{dist}_{C^1}(g, g_5) + \mathrm{dist}_{C^1}(g_5, g_4) + \mathrm{dist}_{C^1}(g_4, f)$$

$$\leq \epsilon + \frac{b\delta}{1-\delta}\|Df^{\pm}(p)\| + 2\epsilon < a\delta\|Df^{\pm}(p)\|$$

as desired. This completes the proof of Theorem A.

PROOF OF COROLLARY B.   It suffices to notice that, in the proof of Theorem A, at the last moment, we obtain an $a\delta\|Df^{\pm}(p)\|$-perturbation g of f such that $H(p_g)$ contains periodic points of index one and two. Thus, under the additional assumption, we are allowed to apply Lemma 2.3 to $H(p_g)$, obtaining a robust homoclinic tangency associated to a hyperbolic set containing p_g.

References

[AS]   R. ABRAHAM and S. SMALE, Nongenericity of Ω-stability, *Global Analysis I* (Proceedings of Symposia in Pure Mathematics, 14) American Mathematical Society, Providence, RI, 1968. pp. 5–8.

[A]   M. ASAOKA, Hyperbolic sets exhibiting C^1-persistent homoclinic tangency for higher dimensions, Proc. Amer. Math. Soc. **136** (2008), 677–686.

[B]   C. BONATTI, Towards a global view of dynamical systems, for the C^1-topology, Ergod. Th. & Dynam. Sys. **31** (2011), 959–993.

[BB]   J. BOCHI and C. BONATTI, Perturbation on the Lyapunov spectra of periodic orbits, Proc. Lond. Math. Soc. (3)**105** (2012), 1–48.

[BC1]   C. BONATTI and S. CROVISIER, Recurrence and genericity, Invent. Math. **158** (2004), 33–104.

[BC2]   C. BONATTI and S. CROVISIER, Center manifolds for partially hyperbolic set without strong unstable connections, arXiv: 1401.2452 (2014).

[BCDG]   C. BONATTI, S. CROVISIER, L. J. DÍAZ and N. GOURMELON, Internal perturbations of homoclinic classes: non-domination, cycles, and self-replication, Ergod. Th. & Dynam. Sys. **33** (2013), 739–776.

[BCGP]   C. BONATTI, S. CROVISIER, N. GOURMELON and R. POTRIE, Tame dynamics and robust transitivity chain-recurrence classes versus homoclinic classes, Tran. Amer. Math. Soc. **366** (2014), 4849–4871.

[BD1]   C. BONATTI and L. J. DÍAZ, Robust heterodimensional cycles and C^1-generic dynamics, J. Inst. Math. Jussieu **73** (2008), 469–525.

[BD2]   C. BONATTI and L. J. DÍAZ, Fragile cycles, J. Diff. Eq. **252** (2012), 4176–4199.

[BD3]   C. BONATTI and L. J. DÍAZ, Abundance of C^1-robust homoclinic tangencies, Tran. Amer. Math. Soc. **364** (2012), 5111–5148.

[BDF]   C. BONATTI, L. J. DÍAZ and T. FISHER, Super-exponential growth of the number of periodic points inside homoclinic classes, Discrete Contin. Dyn. Syst. **20** (2008), 589–604.

[BDK]   C. BONATTI, L. J. DÍAZ and S. KIRIKI, Stabilization of heterodimensional cycles, Nonlinearity **25** (2012), 930–960.

[BDP]   C. BONATTI, L. J. DÍAZ and E. R. PUJALS, A C^1-generic dichotomy for diffeomorphisms: weak forms of hyperbolicity or infinitely many sink or sourses, Ann. of Math. **158** (2003), 355–418.

[BDPR]   C. BONATTI, L. J. DÍAZ, E. PUJALS and J. ROCHA, Robustly transitive sets and heterodimensional cycles, Geometric methods in dynamics I Astérisque **286** (2003), 187–222.

[BS]   M. BRIN and G. STUCK, *Introduction to dynamical systems*, Cambridge University Press, 2002.

[F]   J. FRANKS, Necessary conditions for stability of diffeomorphisms, Trans. Amer. Math. Soc. **158** (1971), 301–308.

[G1]   N. GOURMELON, A Franks' Lemma that preserves invariant manifolds, arXiv: 0912.1121 (2009).

[G2]   N. GOURMELON, An isotopic perturbation lemma along periodic orbits, arXiv: 1212.6638 (2012).

[GY]   S. GAN and D. YANG, Expansive homoclinic classes, Nonlinearity **22** (2009), 729–733.

[H]   S. HAYASHI, Connecting invariant manifolds and the solution of the C^1 stability and Ω-stability conjectures for flows, Ann. of Math. **116** (1997), 81–137.

[KY]   J. KENNEDY and J. YORKE, Topological horseshoes, Trans. Amer. Math. Soc. **353** (2001), 2513–2530.

[LVY]   G. LIAO, M. VIANA and J. YANG, The entropy conjecture for diffeomorphisms away from tangencies, J. Eur. Math. Soc. **15** (2013), 2043–2060.

[M]   R. MAÑÉ, An ergodic closing lemma, Ann. of Math. **145** (1982), 503–540.

[Mo]   C. G. MOREIRA, There are no C^1-stable intersections of regular Cantor sets, Acta Math. **206** (2011), 311–323.

[N1]   S. NEWHOUSE, Diffeomorphisms with infinitely many sinks, Topology **13** (1974), 9–18.

[N2]   S. NEWHOUSE, The abundance of wild hyperbolic sets and nonsmooth stable sets for diffeomorphisms, Publ. Math. Inst. Hautes. Etudes Sci. **50** (1979), 101–151.

[PPV]   M. J. PACIFICO, E. R. PUJALS and J. L. VIEITEZ, Robustly expansive homoclinic classes, Ergod. Th. & Dynam. Sys. **25** (2005), 271–300.

[PV]   M. PALIS and M. VIANA, High dimension diffeomorphisms displaying infinitely many periodic attractors, Ann. of Math. **140** (1994), 207–250.

[PdM]   J. PALIS and W. DE MELO, *Geometric theory of dynamical systems*, Springer-Verlag, 1982.

[PT]   J. PALIS and F. TAKENS, *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations*, Cambridge University Press, 1993.

[S]   K. SHINOHARA, On the index problem of C^1-generic wild homoclinic classes in dimension three, Discrete Contin. Dyn. Syst. **31** (2011), 913–940.

[W]   L. WEN, Homoclinic tangencies and dominated splittings, Nonlinearity **15** (2002), 1445–1469.

Present Address:
School of Mathematics and Statistics,
Huazhong University of Science and Technology,
Wuhan 430074, P. R. China.
e-mail: lixl@hust.edu.cn

Graduate School of Mathematical Sciences,
The University of Tokyo,
3–8–1 Komaba, Tokyo, Japan.
e-mail: lixl@ms.u-tokyo.ac.jp