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Orbital Integrals on Unitary Hyperbolic Spaces Over $\mathfrak{p}$-adic Fields

Dedicated to Professor Ken-ichi SHINODA

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Abstract. For a given étale quadratic algebra E over a $\mathfrak{p}$-adic field F, we establish a transfer of unramified test functions on the symmetric space $\mathrm{GL}(2, F)\backslash\mathrm{GL}(2, E)$ to those on a unitary hyperbolic space so that the orbital integrals match. This is an important step toward a comparison of relative trace formulas of these symmetric spaces, which would yield an example of a non-tempered analogue of a refined global Gross-Prasad conjecture.

1. Introduction

1.1. Motivation. To explain the motivation of our study in this paper, we start with a global setting. Let K/k be a CM-extension of a totally real number field k and $\mathfrak{o}_K$, $\mathfrak{o}_k$ the integer rings of K and k, respectively. The adele rings of K and k are denoted by $\mathbb{A}_K$ and $\mathbb{A}_k$, respectively. Let $(V, \mathbf{h})$ be a non-degenerate hermitian space over K of dimension $m \geqslant 3$ and $G = U(\mathbf{h})$ its unitary group viewed as an algebraic group over k. Let $\ell_0 \in V$ be a unit vector (i.e., $\mathbf{h}(\ell_0, \ell_0) = 1$) and H the stabilizer in G of the one dimensional subspace $K\ell_0$; H is the fixed point set of a k-involution of G, and is isomorphic to the direct product of the unitary group of $\ell_0^{\perp}$ and the k-anisotropic torus K^1 of the norm 1 elements in K. A cuspidal automorphic representation π of $G(\mathbb{A}_k)$ is called *H-distinguished* if π contains a cusp form φ on $G(\mathbb{A}_k)$ such that the H-period integral $\mathcal{P}^H(\varphi) = \int_{H(k)\backslash H(\mathbb{A}_k)} \varphi(h)\mathrm{d}h$ is non-zero, where $\mathrm{d}h$ is the Tamagawa measure on $H(\mathbb{A}_k)$. The totality of H-distinguished cuspidal automorphic representations of $G(\mathbb{A}_k)$ is denoted by $\Pi_{\mathrm{cusp}}^H(G)$. The automorphic forms belonging to such representations of $G(\mathbb{A}_k)$ have been studied in the context of the cycle geometry of the associated unitary Shimura varieties by Kudla and Milson ([15], [16]), especially when the variety is compact; the special cycles arising from the embedding $H \hookrightarrow G$ is an interesting and rich source of H-distinguished automorphic representations. From those works, it is known that the space of harmonic Poincaré dual forms of special cycles is exhausted by the image of the theta lifting from a certain class of holomorphic cusp forms on $U(2)_{K/k}$, the quasi-split unitary group of degree 2. The lifting is studied by means of the Poincaré series dual to the

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Fourier coefficients, which are additive objects in nature. A multiplicative aspect of their works in the context of the functorial transfer and Arthur's classification of non-tempered automorphic spectrum is pursued in a recent work by Burgeron, Milson and Moeglin [2]. Invoking a characterization of the image of the unstable base change from $U(2)_{K/k}$ to $\mathrm{R}_{K/k}(\mathrm{GL}(2))$ in terms of non-vanishing of $\mathrm{GL}(2)_k$-periods (Flicker [5]), we can expect a transfer map from the $\mathrm{GL}(2)_k$-distinguished cuspidal representations of $\mathrm{GL}(2, \mathbb{A}_K)$ to the H-distinguished spectrum $\Pi^H_{\mathrm{cusp}}(G)$. We shall realize the transfer by means of a comparison of two relative trace formulas on algebraic symmetric spaces $\mathcal{S} = \mathrm{GL}(2)_k\backslash\mathrm{R}_{K/k}(\mathrm{GL}(2))$ and $\mathbf{\Sigma} = H\backslash G$ following a common strategy initiated by Jacquet ([11], [13]). The aim of this paper is to construct a transfer of unramified test functions on $\mathcal{S}$ to those on $\mathbf{\Sigma}$ so that the orbital integrals match, which plays a pivotal role in the comparison. If completed, the comparison will yield a yet another example of a non-tempered analogue of the global Gross-Prasad conjecture refined by Ichino and Ikeda [9] and Harris [10]. For a different possible approach, we refer to Flicker [6].

Suppose G is k-anisotropic and fix a maximal $\mathfrak{o}_K$-lattice $\mathcal{L}$ in V. For any finite place v of k, set $K_v = K \otimes_k k_v$, $\mathfrak{o}_{K,v} = \mathfrak{o}_K \otimes_{\mathfrak{o}_k} \mathfrak{o}_{k,v}$ and $\mathcal{L}_v = \mathcal{L} \otimes_{\mathfrak{o}_K} \mathfrak{o}_{K,v}$. Then $\mathcal{U}_v = \mathrm{GL}_{\mathfrak{o}_{K,v}}(\mathcal{L}_v) \cap G(k_v)$ is a maximal compact subgroup of $G(k_v)$. Let $\phi = \otimes_v \phi_v$ be a decomposable smooth compactly supported function on $H(\mathbb{A}_k)\backslash G(\mathbb{A}_k)$ such that $\phi_v = \phi_v^\circ$ for almost all v, where ϕ_v° is the characteristic function of $H(k_v)\,\mathcal{U}_v$. The relative trace formula for $\mathbf{\Sigma}$ is an identity equating two expressions, referred as the spectral side and the geometric side, obtained by computing the H-period integral $\mathcal{P}^H(K^{\mathbf{\Sigma}}_\phi)$ of the Poincaré series $K^{\mathbf{\Sigma}}_\phi(g) = \sum_{\gamma \in H(k)\backslash G(k)} \phi(\gamma g)$ with $g \in G(\mathbb{A}_k)$ in two different ways. By breaking the summation according to the $H(k)$-orbits in $H(k)\backslash G(k)$ ([11, §2]), we have the expression which provides us with the geometric side of the relative trace formula:

$$\mathcal{P}^H(K^{\mathbf{\Sigma}}_\phi) = \sum_{\gamma \in H(k)\backslash G(k)/H(k)} a^H(\gamma)\, J^{\mathbf{\Sigma}}_{\mathbb{A}}(\gamma; \phi)\,, \tag{1.1}$$

where $a^H(\gamma)$ is the Tamagawa number of the group $H_\gamma = H \cap \gamma^{-1} H \gamma$ and

$$J^{\mathbf{\Sigma}}_{\mathbb{A}}(\gamma; \phi) = \int_{H_\gamma(\mathbb{A}_k)\backslash H(\mathbb{A}_k)} \phi(\gamma h)\, \mathrm{d}O_\gamma(h) \tag{1.2}$$

is the orbital integral of ϕ with respect to the Tamagawa measure $\mathrm{d}O_\gamma$ on $H_\gamma(\mathbb{A}_k)\backslash H(\mathbb{A}_k)$.

Let T and Z be the diagonal split torus and the center of $\mathrm{GL}(2)$, respectively. Let η be an idele class character of $K^\times$ such that $\eta|\mathbb{A}_k^\times = \varepsilon_{K/k}^{m-1}$, where $\varepsilon_{K/k}$ is the quadratic idele class character of $k^\times$ corresponding to the field extension K/k by the class field theory. We define automorphic quasi-characters ω and Ω of $Z(\mathbb{A}_K)$ and $T(\mathbb{A}_K)$, respectively by

$$\omega\left(\left[\begin{smallmatrix} \tau & 0 \\ 0 & \tau \end{smallmatrix}\right]\right) = \eta^2(\tau)\,, \quad \Omega\left(\left[\begin{smallmatrix} \tau\alpha & 0 \\ 0 & \tau \end{smallmatrix}\right]\right) = \eta^{-2}(\tau)\,|\mathrm{N}_{K/k}(\alpha)|_{\mathbb{A}_k}^{(m-2)/2}\eta^{-1}(\alpha)\,.$$

A function f on $\mathrm{GL}(2, \mathbb{A}_k)\backslash\mathrm{GL}(2, \mathbb{A}_K)$ is said to have the central character ω if $f(zg) = \omega(z) f(g)$ for all $z \in Z(\mathbb{A}_K)$. Let $f = \otimes_v f_v$ be a decomposable smooth function on $\mathrm{GL}(2, \mathbb{A}_k)\backslash\mathrm{GL}(2, \mathbb{A}_K)$ of the central character ω such that $f_v = f_v^\circ$ for almost all v, where

f_v° is the function on $\mathrm{GL}(2, K_v)$ defined by setting $f_v^\circ(zhk) = \omega_v(z)$ for $z \in Z(K_v)$, $h \in \mathrm{GL}(2, k_v)$ and $k \in \mathrm{GL}(2, \mathfrak{o}_{K,v})$ and $f_v^\circ(g) = 0$ for $g \notin Z(K_v)\mathrm{GL}(2, k_v)\mathrm{GL}(2, \mathfrak{o}_{K,v})$. The relative trace formula for $\mathcal{S}$ to be compared with the previous one is constructed by computing the Hecke zeta integral $Z(\varphi; \Omega) = \int_{T(K)Z(\mathbb{A}_K)\backslash T(\mathbb{A}_K)} \varphi(t)\Omega(t)\mathrm{d}t$ of the Poincaré series $K_f^{\mathcal{S}}(g) = \sum_{\gamma \in Z(K)\mathrm{GL}(2,k)\backslash \mathrm{GL}(2,K)} f(\gamma g)$ with $g \in \mathrm{GL}(2, \mathbb{A}_K)$. Setting aside the issue of convergence of integrals, we formally proceed as above to have the expression, the geometric side of the relative trace formula:

$$Z(K_f^{\mathcal{S}}; \Omega) = \sum_{\delta \in \mathrm{GL}(2,k)\backslash \mathrm{GL}(2,K)/T(K)} a^T(\delta)\, J_{\mathbb{A}}^{\mathcal{S}}(\delta, f; \Omega)\,, \tag{1.3}$$

where $a^T(\delta)$ denotes the integral of Ω on $Z(\mathbb{A}_K)T_\delta(K)\backslash T_\delta(\mathbb{A}_K)$ with $T_\delta(K) = T(K) \cap \delta^{-1}\mathrm{GL}(2, k)\delta$ and $J_{\mathbb{A}}^{\mathcal{S}}(\delta, f; \Omega)$ is the orbital integral

$$J_{\mathbb{A}}^{\mathcal{S}}(\delta, f; \Omega) = \int_{T_\delta(\mathbb{A}_K)Z(\mathbb{A}_K)\backslash T(\mathbb{A}_K)} f(\delta t)\Omega(t)\, \mathrm{d}o_\delta(t)\,, \tag{1.4}$$

where $\mathrm{d}o_\delta$ is the Tamagawa measure on $T_\delta(\mathbb{A}_K)\backslash T(\mathbb{A}_K)$. From [12], we know that, except for a finite number of singular ones, any $T(K)$-orbit in $\mathrm{GL}(2, k)\backslash \mathrm{GL}(2, K)$ is of the form $\delta_b = \mathrm{GL}(2, k) \begin{bmatrix} 1 & \sqrt{\theta}\frac{b+1}{b-1} \\ 1 & \sqrt{\theta} \end{bmatrix}^{-1} T(K)$ with $b \in (K^\times - K^1)/K^1$, where we fix $\theta \in k^\times$ such that $K = k[\sqrt{\theta}]$; those cosets δ_b are referred as regular. It is known that the obvious inclusion $Z(k) \subset T_\delta(K)$ becomes an equality if δ is regular. We say that an $H(k)$-orbit $\gamma \in (H(k)\backslash G(k))/H(k)$ is regular if $\mathrm{N}_{K/k}(\mathbf{h}(\gamma^{-1}\ell_0, \ell_0)) \in F^\times - \{1\}$. A pair of regular orbits (δ_b, γ) from $(\mathrm{GL}(2, k)\backslash \mathrm{GL}(2, K)/T(K)) \times (H(k)\backslash G(k)/H(k))$ is said to be a matching pair if $\mathrm{N}_{K/k}(\mathbf{h}(\gamma^{-1}\ell_0, \ell_0)) = \mathrm{N}_{K/k}(b)$. Moreover, we say that test functions f and ϕ as above match if $J_{\mathbb{A}}^{\mathbf{\Sigma}}(\gamma; \phi) = J_{\mathbb{A}}^{\mathcal{S}}(\delta_b; f, \Omega)$ for all matching pairs of regular orbits (δ_b, γ). Since the singular orbital integrals are retrieved from the germ expansions of the orbital integrals of regular orbits, to match up the geometric sides of two relative trace formulas (1.1) and (1.3), one first needs to show the existence of abundant matching pairs (ϕ, f) of test functions, for which we should get the wanted coincidence of the spectral sides as a consequence. Since the orbital integrals (1.2) and (1.4) are Euler products over all places of k, the search for matching pairs of test functions boils down to a bunch of similar local tasks. For almost all finite places v, where all the global objects we start with are unramified, we accomplish those local tasks (so called the ‘fundamental lemma’) in this paper. Based on the results in this paper, we shall discuss the full comparison of the geometric sides of the relative trace formulas in the forthcoming works ([22], [23]).

1.2. Description of main result. In this article throughout, we consider only those non-archimedean local fields of characteristic 0 whose residual characteristic is different from 2. For such a field F, we use the following notation and convention. The maximal order of F is denoted by $\mathfrak{o}_F$. We fix a prime element ϖ of F once and for all. Let q denote the order of the residue field of F. The Haar measure $\mathrm{d}x$ on F is supposed to be normalized so that $\mathfrak{o}_F$

has volume 1; $\mathrm{d}x$ is self-dual with respect to the duality considered by an additive character $\psi : F \to \mathbb{C}^1$ such that $\psi|\mathfrak{o}_F = 1$ and $\psi|\varpi^{-1}\mathfrak{o}_F \neq 1$, where $\mathbb{C}^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$. The modulus function of F is denoted by $|\,|_F$.

Let $\mathbf{K}^F = \mathrm{GL}(2, \mathfrak{o}_F)$. We fix a Haar measure on $\mathrm{GL}(2, F)$ such that $\mathbf{K}^F$ has measure 1. Let $\mathcal{H}^{\mathrm{ur}}(\mathrm{GL}(2, F))$ denote the Hecke algebra of the pair $(\mathrm{GL}(2, F), \mathbf{K}^F)$, which is defined to be the convolution algebra of finite $\mathbb{C}$-linear combinations of characteristic functions of $\mathbf{K}^F$-double cosets in $\mathrm{GL}(2, F)$. Set $\mathbb{X}_F = \mathbb{C}/2\pi\sqrt{-1}(\log q)^{-1}\mathbb{Z}$.

Let $E = F[X]/(X^2 - \theta)$ with $\theta \in \mathfrak{o}_F^\times$. We have $E = F[\sqrt{\theta}]$ with $\sqrt{\theta}$ the residue class of the monomial X. The F-algebra E is an unramified quadratic extension or is isomorphic to $F \oplus F$ according to $X^2 - \theta$ is F-irreducible or not. The non-trivial F-automorphism of E is denoted by $x \mapsto \bar{x}$. For $\alpha \in E$, we set $|\alpha|_E = |\mathrm{N}\alpha|_F$, where $\mathrm{N}\alpha = \alpha\bar{\alpha}$ denotes the norm of α. Let $E^1 = \{\alpha \in E \,|\, \mathrm{N}\alpha = 1\}$ be the subgroup of $E^\times$ of norm 1 elements. The maximal order of E is $\mathfrak{o}_E = \mathfrak{o}_F + \mathfrak{o}_F\sqrt{\theta}$. Let us define q_E, ε_E and $\zeta_E(\nu)$ by the following table.

$x^2 - \theta$	q_E	ε_E	$\zeta_E(\nu)$
F-irreducible	q^2	-1	$(1 - q^{-2\nu})^{-1}$
F-reducible	q	$+1$	$(1 - q^{-\nu})^{-2}$

Set

$$Q_{E,m} = 1 - \varepsilon_E^{m-1} q^{-(m-1)}, \quad \mathbb{X}_E = \mathbb{C}/2\pi\sqrt{-1}(\log q_E)^{-1}\mathbb{Z}.$$

1.2.1. The $\mathrm{GL}(2)$-side.

Let Z denote the center of $\mathrm{GL}(2)$. Given an unramified unitary character ω of $Z(E)$ trivial on $Z(F)$, let $C_{\mathrm{c}}(Z(E)\mathrm{GL}(2, F)\backslash\mathrm{GL}(2, E)/\mathbf{K}^E, \omega)$ denote the space of all those complex-valued functions f on $\mathrm{GL}(2, E)$ with compact support modulo $Z(E)\mathrm{GL}(2, F)$ satisfying the equivariance $f(zhgk) = \omega(z)f(g)$ for $(z, h, k) \in Z(E) \times \mathrm{GL}(2, F) \times \mathbf{K}^E$.

For any $b \in E^\times - E^1$ and an unramified quasi-character $\xi : E^\times \to \mathbb{C}^\times$, we define

$$\mathbb{J}(b; f, \xi) = \int_{E^\times} f\left(\begin{bmatrix} 1 & \sqrt{\theta}\frac{b+1}{b-1} \\ 1 & \sqrt{\theta} \end{bmatrix}^{-1} \begin{bmatrix} \tau & 0 \\ 0 & 1 \end{bmatrix}\right) \xi(\tau)\, \mathrm{d}^\times\tau\,, \tag{1.5}$$
$$f \in C_{\mathrm{c}}(Z(E)\mathrm{GL}(2, F)\backslash\mathrm{GL}(2, E)/\mathbf{K}^E, \omega)\,,$$

where $\mathrm{d}^\times\tau$ is the Haar measure on $E^\times$ such that $\mathrm{vol}(\mathfrak{o}_E^\times) = 1$.

1.2.2. The unitary side.

Let V be a free E-module of rank $m \geqslant 4$ and $\mathbf{h} : V \times V \to E$ a non-degenerate hermitian form on V (see [20]). For $x \in V$, we set $\mathbf{h}[x] = \mathbf{h}(x, x)$. Let $\mathcal{L}$ be a unimodular $\mathfrak{o}_E$-lattice in V, i.e., $\mathcal{L}$ is a free $\mathfrak{o}_F$-module stable by $\mathfrak{o}_E$-multiplication such that $\{x \in V \,|\, \mathbf{h}(x, \mathcal{L}) \subset \mathfrak{o}_E\}$ coincides with $\mathcal{L}$. Let

$$G = U(\mathbf{h}) = \{g \in \mathrm{GL}(V) \,|\, \mathbf{h}(gx, gy) = \mathbf{h}(x, y) \text{ for all } x,\ y \in V\}$$

be the unitary group of $\mathbf{h}$; then $\mathcal{U} = G \cap \mathrm{GL}_{\mathfrak{o}_E}(\mathcal{L})$ is a maximal compact subgroup of G. We fix a vector $\ell_0 \in \mathcal{L}$ such that $\mathbf{h}[\ell_0] = 1$ once and for all, and let H be the stabilizer in G of the rank one submodule $E\ell_0$. For $\gamma \in G - H$, set

$$b(\gamma) = \mathbf{h}(\gamma^{-1}\ell_0, \ell_0)\,, \quad \ell_0^\gamma = \gamma^{-1}\ell_0 - b(\gamma)\,\ell_0\,, \quad \Delta_\gamma = \mathbf{h}[\ell_0^\gamma]\,.$$

A simple computation reveals that the vector ℓ_0^γ is orthogonal to ℓ_0 and $\Delta_\gamma = 1 - \mathrm{N}b(\gamma)$. Let $C_{\mathrm{c}}(H\backslash G/\mathcal{U})$ be the space of all the finite $\mathbb{C}$-linear combinations of characteristic functions of $(H, \mathcal{U})$-double cosets in G. For $\gamma \in G - H$ such that $\mathrm{N}b(\gamma) \neq 0, 1$, we consider the integral

$$\mathbb{J}_{\mathbf{h}}^{\ell_0}(\gamma; f) = \int_{H\cap\gamma^{-1}H\gamma\backslash H} f(\gamma h)\,\mathrm{d}O_\gamma(h)\,, \quad f \in C_{\mathrm{c}}(H\backslash G/\mathcal{U})\,, \tag{1.6}$$

where $\mathrm{d}O_\gamma$ is the H-invariant measure on $H \cap \gamma^{-1}H\gamma\backslash H$ defined in §4.4.

1.2.3. The statement of the main result.

Let $\eta : E^\times \to \mathbb{C}^1$ be an unramified character of $E^\times$ which extends the character $a \mapsto \varepsilon_E^{(m-1)\,\mathrm{ord}_F(a)}$ of $F^\times$. Then we define unramified quasi-characters $\xi : E^\times \to \mathbb{C}^\times$ and $\omega : Z(E) \to \mathbb{C}^\times$ by the formulas

$$\xi(\tau) = |\mathrm{N}(\tau)|_F^{(m-2)/2}\eta^{-1}(\tau)\,, \quad \omega\left(\left[\begin{smallmatrix}\tau & 0\\ 0 & \tau\end{smallmatrix}\right]\right) = \eta^2(\tau)\,, \quad \tau \in E^\times\,.$$

We note that ω is trivial on $Z(F)$.

Let $\mathcal{A}$ denote the space of Laurent polynomials in $z = q_E^{-s}$ invariant by $z \mapsto z^{-1}$. There exists a unique involutive $\mathbb{C}$-algebra automorphism ι of $\mathcal{A}$ such that $\iota(z) = \varepsilon_E^{m-1}z$. We define the transfer map

$$\mathrm{Trans}_\eta : C_{\mathrm{c}}(Z(E)\mathrm{GL}(2, F)\backslash\mathrm{GL}(2, E)/\mathbf{K}^E; \omega) \longrightarrow C_{\mathrm{c}}(H\backslash G/\mathcal{U})$$

as the composite $\mathcal{F}_{\mathbf{h}}^* \circ \iota \circ \mathcal{F}$, where $\mathcal{F} : C_{\mathrm{c}}(Z(E)\mathrm{GL}(2, F)\backslash\mathrm{GL}(2, E)/\mathbf{K}^E; \omega) \to \mathcal{A}$ is the Fourier transform on $\mathrm{GL}(2, E)$ to be defined in §2.2 and §3.2, and $\mathcal{F}_{\mathbf{h}}^* : \mathcal{A} \longrightarrow C_{\mathrm{c}}(H\backslash G/\mathcal{U})$ is the inverse Fourier transform on G to be defined in §4.3. Our main theorem is the following.

THEOREM 1.1. (1) *Let us define a function $f^\circ : \mathrm{GL}(2, E) \to \mathbb{C}$ by setting $f^\circ(zhk) = \omega(z)$ for all $(z, h, k) \in Z(E) \times \mathrm{GL}(2, F) \times \mathbf{K}^E$ and $f^\circ(g) = 0$ for all $g \notin Z(E)\mathrm{GL}(2, F)\mathbf{K}^E$. Let $\phi^\circ : G \to \mathbb{C}$ be the characteristic function of $H\mathcal{U}$ on G. Then*

$$\mathrm{Trans}_\eta(f^\circ) = \phi^\circ.$$

(2) *Let $b \in E^\times - E^1$. Let $\gamma \in G - H$ be such that $\mathrm{N}(\mathbf{h}(\gamma^{-1}\ell_0, \ell_0)) \in F^\times - \{1\}$. The integrals* (1.5) *and* (1.6) *converge absolutely. If $\mathrm{N}b = \mathrm{N}(\mathbf{h}(\gamma^{-1}\ell_0, \ell_0))$, then we have*

$$\xi(b-1)\,\mathbb{J}(b; f, \xi) = (1 - \varepsilon_E^{m-1}q^{-(m-1)})^{-1}\,\mathbb{J}_{\mathbf{h}}^{\ell_0}(\gamma; \mathrm{Trans}_\eta(f))$$

for all $f \in C_{\mathrm{c}}(Z(E)\mathrm{GL}(2, F)\backslash\mathrm{GL}(2, E)/\mathbf{K}^E; \omega)$,

1.3. Organization of paper. To prove Theorem 1.1, we compute explicitly the orbital integrals (1.5) and (1.6). Our basic strategy for that is a thorough use of the Fourier inversion formula for a symmetric space; the inversion formula is written by a contour integral of the Fourier transform of a test function on the symmetric space along the purely imaginary locus $\mathbb{X}_E^0$ of $\mathbb{X}_E$. We perform the orbital integral computation for the kernel function (the "Green function") which appears in the inversion formula when we shift the contour $\mathbb{X}_E^0$ to $\mathbb{X}_E^\delta = \{s \in \mathbb{X}_E \,|\, \mathrm{Re}\, s = \delta\}$ with a positive δ. The first two sections §2 and §3 after introduction are devoted to the analysis of the integral (1.5). We discuss the inert case (i.e., E is a field) in §2 and the split case (i.e., $E \cong F \oplus F$) in §3. In §2, after recalling the basic facts on harmonic analysis of $\mathrm{GL}(2, F)\backslash\mathrm{GL}(2, E)$ following [13], we introduce the Green function, denoted by $\Psi_s^{\mathcal{S}}$, to write the spherical function explicitly as a linear combination of the Green functions $\Psi_s^{\mathcal{S}}$ and $\Psi_{-s}^{\mathcal{S}}$ as shown in (2.4). Then our main task is reduced to computations of the orbital integrals for the Green function $\Psi_s^{\mathcal{S}}$, whose details are given in the proof of Theorems 2.2. The same method is applied to the split case to prove Theorem 3.2 in §3. In §4, we solely work on the unitary group G and its symmetric space $H\backslash G$. For our purpose, we need an explicit formula of the spherical function and an explicit Fourier inversion formula for unramified functions on the hyperbolic space $H\backslash G$, which may be deducible from a general theory developed in [18], [19] for split groups and its expected extension to quasi-split groups. Due to a lack of a proper reference and for the later convenience, we include a rather thorough treatment for basic ingredients of harmonic analysis on $H\backslash G$ in §5 Appendix 1. We introduce the Green function Ψ_ν in §4.2 to write the spherical function as a linear combination of Ψ_ν and $\Psi_{-\nu}$. As in the case of GL(2), the problem is reduced to calculations of the orbital integrals of the Green function Ψ_ν, which are described in the proof of Theorems 4.8 and 4.9; in the course of the proof, we need a Cartan type decomposition (4.2) of G, which is proved in [8] when E is a field. In §6, we provide a proof of the decomposition (4.2) when E is not a field. In this paper, the assumption $m \geqslant 4$ is made to simplify the exposition in §5, despite we believe the final results are true even for $m = 3$.

NOTATION: In this paper throughout, $\mathbb{N}$ denotes the set of all non-negative integers and $\mathbb{N}^* = \mathbb{N} - \{0\}$. For a given condition P, $\delta(\mathrm{P})$ is 1 if P is true and is 0 if P is false. The identity matrix of degree 2 is denoted by 1_2.

2. Orbital integrals on $\mathrm{GL}(2, F)\backslash\mathrm{GL}(2, E)$: the inert case

In this section, we keep notation in §1.2 and suppose $E = F[\sqrt{\theta}]$ is a field; then ϖ is also a prime element of E and $q_E = q^2 = |\varpi|_E^{-1}$. We set $G = \mathrm{GL}(2)$ and B to be the Borel subgroup of G of upper triangular matrices in G. We let Z denote the center of G, and suppose $\omega = 1$ to prevent the space $C_c(Z(E)G(F)\backslash G(E), \omega)$ defined in §1.2.1 from being $\{0\}$ by a trivial reason. For any $\delta \in \mathbb{R}$, put $\mathbb{X}_E^\delta = \{s \in \mathbb{X}_E \,|\, \mathrm{Re}\, s = \delta\}$, which is mapped homeomorphically to the circle $|z| = q_E^\delta$ on the z-plane by the relation $z = q_E^s$.

2.1. Spherical functions. Let us recall what we need on spherical functions on the space of split hermitian forms of degree 2 ([13, §3]). Let H be the unitary similitude group of the split hermitian form $w = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. The F-points

$$H(F) = \{h \in G(E) \,|\, {}^t\bar{h}wh = \kappa(h)w \text{ for some } \kappa(h) \in F^\times\}$$

is a subgroup of $G(E)$ conjugate to $Z(E)G(F)$ by $\left[\begin{smallmatrix} \sqrt{\theta} & 0 \\ 0 & 1 \end{smallmatrix}\right]$, i.e.,

$$\left[\begin{smallmatrix} \sqrt{\theta} & 0 \\ 0 & 1 \end{smallmatrix}\right] Z(E)G(F) \left[\begin{smallmatrix} \sqrt{\theta} & 0 \\ 0 & 1 \end{smallmatrix}\right]^{-1} = H(F)\,. \tag{2.1}$$

The group $G(E)$ acts from the right on the F-vector space of hermitian matrices $\mathrm{Her}_2 = \{x \in \mathrm{M}_2(E) \,|\, {}^t\bar{x} = x\}$ by the rule

$$x \bullet g = {}^t\bar{g}xg\,, \quad g \in G(E),\ x \in \mathrm{Her}_2\,.$$

For $x \in \mathrm{Her}_2$, let $\langle x \rangle$ denote the $F^\times$-homothety class of x. By passing to the quotient, $G(E)$ acts on the projective space $\mathbb{P}(\mathrm{Her}_2) = \mathrm{Her}_2/F^\times$ naturally. The $G(E)$ orbit of $\langle w \rangle$ is $\mathcal{S}/F^\times$ with $\mathcal{S} = \{x \in \mathrm{Her}_2 \,|\, -\det x \in \mathrm{N}_{E/F}(E^\times)\}$ and the stabilizer of $\langle w \rangle$ coincides with $H(F)$. By the theory of elementary divisors, we have the disjoint decomposition

$$\mathcal{S} = \bigcup_{n=0}^{\infty} F^\times \left[\begin{smallmatrix} -\varpi^{2n} & 0 \\ 0 & 1 \end{smallmatrix}\right] \bullet \mathbf{K}^E\,.$$

Let $s \in \mathbb{X}_E$. From [13, §3.1], there exists a unique function $\Omega_s^{\mathcal{S}} : \mathcal{S} \to \mathbb{C}$ with the properties:

(a) $\Omega_s^{\mathcal{S}}(w) = 1$.
(b) It has the invariance $\Omega_s^{\mathcal{S}}(zx \bullet k) = \Omega_s^{\mathcal{S}}(x)$ for all $z \in F^\times$, $x \in \mathcal{S}$ and $k \in \mathbf{K}^E$.
(c) It satisfies the Hecke eigenequation

$$\int_{G(E)} \Omega_s^{\mathcal{S}}(x \bullet g)\phi(g)\mathrm{d}g = \hat{\phi}(s)\,\Omega_s^{\mathcal{S}}(x)\,, \quad x \in \mathcal{S},\ \phi \in \mathcal{H}^{\mathrm{ur}}(G(E))\,,$$

where $\hat{\phi}(s)$ denotes the Satake transform of ϕ on the spherical principal series $\mathrm{Ind}_{B(E)}^{G(E)}\big(|\,|_E^s \boxtimes |\,|_E^{-s}\big)$.

From the property (b), the spherical function $\Omega_s^{\mathcal{S}}$ is determined by its values for the diagonal matrix $\left[\begin{smallmatrix} -\varpi^{2n} & 0 \\ 0 & 1 \end{smallmatrix}\right]$ ($n \geqslant 0$), which are given by

$$\Omega_s^{\mathcal{S}}\left(\left[\begin{smallmatrix} -\varpi^{2n} & 0 \\ 0 & 1 \end{smallmatrix}\right]\right) = \frac{qX - 1}{(X+1)(q-1)}(Xq^{-1})^n + \frac{qX^{-1} - 1}{(X^{-1}+1)(q-1)}(X^{-1}q^{-1})^n \tag{2.2}$$

with $X = q_E^{-s}$. If we define $\Psi_s^{\mathcal{S}} : \mathcal{S} \to \mathbb{C}$ by setting

$$\Psi_s^{\mathcal{S}}\left(z\left[\begin{smallmatrix} -\varpi^{2n} & 0 \\ 0 & 1 \end{smallmatrix}\right] \bullet k\right) = q^{-s}(1 - q^{-2s-1})^{-1}\, q^{-(2s+1)n}\,, \quad n \in \mathbb{N},\ z \in F^\times,\ k \in \mathbf{K}^E\,, \tag{2.3}$$

then

$$\Omega_s^{\mathcal{S}} = \frac{(1-q^{-2s-1})(1-q^{2s-1})}{(1-q^{-1})(q^{-s}+q^{s})}(\Psi_s^{\mathcal{S}}+\Psi_{-s}^{\mathcal{S}})\,. \tag{2.4}$$

We admit an ambiguity of square root q^s of q^{2s} in $\Psi_s^{\mathcal{S}}$, which disappears in (2.4).

2.2. Spectral decomposition. We have a bijection from $Z(E)G(F)\backslash G(E)/\mathbf{K}^E$ onto the $\mathbf{K}^E$-orbits in $\mathcal{S}/F^\times$ by sending the double coset $Z(E)G(F)g\mathbf{K}^E$ to the class $F^\times w \bullet \left[\begin{smallmatrix}\sqrt{\theta} & 0\\ 0 & 1\end{smallmatrix}\right] g \left[\begin{smallmatrix}\sqrt{\theta} & 0\\ 0 & 1\end{smallmatrix}\right]^{-1}$. By this identification of spaces, a function $f : Z(E)G(F)\backslash G(E)/\mathbf{K}^E \to \mathbb{C}$ determines a $\mathbf{K}^E$-invariant and $F^\times$-invariant function on $\mathcal{S}$, which is denoted by $f^{\mathcal{S}}$; explicitly, $f^{\mathcal{S}}$ is defined by the relation

$$f^{\mathcal{S}}\left(w \bullet \left[\begin{smallmatrix}\sqrt{\theta} & 0\\ 0 & 1\end{smallmatrix}\right] g \left[\begin{smallmatrix}\sqrt{\theta} & 0\\ 0 & 1\end{smallmatrix}\right]^{-1}\right) = f(g)\,, \quad g \in G(E)\,. \tag{2.5}$$

In particular, we have functions Ω_s and Ψ_s on the group $G(E)$ corresponding to $\Omega_s^{\mathcal{S}}$ and $\Psi_s^{\mathcal{S}}$, respectively. From (b) and (2.1), the function Ω_s has the equivariance property

$$\Omega_s(zhgk) = \Omega_s(g)\,, \quad z \in Z(E)\,,\ \ h \in G(F)\,,\ \ g \in G(E)\,,\ \ k \in \mathbf{K}^E\,.$$

From (a) and (c), we have $\Omega_s(1_2) = 1$ and the Hecke eigenequation

$$R(\phi)\Omega_s = \hat{\phi}(s)\,\Omega_s\,, \quad \phi \in \mathcal{H}^{\mathrm{ur}}(G(E))\,,$$

where R means the right translation on $G(E)$. For a function $f \in C_{\mathrm{c}}(Z(E)G(F)\backslash G(E)/\mathbf{K}^E, \omega)$, we define its Fourier transform $\mathcal{F}f$ by setting

$$\mathcal{F}f(s) = \int_{Z(E)G(F)\backslash G(E)} \Omega_s(g) f(g)\,\mathrm{d}\dot{g}\,, \quad s \in \mathbb{X}_E\,, \tag{2.6}$$

where $\mathrm{d}\dot{g}$ is the right $G(E)$-invariant measure on $Z(E)G(F)\backslash G(E)$ such that the measure of the image of $\mathbf{K}^E$ by the canonical surjection $G(E) \to Z(E)G(F)\backslash G(E)$ equals 1. Since f is of finite support on $Z(E)G(F)\backslash G(E)/\mathbf{K}^E$, the integral reduces to a finite summation. Indeed, by the integration formula [13, p. 320], we have

$$\mathcal{F}f(s) = \sum_{n=0}^{N} \Omega_s^{\mathcal{S}}\left(\left[\begin{smallmatrix}-\varpi^{2n} & 0\\ 0 & 1\end{smallmatrix}\right]\right) f^{\mathcal{S}}\left(\left[\begin{smallmatrix}-\varpi^{2n} & 0\\ 0 & 1\end{smallmatrix}\right]\right)\gamma_n \quad \text{with } \gamma_n = \begin{cases} 1 & (n=0)\,,\\ q^{2n}(1-q^{-1}) & (n>0)\,,\end{cases}$$

where N is such that $f^{\mathcal{S}}\left(\left[\begin{smallmatrix}-\varpi^{2n} & 0\\ 0 & 1\end{smallmatrix}\right]\right) = 0$ for all $n > N$. By the formula (2.2), it is easily confirmed that the values of $\Omega_s^{\mathcal{S}}$ occurring in the formula belong to the space

$$\mathcal{A} = \{\alpha(s) \in \mathbb{C}[q_E^{s}, q_E^{-s}] \,|\, \alpha(s) = \alpha(-s)\}\,.$$

As a finite linear combination of such, $\mathcal{F}f(s)$ itself can be viewed as an element of $\mathcal{A}$. For $\alpha(s) \in \mathcal{A}$, we define

$$\mathcal{F}^*\alpha(g) = \int_{\mathbb{X}_E^0} \Omega_s(g)\alpha(s)\,\mathrm{d}\Lambda(s)\,, \quad g \in G(E)\,, \tag{2.7}$$

where $\mathbb{X}_E^0$ denotes the purely imaginary locus of $\mathbb{X}_E$ and $\mathrm{d}\Lambda(s)$ is the Radon measure on $\mathbb{X}_E^0$ given by

$$\mathrm{d}\Lambda(\sqrt{-1}y) = \frac{1-q^{-1}}{2\pi}\left|\frac{1+q_E^{-\sqrt{-1}y}}{1-q^{-1}q_E^{-\sqrt{-1}y}}\right|^2 (\log q)\,\mathrm{d}y\,.$$

Since $s \mapsto \Omega_s(g)$ is holomorphic on $\mathbb{X}_E^0$, the integral converges absolutely defining a function on $G(E)$ which obviously has the left $Z(E)G(F)$-invariance and the right $\mathbf{K}^E$-invariance.

Let $L^2(Z(E)G(F)\backslash G(E)/\mathbf{K}^E)$ be the Hilbert space completion of $C_{\mathrm{c}}(Z(E)G(F)\backslash G(E)/\mathbf{K}^E, \omega)$ with respect to the inner-product

$$(f\,|\,f_1)_{G(E)} = \int_{Z(E)G(F)\backslash G(E)} f(g)\overline{f_1(g)}\,\mathrm{d}g\,.$$

(We remind readers that ω is trivial.) Let $L^2(\mathbb{X}_E^0; \mathrm{d}\Lambda)$ be the L^2-space of the compact space $\mathbb{X}_E^0$ with the inner-product

$$(\alpha|\alpha_1)_{\mathbb{X}_E^0} = \int_{\mathbb{X}_E^0} \alpha(s)\overline{\alpha_1(s)}\,\mathrm{d}\Lambda(s)\,.$$

THEOREM 2.1. *The integrals* (2.6) *and* (2.7) *define linear bijections*

$$\mathcal{F} : C_{\mathrm{c}}(Z(E)G(F)\backslash G(E)/\mathbf{K}^E, \omega) \longrightarrow \mathcal{A}\,, \quad \mathcal{F}^* : \mathcal{A} \longrightarrow C_{\mathrm{c}}(Z(E)G(F)\backslash G(E)/\mathbf{K}^E, \omega)\,,$$

each of which inverts the other one. Moreover, $\mathcal{F}$ is extended to an isometry from $L^2(Z(E)G(F)\backslash G(E)/\mathbf{K}^E)$ onto $L^2(\mathbb{X}_E^0; \mathrm{d}\Lambda)$, whose inverse isometry extends $\mathcal{F}^*$. We have

$$(\mathcal{F}f|\alpha)_{\mathbb{X}_E^0} = (f|\mathcal{F}^*\alpha)_{G(E)}\,.$$

PROOF. We refer to §5.4. □

2.3. Orbital integrals. The following is the main result of this section.

THEOREM 2.2. *Let $b \in E^\times - E^1$ and ξ an unramified quasi-character of $E^\times$ such that $\xi(\varpi) \neq 1$. Then, for any $f \in C_{\mathrm{c}}(Z(E)\mathrm{GL}(2, F)\backslash \mathrm{GL}(2, E)/\mathbf{K}^E, \omega)$, the integral* (1.5) *converges absolutely and has the contour integral expression*

$$\mathbb{J}(b; f, \xi) = \frac{1}{\pi\sqrt{-1}} \int_{\mathbb{X}_E^\delta} \hat{\mathbb{J}}(b; s, \xi)\, \mathcal{F}f(s)\,\mathrm{d}\mu(s) \tag{2.8}$$

with $\mathrm{d}\mu(s) = (q^s + q^{-s})(\log q)\,\mathrm{d}s$, *where* $\delta > 0$ *is taken so that* $q^{-2\delta-1} < \min\{|\xi(\varpi)|, |\xi(\varpi)|^{-1}\}$ *and* $\hat{\mathbb{J}}(b; s, \xi)$ *is defined as follows. If* $\mathrm{ord}_E(b) \geqslant 0$, *then*

$$\hat{\mathbb{J}}(b; s, \xi) = \xi(b-1)^{-1} q^{-s} \frac{1 - \xi(\varpi(1-\mathrm{N}b)) - (\xi(\varpi) - \xi(1-\mathrm{N}b))q^{-2s-1}}{(1-\xi^{-1}(\varpi)q^{-2s-1})(1-\xi(\varpi)q^{-2s-1})(1-\xi(\varpi))}. \tag{2.9}$$

If $\mathrm{ord}_E(b) < 0$, *then*

$$\hat{\mathbb{J}}(b; s, \xi) = \xi(b-1)^{-1}\xi(b) q^{-s} |b|_E^{-s-1/2} \frac{1+q^{-2s-1}}{(1-\xi^{-1}(\varpi)q^{-2s-1})(1-\xi(\varpi)q^{-2s-1})}. \tag{2.10}$$

PROOF. Set $\alpha(s) = \mathcal{F}f(s)$. From Theorem 2.1, we have the first equality of

$$f(g) = \mathcal{F}^*\alpha(g) = \int_{\mathbb{X}_E^0} \alpha(s)\Omega_s(g)\mathrm{d}\Lambda(s) = \frac{1}{\pi\sqrt{-1}} \int_{\mathbb{X}_E^0} \alpha(s)\Psi_s(g)\,\mathrm{d}\mu(s),$$

where the last equality follows from (2.4) by the invariance $\alpha(s) = \alpha(-s)$. Since $\alpha(s)\Psi_s(g)\mathrm{d}\mu(s)$ is a well-defined holomorphic 1-form on the region $\mathrm{Re}(s) > 0$ of $\mathbb{X}_E$, we can shift the contour $\mathbb{X}_E^0$ rightward to $\mathbb{X}_E^\delta$ ($\delta > 0$) by Cauchy's theorem. Thus the integral (1.5) becomes

$$\mathbb{J}(b; f, \xi) = \int_{E^\times} \left\{ \frac{1}{\pi\sqrt{-1}} \int_{\mathbb{X}_E^\delta} \alpha(s)\Psi_s\left(\begin{bmatrix} 1 & \sqrt{\theta}\beta \\ 1 & \sqrt{\theta} \end{bmatrix}^{-1} \begin{bmatrix} \tau & 0 \\ 0 & 1 \end{bmatrix} \right) \mathrm{d}\mu(s) \right\} \xi(\tau)\mathrm{d}^\times\tau. \tag{2.11}$$

where we set $\beta = (b+1)/(b-1)$. We shall show that the integral

$$\hat{\mathbb{J}}(b; s, \xi) = \int_{E^\times} \Psi_s\left(\begin{bmatrix} 1 & \sqrt{\theta}\beta \\ 1 & \sqrt{\theta} \end{bmatrix}^{-1} \begin{bmatrix} \tau & 0 \\ 0 & 1 \end{bmatrix} \right) \xi(\tau)\mathrm{d}^\times\tau$$

is absolutely convergent for all s in the region $\mathrm{Re}(s) \gg 0$ and is evaluated as in the theorem. Then, due to Fubini's theorem, the formula (2.8) is obtained by changing the order of integrals in (2.11). By (2.5) and by $\begin{bmatrix} \sqrt{\theta} & 0 \\ 0 & 1 \end{bmatrix} \in \mathbf{K}^E$, we have

$$\begin{aligned} \Psi_s\left(\begin{bmatrix} 1 & \sqrt{\theta}\beta \\ 1 & \sqrt{\theta} \end{bmatrix}^{-1} \begin{bmatrix} \tau & 0 \\ 0 & 1 \end{bmatrix} \right) &= \Psi_s^{\mathcal{S}}\left(\begin{bmatrix} \bar{\tau} & 0 \\ 0 & 1 \end{bmatrix} {}^t\overline{\begin{bmatrix} 1 & \sqrt{\theta}\beta \\ 1 & \sqrt{\theta} \end{bmatrix}}^{-1} \begin{bmatrix} -\sqrt{\theta} & 0 \\ 0 & 1 \end{bmatrix} w \begin{bmatrix} \sqrt{\theta} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \sqrt{\theta}\beta \\ 1 & \sqrt{\theta} \end{bmatrix}^{-1} \begin{bmatrix} \tau & 0 \\ 0 & 1 \end{bmatrix} \right) \\ &= \Psi_s^{\mathcal{S}}\left(\mathrm{N}\left(\frac{\tau}{1-\beta} \right) \begin{bmatrix} 2 & -(1+\beta)/\tau \\ -(1+\bar{\beta})/\bar{\tau} & (\beta+\bar{\beta})/\mathrm{N}(\tau) \end{bmatrix} \right). \end{aligned}$$

Since $\Psi_s^{\mathcal{S}}$ is $F^\times$-invariant, the factor $\mathrm{N}(\tau/(1-\beta))$ in the argument can be omitted. Thus,

$$\begin{aligned} \hat{\mathbb{J}}(b; s, \xi) &= \int_{E^\times} \Psi_s^{\mathcal{S}}\left(\begin{bmatrix} 2 & -(1+\beta)/\tau \\ -(1+\bar{\beta})/\bar{\tau} & (\beta+\bar{\beta})/\mathrm{N}(\tau) \end{bmatrix} \right) \xi(\tau)\mathrm{d}^\times\tau \\ &= \int_{E^\times} \Psi_s^{\mathcal{S}}\left(2\begin{bmatrix} 1 & \frac{-b}{\tau(b-1)} \\ \frac{-\bar{b}}{\bar{\tau}(\bar{b}-1)} & \frac{\mathrm{N}b-1}{\mathrm{N}(\tau(b-1))} \end{bmatrix} \right) \xi(\tau)\mathrm{d}^\times\tau \\ &= \xi(b-1)^{-1}\xi(b) \int_{E^\times} \Psi_s^{\mathcal{S}}\left(\begin{bmatrix} 1 & \tau \\ \bar{\tau} & (1-\mathrm{N}b^{-1})\mathrm{N}(\tau) \end{bmatrix} \right) \xi^{-1}(\tau)\mathrm{d}^\times\tau, \end{aligned}$$

where the last equality is due to the obvious variable change. Note $\xi(-1)=1$. We compute the last integral by breaking it to the sum $J_1(b)+J_2(b)$, where $J_1(b)$ and $J_2(b)$ denote the integrals over $|\tau|_E \leqslant \inf(|b|_E,1)$ and over $|\tau|_E > \inf(|b|_E,1)$, respectively. In this proof, we use the symbol $\cong$ in the following sense: for two hermitian matrices X and Y in $\mathcal{S}$, $X\cong Y$ if and only if $X = zY\bullet k$ with some $z\in F^\times$ and $k\in \mathbf{K}^E$. For any $\tau\in\mathfrak{o}_E$,

$$\begin{bmatrix} 1 & \tau \\ \bar\tau & (1-\mathrm{N}b^{-1})\mathrm{N}(\tau)\end{bmatrix} \cong \begin{bmatrix} 1 & \tau \\ \bar\tau & (1-\mathrm{N}b^{-1})\mathrm{N}(\tau)\end{bmatrix}\bullet\begin{bmatrix} 1 & -\tau \\ 0 & 1\end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -\mathrm{N}b^{-1}\mathrm{N}\tau\end{bmatrix}\cong\begin{bmatrix} -\varpi^{2n-2\,\mathrm{ord}_E(b)} & 0 \\ 0 & 1\end{bmatrix}$$

with $n=\mathrm{ord}_E(\tau)$. We first consider the case $\mathrm{ord}_E(b)\geqslant 0$. By (2.3),

$$\begin{aligned} J_1(b) &= \sum_{n=\mathrm{ord}_E(b)}^{\infty}\int_{\tau\in\varpi^n\mathfrak{o}_E^\times}\Psi_s^{\mathcal{S}}\left(\begin{bmatrix} -\varpi^{2n-2\,\mathrm{ord}_E(b)} & 0 \\ 0 & 1\end{bmatrix}\right)\xi(\tau)^{-1}\mathrm{d}^\times\tau \\ &= \frac{\mathrm{vol}(\mathfrak{o}_E^\times)q^{-s}}{1-q^{-2s-1}}\sum_{n=\mathrm{ord}_E(b)}^{\infty} q^{-(n-\mathrm{ord}_E(b))(2s+1)}\xi^{-1}(\varpi^n) \\ &= \frac{\mathrm{vol}(\mathfrak{o}_E^\times)q^{-s}}{1-q^{-2s-1}}\frac{\xi^{-1}(b)}{1-q^{-(2s+1)}\xi^{-1}(\varpi)} \quad (\text{for } |q^{-2s-1}\xi(\varpi)^{-1}|<1)\,. \end{aligned}$$

We have

$$J_2(b)=\int_{|\tau|_E<|b^{-1}|_E}\Psi_s^{\mathcal{S}}\left(\begin{bmatrix} 1 & 1/\tau \\ 1/\bar\tau & (1-\mathrm{N}b^{-1})/\mathrm{N}(\tau)\end{bmatrix}\right)\xi(\tau)\mathrm{d}^\times\tau = J_2'(b)+J_2''(b)\,,$$

where $J_2'(b)$ is the integral over $|\tau|_E < |1-\mathrm{N}b^{-1}|_F^2|b|_E$ and $J_2''(b)$ is the integral over $|1-\mathrm{N}b^{-1}|_F^2|b|_E \leqslant |\tau|_E < |b|_E^{-1}$. Let $|\tau|_E < |1-\mathrm{N}b^{-1}|_F^2|b|_E$. Then since the element $x=-\tau/(1-\mathrm{N}b^{-1})$ belongs to $\mathfrak{o}_E$,

$$\begin{bmatrix} 1 & 1/\tau \\ 1/\bar\tau & (1-\mathrm{N}b^{-1})/\mathrm{N}(\tau)\end{bmatrix}\cong\begin{bmatrix} \mathrm{N}\tau & \bar\tau \\ \tau & 1-\mathrm{N}b^{-1}\end{bmatrix}\bullet\begin{bmatrix} 1 & 0 \\ x & 1\end{bmatrix}=\begin{bmatrix} \mathrm{N}\tau/(1-\mathrm{N}b) & 0 \\ 0 & 1-\mathrm{N}b^{-1}\end{bmatrix}\cong\begin{bmatrix} \frac{-\mathrm{N}\tau\mathrm{N}b^{-1}}{(1-\mathrm{N}b^{-1})2} & 0 \\ 0 & 1\end{bmatrix}.$$

We have $\mathrm{ord}_F\left(\frac{\mathrm{N}\tau\mathrm{N}b^{-1}}{(1-\mathrm{N}b^{-1})^2}\right)=2\{\mathrm{ord}_E(\tau)-\mathrm{ord}_E(b)-\mathrm{ord}_F(1-\mathrm{N}b^{-1})\}\geqslant 1$. Hence by (2.3),

$$\begin{aligned} J_2'(b) &= q^{-s}(1-q^{-2s-1})^{-1} \\ &\quad\times\sum_{n=\mathrm{ord}_E(b)+\mathrm{ord}_F(1-\mathrm{N}b^{-1})+1}^{\infty}\int_{\varpi^n\mathfrak{o}_E^\times} q^{-(2s+1)(n-\mathrm{ord}_E(b)-\mathrm{ord}_F(1-\mathrm{N}b^{-1})}\xi(\tau)\mathrm{d}^\times\tau \\ &= \mathrm{vol}(\mathfrak{o}_E^\times)\frac{q^{-s}(1-q^{-2s-1})^{-1}q^{-(2s+1)}\xi\big(\varpi b(1-\mathrm{N}b^{-1})\big)}{1-q^{-(2s+1)}\xi(\varpi)} \quad (\text{for } |q^{-2s-1}\xi(\varpi)|<1)\,. \end{aligned}$$

Let $|1-\mathrm{N}b^{-1}|_F^2|b|_E \leqslant |\tau|_E < |b|_E^{-1}$. If $\mathrm{ord}_E(b)>0$, then there is no τ satisfying this inequality; thus $J_2''(b)=0$. Suppose $\mathrm{ord}_E(b)=0$. Then we can write $\tau=\varpi^l u$ and $1-\mathrm{N}b^{-1}=\varpi^m u_1$ with $m\geqslant l\geqslant 1$, $u\in\mathfrak{o}_E^\times$ and $u_1\in\mathfrak{o}_F^\times$. Suppose $m>l$. Then

$$\begin{bmatrix} 1 & 1/\tau \\ 1/\bar\tau & (1-\mathrm{N}b^{-1})/\mathrm{N}(\tau)\end{bmatrix}\cong\begin{bmatrix} \mathrm{N}\tau & \bar\tau \\ \tau & 1-\mathrm{N}b^{-1}\end{bmatrix}\cong\begin{bmatrix} \varpi^l u\bar u & \bar u \\ u & \varpi^{m-l}u_1\end{bmatrix}\cong\begin{bmatrix} \varpi^l u\bar u & \bar u \\ u & \varpi^{m-l}u_1\end{bmatrix}\bullet\begin{bmatrix} 1 & 0 \\ u & 1\end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{N}u(2+\varpi^l+u_1\varpi^{m-l}) & \bar{u}(1+u_1\varpi^{m-l}) \\ u(1+u_1\varpi^{m-l}) & \varpi^{m-l}u_1 \end{bmatrix}.$$

Then by multiplying the inverse of the upper-left entry, which is a unit due to $2 \in \mathfrak{o}_F^\times$ and $1 \leqslant l < m$, we see that the last matrix is equivalent to a matrix of the form $\begin{bmatrix} 1 & u_2 \\ \bar{u}_2 & \varpi^{m-l}u_3 \end{bmatrix}$ with $u_2, u_3 \in \mathfrak{o}_E^\times$. Since $\mathrm{N}(\mathfrak{o}_E^\times) = \mathfrak{o}_F^\times$, we can write $-(\varpi^{m-l}u_3 - \mathrm{N}u_2) = \mathrm{N}u_4$ with $u_4 \in \mathfrak{o}_E^\times$. By transforming by $\begin{bmatrix} 1 & -u_2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & u_4^{-1} \end{bmatrix} \in \mathbf{K}^E$, we have $\begin{bmatrix} 1 & u_2 \\ \bar{u}_2 & \varpi^{m-l}u_3 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cong w$. Suppose $l = m$. Then

$$\begin{bmatrix} 1 & 1/\tau \\ 1/\bar{\tau} & (1-\mathrm{N}b^{-1})/\mathrm{N}(\tau) \end{bmatrix} \cong \begin{bmatrix} \varpi^l u\bar{u} & \bar{u} \\ u & u_1 \end{bmatrix} \cong \begin{bmatrix} \varpi^l u\bar{u}u_1^{-1} & \bar{u}u_1^{-1} \\ uu_1^{-1} & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 \\ -u\bar{u}_1^{-1} & 1 \end{bmatrix} = \begin{bmatrix} u\bar{u}u_1^{-2}(u_1\varpi^l-1) & 0 \\ 0 & 1 \end{bmatrix}.$$

Since $l \geqslant 1$, the last matrix is equivalent to w. Therefore, noting $\xi(\varpi) \neq 1$, we obtain

$$\begin{aligned} J_2''(b) &= \int_{|1-\mathrm{N}b^{-1}|_E \leqslant |\tau|_E < 1} \Psi_s^{\mathcal{S}}\left(\begin{bmatrix} 1 & 1/\tau \\ 1/\bar{\tau} & (1-\mathrm{N}b^{-1})/\mathrm{N}(\tau) \end{bmatrix}\right) \xi(\tau)\mathrm{d}^\times\tau \\ &= \Psi_s^{\mathcal{S}}(w) \int_{|1-\mathrm{N}b^{-1}|_F^2 \leqslant |\tau|_E < 1} \xi(\tau)\mathrm{d}^\times\tau \\ &= \mathrm{vol}(\mathfrak{o}_E^\times)\, q^{-s}(1-q^{-2s-1})^{-1} \frac{\xi(\varpi) - \xi\big(\varpi(1-\mathrm{N}b^{-1})\big)}{1-\xi(\varpi)}. \end{aligned}$$

When $\mathrm{ord}_E(b) = 0$, $\hat{\mathbb{J}}(b; s, \xi)$ equals

$$\begin{aligned} &\xi(b-1)^{-1}\xi(b)\{J_1(b) + J_2'(b) + J_2''(b)\} \\ &\quad = \xi(b-1)^{-1} \frac{\mathrm{vol}(\mathfrak{o}_E^\times)q^{-s}}{1-q^{-2s-1}} \\ &\qquad \times \left\{ \frac{1}{1-q^{-(2s+1)}\xi^{-1}(\varpi)} + \frac{q^{-(2s+1)}\xi\big(\varpi(1-\mathrm{N}b^{-1})\big)}{1-q^{-(2s+1)}\xi(\varpi)} + \frac{\xi(\varpi) - \xi\big(\varpi(1-\mathrm{N}b^{-1})\big)}{1-\xi(\varpi)} \right\} \\ &\quad = \mathrm{vol}(\mathfrak{o}_E^\times)\xi(b-1)^{-1}q^{-s} \frac{1-\xi\big(\varpi(1-\mathrm{N}b^{-1})\big) - (\xi(\varpi) - \xi(1-\mathrm{N}b^{-1}))\,q^{-(2s+1)}}{(1-\xi(\varpi)^{-1}q^{-(2s+1)})(1-\xi(\varpi)q^{-(2s+1)})(1-\xi(\varpi))}. \end{aligned}$$

To obtain (2.9), it suffices to apply the relation $\xi(1-\mathrm{N}b^{-1}) = \xi(1-\mathrm{N}b)$, which follows from $-\mathrm{N}b \in \mathfrak{o}_F^\times$. When $\mathrm{ord}_E(b) > 0$, $\hat{\mathbb{J}}(b; s, \xi)$ equals

$$\begin{aligned} &\xi(b-1)^{-1}\xi(b)\{J_1(b) + J_2'(b)\} \\ &\quad = \xi(b-1)^{-1}\xi(b)\frac{\mathrm{vol}(\mathfrak{o}_E^\times)q^{-s}}{1-q^{-2s-1}} \left\{ \frac{\xi(b)^{-1}}{1-q^{-(2s+1)}\xi^{-1}(\varpi)} + \frac{q^{-(2s+1)}\xi\big(\varpi b(1-\mathrm{N}b^{-1})\big)}{1-q^{-(2s+1)}\xi(\varpi)} \right\} \\ &\quad = \xi(b-1)^{-1}\mathrm{vol}(\mathfrak{o}_E^\times)q^{-s} \frac{1+q^{-(2s+1)}}{\big(1-\xi(\varpi)^{-1}q^{-(2s+1)}\big)\big(1-\xi(\varpi)q^{-(2s+1)}\big)} \end{aligned}$$

due to the relation $\xi(b) = \xi(\bar{b})$ valid for any unramified character ξ. Since $\xi(1-\mathrm{N}b) = 1$, this agrees with (2.9). In the course of the proof, we see that $|q^{-2s-1}| < \min\{|\xi(\varpi)|, |\xi(\varpi)|^{-1}\}$ is the absolute convergence region of the integral $\hat{\mathbb{J}}(b; s, \xi)$.

Let us consider the case $\mathrm{ord}_E(b) < 0$. In the same way as above, we have

$$J_1(b) = \frac{\mathrm{vol}(\mathfrak{o}_E^\times)q^{-s}}{1-q^{-2s-1}} \sum_{n=0}^{\infty} q^{-(n-\mathrm{ord}_E(b))(2s+1)}\xi^{-1}(\varpi^n)$$
$$= \frac{\mathrm{vol}(\mathfrak{o}_E^\times)q^{-s}}{1-q^{-2s-1}} \frac{|b|_E^{-s-1/2}}{1-q^{-(2s+1)}\xi^{-1}(\varpi)} \quad \big(\text{for } |q^{-2s-1}\xi(\varpi)^{-1}| < 1\big)$$

and

$$J_2(b) = \int_{|\tau|_E<1} \Psi_s^{\mathcal{S}}\left(\left[\begin{smallmatrix} 1 & 1/\tau \\ 1/\bar{\tau} & (1-\mathrm{N}b^{-1})/\mathrm{N}(\tau) \end{smallmatrix}\right]\right)\xi(\tau)\mathrm{d}^\times\tau$$
$$= q^{-s}(1-q^{-2s-1})^{-1}\sum_{n=1}^{\infty}\int_{\varpi^n\mathfrak{o}_E^\times} q^{-(2s+1)(n-\mathrm{ord}_E(b)-\mathrm{ord}_F(1-\mathrm{N}b^{-1}))}\xi(\tau)\mathrm{d}^\times\tau$$
$$= \frac{\mathrm{vol}(\mathfrak{o}_E^\times)q^{-s}}{1-q^{-2s-1}} \frac{q^{-(2s+1)}|b|_E^{-s-1/2}\xi(\varpi)}{1-q^{-(2s+1)}\xi(\varpi)} \quad \big(\text{for } |q^{-2s-1}\xi(\varpi)| < 1\big)$$

due to $\mathrm{ord}_F(1-\mathrm{N}b^{-1}) = 0$. Thus,

$$\hat{\mathbb{J}}(b; s, \xi) = \xi(b-1)^{-1}\xi(b)\{J_1(b)+J_2(b)\}$$
$$= \frac{\mathrm{vol}(\mathfrak{o}_E^\times)q^{-s}}{1-q^{-2s-1}}|b|_E^{-s-1/2}\xi(b-1)^{-1}\xi(b)\left\{\frac{1}{1-q^{-(2s+1)}\xi^{-1}(\varpi)} + \frac{q^{-(2s+1)}\xi(\varpi)}{1-q^{-(2s+1)}\xi(\varpi)}\right\}$$
$$= \mathrm{vol}(\mathfrak{o}_E^\times)\xi(b-1)^{-1}\xi(b)q^{-s}|b|_E^{-s-1/2}\frac{1+q^{-(2s+1)}}{(1-q^{-(2s+1)}\xi^{-1}(\varpi))(1-q^{-(2s+1)}\xi(\varpi))}$$

as desired. This completes the proof. □

3. Orbital integrals on $\mathrm{GL}(2, F)\backslash\mathrm{GL}(2, E)$: the split case

In this section, we suppose $E = F[\sqrt{\theta}]$ is isomorphic to $F \oplus F$. We use the symbols G, B, Z and $\mathbb{X}_E^\delta$ by the same meaning as in §2. In particular, $G = \mathrm{GL}(2)$ in this section. We fix a character $\omega : Z(E) \to \mathbb{C}^1$ trivial on $Z(F)$ and define $C_c(Z(E)G(F)\backslash G(E)/\mathbf{K}^E, \omega)$ as in §1.2.1.

3.1. Spherical functions. For any $(s, c) \in \mathbb{X}_F \times 2\mathbb{X}_F$, let $\Omega_{s,c}(g)$ be the unramified matrix coefficient for the principal series $\mathrm{Ind}_{B(F)}^{G(F)}(|\,|_F^{s+c/2} \boxtimes |\,|_F^{-s+c/2})$; c parametrizes the central character of the representation. From the Cartan decomposition,

$$G(F) = \bigcup_{n=0}^{\infty} Z(F)\mathbf{K}^F \left[\begin{smallmatrix} \varpi^n & 0 \\ 0 & 1 \end{smallmatrix}\right] \mathbf{K}^F, \tag{3.1}$$

where the right-hand side is a disjoint union. We have the Macdonald formula

$$\Omega_{s,c}\left(z\left[\begin{smallmatrix}\varpi^n & 0\\ 0 & 1\end{smallmatrix}\right]\right)=|\det z|_F^{c/2}\left\{\frac{1-qX^2}{(q+1)(1-X^2)}(q^{-(c+1)/2}X)^n + \frac{1-qX^{-2}}{(q+1)(1-X^{-2})}(q^{-(c+1)/2}X^{-1})^n\right\} \tag{3.2}$$

for all $z\in Z(F)$ and $n\in\mathbb{N}$, where $X=q^{-s}$ ([1, Theorem 4.6.6]). Using (3.1), we define a function $\Psi_{s,c}: G(F)\to\mathbb{C}$ by the formula

$$\Psi_{s,c}\left(zk\left[\begin{smallmatrix}\varpi^n & 0\\ 0 & 1\end{smallmatrix}\right]k'\right)=|\det z|_F^{c/2}\,q^{-s}(1-q^{-(2s+1)})^{-1}(q^{-s-(c+1)/2})^n\,, \qquad z\in Z(F),\quad k,k'\in\mathbf{K}^F,\quad n\in\mathbb{N}. \tag{3.3}$$

Then from (3.2), we obtain

$$\Omega_{s,c}=\frac{(q^{2s-1}-1)(q^{-2s-1}-1)}{(1+q^{-1})(q^{-s}-q^{s})}(\Psi_{s,c}-\Psi_{-s,c})\,. \tag{3.4}$$

3.2. Spectral decomposition. For $c\in 2\mathbb{X}_F^0$, let $C_{\mathrm{c}}(Z(F)\mathbf{K}^F\backslash G(F)/\mathbf{K}^F,c)$ denote the space of all functions $f: G(F)\to\mathbb{C}$ with compact support modulo $Z(F)$ having the equivariance

$$f(zkhk')=|\det z|_F^{-c/2}f(h)\,,\quad z\in Z(F),\ h\in G(F),\ k,k'\in\mathbf{K}^F\,.$$

The space endowed with the hermitian inner-product

$$(f|f')_{G(F)}=\int_{Z(F)\backslash G(F)}f(h)\overline{f'(h)}\,\mathrm{d}h\,,\quad f,f'\in C_{\mathrm{c}}\big(Z(F)\mathbf{K}^F\backslash G(F)/\mathbf{K}^F,c\big)\,,$$

is completed to the Hilbert space $L^2\big(Z(F)\mathbf{K}^F\backslash G(F)/\mathbf{K}^F,c\big)$. The Fourier transform of $f\in C_{\mathrm{c}}(Z(F)\mathbf{K}^F\backslash G(F)/\mathbf{K}^F,c)$ is defined by

$$\mathcal{F}f(s)=\int_{Z(F)\backslash G(F)}\Omega_{s,c}(h)f(h)\,\mathrm{d}h\,,\quad s\in\mathbb{X}_F\,, \tag{3.5}$$

where $\mathrm{d}h$ is the Haar measure on $G(F)$ such that $\mathrm{vol}(\mathbf{K}^F)=1$. As in §2.2, $\mathcal{F}f(s)$ belongs to the space of invariant Laurent polynomials $\mathcal{A}=\{\alpha(s)\in\mathbb{C}[z,z^{-1}]\,|\,\alpha(s)=\alpha(-s)\}$ with $z=q^{-s}$. For any $\alpha\in\mathcal{A}$, define

$$\mathcal{F}^*\alpha(h)=\int_{\mathbb{X}_F^0}\Omega_{s,-c}(h)\,\alpha(s)\,\mathrm{d}\Lambda(s)\,,\quad h\in G(F)\,, \tag{3.6}$$

where $\mathrm{d}\Lambda(s)$ is a Radon measure on the purely imaginary locus $\mathbb{X}_F^0$ of $\mathbb{X}_F$ defined by

$$\mathrm{d}\Lambda(\sqrt{-1}y)=\frac{1+q^{-1}}{4\pi}\left|\frac{1-q^{-2\sqrt{-1}y}}{1-q^{-2\sqrt{-1}y-1}}\right|^2(\log q)\,\mathrm{d}y\,.$$

Let $L^2(\mathbb{X}_F^0; \mathrm{d}\Lambda)$ be the L^2-space for the measure space $(\mathbb{X}_F^0, \mathrm{d}\Lambda)$, whose inner-product is denoted by $(\,|\,)_{\mathbb{X}_F^0}$.

THEOREM 3.1. *Let $c \in 2\mathbb{X}_F^0$. The integrals* (3.5) *and* (3.6) *define linear bijections*

$$\mathcal{F} : C_{\mathrm{c}}\big(Z(F)\mathbf{K}^F\backslash G(F)/\mathbf{K}^F, c\big) \longrightarrow \mathcal{A}, \quad \mathcal{F}^* : \mathcal{A} \longrightarrow C_{\mathrm{c}}\big(Z(F)\mathbf{K}^F\backslash G(F)/\mathbf{K}^F, c\big),$$

each of which inverts the other one. Moreover, $\mathcal{F}$ is extended to an isometry from $L^2\big(Z(F)\mathbf{K}^F\backslash G(F)/\mathbf{K}^F, c\big)$ onto $L^2(\mathbb{X}_F^0; \mathrm{d}\Lambda)$, whose inverse isometry extends $\mathcal{F}^*$. We have

$$(\mathcal{F} f|\alpha)_{\mathbb{X}_F^0} = (f|\mathcal{F}^*\alpha)_{G(F)}\,.$$

PROOF. This follows from [21, Theorem 4.7] immediately. Alternatively, a direct argument is possible along the same line as in §5.4. □

3.3. Orbital integrals. We identify $G(F)$ with the diagonal subgroup of $G(E) = G(F) \times G(F)$. Since $\omega|Z(F) = 1$, the character ω is of the form $\omega\big(\big[\begin{smallmatrix} t & 0 \\ 0 & t \end{smallmatrix}\big]\big) = |t_1/t_2|_F^c$ $(t = (t_1, t_2) \in E^\times)$ with a unique $c \in \mathbb{X}_F^0$. We suppose $c \in 2\mathbb{X}_F^0$. For $f \in C_{\mathrm{c}}(Z(E)G(F)\backslash G(E)/\mathbf{K}^E, \omega)$, we define $f_0 : G(F) \to \mathbb{C}$ by setting

$$f_0(h) = f(1, h)\,, \quad h \in G(F)\,.$$

Then the mapping $f \mapsto f_0$ is a linear bijection from $C_{\mathrm{c}}\big(Z(E)G(F)\backslash G(E)/\mathbf{K}^E; \omega\big)$ onto the space $C_{\mathrm{c}}\big(Z(F)\mathbf{K}^F\backslash G(F)/\mathbf{K}^F, c\big)$. The following is the main result of this section.

THEOREM 3.2. *Let $\xi : E^\times \to \mathbb{C}^\times$ be an unramified quasi-character such that $\xi(t_1, t_2) = |t_1/t_2|_F^{-c/2}\xi_0(t_1 t_2)$ for all $(t_1, t_2) \in E^\times$ with a quasi-character ξ_0 of $F^\times$ such that $\xi_0(\varpi) \neq \pm 1$. Then for any $b = (b_1, b_2) \in E^\times - E^1$, the integral* (1.5) *converges absolutely and has the contour integral expression*

$$\mathbb{J}(b; f, \xi) = \frac{1}{\pi\sqrt{-1}} \int_{\mathbb{X}_F^\delta} \hat{\mathbb{J}}(b; s, \xi)\mathcal{F} f_0(s)\,\mathrm{d}\mu(s)\,,$$

with $\mathrm{d}\mu(s) = 2^{-1}(q^s - q^{-s})(\log q)\mathrm{d}s$, where $\delta > 0$ is taken so that $q^{-\delta-1/2} < \min\big\{|\xi_0(\varpi)|, |\xi_0(\varpi)|^{-1}\big\}$, and $\hat{\mathbb{J}}(b; s, \xi)$ is defined as follows. If $e = \mathrm{ord}_F(\mathrm{N}b) > 0$, then

$$\begin{aligned}\hat{\mathbb{J}}(b; s, \xi) = {} & |b_1/b_2|_F^{c/2}\xi(b-1)^{-1}\xi(b) \\ & \times \frac{q^{-s}\xi_0(\mathrm{N}b)^{-1}\big\{(e-1)q^{-2s-1} - e(\xi_0(\varpi) + \xi_0(\varpi)^{-1})q^{-s-1/2} + (e+1)\big\}}{\big(1 - \xi_0(\varpi)q^{-s-1/2}\big)^2\big(1 - \xi_0(\varpi)^{-1}q^{-s-1/2}\big)^2}\,.\end{aligned}$$

If $e = \mathrm{ord}_F(\mathrm{N}b) \leqslant 0$, then

$$\begin{aligned}\hat{\mathbb{J}}(b; s, \xi) = {} & |b_1/b_2|_F^{c/2}\xi(b-1)^{-1}\xi(b) \\ & \times \frac{q^{-s}|\mathrm{N}b|_F^{-s-1/2}}{1 - \xi_0(\varpi)^2}\left\{\frac{1}{\big(1 - \xi_0(\varpi)^{-1}q^{-s-1/2}\big)^2} - \frac{\xi_0\big(\varpi(1 - \mathrm{N}b^{-1})\big)^2}{\big(1 - \xi_0(\varpi)q^{-s-1/2}\big)^2}\right\}.\end{aligned}$$

PROOF. In the proof, we set $\sqrt{\theta} = (\theta_1, -\theta_1)$. Set $\alpha(s) = \mathcal{F} f_0(s)$. By Theorem 3.1 and the formula (3.4), we have

$$f(h_1, h_2) = f_0(h_1^{-1}h_2) = \int_{\mathbb{X}_F^0} \alpha(s)\Omega_{s,-c}(h_1^{-1}h_2)\,\mathrm{d}\Lambda(s)$$
$$= \frac{1}{\pi\sqrt{-1}} \int_{\mathbb{X}_F^0} \alpha(s)\Psi_{s,-c}(h_1^{-1}h_2)\,\mathrm{d}\mu(s)\,.$$

Since the function $\alpha(s)\Psi_{s,-c}(h_1^{-1}h_2)(q^s - q^{-s})$ is holomorphic on $\mathrm{Re}(s) > 0$, we can shift the contour $\mathbb{X}_F^0$ rightward to $\mathbb{X}_F^\delta$ ($\delta > 0$). Plugging this contour integral representation of f, we obtain from (1.5)

$$\begin{aligned}
&\mathbb{J}(b; f, \xi)\\
&= \int_{E^\times} \left\{ \frac{1}{\pi\sqrt{-1}} \int_{\mathbb{X}_F^\delta} \alpha(s)\Psi_{s,-c}\left(\begin{bmatrix} \tau_1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & \theta_1 \frac{b_1+1}{b_1-1} \\ 1 & \theta_1 \end{bmatrix} \begin{bmatrix} 1 & -\theta_1 \frac{b_2+1}{b_2-1} \\ 1 & -\theta_1 \end{bmatrix}^{-1} \begin{bmatrix} \tau_2 & 0 \\ 0 & 1 \end{bmatrix} \right) \mathrm{d}\mu(s) \right\} \xi(\tau)\mathrm{d}^\times\tau\\
&= \int_{E^\times} \left\{ \frac{1}{\pi\sqrt{-1}} \int_{\mathbb{X}_F^\delta} \alpha(s)\Psi_{s,-c}\left(\tfrac{-b_2}{t_1} \begin{bmatrix} t_2 & \mathrm{N}b^{-1}-1 \\ t_1t_2 & -t_1 \end{bmatrix} \right) \mathrm{d}\mu(s) \right\} \xi(\tau)\mathrm{d}^\times\tau\\
&= \xi(b-1)^{-1}\xi(b) \int_{E^\times} \left\{ \frac{1}{\pi\sqrt{-1}} \int_{\mathbb{X}_F^\delta} \alpha(s)\Psi_{s,-c}\left(\tfrac{-b_2}{t_1} \begin{bmatrix} t_2 & \mathrm{N}b^{-1}-1 \\ t_1t_2 & -t_1 \end{bmatrix} \right) \mathrm{d}\mu(s) \right\} \xi(t)\mathrm{d}^\times t\,,
\end{aligned}$$

where we made a variable change by setting $\tau_1 = \frac{b_1}{b_1-1}t_1$, $\tau_2 = \frac{b_2}{b_2-1}t_2$. To conclude the proof, we have only to show that the integral

$$\hat{\mathbb{J}}(b; s, \xi) = \xi(b-1)^{-1}\xi(b) \int_{E^\times} \Psi_{s,-c}\left(\tfrac{-b_2}{t_1} \begin{bmatrix} t_2 & \mathrm{N}b^{-1}-1 \\ t_1t_2 & -t_1 \end{bmatrix} \right) \xi(t)\mathrm{d}^\times t$$

converges absolutely for $\mathrm{Re}(s) \gg 0$ and is evaluated as in the theorem. If we write

$$\tfrac{-b_2}{t_1} \begin{bmatrix} t_2 & \mathrm{N}b^{-1}-1 \\ t_1t_2 & -t_1 \end{bmatrix} = k \begin{bmatrix} \varpi^{n+l} & 0 \\ 0 & \varpi^l \end{bmatrix} k', \quad k,\, k' \in \mathbf{K}^F\,, \quad n \in \mathbb{N}\,,\ l \in \mathbb{Z}\,,$$

then it is easy to see that

$$n + 2l = \mathrm{ord}_F\left(\tfrac{b_2t_2}{b_1t_1}\right), \quad n = \mathrm{ord}_F\left(\tfrac{t_1t_2}{b_1b_2}(t_1, t_2, t_1t_2, 1-\mathrm{N}b^{-1})^{-2}\right).$$

Hence from (3.3),

$$\begin{aligned}
&\Psi_{s,-c}\left(\tfrac{-b_2}{t_1} \begin{bmatrix} t_2 & \mathrm{N}b^{-1}-1 \\ t_1t_2 & -t_1 \end{bmatrix} \right)\\
&\quad = q^{-s}(1-q^{-(2s+1)})^{-1} q^{(n+2l)c/2} q^{-n(s+1/2)}\\
&\quad = q^{-s}(1-q^{-(2s+1)})^{-1} \left| \tfrac{b_2t_2}{b_1t_1} \right|_F^{-c/2} q^{-(s+1/2)\,\mathrm{ord}_F\left(\frac{t_1t_2}{b_1b_2}(t_1,t_2,t_1t_2,1-\mathrm{N}b^{-1})^{-2}\right)}\,.
\end{aligned}$$

Since $\xi(t) = |t_1/t_2|_F^{-c/2}\xi_0(t_1t_2)$, we have that $\hat{\mathbb{J}}(b; s, \xi)$ equals

$$\xi(b-1)^{-1}\xi(b)q^{-s}(1-q^{-(2s+1)})^{-1}\left|\frac{b_2}{b_1}\right|_F^{-c/2}$$
$$\times \iint_{F^\times\times F^\times} q^{-(s+1/2)\,\mathrm{ord}_F\left(\frac{t_1t_2}{b_1b_2}(t_1,t_2,t_1t_2,1-\mathrm{N}b^{-1})^{-2}\right)}\xi_0(t_1t_2)\mathrm{d}^\times t_1\mathrm{d}^\times t_2$$
$$= \frac{\xi(b-1)^{-1}\xi(b)q^{-s}\mathrm{vol}(\mathfrak{o}_F^\times)^2}{1-q^{-(2s+1)}}\,|b_2/b_1|_F^{-c/2}\,J(b;\xi_0,\xi_0)\,,$$

where we set

$$J(b;\eta_1,\eta_2) = \sum_{l_1,l_2\in\mathbb{Z}} q^{-(s+1/2)\,\mathrm{ord}_F\left(\frac{\varpi^{l_1+l_2}}{b_1b_2}(\varpi^{l_1},\varpi^{l_2},\varpi^{l_1+l_2},1-\mathrm{N}b^{-1})^{-2}\right)}\eta_1(\varpi)^{l_1}\eta_2(\varpi)^{l_2}$$

for any pair (η_1,η_2) of unramified quasi-characters of $F^\times$. For $l\in\mathbb{Z}$, set $\mathrm{sgn}(l) = +$ if $l\geqslant 0$ and $\mathrm{sgn}(l) = -$ if $l<0$. For $\varepsilon,\varepsilon'\in\{+,-\}$, let $J_{\varepsilon,\varepsilon'} = J_{\varepsilon,\varepsilon'}(b;\eta_1,\eta_2)$ denote the sub-series of $J(b;\eta_1,\eta_2)$ for terms with $\mathrm{sgn}(l_1)=\varepsilon$, $\mathrm{sgn}(l_2)=\varepsilon'$. Put $\beta = 1-\mathrm{N}b^{-1}$. A tedious but straightforward computation reveals the following identities.

$$J_{++} = \frac{|b_1b_2|_F^{-s-1/2}\big(1-q^{-2s-1}\eta_1\eta_2(\varpi)\big)}{\big(1-q^{-s-1/2}\eta_1(\varpi)\big)\big(1-q^{-s-1/2}\eta_2(\varpi)\big)}$$
$$\times\left\{\delta(\beta\in\mathfrak{o}_F)\frac{1-\eta_1\eta_2(\beta)}{1-\eta_1\eta_2(\varpi)} + \frac{|\beta|_F^{-2s-1}\big(|\beta|_F^{2s+1}\eta_1\eta_2(\beta)\big)^{\delta(\beta\in\mathfrak{o}_F)}}{1-q^{-2s-1}\eta_1\eta_2(\varpi)}\right\},$$

$$J_{-+} = \frac{|b_1b_2|_F^{-s-1/2}q^{-s-1/2}\eta_1(\beta)^{-1}}{1-q^{-s-1/2}\eta_2(\varpi)}$$
$$\times\left\{\frac{\big(|\beta|^{-s-1/2}\eta_1(\beta)\big)^{\delta(\beta\notin\mathfrak{o}_F)}}{1-q^{-s-1/2}\eta_1(\varpi)^{-1}} + \frac{\delta(\beta\notin\mathfrak{o}_F)|\beta|_F^{-2s-1}q^{2s+1}\big(1-|\beta|_F^{s+1/2}\eta_1(\beta)\big)}{1-q^{s+1/2}\eta_1(\varpi)^{-1}}\right\},$$

$$J_{+-} = \frac{|b_1b_2|_F^{-s-1/2}q^{-s-1/2}\eta_2(\varpi)^{-1}}{1-q^{-s-1/2}\eta_1(\varpi)}$$
$$\times\left\{\frac{\big(|\beta|_F^{-s-1/2}\eta_2(\beta)\big)^{\delta(\beta\notin\mathfrak{o}_F)}}{1-q^{-s-1/2}\eta_2(\varpi)^{-1}} + \frac{\delta(\beta\notin\mathfrak{o}_F)|\beta|_F^{-2s-1}q^{2s+1}\big(1-|\beta|_F^{s+1/2}\eta_2(\beta)\big)}{1-q^{s+1/2}\eta_2(\varpi)^{-1}}\right\},$$

$$J_{--} = \frac{|b_1b_2|_F^{-s-1/2}q^{-s-1/2}\eta_1(\varpi)^{-1}}{1-\eta_1^{-1}\eta_2(\varpi)}$$
$$\times\left\{\left(\frac{\big(|\beta|_F^{-s-1/2}\eta_2(\beta)\big)^{\delta(\beta\notin\mathfrak{o}_F)}}{1-q^{-s-1/2}\eta_2(\varpi)^{-1}} + \frac{\delta(\beta\notin\mathfrak{o}_F)|\beta|_F^{-2s-1}q^{2s+1}\big(1-|\beta|_F^{s+1/2}\eta_2(\beta)\big)}{1-q^{s+1/2}\eta_2(\varpi)^{-1}}\right)\right.$$

$$-\left(\frac{(|\beta|_F^{-s-1/2}\eta_1(\beta))^{\delta(\beta\notin\mathfrak{o}_F)}}{1-q^{-s-1/2}\eta_1(\varpi)^{-1}}+\frac{\delta(\beta\notin\mathfrak{o}_F)|\beta|_F^{-2s-1}q^{2s+1}(1-|\beta|_F^{s+1/2}\eta_1(\beta))}{1-q^{s+1/2}\eta_1(\varpi)^{-1}}\right)\Bigg\}\,.$$

While the second and the third formulas are valid for all (η_1,η_2) satisfying

$$|q^{-s-1/2}|\max\{|\eta_1(\varpi)|,|\eta_1(\varpi)^{-1}|,|\eta_2(\varpi)|,|\eta_2(\varpi)^{-1}|\}<1\,, \tag{3.7}$$

the first and the last formulas are valid only for those (η_1,η_2) inside this region with $\eta_1\neq\eta_2^{\pm1}$. By summing up J_{++}, J_{+-}, J_{-+}, and J_{--}, after a tedious computation, we have that $J(b;\eta_1,\eta_2)$ is equal to

$$\frac{|b_1b_2|_F^{-s-1/2}(1-q^{-2s-1})}{1-\eta_1\eta_2(\varpi)}\times\left\{\frac{1}{(1-q^{-s-1/2}\eta_1(\varpi)^{-1})(1-q^{-s-1/2}\eta_2(\varpi)^{-1})}-\frac{\eta_1\eta_2(\varpi\beta)}{(1-q^{-s-1/2}\eta_1(\varpi))(1-q^{-s-1/2}\eta_2(\varpi))}\right\}$$

if $\beta\in\mathfrak{o}_F$, and to

$$\frac{|b_1b_2|_F^{-s-1/2}|\beta|_F^{-s-1/2}(q^{-2s-1}-1)}{\eta_1(\varpi)-\eta_2(\varpi)}\times\frac{(\eta_1(\varpi\beta)-\eta_2(\varpi\beta))q^{-2s-1}-(1+\eta_1\eta_2(\varpi))(\eta_1(\beta)-\eta_2(\beta))q^{-s-1/2}+\eta_1(\beta)\eta_2(\varpi)-\eta_1(\varpi)\eta_2(\beta)}{(1-q^{-s-1/2}\eta_1(\varpi)^{-1})(1-q^{-s-1/2}\eta_2(\varpi)^{-1})(1-q^{-s-1/2}\eta_1(\varpi))(1-q^{-s-1/2}\eta_2(\varpi))}$$

if $\beta\notin\mathfrak{o}_F$. These formulas are valid for all pairs (η_1,η_2) inside the region (3.7) with $\eta_1\neq\eta_2^{\pm1}$. Suppose ξ_0 and s satisfy $|q^{-s-1/2}|\max\{|\xi_0(\varpi)|,|\xi_0(\varpi)|^{-1}\}<1$. We apply the above evaluation of $J(b;\eta_1,\eta_2)$ for $(\eta_1,\eta_2)=(\xi_0,\xi_0|\,|_F^{\lambda})$ ($\lambda\in\mathbb{C}$) with sufficiently small $|\lambda|>0$ and then take the limit as $\lambda\to0$. If $\beta\in\mathfrak{o}_F$, we immediately have

$$J(b;\xi_0,\xi_0)=\frac{|b_1b_2|_F^{-s-1/2}(1-q^{-2s-1})}{1-\xi_0(\varpi)^2}\left\{\frac{1}{(1-q^{-s-1/2}\xi_0(\varpi)^{-1})^2}-\frac{\xi_0(\varpi\beta)^2}{(1-q^{-s-1/2}\xi_0(\varpi))^2}\right\}.$$

If $\beta\notin\mathfrak{o}_F$, then by the relations

$$\lim_{\lambda\to0}\frac{\eta_1(\varpi\beta)-\eta_2(\varpi\beta)}{\eta_1(\varpi)-\eta_2(\varpi)}=\xi_0(\beta)(1-e)\,,\quad\lim_{\lambda\to0}\frac{\eta_1(\varpi)\eta_2(\beta)-\eta_1(\beta)\eta_2(\varpi)}{\eta_1(\varpi)-\eta_2(\varpi)}=\xi_0(\beta)(e+1)\,,$$

$$\lim_{\lambda\to0}\frac{(1+\eta_1\eta_2(\varpi))(\eta_2(\beta)-\eta_1(\beta))}{\eta_1(\varpi)-\eta_2(\varpi)}=\xi_0(\beta)(\xi_0(\varpi)+\xi_0(\varpi)^{-1})e$$

with $e=-\operatorname{ord}_F(\beta)=\operatorname{ord}_F\mathrm{N}b$, we have

$$J(b;\xi_0,\xi_0)=\frac{|b_1b_2|_F^{-s-1/2}|\beta|_F^{-s-1/2}\xi_0(\beta)\{(e-1)q^{-2s-1}-e(\xi_0(\varpi)+\xi_0(\varpi)^{-1})q^{-s-1/2}+(e+1)\}}{(1-q^{-2s-1})^{-1}(1-q^{-s-1/2}\xi_0(\varpi))^2(1-q^{-s-1/2}\xi_0(\varpi)^{-1})^2}.$$

Since $\beta=1-\mathrm{N}b^{-1}\in\mathrm{N}b^{-1}\mathfrak{o}_F^\times$, this can be further simplified by the relations $\xi_0(\beta)=\xi_0(\mathrm{N}b)^{-1}$ and $|b_1b_2|_F^{-s-1/2}|\beta|_F^{-s-1/2}=1$. □

4. Orbital integrals on a unitary hyperbolic space

In this section, we work with the notation in §1.2.2 keeping all the assumptions there. In particular, $G=U(\mathbf{h})$, $H=\mathrm{Stab}_G(E\ell_0)$ in this section. We let H_0 to be the stabilizer of ℓ_0 in

G. We easily have the orthogonal decomposition $\mathcal{L} = \mathfrak{o}_E\ell_0 \oplus (\ell_0^\perp \cap \mathcal{L})$, from which $\ell_0^\perp \cap \mathcal{L}$ is seen to be a unimodular $\mathfrak{o}_E$-lattice in the hermitian space $\ell_0^\perp$. If m is even, then we have an orthogonal decomposition $\mathcal{L} = \bigoplus_{j=1}^{m/2}(\mathfrak{o}_E e_j + \mathfrak{o}_E e_{m-j+1})$ with an $\mathfrak{o}_E$-basis $\mathcal{B} = \{e_j\}_{j=1}^m$ such that $\mathbf{h}(e_j, e_{m-j+1}) = 1$, $\mathbf{h}[e_j] = \mathbf{h}[e_{m-j+1}] = 0$ for all $1 \leqslant j \leqslant m/2$. If m is odd, we have an orthogonal decomposition $\mathcal{L} = \mathfrak{o}_E e_{(m+1)/2} \oplus \{\bigoplus_{j=1}^{(m-1)/2}(\mathfrak{o}_E e_j + \mathfrak{o}_E e_{m-j+1})\}$ with an $\mathfrak{o}_E$-basis $\mathcal{B} = \{e_j\}_{j=1}^m$ such that $\mathbf{h}(e_j, e_{m-j+1}) = 1$, $\mathbf{h}[e_j] = \mathbf{h}[e_{m-j+1}] = 0$ for all $1 \leqslant j \leqslant (m-1)/2$ and $\mathbf{h}[e_{(m+1)/2}] \in \mathfrak{o}_F^\times$. Since $\ell_0 \in \mathcal{L}$ is a unit vector, it is primitive in $\mathcal{L}$; hence, we can choose $\mathcal{B}$ such that

$$\ell_0 = ae_1 + e_m \tag{4.1}$$

with some $a \in E$ such that $a + \bar{a} = 1$. We fix such an $\mathfrak{o}_E$-basis $\mathcal{B}$ once and for all, and set $\kappa = \mathbf{h}[e_{(m+1)/2}]$ if m is odd.

For $a_j \in E^\times$ $(1 \leqslant j \leqslant m)$, let $\mathsf{d}(a_1, \dots, a_m)$ be the element of $\mathrm{GL}_E(V)$ defined by

$$\mathsf{d}(a_1, \dots, a_m)e_j = a_j e_j \quad (1 \leqslant j \leqslant m)\,.$$

For $t \in E^\times$, we designate the element $\mathsf{d}(a_1, \dots, a_m)$ with $a_1 = t$, $a_m = \bar{t}^{-1}$ and $a_j = 1$ $(1 < j < m)$ as $\mathsf{d}[t]$, which is indeed belongs to G. Let ϖ_E be ϖ if E is a field and denote an element of $\mathfrak{o}_E$ such that $\mathrm{N}\varpi_E = \varpi$ if $E \cong F \oplus F$. We have the disjoint decomposition

$$G = \bigcup_{l=0}^{\infty} H\,\mathsf{d}[\varpi_E^{-l}]\mathcal{U}\,. \tag{4.2}$$

When E is a field, this decomposition follows from [8, Proposition 3.9] due to the fact that the center Z_G of G is contained in $\mathcal{U}$ and $H = H_0 Z_G$. When $E \cong F \oplus F$, we give a proof in §6 for completeness.

4.1. Hyperboloids. For $\Delta \in F$, set

$$\mathbf{\Sigma}(V, \Delta) = \{x \in V - \{0\} | \mathbf{h}[x] = \Delta\}\,,$$

which we regard as an F-algebraic variety (identified with its F-points) with rational G-action. Obviously $\Sigma(V, \Delta)$ is a G-stable subset of V; Witt's theorem tells us that it is a G-orbit in V. Let G_ℓ be the stabilizer in G of a vector $\ell \in \mathbf{\Sigma}(V, \Delta)$. By sending a coset $G_\ell g$ to the vector $g^{-1}\ell$, we have a G-isomorphism

$$G_\ell \backslash G \to \mathbf{\Sigma}(V, \Delta)\,.$$

There exists a unique gauge form ω_V on V ([24]), viewed as a $2m$-dimensional affine space over F, such that

$$\omega_V(\xi) = \det\big(\mathbf{h}(\xi_i, \xi_j)\big) \prod_{j=1}^{m} \frac{\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j}{2\sqrt{\theta}} \tag{4.3}$$

for any E-basis $\{\xi_j\}$ of V, where z_j is the E-coordinate functions on V corresponding to the basis $\{\xi_j\}$. Let $V^\times = \{\xi \in V \,|\, \mathbf{h}[\xi] \neq 0\}$ and $\nu : V^\times \to \mathrm{GL}(1)$ the map $\nu(\xi) = \mathbf{h}[\xi]$. Then there exists a unique G-invariant gauge form $\omega_{V,\Delta}$ on $\mathbf{\Sigma}(V, \Delta)$ such that $\omega_V(\xi) = \nu^*(\mathrm{d}t)_\xi \wedge \omega_{V,\Delta}(\xi)$ at all $\xi \in \mathbf{\Sigma}(V, \Delta)$, where $\nu^*(\mathrm{d}t)$ denotes the pull-back of the differential $\mathrm{d}t$ of the standard coordinate t on GL(1) by ν. Let $|\omega_V|_F$ and $|\omega_{V,\Delta}|_F$ be the measures on V and $\mathbf{\Sigma}(V, \Delta)$ defined by ω_V and $\omega_{V,\Delta}$, respectively. The image $\nu(V^\times)$ coincides with $F^\times$. We have the integration formula

$$\int_V f(\xi)\,|\omega_V|_F = \int_{F^\times} \mathrm{d}\Delta \int_{\mathbf{\Sigma}(V,\Delta)} f(\xi)|\omega_{V,\Delta}|_F \tag{4.4}$$

for any $f \in L^1(V)$.

LEMMA 4.1. *For any $\alpha \in E^\times$, we have*

$$\int_{\mathbf{\Sigma}(V,\Delta)} f(\xi)|\omega_{V,\Delta}|_F = |\alpha|_E^{1-m} \int_{\mathbf{\Sigma}(V,\Delta \mathrm{N}\alpha)} f(\alpha^{-1}\xi)|\omega_{V,\Delta\mathrm{N}\alpha}|_F$$

for any $f \in C_{\mathrm{c}}^\infty(V)$.

PROOF. From (4.4), replacing $f(\xi)$ with $f(\xi)\psi(\mathbf{h}[\xi]\tau)$, we have

$$\int_V f(\xi)\psi(\mathbf{h}[\xi]\tau)\,|\omega_V|_F = \int_{F^\times} \left\{ \int_{\mathbf{\Sigma}(V,\Delta)} f(\xi)|\omega_{V,\Delta}|_F \right\} \psi(\Delta\tau)\mathrm{d}\Delta$$

for any $\tau \in F$. By the Fourier inversion formula,

$$\int_{\mathbf{\Sigma}(V,\Delta)} f(\xi)|\omega_{V,\Delta}|_F = \int_F \left\{ \int_V f(\xi)\psi(\mathbf{h}[\xi]\tau)\,|\omega_V|_F \right\} \psi(-\tau\Delta)\mathrm{d}\tau.$$

We have $|\omega_V|_F(\alpha\xi) = |\alpha|_E^m |\omega_V|_F(\xi)$ and $\mathrm{d}\big((\mathrm{N}\alpha)^{-1}\tau\big) = |\alpha|_E^{-1}\mathrm{d}\tau$. From these, the desired formula follows immediately. □

LEMMA 4.2. *Let $\mathcal{L}$ be any self-dual $\mathfrak{o}_E$-lattice in V. For any $\Delta \in F^\times$, we have*

$$\mathrm{vol}(\mathbf{\Sigma}(V, \Delta) \cap \mathcal{L};\, |\omega_{V,\Delta}|_F) = \delta(\Delta \in \mathfrak{o}_F)\, \frac{1 - \varepsilon_E^m q^{-m}}{1 - \varepsilon_E^m q^{-(m-1)}} (1 - \varepsilon_E^{m\, \mathrm{ord}_F(\varpi\Delta)} q^{-(m-1)} |\Delta|_F^{m-1}) \,.$$

PROOF. Since $\mathbf{h}[\mathcal{L}] \subset \mathfrak{o}_F$, the intersection $\mathbf{\Sigma}(V, \Delta) \cap \mathcal{L}$ is empty unless $\Delta \in \mathfrak{o}_F$. In the remaining part of the proof, we suppose $\Delta \in \mathfrak{o}_F$. If we write a general point $\xi \in V$ in the form $\xi = \sum_{j=1}^m z_j e_j$ with $(z_j) \in E^m$, then $\xi \in \mathcal{L}$ if and only if $z_j \in \mathfrak{o}_E$ for all j. From (4.3), we have $|\omega_V|_F = \prod_{j=1}^m \mathrm{d}\mu(z_j)$ with $\mathrm{d}\mu(z) = |\mathrm{d}z \wedge \mathrm{d}\bar{z}|_F$. For any $t \in F^\times$, set $v(t) = \mathrm{vol}(\mathbf{\Sigma}(V, t) \cap \mathcal{L};\, |\omega_{V,t}|_F)$ and $\hat{v}(\tau)$ its Fourier transform. Suppose m is even. From (4.4),

$$\hat{v}(\tau) = \int_{F^\times} \int_{\mathbf{\Sigma}(V,\Delta)\cap\mathcal{L}} \psi(\tau\mathbf{h}[\xi])\,|\omega_{V,\Delta}|_F\,\mathrm{d}\Delta = \int_{\mathcal{L}} \psi(\tau\mathbf{h}[\xi])|\omega_V|_F$$

$$= \prod_{j=1}^{m/2} \int_{\mathfrak{o}_E^2} \psi(\tau \mathrm{tr}_{E/F}(z_j \bar{z}_{m-j+1}))\, \mathrm{d}\mu(z_j)\mathrm{d}\mu(z_{m-j+1})$$

$$= \prod_{j=1}^{m/2} \int_{z_j \in \mathfrak{o}_E} \delta(\tau z_j \in \mathfrak{o}_E)\, \mathrm{d}\mu(z_j) = \prod_{j=1}^{m/2} \mathrm{vol}(\mathfrak{o}_E \cap \tau^{-1}\mathfrak{o}_E) = \inf(1, |\tau|_E^{-1})^{m/2}\,.$$

Note that $\mathrm{d}\mu(z)$ coincides with the Haar measure on E such that $\mathrm{vol}(\mathfrak{o}_E) = 1$. By the Fourier inversion formula,

$$\begin{aligned} v(\Delta) = \int_F \hat{v}(\tau)\psi(-\Delta\tau)\mathrm{d}\tau &= \int_F \inf(1, |\tau|_E^{-1})^{m/2}\psi(-\Delta\tau)\mathrm{d}\tau \\ &= \int_{\tau\in\mathfrak{o}_F} \psi(-\Delta\tau)\mathrm{d}\tau + \int_{\tau\in F-\mathfrak{o}_F} |\tau|_E^{-m/2}\psi(-\Delta\tau)\mathrm{d}\tau \\ &= 1 + \sum_{l=1}^{\infty} q^{-(m-1)l} \int_{\mathfrak{o}_F^\times} \psi(-\Delta\varpi^{-l}u)\mathrm{d}u\,. \end{aligned}$$

The u-integral is computed as $\delta(\Delta\varpi^{-l} \in \mathfrak{o}_F) - q^{-1}\delta(\Delta\varpi^{-l+1} \in \mathfrak{o}_F)$. By this,

$$\begin{aligned} v(\Delta) &= 1 + \sum_{l=1}^{\mathrm{ord}_F(\Delta)} (1-q^{-1})q^{-l(m-1)} + (-q^{-1})\, q^{-(m-1)(\mathrm{ord}_F(\Delta)+1)} \\ &= \frac{1-q^{-m}}{1-q^{-(m-1)}}(1 - q^{-(m-1)}|\Delta|_F^{m-1}) \end{aligned}$$

as desired for an even m. Suppose m is odd. Since $\mathbf{h}[\xi] = \sum_{j=1}^{(m-1)/2} \mathrm{tr}_{E/F}(z_j\bar{z}_{m-j+1}) + \kappa z_{(m+1)/2}\bar{z}_{(m+1)/2}$ with $\kappa = \mathbf{h}[e_{(m+1)/2}]$, in the same way as above, we have

$$\begin{aligned} \hat{v}(\tau) &= \left\{ \prod_{j=1}^{(m-1)/2} \int_{\mathfrak{o}_E^2} \psi(\tau\mathrm{tr}_{E/F}(z_j\bar{z}_{m-j+1}))\, \mathrm{d}\mu(z_j)\mathrm{d}\mu(z_{m-j+1}) \right\} \int_{\mathfrak{o}_E} \psi(\kappa z\bar{z}\tau)\, \mathrm{d}\mu(z) \\ &= \inf(1, |\tau|_E^{-1})^{(m-1)/2} \int_{\mathfrak{o}_E} \psi(\kappa z\bar{z}\tau)\, \mathrm{d}\mu(z)\,. \end{aligned}$$

When $E = F \oplus F$ and $\mathfrak{o}_E = \mathfrak{o}_F \oplus \mathfrak{o}_F$, then the last integral is easily seen to be equal to $\inf(1, |\tau|_F^{-1})$. Thus, $\hat{v}(\tau) = \inf(1, |\tau|_F^{-1})^{(m+1)/2}$ in this case. In the same way as above, by the Fourier inversion, we have the formula of $v(\Delta)$ as desired. In the remaining part of the proof, we assume that E is a field. By the Fourier inversion,

$$\begin{aligned} v(\Delta) &= \int_F \inf(1, |\tau|_E^{-1})^{(m-1)/2} \left\{ \int_{\mathfrak{o}_E} \psi(\kappa z\bar{z}\tau)\, \mathrm{d}\mu(z) \right\} \psi(-\Delta\tau)\mathrm{d}\tau \\ &= \int_{\mathfrak{o}_F}\int_{\mathfrak{o}_E} \psi(-\tau(\Delta - \kappa z\bar{z}))\mathrm{d}\tau\mathrm{d}\mu(z) \end{aligned}$$

$$+\int_{z\in\mathfrak{o}_E}\left\{\int_{\tau\in F-\mathfrak{o}_F}|\tau|_E^{-(m-1)/2}\psi(-\tau(\Delta-\kappa z\bar{z}))\,\mathrm{d}\tau\right\}\mathrm{d}\mu(z)$$

$$=1+\int_{z\in\mathfrak{o}_E}\mathrm{d}\mu(z)\sum_{l=1}^{\infty}q^{-(m-1)l+l}\int_{\mathfrak{o}_F^\times}\psi(-(\Delta-\kappa z\bar{z})\varpi^{-l}u)\mathrm{d}u$$

$$=1+\int_{z\in\mathfrak{o}_E}\left\{\sum_{l=1}^{\mathrm{ord}_F(\Delta-\kappa z\bar{z})}(1-q^{-1})q^{-(m-2)l}+(-q^{-1})\,q^{-(m-2)(\mathrm{ord}_F(\Delta-\kappa z\bar{z})+1)}\right\}\mathrm{d}\mu(z)$$

$$=1+\int_{z\in\mathfrak{o}_E}\frac{1-q^{-1}-(1-q^{-(m-1)})|\Delta-\kappa z\bar{z}|_F^{m-2}}{q^{m-2}-1}\mathrm{d}\mu(z)$$

$$=\frac{1-q^{-(m-1)}}{1-q^{-(m-2)}}\left(1-q^{-(m-2)}\int_{z\in\mathfrak{o}_E}|\Delta-\kappa z\bar{z}|_F^{m-2}\mathrm{d}\mu(z)\right).$$

By Lemma 4.3, we are done. □

LEMMA 4.3. *Suppose E is a field and m is odd. For $\Delta\in\mathfrak{o}_F-\{0\}$,*

$$\int_{z\in\mathfrak{o}_E}|\Delta-\kappa z\bar{z}|_F^{m-2}\,\mathrm{d}\mu(z)$$

$$=-(-1)^{\mathrm{ord}_F(\Delta)}|\Delta|_F^{m-1}\frac{q^{-1}(1-q^{-m+2})(1+q^{-m})}{1-q^{-2(m-1)}}+\frac{1-q^{-2}}{1-q^{-2(m-1)}},$$

where $\mathrm{d}\mu(z)=|\mathrm{d}z\wedge\mathrm{d}\bar{z}|_F$.

PROOF. Since $\kappa\in\mathfrak{o}_F^\times$, by replacing Δ with $-\kappa\Delta$, we may suppose $\kappa=-1$. Set $l=\mathrm{ord}_F(\Delta)$. Let I_1, I_2 and I_3 be the integrals of $|\Delta+z\bar{z}|_F^{m-2}$ over the subsets of $\mathfrak{o}_E$ defined by $|z|_E<|\Delta|_F$, $|z|_E=|\Delta|_F$ and $|z|_E>|\Delta|_F$, respectively. We have

$$I_1=\int_{|z|_E<|\Delta|_F}|\Delta|_F^{m-2}\mathrm{d}\mu(z)=|\Delta|_F^{m-2}\mathrm{vol}\{z\in\mathfrak{o}_E\,|\,|z|_E<|\Delta|_F\}=|\Delta|_F^{m-1}\begin{cases}q^{-2} & (l \text{ is even}),\\ q^{-1} & (l \text{ is odd}),\end{cases}$$

and

$$I_3=\int_{|\Delta|_F<|z|_E\leqslant 1}|z\bar{z}|_F^{m-2}\mathrm{d}\mu(z)=\sum_{k=0}^{[\frac{l-1}{2}]}q^{-2k(m-2)}\,\mathrm{vol}(\varpi^k\mathfrak{o}_E^\times)$$

$$=\frac{1-q^{-2}}{1-q^{-2(m-1)}}\begin{cases}1-q^{-l(m-1)} & (l \text{ is even}),\\ 1-q^{-(l+1)(m-1)} & (l \text{ is odd}).\end{cases}$$

Since $|z|_E$ is an even power of q, the set $|z|_E=|\Delta|_F$ is empty unless $l=\mathrm{ord}_F(\Delta)$ is even. Thus $I_2=0$ if l is odd. Suppose l is even and set $\Delta=\varpi^l\Delta_0$ with $\Delta_0\in\mathfrak{o}_F^\times$. Then

$$I_2=\int_{\varpi^{l/2}\mathfrak{o}_E^\times}|\Delta+z\bar{z}|_F^{m-2}\mathrm{d}\mu(z)=q^{-(m-1)l}\int_{\mathfrak{o}_E^\times}|\Delta_0+u\bar{u}|_F^{m-2}\mathrm{d}u,$$

where $\mathrm{d}u$ is the Haar measure on $\mathfrak{o}_E$ such that $\mathrm{vol}(\mathfrak{o}_E) = 1$. Since E/F is unramified, the map $u \mapsto v = u\bar{u}$ from $\mathfrak{o}_E^\times$ to $\mathfrak{o}_F^\times$ is surjective. If $\mathrm{d}v$ denotes the Haar measure on $\mathfrak{o}_F$ such that $\mathrm{vol}(\mathfrak{o}_F) = 1$, then there exists a Haar measure $\mathrm{d}^1 z$ on $E^1 = \{z \in \mathfrak{o}_E \mid z\bar{z} = 1\}$ such that $\mathrm{d}u/\mathrm{d}^1 z = \mathrm{d}v$ and $\mathrm{vol}(E^1; \mathrm{d}^1 z) = \mathrm{vol}(\mathfrak{o}_E^\times)/\mathrm{vol}(\mathfrak{o}_F^\times) = (1-q^{-2})/(1-q^{-1}) = 1+q^{-1}$. Thus

$$I_2 = q^{-(m-1)l} \int_{\mathfrak{o}_F^\times} \int_{E^1} |\Delta_0 + v|_F^{m-2} \mathrm{d}^1 z \, \mathrm{d}v = q^{-(m-1)l}(1+q^{-1}) \int_{\mathfrak{o}_F^\times} |\Delta_0 + v|_F^{m-2} \mathrm{d}v \,.$$

In the last integral, by a variable change, we may assume $\Delta_0 = 1$. Then,

$$\begin{aligned} \int_{\mathfrak{o}_F^\times} |1+v|_F^{m-2} \mathrm{d}v &= \sum_{\eta \in \mathfrak{o}_F^\times/(1+\varpi\mathfrak{o}_F)} \int_{1+\varpi\mathfrak{o}_F} |1+\eta v|_F^{m-2} \mathrm{d}v \\ &= q^{-1} \sum_{\eta \in \mathfrak{o}_F^\times/(1+\varpi\mathfrak{o}_F)} \int_{\mathfrak{o}_F} |1+\eta(1+\varpi x)|_F^{m-2} \mathrm{d}x \,. \end{aligned}$$

If η is not the class $-1 \pmod{\varpi}$, then the integral becomes $\mathrm{vol}(\mathfrak{o}_F) = 1$; the number of such η is $q-2$. If $\eta \equiv -1 \pmod{\varpi}$, then the x-integral is

$$\int_{\mathfrak{o}_F} |\varpi x|_F^{m-2} \mathrm{d}x = q^{-(m-2)} \sum_{j=0}^{\infty} q^{-j(m-2)} q^{-j}(1-q^{-1}) = \frac{q^{-(m-2)}(1-q^{-1})}{1-q^{-(m-1)}} \,.$$

Hence for an even l,

$$\begin{aligned} I_2 &= (1+q^{-1}) q^{-(m-1)l} \int_{\mathfrak{o}_F^\times} |1+v|_F^{m-2} \mathrm{d}v \\ &= (1+q^{-1}) q^{-(m-1)l} \times q^{-1} \left\{ (q-2) + \frac{q^{-(m-2)}(1-q^{-1})}{1-q^{-(m-1)}} \right\} \\ &= q^{-(m-1)l} (q^{-1} + q^{-2}) \frac{q-2+q^{-(m-1)}}{1-q^{-(m-1)}} \,. \end{aligned}$$

The formula in the lemma is obtained as $I_1 + I_2 + I_3$ by using the above evaluation of each summand. □

4.2. Spherical functions. Let P be an F-parabolic subgroup of G defined as the stabilizer of the rank one submodule Ee_1, and N the unipotent radical of P. Let P^1 be the stabilizer in G of e_1; then P^1 is a subgroup of P and $P = \{\mathsf{d}[t] \mid t \in E^\times\} P^1$. For any $\nu \in \mathbb{X}_E$, by letting the group G act by the right translation on the $\mathbb{C}$-vector space I_ν of all smooth functions $f : G \to \mathbb{C}$ such that

$$f(\mathsf{d}[t] p g) = |t|_E^{\nu + (m-1)/2} f(g) \,, \quad t \in E^\times \,, \quad p \in P^1 \,, \quad g \in G \,,$$

we define a smooth representation of G, denoted by (π_ν, I_ν). By the decomposition $G = P\mathcal{U}$, we have $I_\nu^{\mathcal{U}} = \mathbb{C} f_0^{(\nu)}$ with $f_0^{(\nu)}$ the unique element of I_ν such that $f_0^{(\nu)}(k) = 1$ for all $k \in \mathcal{U}$. Let us recall that a smooth G-module π is said to be H-distinguished if $\mathrm{Hom}_H(\pi, \mathbb{C}) \neq \{0\}$.

The representation I_ν is H-distinguished for all ν in an open and dense subset of $\mathbb{X}_E$. Indeed, we have

THEOREM 4.4. *There exists a unique meromorphic family* $\Xi^0(\nu) \in \mathrm{Hom}_H(I_\nu, \mathbb{C})$ *over* $\nu \in \mathbb{X}_E$ *such that* $\langle \Xi^0(\nu), f_0^{(\nu)} \rangle = 1$.

PROOF. We refer to § 5.3. □

Now, we define the spherical function $\Omega_\nu : G \to \mathbb{C}$ by

$$\Omega_\nu(g) = \langle \Xi^0(\nu), \pi_\nu(g)\, f_0^{(\nu)} \rangle, \quad g \in G. \tag{4.5}$$

Obviously, Ω_ν is left H-invariant and right $\mathcal{U}$-invariant. By the decomposition (4.2), we have a well-defined smooth function Ψ_ν on G by requiring that it is left H-invariant and right $\mathcal{U}$-invariant, and that it satisfies

$$\Psi_\nu(\mathsf{d}[\varpi_E^{-l}]) = q^{-\nu}\zeta_E(\nu + (m-1)/2)\, q_E^{-l(\nu+\frac{m-1}{2})}, \quad l \in \mathbb{N}. \tag{4.6}$$

We admit an ambiguity of the choice of a square root q^ν of q_E^ν when E is a field, which disappears in the following formula.

THEOREM 4.5. *For* $\nu \in \mathbb{X}_E$ *and* $g \in H\mathsf{d}[\varpi_E^{-l}]\mathcal{U}$ $(l \in \mathbb{N})$, *we have*

$$\begin{aligned}\Omega_\nu(g) &= \frac{1}{Q_{E,m}}\left\{\frac{\zeta_E(-\nu+(m-1)/2)^{-1}}{1-\varepsilon_E^m q^{2\nu}}(q_E^{-\nu-\frac{m-1}{2}})^l + \frac{\zeta_E(\nu+(m-1)/2)^{-1}}{1-\varepsilon_E^m q^{-2\nu}}(q_E^{\nu-\frac{m-1}{2}})^l\right\}\\ &= \frac{\zeta_E(\nu+(m-1)/2)^{-1}\zeta_E(-\nu+(m-1)/2)^{-1}}{Q_{E,m}(q^\nu-\varepsilon_E^m q^{-\nu})}\{-\varepsilon_E^m\Psi_\nu(g)+\Psi_{-\nu}(g)\}.\end{aligned}$$

PROOF. We refer to § 5.4. □

4.3. Spectral decomposition. Let $C_c(H\backslash G/\mathcal{U})$ be the space of all the finite $\mathbb{C}$-linear combinations of characteristic functions of double $(H, \mathcal{U})$-cosets. The spherical Fourier transform of a function $f \in C_c(H\backslash G/\mathcal{U})$ is defined to be the integral

$$\mathcal{F}_{\mathbf{h}} f(\nu) = \int_{H\backslash G} f(g)\Omega_\nu(g)\, \mathrm{d}\dot{g}, \quad \nu \in \mathbb{X}_E^0 \tag{4.7}$$

with $\mathrm{d}\dot{g}$ the G-invariant measure on $H\backslash G$ such that $\mathrm{vol}(H\backslash H\mathcal{U}) = 1$. By (4.2), the integral $\mathcal{F}_{\mathbf{h}} f(\nu)$ reduces to a sum of $f(\mathsf{d}[\varpi^{-l}])\Omega_\nu(\mathsf{d}[\varpi^{-l}])\,\mathrm{vol}(H\backslash H\mathsf{d}[\varpi^{-l}]\mathcal{U})$ for $0 \leqslant l \leqslant N$, where N is an integer such that $f(\mathsf{d}[\varpi^{-l}]) = 0$ for all $l > N$. From the first expression of Ω_ν in Theorem 4.5, the values $\Omega_\nu(\mathsf{d}[\varpi^{-l}])$ are seen to be Laurent polynomials of $X = q_E^{-\nu}$ invariant by $X \mapsto X^{-1}$. Hence $\mathcal{F}_{\mathbf{h}} f(\nu)$ also can be viewed as an element of $\mathcal{A} = \{\alpha(\nu) \in \mathbb{C}[X, X^{-1}] \mid \alpha(\nu) = \alpha(-\nu)\}$. Conversely, for any $\alpha \in \mathcal{A}$, we set

$$\mathcal{F}_{\mathbf{h}}^*\alpha(g) = \int_{\mathbb{X}_E^0} \alpha(\nu)\Omega_\nu(g)\, \mathrm{d}\Lambda_{\mathbf{h}}(\nu), \quad g \in G, \tag{4.8}$$

where $\mathrm{d}\Lambda_{\mathbf{h}}$ is a positive Radon measure on the imaginary axis $\mathbb{X}_E^0$ defined as

$$\mathrm{d}\Lambda_{\mathbf{h}}(\sqrt{-1}y) = \frac{Q_{E,m}}{4\pi}\left|\frac{1-\varepsilon_E^m q^{2\sqrt{-1}y}}{\zeta_E(\sqrt{-1}y+(m-1)/2)^{-1}}\right|^2 (\log q_E)\,\mathrm{d}y\,. \tag{4.9}$$

Let $L^2(\mathbb{X}_E^0; \mathrm{d}\Lambda_{\mathbf{h}})$ be the L^2-space for the measure space $(\mathbb{X}_E^0, \mathrm{d}\Lambda_{\mathbf{h}})$.

THEOREM 4.6. *The integrals* (4.7) *and* (4.8) *define* $\mathbb{C}$*-linear bijections*

$$\mathcal{F}_{\mathbf{h}} : C_c(H\backslash G/\mathcal{U}) \to \mathcal{A}\,, \quad \mathcal{F}_{\mathbf{h}}^* : \mathcal{A} \to C_c(H\backslash G/\mathcal{U})\,,$$

each of which inverts the other one. Moreover, $\mathcal{F}_{\mathbf{h}}$ *extends to an isometry from the Hilbert space* $L^2(H\backslash G/\mathcal{U})$ *onto the Hilbert space* $L^2(\mathbb{X}_E^0; \mathrm{d}\Lambda_{\mathbf{h}})$*. We have* $(\mathcal{F}_{\mathbf{h}} f|\alpha)_{\mathbb{X}_E^0} = (f|\mathcal{F}_{\mathbf{h}}^*\alpha)_{H\backslash G}$ *for all* $f \in L^2(H\backslash G/\mathcal{U})$ *and* $\alpha \in L^2(\mathbb{X}_E^0; \mathrm{d}\Lambda_{\mathbf{h}})$.

PROOF. See § 5.4. □

4.4. Orbital integrals. Recall that $H = \{g \in G|\, g\ell_0 \in E\ell_0\}$, $H_0 = \{g \in G|\, g\ell_0 = \ell_0\}$. As in § 1.2.2, we set $b(\gamma) = \mathbf{h}(\gamma^{-1}\ell_0, \ell_0)$, $\ell_0^\gamma = \gamma^{-1}\ell_0 - b(\gamma)\ell_0$, and $\Delta_\gamma = \mathbf{h}[\ell_0^\gamma]$ for $\gamma \in G - H$.

LEMMA 4.7. *Suppose* $\mathrm{N}b(\gamma) \neq 0$*. Then* $H \cap \gamma^{-1}H\gamma$ *consists of all* $g \in G$ *such that* $g\ell_0 = a\ell_0$, $g\ell_0^\gamma = a\ell_0^\gamma$ *with some* $a \in E^1$*. The subgroup* $H_0 \cap \gamma^{-1}H_0\gamma$ *consists of* $g \in G$ *such that* $g\ell_0 = \ell_0$ *and* $g\ell_0^\gamma = \ell_0^\gamma$*. In particular, the inclusion* $H_0 \hookrightarrow H$ *induces a bijection*

$$H_0 \cap \gamma^{-1}H_0\gamma\backslash H_0 \cong H \cap \gamma^{-1}H\gamma\backslash H\,.$$

PROOF. For $g \in G$ to belong to $\gamma^{-1}H\gamma \cap H$ is equivalent to the condition

$$g\ell_0 = a\ell_0\,, \quad g\gamma^{-1}\ell_0 = a_1\gamma^{-1}\ell_0 \quad \text{for some } a,\, a_1 \in E^1.$$

The first equation implies that g preserves the decomposition $V = E\ell_0 \oplus \ell_0^\perp$. From $\gamma^{-1}\ell_0 = b(\gamma)\ell_0 + \ell_0^\gamma$ and $\mathrm{N}b(\gamma) \neq 0$, the equation $g\gamma^{-1}\ell_0 = a_1\gamma^{-1}\ell_0$ yields $g\ell_0 = a_1\ell_0$ and $g\ell_0^\gamma = a_1\ell_0^\gamma$. Hence $a = a_1$. This shows the first two claims, which implies $H\cap\gamma^{-1}H\gamma = Z_G(H_0\cap\gamma^{-1}H_0\gamma)$ with Z_G the center of G. Since $H = Z_G H_0$, we have the last assertion. □

The unitary group $U(\mathbf{h}|\ell_0^\perp)$ of the hermitian space $(\ell_0^\perp, \mathbf{h}|\ell_0^\perp)$ is identified with H_0. The stabilizer $H_0(\ell_0^\gamma)$ in H_0 of ℓ_0^γ coincides with $H_0 \cap \gamma^{-1}H_0\gamma$. By the H_0-isomorphism

$$H_0(\ell_0^\gamma)\backslash H_0 \ni H(\ell_0^\gamma)h \mapsto h^{-1}\ell_0^\gamma \in \mathbf{\Sigma}(\ell_0^\perp, \Delta_\gamma)\,,$$

we transport the measure $|\omega_{\ell_0^\perp,\Delta_\gamma}|_F$ on $\mathbf{\Sigma}(\ell_0^\perp, \Delta_\gamma)$ to $H_0(\ell_0^\gamma)\backslash H_0$. By Lemma 4.7, we have an H-invariant measure on $H \cap \gamma^{-1}H\gamma\backslash H$ to be denoted by $\mathrm{d}O_\gamma(h)$ when $\mathrm{N}b(\gamma) \neq 0$.

For any $\gamma \in G - H$ such that $\mathrm{N}b(\gamma) \neq 0, 1$ and for any function $f \in C_c(H\backslash G/\mathcal{U})$, we consider the integral (1.6)

THEOREM 4.8. *Suppose E is a field. Let $\gamma \in G - H$ with $\mathrm{N}b(\gamma) \neq 0, 1$. Then for any $f \in C_{\mathrm{c}}(H\backslash G/\mathcal{U})$, the integral* (1.6) *converges absolutely and has the contour integral expression*

$$\mathbb{J}_{\mathbf{h}}^{\ell_0}(\gamma; f) = \frac{1}{\pi\sqrt{-1}} \int_{\mathbb{X}_E^{\delta}} \hat{\mathbb{J}}_{\mathbf{h}}^{\ell_0}(\gamma; \nu)\, \mathcal{F}_{\mathbf{h}} f(\nu)\, \mathrm{d}\mu_m(\nu)$$

with $\mathrm{d}\mu_m(\nu) = (q^{\nu} - (-1)^m q^{-\nu})(\log q)\mathrm{d}\nu$, *where* $\delta > \frac{m-3}{2}$ *and* $\hat{\mathbb{J}}_{\mathbf{h}}^{\ell_0}(\gamma; \nu)$ *is given as follows. If* $\mathrm{ord}_F \mathrm{N}b(\gamma) \geqslant 0$, *then*

$$\hat{\mathbb{J}}_{\mathbf{h}}^{\ell_0}(\gamma; \nu) = \frac{q^{-\nu}\{1 - \varepsilon_{\gamma}\, q^{-(m-2)}|1 - \mathrm{N}b(\gamma)|_F^{m-2} - \big(q^{-(m-2)} - \varepsilon_{\gamma}|1 - \mathrm{N}b(\gamma)|_F^{m-2}\big)X\}}{(1 - \varepsilon q^{-(m-1)})^{-1}(1 - \varepsilon q^{-(m-2)})(1 - q^{-(m-2)}X)(1 - q^{m-2}X)},$$

and if $\mathrm{ord}_F \mathrm{N}b(\gamma) < 0$, *then*

$$\hat{\mathbb{J}}_{\mathbf{h}}^{\ell_0}(\gamma; \nu) = \frac{q^{-\nu}|\mathrm{N}b(\gamma)|_F^{(m-2)/2-\nu-1/2}}{(1 - \varepsilon q^{-(m-1)})^{-1}} \frac{1 + \varepsilon X}{(1 - q^{m-2}X)(1 - q^{-(m-2)}X)},$$

where $X = q^{-(2\nu+1)}$, $\varepsilon = (-1)^{m-1}$ *and* $\varepsilon_{\gamma} = \varepsilon^{\mathrm{ord}_F(\varpi(1 - \mathrm{N}b(\gamma)))}$.

PROOF. Set $\alpha = \mathcal{F}_{\mathbf{h}} f$. Then from Theorem 4.6 and Lemma 4.5,

$$f(g) = \int_{\mathbb{X}_E^0} \alpha(\nu)\Omega_{\nu}(g)\mathrm{d}\Lambda_{\mathbf{h}}(\nu) = \frac{1}{\pi\sqrt{-1}} \int_{\mathbb{X}_E^0} \alpha(\nu)\Psi_{\nu}(g)\mathrm{d}\mu_m(\nu).$$

Since the integrand is holomorphic on $\mathrm{Re}(\nu) > 0$, we shift the contour $\mathbb{X}_E^0$ to $\mathbb{X}_E^{\delta}$ for any $\delta > 0$. By substituting the contour integral expression, we have

$$\mathbb{J}_{\mathbf{h}}^{\ell_0}(\gamma; f) = \int_{H\cap\gamma^{-1}H\gamma\backslash H} \left\{ \frac{1}{\pi\sqrt{-1}} \int_{\mathbb{X}_E^{\delta}} \alpha(\nu)\Psi_{\nu}(\gamma h)\mathrm{d}\mu_m(\nu) \right\} \mathrm{d}O_{\gamma}(h).$$

We shall show that the integral

$$\hat{\mathbb{J}}_{\mathbf{h}}^{\ell_0}(\gamma; \nu) = \int_{H\cap\gamma^{-1}H\gamma\backslash H} \Psi_{\nu}(\gamma h)\, \mathrm{d}O_{\gamma}(h)$$

converges absolutely for $\mathrm{Re}(\nu) > \frac{m-3}{2}$ and is evaluated as in the theorem. From the proof of [8, Proposition 3.9], an element $g \in G$ belongs to $H\mathsf{d}[\varpi^{-l}]\mathcal{U}$ if and only if $g^{-1}\ell_0 \in \varpi^{-l}\mathcal{L}_{\mathrm{prim}}$, where $\mathcal{L}_{\mathrm{prim}} = \mathcal{L} - \varpi\mathcal{L}$ is the set of primitive vectors in $\mathcal{L}$. Thus, by (4.2) and (4.6),

$$\hat{\mathbb{J}}_{\mathbf{h}^{\ell_0}}(\gamma; \nu) = q^{-\nu}(1 - q_E^{-\nu-(m-1)/2})^{-1} \sum_{l=0}^{\infty} q_E^{-l(\nu+(m-1)/2)} A_l \tag{4.10}$$

with

$$A_l = \mathrm{vol}(\{h \in H_0 \cap \gamma^{-1}H_0\gamma\backslash H_0 |\, h^{-1}\gamma^{-1}\ell_0 \in \varpi^{-l}\mathcal{L}_{\mathrm{prim}}\}; \mathrm{d}O_{\gamma}).$$

Set $\mathcal{M} = \ell_0^\perp \cap \mathcal{L}$. From $\gamma^{-1}\ell_0 = b(\gamma)\ell_0 + \ell_0^\gamma$ and $\mathcal{L} = \mathfrak{o}_E\ell_0 \oplus \mathcal{M}$, for $h \in H_0$, we have that $h^{-1}\gamma^{-1}\ell_0 \in \varpi^{-l}\mathcal{L}_{\mathrm{prim}}$ if and only if

(i) $b(\gamma) \in \varpi^{-l}\mathfrak{o}_E^\times$, $\quad h^{-1}\ell_0^\gamma \in \varpi^{-l}\mathcal{M}$, or

(ii) $b(\gamma) \in \varpi^{-l}\mathfrak{o}_E$, $\quad h^{-1}\ell_0^\gamma \in \varpi^{-l}\mathcal{M}_{\mathrm{prim}}$, where $\mathcal{M}_{\mathrm{prim}} = \mathcal{M} - \varpi\mathcal{M}$.

From assumption, $b(\gamma) \neq 0$ and $\Delta_\gamma = 1 - \mathrm{N}b(\gamma) \neq 0$. Set $e = \mathrm{ord}_E\, b(\gamma)$. From this, $A_l = 0$ for $0 \leqslant l < -e$ is obvious. If $l = -e$, then we have the case (i); thus from the construction of the measure $\mathrm{d}O_\gamma$ and by Lemma 4.2 applied to the unimodular lattice $\mathcal{M}$ ($\subset \ell_0^\perp$),

$$\begin{aligned}
A_{-e} &= \mathrm{vol}\left(\boldsymbol{\Sigma}(\ell_0^\perp, \Delta_\gamma) \cap \varpi^e\mathcal{M}; |\omega_{\ell_0^\perp,\Delta_\gamma}|_F\right)\\
&= q^{-2e(m-2)}\,\mathrm{vol}\left(\boldsymbol{\Sigma}(\ell_0^\perp, \varpi^{-2e}\Delta_\gamma) \cap \mathcal{M}; |\omega_{\ell_0^\perp,\varpi^{-2e}\Delta_\gamma}|_F\right) \quad \text{(by Lemma 4.1)}\\
&= q^{-2e(m-2)}\,\delta(\varpi^{-2e}\Delta_\gamma \in \mathfrak{o}_F)\frac{1-\varepsilon q^{-(m-1)}}{1-\varepsilon q^{-(m-2)}}\left(1 - \varepsilon^{\mathrm{ord}_F(\varpi\Delta_\gamma)}q^{-(m-2)}|\varpi^{-2e}\Delta_\gamma|_F^{m-2}\right),\\
&= q^{-2e(m-2)}\,\delta(\varpi^{-2e}\Delta_\gamma \in \mathfrak{o}_F)\frac{1-\varepsilon q^{-(m-1)}}{1-\varepsilon q^{-(m-2)}}\left(1 - \varepsilon^{\mathrm{ord}_F(\varpi\Delta_\gamma)}q^{(2e-1)(m-2)}|\Delta_\gamma|_F^{m-2}\right),
\end{aligned}$$

where $\varepsilon = (-1)^{m-1}$. If $l > -e$, then we have the case (ii); thus in the same way as above,

$$\begin{aligned}
A_l &= \mathrm{vol}(\boldsymbol{\Sigma}(\ell_0^\perp, \Delta_\gamma) \cap \varpi^{-l}\mathcal{M}; |\omega_{\ell_0^\perp,\Delta_\gamma}|_F) - \mathrm{vol}(\boldsymbol{\Sigma}(\ell_0^\perp, \Delta_\gamma) \cap \varpi^{-(l-1)}\mathcal{M}; |\omega_{\ell_0^\perp,\Delta_\gamma}|_F)\\
&= q^{2l(m-2)}\mathrm{vol}(\boldsymbol{\Sigma}(\ell_0^\perp, \varpi^{2l}\Delta_\gamma) \cap \mathcal{M}; |\omega_{\ell_0^\perp,\varpi^{2l}\Delta_\gamma}|_F)\\
&\quad - q^{2(l-1)(m-2)}\mathrm{vol}(\boldsymbol{\Sigma}(\ell_0^\perp, \varpi^{2(l-1)}\Delta_\gamma) \cap \mathcal{M}; |\omega_{\ell_0^\perp,\varpi^{2(l-1)}\Delta_\gamma}|_F) \qquad \text{(by Lemma 4.1)}\\
&= \frac{1-\varepsilon q^{-(m-1)}}{1-\varepsilon q^{-(m-2)}}\left\{q^{2l(m-2)}\delta(\varpi^{2l}\Delta_\gamma \in \mathfrak{o}_F)\left(1 - \varepsilon^{\mathrm{ord}_F(\varpi\Delta_\gamma)}q^{-(m-2)}|\varpi^{2l}\Delta_\gamma|_F^{m-2}\right)\right.\\
&\quad \left. - q^{2(l-1)(m-2)}\delta(\varpi^{2(l-1)}\Delta_\gamma \in \mathfrak{o}_F)\left(1 - \varepsilon^{\mathrm{ord}_F(\varpi\Delta_\gamma)}q^{-(m-2)}|\varpi^{2(l-1)}\Delta_\gamma|_F^{m-2}\right)\right\}.
\end{aligned}$$

If $e = \mathrm{ord}_E\, b(\gamma) < 0$, then $\mathrm{ord}_F(\Delta_\gamma) = \mathrm{ord}_F(1 - \mathrm{N}b(\gamma)) = 2e$ and $\mathrm{ord}_F(\varpi^{2l}\Delta_\gamma) = 2(l + e) \geqslant 0$ for $l > -e$. If $e \geqslant 0$, then $\mathrm{ord}_F(\Delta_\gamma) = \mathrm{ord}_F(1 - \mathrm{N}b(\gamma)) \geqslant 0$ and $\mathrm{ord}_F(\varpi^{2l}\Delta_\gamma) \geqslant 2l \geqslant 0$. Thus, $\mathrm{ord}_F(\varpi^{2(l-1)}\Delta_\gamma)$ ($l \in \mathbb{N}$, $l > -e$) is negative only when $\Delta_\gamma \in \mathfrak{o}_F^\times$, $e > 0$, $l = 0$. Hence, for $l \in \mathbb{N}$ with $l > -e$, A_l is equal to

$$\begin{aligned}
&\frac{1-\varepsilon q^{-(m-1)}}{1-\varepsilon q^{-(m-2)}}\left\{q^{2l(m-2)}\left(1 - \varepsilon^{\mathrm{ord}_F(\varpi\Delta_\gamma)}q^{-(m-2)}|\varpi^{2l}\Delta_\gamma|_F^{m-2}\right)\right.\\
&\quad - q^{2(l-1)(m-2)}\left(1 - \varepsilon^{\mathrm{ord}_F(\varpi\Delta_\gamma)}q^{-(m-2)}|\varpi^{2(l-1)}\Delta_\gamma|_F^{m-2}\right)\\
&\quad \left. + \delta(e > 0, l = 0, \Delta_\gamma \in \mathfrak{o}_F^\times)q^{-2(m-2)}(1 - \varepsilon q^{m-2})\right\}\\
&= \frac{1-\varepsilon q^{-(m-1)}}{1-\varepsilon q^{-(m-2)}}\left\{(1 - q^{-2(m-2)})q^{2l(m-2)} + \delta(e > 0, l = 0, \Delta_\gamma \in \mathfrak{o}_F^\times)q^{-2(m-2)}(1 - \varepsilon q^{m-2})\right\}.
\end{aligned}$$

Suppose $e > 0$. Then $\mathrm{N}b(\gamma) \in \varpi\mathfrak{o}_F$ and $\Delta_\gamma = 1 - \mathrm{N}b(\gamma) \in \mathfrak{o}_F^\times$. Hence,

$$\begin{aligned}
&\sum_{l=0}^{\infty} q_E^{-l(\nu+(m-1)/2)} A_l \\
&= \frac{1-\varepsilon q^{-(m-1)}}{1-\varepsilon q^{-(m-2)}} \sum_{l=0}^{\infty} q^{-2l(\nu+(m-1)/2)} \\
&\quad \times \left\{(1-q^{-2(m-2)})q^{2l(m-2)} + \delta(l=0)q^{-2(m-2)}(1-\varepsilon q^{m-2})\right\} \\
&= \frac{1-\varepsilon q^{-(m-1)}}{1-\varepsilon q^{-(m-2)}} \left\{ \frac{1-q^{-2(m-2)}}{1-q^{-2(\nu-(m-3)/2)}} + q^{-2(m-2)}(1-\varepsilon q^{m-2}) \right\} \\
&= \frac{1-\varepsilon q^{-(m-1)}}{1-\varepsilon q^{-(m-2)}} \times \frac{(1-\varepsilon q^{-(m-2)})(1+\varepsilon q^{-2\nu-1})}{1-q^{-2(\nu-(m-3)/2)}} = \frac{(1-\varepsilon q^{-(m-1)})(1+\varepsilon q^{-2\nu-1})}{1-q^{-2(\nu-(m-3)/2)}}.
\end{aligned}$$

By plugging this to (4.10), we obtain the formula of $\hat{\mathbb{J}}_{\mathbf{h}}^{\ell_0}(\gamma;\nu)$ as desired. Suppose $e = \mathrm{ord}_E\, b(\gamma) \leqslant 0$. Then

$$\begin{aligned}
&\sum_{l=0}^{\infty} q_E^{-l(\nu+(m-1)/2)} A_l \\
&= \frac{1-\varepsilon q^{-(m-1)}}{1-\varepsilon q^{-(m-2)}} \left\{ q^{2e(\nu-(m-3)/2)} \left(1 - \varepsilon^{\mathrm{ord}_F(\varpi\Delta_\gamma)} q^{(2e-1)(m-2)} |\Delta_\gamma|_F^{m-2}\right)\right. \\
&\quad \left. + \sum_{l=-e+1}^{\infty} q^{-2l(\nu+(m-1)/2)}(1-q^{-2(m-2)})q^{2l(m-2)} \right\} \\
&= \frac{1-\varepsilon q^{-(m-1)}}{1-\varepsilon q^{-(m-2)}} \left\{ q^{2e(\nu-(m-3)/2)} \left(1 - \varepsilon^{\mathrm{ord}_F(\varpi\Delta_\gamma)} q^{(2e-1)(m-2)} |\Delta_\gamma|_F^{m-2}\right)\right. \\
&\quad \left. + \frac{(1-q^{-2(m-2)})q^{2(\nu-(m-3)/2)(e-1)}}{1-q^{-2(\nu-(m-3)/2)}} \right\} \\
&= \frac{1-\varepsilon q^{-(m-1)}}{1-\varepsilon q^{-(m-2)}} \left\{ \frac{q^{2(\nu-(m-3)/2)e}}{1-q^{-2(\nu-(m-3)/2)}} - q^{-(m-2)}\varepsilon^{\mathrm{ord}_F(\varpi\Delta_\gamma)}|\Delta_\gamma|_F^{m-2} \frac{q^{2(\nu+(m-1)/2)e}}{1-q^{-2(\nu+(m-1)/2)}} \right\} \\
&\quad \times (1-q^{-2(\nu+(m-1)/2)}).
\end{aligned}$$

By plugging this to (4.10), we obtain the formula of $\hat{\mathbb{J}}_{\mathbf{h}}^{\ell_0}(\gamma;\nu)$ as desired. We note that when $e < 0$, then $\mathrm{ord}_F\, \mathrm{N}b(\gamma) = \mathrm{ord}_F(\Delta_\gamma) \in 2\mathbb{Z}$ and $|\Delta_\gamma|_F^{m-2} = q^{-2e(m-2)}$. □

THEOREM 4.9. *Suppose E is isomorphic to $F \oplus F$. Let $\gamma \in G - H$ with* $\mathrm{N}b(\gamma) \neq 0, 1$. *Then for any $f \in C_c(H\backslash G/\mathcal{U})$, the integral* (1.6) *converges absolutely and has the contour integral expression*

$$\mathbb{J}_{\mathbf{h}}^{\ell_0}(\gamma; f) = \frac{1}{\pi\sqrt{-1}} \int_{\mathbb{X}_E^\delta} \hat{\mathbb{J}}_{\mathbf{h}}^{\ell_0}(\gamma;\nu)\, \mathcal{F}_{\mathbf{h}} f(\nu)\, \mathrm{d}\mu(\nu)$$

with $\mathrm{d}\mu(\nu) = 2^{-1}(q^{\nu} - q^{-\nu})(\log q)\mathrm{d}\nu$, *where* $\delta > \frac{m-3}{2}$ *and* $\hat{\mathbb{J}}_{\mathbf{h}}^{\ell_0}(\gamma;\nu)$ *is defined as follows*: *If* $e = \mathrm{ord}_F \mathrm{N}b(\gamma) > 0$,

$$\hat{\mathbb{J}}_{\mathbf{h}}^{\ell_0}(\gamma;\nu) = q^{-\nu}(1-q^{-(m-1)})\frac{(e-1)X^2 - e(q^{(m-2)/2}+q^{-(m-2)/2})X + (e+1)}{(1-q^{(m-2)/2}X)^2(1-q^{-(m-2)/2}X)^2},$$

and if $e = \mathrm{ord}_F \mathrm{N}b(\gamma) \leqslant 0$,

$$\hat{\mathbb{J}}_{\mathbf{h}}^{\ell_0}(\gamma;\nu) = q^{-\nu}\frac{1-q^{-(m-1)}}{1-q^{-(m-2)}}(q^{(m-2)/2}X)^{-e} \times \left\{\frac{1}{(1-q^{(m-2)/2}X)^2} - \frac{q^{-(m-2)}|1-\mathrm{N}b(\gamma)^{-1}|_F^{m-2}}{(1-q^{-(m-2)/2}X)^2}\right\},$$

where $X = q^{-(\nu+1/2)}$.

PROOF. In the same way as the proof of the previous theorem, we have

$$\mathbb{J}_{\mathbf{h}}^{\ell_0}(\gamma;f) = \int_{H\cap\gamma^{-1}H\gamma\backslash H}\left\{\frac{1}{\pi\sqrt{-1}}\int_{\mathbb{X}_E^{\delta}}\alpha(\nu)\Psi_\nu(\gamma h)\mathrm{d}\mu(\nu)\right\}\mathrm{d}O_\gamma(h),$$

where $\alpha = \mathcal{F}_{\mathbf{h}}f(\nu)$. By Lemma 4.7, in the h-integral, we may replace the integration domain with $H_0\cap\gamma^{-1}H_0\gamma\backslash H_0$. We shall show that the integral

$$\hat{\mathbb{J}}_{\mathbf{h}}^{\ell_0}(\gamma;\nu) = \int_{H_0\cap\gamma^{-1}H_0\gamma\backslash H_0}\Psi_\nu(\gamma h)\,\mathrm{d}O_\gamma(h)$$

converges absolutely for $\mathrm{Re}(\nu) > \frac{m-3}{2}$ and is evaluated as in the theorem. We may take our ℓ_0 of the form $\ell_0 = ae_1 + e_m$ with $a \in \mathfrak{o}_E$. Let $\theta_1 \in \mathfrak{o}_F$ be a square root of θ in F. Then we fix the identification $E = F[\sqrt{\theta}] \cong F\oplus F$ by the map sending $a+b\sqrt{\theta}$ to $(a+\theta_1 b, a-\theta_1 b)$; thus $\sqrt{\theta} = (\theta_1,-\theta_1)$ and the norm of $b = (b_1,b_2) \in E$ is given as $\mathrm{N}b = b_1b_2$. Set $\varpi_1 = (\varpi,1)$ and $\varpi_2 = (1,\varpi)$. Then, from the proof of Lemma 6.2, $g\in G$ belongs to the coset $H_0\mathsf{d}[\varpi_1^{-l_1}\varpi_2^{-l_2}]\mathcal{U}$ if and only if $g^{-1}\ell_0 \in \varpi_1^{-l_1}\varpi_2^{-l_2}\mathcal{L}_{\mathrm{prim}}$, where $\mathcal{L}_{\mathrm{prim}} = \mathcal{L} - (\varpi_1\mathcal{L}\cup\varpi_2\mathcal{L})$. Since G is a disjoint union of $H_0\mathsf{d}[\varpi_1^{-l_1}\varpi_2^{-l_2}]\mathcal{U}$ ($l_1,l_2\in\mathbb{Z}$, $l_1+l_2\geqslant 0$) and since $H_0\mathsf{d}[\varpi_1^{-l_1}\varpi_2^{-l_2}]\mathcal{U} = \{g\in G|\,g^{-1}\ell_0\in\varpi_1^{-l_1}\varpi_2^{-l_2}\mathcal{L}_{\mathrm{prim}}\}$ as seen in §6, we have

$$\hat{\mathbb{J}}_{\mathbf{h}}^{\ell_0}(\gamma;\nu) = q^{-\nu}(1-q^{-(\nu+(m-1)/2)})^{-2}\sum_{l=0}^{\infty}q^{-l(\nu+(m-1)/2)}\sum_{\substack{(l_1,l_2)\in\mathbb{Z}^2\\ l_1+l_2=l}}A_{l_1,l_2} \tag{4.11}$$

with

$$A_{l_1,l_2} = \mathrm{vol}\left(\{h\in H_0\cap\gamma^{-1}H_0\gamma\backslash H_0|\,h^{-1}\gamma^{-1}\ell_0\in\varpi_1^{-l_1}\varpi_2^{-l_2}\mathcal{L}_{\mathrm{prim}}\};\mathrm{d}O_\gamma\right).$$

Set $\mathfrak{n} = \varpi_1^{l_1}\varpi_2^{l_2}\mathfrak{o}_E$ and $\mathcal{M} = \ell_0^{\perp} \cap \mathcal{L}$. Since $h^{-1}\gamma^{-1}\ell_0 = b(\gamma)\ell_0 + h^{-1}\ell_0^{\gamma}$ is the decomposition by the direct sum $\mathcal{L} = \mathfrak{o}_E\ell_0 \oplus \mathcal{M}$, the vector $h^{-1}\gamma^{-1}\ell_0$ belongs to $\mathfrak{n}^{-1}\mathcal{L}_{\mathrm{prim}}$ if and only if one of the following 4 conditions is satisfied.

(i) $b(\gamma) \in \mathfrak{n}^{-1}\mathfrak{o}_E^{\times}$, $h^{-1}\ell_0^{\gamma} \in \mathfrak{n}^{-1}\mathcal{M}$,

(ii) $b(\gamma) \in \mathfrak{n}^{-1}(\varpi_1\mathfrak{o}_E \cap \varpi_2\mathfrak{o}_E)$, $h^{-1}\ell_0^{\gamma} \in \mathfrak{n}^{-1}(\mathcal{M} - \varpi_1\mathcal{M} - \varpi_2\mathcal{M})$,

(iii) $b(\gamma) \in \mathfrak{n}^{-1}(\varpi_1\mathfrak{o}_E - \varpi_2\mathfrak{o}_E)$, $h^{-1}\ell_0^{\gamma} \in \mathfrak{n}^{-1}(\mathcal{M} - \varpi_1\mathcal{M})$,

(iv) $b(\gamma) \in \mathfrak{n}^{-1}(\varpi_2\mathfrak{o}_E - \varpi_1\mathfrak{o}_E)$, $h^{-1}\ell_0^{\gamma} \in \mathfrak{n}^{-1}(\mathcal{M} - \varpi_2\mathcal{M})$.

Set $b(\gamma) = (b_1, b_2)$ and $e_j = \mathrm{ord}_F(b_j)$ $(j = 1, 2)$; then $\mathrm{N}b(\gamma) = b_1b_2 \neq 0$, $\Delta_\gamma = 1 - b_1b_2 \neq 0$ from assumption, and $e = \mathrm{ord}_F\,\mathrm{N}b(\gamma) = e_1 + e_2$. Set $l = l_1 + l_2$. The case (i) occurs if and only if $l_1 = -e_1$, $l_2 = -e_2$. By Lemmas 4.1 and 4.2, we have

$$\begin{aligned}A_{-e_1,-e_2} &= \mathrm{vol}\big(\boldsymbol{\Sigma}(\ell_0^{\perp}, \Delta_\gamma) \cap \varpi_1^{e_1}\varpi_2^{e_2}\mathcal{M};\, |\omega_{\ell_0^{\perp},\Delta_\gamma}|_F\big)\\ &= q^{-e(m-2)}\delta(\varpi^{-e}\Delta_\gamma \in \mathfrak{o}_F)\frac{1-q^{-(m-1)}}{1-q^{-(m-2)}}\big(1 - q^{-(m-2)}|\varpi^{-e}\Delta_\gamma|_F^{m-2}\big).\end{aligned}$$

If $e > 0$, then $\mathrm{ord}_F(\Delta_\gamma) = \mathrm{ord}_F(1 - \mathrm{N}b(\gamma)) = 0$; thus $\delta(\varpi^{-e}\Delta_\gamma \in \mathfrak{o}_F) = 0$ in this case. If $e \leqslant 0$, then $\mathrm{ord}_F(\Delta_\gamma) \geqslant \mathrm{ord}_F\,\mathrm{N}b(\gamma) = e$ and $\delta(\varpi^{-e}\Delta_\gamma \in \mathfrak{o}_F) = 1$.

The case (ii) occurs if and only if $l_1 \geqslant -e_1 + 1$, $l_2 \geqslant -e_2 + 1$. By Lemmas 4.1 and 4.2,

$$\begin{aligned}A_{l_1,l_2} &= q^{l(m-2)}\bigg\{\int_{\boldsymbol{\Sigma}(\ell_0^{\perp},\varpi^{l}\Delta_\gamma)\cap\mathcal{M}}|\omega_{\ell_0^{\perp},\varpi^{l}\Delta_\gamma}|_F - q^{-(m-2)}\int_{\boldsymbol{\Sigma}(\ell_0^{\perp},\varpi^{l-1}\Delta_\gamma)\cap\mathcal{M}}|\omega_{\ell_0^{\perp},\varpi^{l-1}\Delta_\gamma}|_F\\ &\quad - q^{-(m-2)}\int_{\boldsymbol{\Sigma}(\ell_0^{\perp},\varpi^{l-1}\Delta_\gamma)\cap\mathcal{M}}|\omega_{\ell_0^{\perp},\varpi^{l-1}\Delta_\gamma}|_F + q^{-2(m-2)}\\ &\quad \times\int_{\boldsymbol{\Sigma}(\ell_0^{\perp},\varpi^{l-2}\Delta_\gamma)\cap\mathcal{M}}|\omega_{\ell_0^{\perp},\varpi^{l-2}\Delta_\gamma}|_F\bigg\}\\ &= q^{l(m-2)}\frac{1-q^{-(m-1)}}{1-q^{-(m-2)}}\bigg\{\delta(\varpi^{l}\Delta_\gamma \in \mathfrak{o}_F)\Big(1 - q^{-(m-2)}|\varpi^{l}\Delta_\gamma|_F^{m-2}\Big)\\ &\quad - 2q^{-(m-2)}\delta(\varpi^{l-1}\Delta_\gamma \in \mathfrak{o}_F)\Big(1 - q^{-(m-2)}|\varpi^{l-1}\Delta_\gamma|_F^{m-2}\Big)\\ &\quad + q^{-2(m-2)}\delta(\varpi^{l-2}\Delta_\gamma \in \mathfrak{o}_F)\Big(1 - q^{-(m-2)}|\varpi^{l-2}\Delta_\gamma|_F^{m-2}\Big)\bigg\}.\end{aligned}$$

The case (iii) occurs if and only if $l_1 \geqslant -e_1 + 1$, $l_2 = -e_2$; the case (iv) occurs if and only if $l_1 = -e_1$, $l_2 \geqslant -e_2 + 1$. In the same way as above, we have

$$\begin{aligned}A_{l_1,l_2} &= q^{l(m-2)}\bigg\{\int_{\boldsymbol{\Sigma}(\ell_0^{\perp},\varpi^{l}\Delta_\gamma)\cap\mathcal{M}}|\omega_{\ell_0^{\perp},\varpi^{l}\Delta_\gamma}|_F - q^{-(m-2)}\int_{\boldsymbol{\Sigma}(\ell_0^{\perp},\varpi^{l-1}\Delta_\gamma)\cap\mathcal{M}}|\omega_{\ell_0^{\perp},\varpi^{l-1}\Delta_\gamma}|_F\bigg\}\\ &= q^{l(m-2)}\frac{1-q^{-(m-1)}}{1-q^{-(m-2)}}\bigg\{\delta(\varpi^{l}\Delta_\gamma \in \mathfrak{o}_F)\Big(1 - q^{-(m-2)}|\varpi^{l}\Delta_\gamma|_F^{m-2}\Big)\\ &\quad - q^{-(m-2)}\delta(\varpi^{l-1}\Delta_\gamma \in \mathfrak{o}_F)\Big(1 - q^{-(m-2)}|\varpi^{l-1}\Delta_\gamma|_F^{m-2}\Big)\bigg\}.\end{aligned}$$

Thus,

$$\sum_{l=0}^{\infty} q^{-l(\nu+(m-1)/2)} \sum_{\substack{(l_1,l_2)\in\mathbb{Z}^2\\ l_1+l_2=l}} A_{l_1,l_2} = \frac{1-q^{-(m-1)}}{1-q^{-(m-2)}}(I_1+I_2+I_3+I_4), \tag{4.12}$$

where

$$I_1 = q^{e(\nu-(m-3)/2)}\delta(e\leqslant 0)\left(1-q^{-(m-2)}|\varpi^{-e}\Delta_\gamma|_F^{m-2}\right),$$

$$\begin{aligned}I_2 = \sum_{\substack{l_1>-e_1,l_2>-e_2\\ l=l_1+l_2\geqslant 0}} q^{-l(\nu-(m-3)/2)}\Big\{&\delta(\varpi^l\Delta_\gamma\in\mathfrak{o}_F)\left(1-q^{-(m-2)}|\varpi^l\Delta_\gamma|_F^{m-2}\right)\\ &-2q^{-(m-2)}\delta(\varpi^{l-1}\Delta_\gamma\in\mathfrak{o}_F)\left(1-q^{-(m-2)}|\varpi^{l-1}\Delta_\gamma|_F^{m-2}\right)\\ &+q^{-2(m-2)}\delta(\varpi^{l-2}\Delta_\gamma\in\mathfrak{o}_F)\left(1-q^{-(m-2)}|\varpi^{l-2}\Delta_\gamma|_F^{m-2}\right)\Big\},\end{aligned}$$

$$\begin{aligned}I_3 = \sum_{\substack{l_1>-e_1,\\ l=l_1-e_2\geqslant 0}} q^{-(l_1-e_2)(\nu-(m-3)/2)}\Big\{&\delta(\varpi^{l_1-e_2}\Delta_\gamma\in\mathfrak{o}_F)\left(1-q^{-(m-2)}|\varpi^{l_1-e_2}\Delta_\gamma|_F^{m-2}\right)\\ &-q^{-(m-2)}\delta(\varpi^{l_1-e_2-1}\Delta_\gamma\in\mathfrak{o}_F)\left(1-q^{-(m-2)}|\varpi^{l_1-e_2-1}\Delta_\gamma|_F^{m-2}\right)\Big\},\end{aligned}$$

$$\begin{aligned}I_4 = \sum_{\substack{l_2>-e_2,\\ l=l_2-e_1\geqslant 0}} q^{-(l_2-e_1)(\nu-(m-3)/2)}\Big\{&\delta(\varpi^{l_2-e_1}\Delta_\gamma\in\mathfrak{o}_F)\left(1-q^{-(m-2)}|\varpi^{l_2-e_1}\Delta_\gamma|_F^{m-2}\right)\\ &-q^{-(m-2)}\delta(\varpi^{l_2-e_1-1}\Delta_\gamma\in\mathfrak{o}_F)\left(1-q^{-(m-2)}|\varpi^{l_2-e_1-1}\Delta_\gamma|_F^{m-2}\right)\Big\}.\end{aligned}$$

Suppose $e=\mathrm{ord}_F\,\mathrm{N}b(\gamma)>0$. Then $I_1=0$ obviously, and $\mathrm{ord}_F\,\Delta_\gamma=\mathrm{ord}_F(1-\mathrm{N}b(\gamma))=0$. Hence

$$\begin{aligned}I_2 &= \sum_{\substack{l_1>-e_1,l_2>-e_2\\ l=l_1+l_2\geqslant 0}} q^{-l(\nu-(m-3)/2)}\Big\{(1-q^{-(l+1)(m-2)})-2q^{-(m-2)}\delta(l\geqslant 1)(1-q^{-l(m-2)})\\ &\qquad +q^{-2(m-2)}\delta(l\geqslant 2)(1-q^{-(l-1)(m-2)})\Big\}\\ &= \sum_{\substack{l_1>-e_1,l_2>-e_2\\ l=l_1+l_2\geqslant 0}} q^{-l(\nu-(m-3)/2)}\Big\{\delta(l=0)(1-q^{-(m-2)})+\delta(l\geqslant 1)(1-q^{-(m-2)})^2\Big\}\\ &= (e-1)(1-q^{-(m-2)})+(1-q^{-(m-2)})^2\sum_{l\geqslant 1}(l+e-1)Y^l\end{aligned}$$

with $Y = q^{-(\nu-(m-3)/2)}$. By applying the formula

$$\sum_{l=a}^{\infty}(l+e-1)Y^l = \frac{(e-1)Y^a}{1-Y} + \frac{Y\{(1-a)Y^a + aY^{a-1}\}}{(1-Y)^2}, \quad (|Y|<1,\ a\in\mathbb{N}),$$

we obtain

$$I_2 = (e-1)(1-q^{-(m-2)}) + \left\{\frac{(e-1)Y}{1-Y} + \frac{Y}{(1-Y)^2}\right\}(1-q^{-(m-2)})^2.$$

It is evident that $I_3 = I_4$. Since $l_1 \geqslant e_2$ implies $l_1 > -e_1$ for $e > 0$, we have that I_3 equals

$$\begin{aligned}
&\sum_{\substack{l_1>-e_1,\\ l=l_1-e_2\geqslant 0}} q^{-(l_1-e_2)(\nu-(m-3)/2)}\Big\{\delta(l_1\geqslant e_2)(1-q^{-(l_1-e_2+1)(m-2)}) \\
&\quad - q^{-(m-2)}\delta(l_1\geqslant e_2+1)(1-q^{-(l_1-e_2)(m-2)})\Big\} \\
&= \sum_{l\geqslant 0} Y^l(1-q^{-(l+1)(m-2)}) - q^{-(m-2)}\sum_{l\geqslant 1} Y^l(1-q^{-l(m-2)}) \\
&= \frac{1}{1-Y} - \frac{q^{-(m-2)}}{1-q^{-(m-2)}Y} - \frac{q^{-(m-2)}Y}{1-Y} + \frac{q^{-2(m-2)}Y}{1-q^{-(m-2)}Y} = -q^{-(m-2)} + \frac{1-q^{-(m-2)}Y}{1-Y}.
\end{aligned}$$

Hence $I_1 + I_2 + I_3 + I_4$ becomes

$$\begin{aligned}
&(1-q^{-(m-2)})(e-1) + \left\{\frac{(e-1)Y}{1-Y} + \frac{Y}{(1-Y)^2}\right\}(1-q^{-(m-2)})^2 \\
&\quad + 2\left(-q^{-(m-2)} + \frac{1-q^{-(m-2)}Y}{1-Y}\right) \\
&= (1-q^{-(m-2)})\,\frac{(e-1)q^{-(m-2)}Y^2 - e(1+q^{-(m-2)})Y + (e+1)}{(1-Y)^2}.
\end{aligned}$$

From this, combined with (4.11) and (4.12), we have the desired formula.

Consider the case when $e = \mathrm{ord}_F\,\mathrm{N}b(\gamma) < 0$. Then $\mathrm{ord}_F(\Delta_\gamma) = e$. In the same way as before, we have $I_1 = (1-q^{-(m-2)})Y^{-e}$,

$$\begin{aligned}
I_2 &= \sum_{\substack{l_1>-e_1,l_2>-e_2\\ l=l_1+l_2\geqslant -e}} q^{-l(\nu-(m-3)/2)}\left\{\delta(l=-e)(1-q^{-(m-2)}) + \delta(l\geqslant -e+1)(1-q^{-(m-2)})^2\right\} \\
&= (1-q^{-(m-2)})^2 \sum_{l\geqslant -e+1}(l+e-1)Y^l \\
&= \left\{\frac{(e-1)Y^{-e+1}}{1-Y} + \frac{Y\{eY^{-e+1} + (-e+1)Y^{-e}\}}{(1-Y)^2}\right\}(1-q^{-(m-2)})^2.
\end{aligned}$$

and

$$I_3 = I_4 = \sum_{l \geqslant -e+1} Y^l \left\{(1 - q^{-(l+e+1)(m-2)}) - q^{-(m-2)}(1 - q^{-(l+e)(m-2)})\right\}$$
$$= (1 - q^{-(m-2)}) \frac{Y^{-e+1}}{1-Y}.$$

Hence, $I_1 + I_2 + I_3 + I_4$ becomes

$$(1 - q^{-(m-2)}) \left\{ 2\frac{Y^{-e+1}}{1-Y} + Y^{-e} + \left(\frac{(e-1)Y^{-e+1}}{1-Y} + \frac{Y\{eY^{-e+1} + (-e+1)Y^{-e}\}}{(1-Y)^2} \right)(1 - q^{-(m-2)}) \right\}$$
$$= (1 - q^{-(m-2)}) Y^{-e} \frac{1 - q^{-(m-2)}Y^2}{(1-Y)^2}.$$

From this, combined with (4.11) and (4.12), we have the following as desired:

$$\hat{\mathbb{J}}^{\ell_0}_{\mathbf{h}}(\gamma;\nu) = q^{-\nu}(1 - q^{-(m-1)}) \frac{Y^{-e}(1 - q^{-(m-2)}Y^2)}{(1 - q^{-(m-2)}Y)^2(1-Y)^2}$$
$$= q^{-\nu} \frac{1 - q^{-(m-1)}}{1 - q^{-(m-2)}} Y^{-e} \left\{ \frac{1}{(1-Y)^2} - \frac{q^{-(m-2)}}{(1 - q^{-(m-2)}Y)^2} \right\}.$$

Consider the case $e = \operatorname{ord}_F \mathrm{N}b(\gamma) = 0$. Set $a = \operatorname{ord}_F(\Delta_\gamma)$; then $a \geqslant 0$. We have $I_1 = 1 - q^{-(a+1)(m-2)}$,

$$I_2 = \sum_{\substack{l_1 > -e_1, l_2 > -e_2 \\ l = l_1 + l_2 \geqslant 0}} Y^l \Big\{ 1 - q^{-(l+a+1)(m-2)} - 2\delta(l+a \geqslant 1) q^{-(m-2)}(1 - q^{-(l+a)(m-2)})$$
$$+ \delta(l+a \geqslant 2) q^{-2(m-2)}(1 - q^{-(l+a-1)(m-2)}) \Big\}$$
$$= \sum_{\substack{l_1 > -e_1, l_2 > -e_2 \\ l = l_1 + l_2 \geqslant 0}} Y^l \left\{ \delta(l+a=0)(1 - q^{-(m-2)}) + \delta(l+a \geqslant 1)(1 - q^{-(m-2)})^2 \right\}$$
$$= (1 - q^{-(m-2)})^2 \sum_{l \geqslant 1} (l-1) Y^l = (1 - q^{-(m-2)})^2 \left\{ \frac{-Y}{1-Y} + \frac{Y}{(1-Y)^2} \right\},$$

and

$$I_3 = I_4 = \sum_{l \geqslant 1} Y^l \{\delta(l+a \geqslant 0)(1 - q^{-(m-2)(l+a+1)}) - q^{-2(m-1)}\delta(l+a \geqslant 1)(1 - q^{-(m-2)(l+a)})\}$$
$$= (1 - q^{-(m-2)}) \sum_{l \geqslant 1} Y^l = (1 - q^{-(m-2)}) \frac{Y}{1-Y}.$$

Thus $I_1 + I_2 + I_3 + I_4$ becomes

$$1 - q^{-(a+1)(m-2)} + (1 - q^{-(m-2)})^2 \left\{ \frac{-Y}{1-Y} + \frac{Y}{(1-Y)^2} \right\} + 2(1 - q^{-(m-2)}) \frac{Y}{1-Y}$$
$$= -q^{-(a+1)(m-2)} + \frac{(1 - q^{-(m-2)}Y)^2}{(1-Y)^2}.$$

From this, combined with (4.11) and (4.12), we have the following as desired:

$$\hat{\mathbb{J}}_{\mathbf{h}}^{\ell_0}(\gamma; \nu) = q^{-\nu} \frac{1 - q^{-(m-1)}}{1 - q^{-(m-2)}} \left\{ -\frac{q^{-(a+1)(m-2)}}{(1 - q^{-(m-2)}Y)^2} + \frac{1}{(1-Y)^2} \right\}.$$

□

4.5. The proof of Theorem 1.1. We work with notation in §1.2.3, holding all the assumptions made there. Due to the normalization of measures, it is easy to have $\mathcal{F} f^\circ = 1$ and $\mathcal{F}_{\mathbf{h}} \phi^\circ = 1$, which evidently yields the identity $\mathrm{Trans}_\eta(f^\circ) = \phi^\circ$ by definition. From the definitions, we have $\xi(\varpi) = \varepsilon_E^{m-1} q^{-(m-2)}$, $z = q_E^{-s}$ and $\iota(z) = \varepsilon_E^{m-1} z$. Note that $\xi(\varpi) \neq 1$ from $m \geqslant 4$. The required relation reduces to the identity

$$\xi(b-1)\, q^s \hat{\mathbb{J}}(b; s, \xi) = (1 - \varepsilon_E^{m-1} q^{-(m-1)})^{-1}\, q^\nu \hat{\mathbb{J}}_{\mathbf{h}}^{\ell_0}(\gamma; \nu)$$

for all $(s, \nu) \in \mathbb{X}_E \times \mathbb{X}_E$ such that $q_E^{-\nu} = \varepsilon_E^{m-1} q_E^{-s}$. This in turn follows from Theorems 2.2 and 4.8 if E is a field and from Theorems 3.2 and 4.9 is $E \cong F \oplus F$.

5. Appendix 1 : Harmonic analysis on hyperboloids

In this section, we use the same symbols and notation for objects introduced in §1.2.2 and §4. Beside these, we further need the following ingredients. Set $V_1 = \sum_{j=2}^{m-1} E e_j$; then V is decomposed to orthogonal direct sum of $Ee_1 + Ee_m$ and V_1. Let G_1 denote the unitary group $U(\mathbf{h}|V_1)$. For $X \in V_1$ and $b \in F$, let $\mathsf{n}(X; b)$ be the element of G defined by

$$\mathsf{n}(X; b)e_1 = e_1, \quad \mathsf{n}(X; b)e_j = e_j - \mathbf{h}(e_j, X)e_1 \ (1 < j < m),$$
$$\mathsf{n}(X; b)e_m = e_m + X + (-2^{-1}\mathbf{h}[X] + \sqrt{\theta} b)e_1.$$

For $h \in G_1$, let us define $\mathsf{m}[h] \in G$ by

$$\mathsf{m}[h]e_1 = e_1, \quad \mathsf{m}[h]e_m = e_m, \quad \mathsf{m}[h]|V_1 = h.$$

Then $N = \{\mathsf{n}(X; b) |\, X \in V_1,\ b \in F\}$ is the unipotent radical of P and $M = \{\mathsf{m}[h]\mathsf{d}[t] |\, h \in G_1,\ t \in E^\times\}$ is a Levi subgroup of P with the Levi decomposition $P = MN$. Moreover, $P^1 = \{\mathsf{m}[h] |\, h \in G_1\}N$. Let w_0 and w_1 be the elements of G defined by

$$w_0(e_1) = e_m, \quad w_0(e_m) = e_1, \quad w_0(e_j) = e_j \ (1 < j < m),$$
$$w_1(e_1) = e_2, \quad w_1(e_2) = e_1, \quad w_1(e_{m-1}) = e_m, \quad w_1(e_m) = e_{m-1},$$

$$w_1(e_j) = e_j \ (2 < j < m-1) \,.$$

Set $\bar{\mathsf{n}}(X; b) = w_0 \mathsf{n}(X; b) w_0$ for $(X, b) \in V_1 \times F$ and $\bar{N} = w_0 N w_0$. Then $\bar{P} = M\bar{N}$ is the parabolic subgroup opposite to P. Let s_{ℓ_0} be the reflexion of the hermitian space V with respect to the vector ℓ_0 and $\sigma : G \to G$ denote the inner automorphism $g \mapsto s_{\ell_0}^{-1} g s_{\ell_0}$ of G. It turns out that the F-torus $S_0 = \{\mathsf{d}[\tau] |\ \tau \in F^\times\}$ is a (σ, F)-split torus and P is σ-split in the sense that $P \cap \sigma(P) = M = Z_G(S_0)$ ([14]). The diagonal elements of G with entries from F form a maximal F-split torus, say T, containing S_0. Let Σ be the root system of (G, T) and Σ_M that of (M, T). Let ε_j $(1 \leqslant j \leqslant [\frac{m}{2}])$ be the character of T which sends a diagonal matrix to its j-th diagonal entry. Then the character group of T is a free $\mathbb{Z}$-module with basis $\{\varepsilon_j\}_{j=1}^{[\frac{m}{2}]}$. Set $\Sigma^+_{(\mathrm{l})} = \{\varepsilon_i \pm \varepsilon_j | 1 \leqslant i < j \leqslant [\frac{m}{2}]\} \cup \{2\varepsilon_i | 1 \leqslant i \leqslant [\frac{m}{2}]\}$ and $\Sigma^+_{(\mathrm{s})} = \{\varepsilon_j | 1 \leqslant j \leqslant [\frac{m}{2}]\}$. Then $\Sigma^+ = \Sigma^+_{(\mathrm{l})}$ is a positive system of Σ, a root system of type C_n if $m = 2n$, and $\Sigma^+ = \Sigma^+_{(\mathrm{l})} \cup \Sigma^+_{(\mathrm{s})}$ is a positive system of Σ, a root system of type BC_n if $m = 2n+1$ is odd. Note that any T-root occurring in N belongs to Σ^+. Set $\Sigma^- = -\Sigma^+$, $\Sigma^{\pm}_M = \Sigma_M \cap \Sigma^{\pm}$. For $\alpha \in \Sigma$, let $m(\alpha)$ denote its multiplicity. Let W_G be the Weyl group of (G, T) and W_M that of (M, T).

To make the presentation clear, we solely describe proofs for the case when E is a field. In the remark in §5.3, we indicate necessary modifications to treat the case when $E \cong F \oplus F$.

5.1. Hecke operators in paraboric level. Set

$$\mathcal{C} = \{u \in \mathcal{U} |\ u(e_1) - te_1 \in \varpi\mathcal{L} \quad \text{for some } t \in \mathfrak{o}_E^\times\} \,.$$

Then $\mathcal{C}$ is an open subgroup of $\mathcal{U}$ admitting the Iwahori factorization $\mathcal{C} = \mathcal{N}_1^- \mathcal{M} \mathcal{N}_0 = \mathcal{N}_0 \mathcal{M} \mathcal{N}_1^-$ with $\mathcal{N}_1^- = \{\bar{\mathsf{n}}(X; b) |\ X \in \varpi\mathcal{L} \cap V_1,\ b \in \varpi\mathfrak{o}_F\}$, $\mathcal{N}_0 = N \cap \mathcal{U}$ and $\mathcal{M} = M \cap \mathcal{U}$. Let $\mathrm{d}c$ be the restriction to $\mathcal{C}$ of the Haar measure on G. Then $\mathrm{d}c = \mathrm{vol}(\mathcal{C}; \mathrm{d}g)\, \mathrm{d}v\, \mathrm{d}m\, \mathrm{d}u = \mathrm{vol}(\mathcal{C}; \mathrm{d}g) \mathrm{d}u\, \mathrm{d}m\, \mathrm{d}v$ with the probability Haar measures $\mathrm{d}u$, $\mathrm{d}v$ and $\mathrm{d}m$ on $\mathcal{N}_1^-$, $\mathcal{N}$ and $\mathcal{M}$, respectively. For $a \in E^\times$, we set

$$\gamma_a = \mathrm{vol}(\mathcal{C}\mathsf{d}[a]\mathcal{C})^{-1} 1_{\mathcal{C}\mathsf{d}[a]\mathcal{C}} \,,$$

where 1_Y denotes the characteristic function of $Y \subset G$. Then the operator $\pi_\nu(\gamma_a)$ on the principal series (π_ν, I_ν) (see §4.2) preserves its $\mathcal{C}$-invariant vectors $I_\nu^{\mathcal{C}}$, which is a three dimensional space with basis $\varphi_e^{(\nu)}$, $\varphi_{w_0}^{(\nu)}$ and $\varphi_{w_1}^{(\nu)}$ whose restrictions to $\mathcal{U}$ are the characteristic functions of $\mathcal{C} = \mathcal{C}e\mathcal{C}$, $\mathcal{C}w_0\mathcal{C}$ and $\mathcal{C}w_1\mathcal{C}$, respectively, where e denotes the identity element of G. We confirm the latter fact by the disjoint decomposition $\mathcal{U} = \bigcup_{w \in W} \mathcal{C}w\mathcal{C}$ with $W = \{e, w_0, w_1\}$. (For this decomposition to be true, the condition $m \geqslant 4$ is necessary.) Moreover, the operators $\pi_\nu(\gamma_a)$ $(a \in E - \mathfrak{o}_E)$ on $I_\nu^{\mathcal{C}}$ are mutually commuting ([4, Lemma 4.1.5]). By the theory of canonical lifting [4, §4], there exists a subspace $I_\nu^{\mathcal{C},\mathrm{can}}$ of $I_\nu^{\mathcal{C}}$ such that $I_\nu^{\mathcal{C},\mathrm{can}} = \pi_\nu(\gamma_a) I_\nu^{\mathcal{C}}$ for all $a \in E^\times$ with sufficiently large $|a|_E$, and the canonical map $j_{\bar{P}} : I_\nu \to (I_\nu)_{\bar{N}}$ yields a bijection $I_\nu^{\mathcal{C},\mathrm{can}} \cong (I_\nu)_{\bar{N}}^{\mathcal{M}}$ ([4, Proposition 4.1.4]), where

$((\pi_\nu)_{\bar{P}}, (I_\nu)_{\bar{N}})$ denotes the (unnormalized) Jacquet module of (π_ν, I_ν) along the parabolic $\bar{P}$ ([4, §6]). If we define the G-invariant pairing $\langle\, ,\, \rangle_{\mathcal{U}} : I_\nu \times I_{-\nu} \to \mathbb{C}$ by

$$\langle \varphi, \varphi' \rangle_{\mathcal{U}} = \int_{\mathcal{U}} \varphi(k)\varphi'(k)\, \mathrm{d}k\,, \quad \varphi \in I_\nu,\ \varphi' \in I_{-\nu}\,,$$

with $\mathrm{d}k$ being the Haar measure on $\mathcal{U}$ such that $\mathrm{vol}(\mathcal{U}) = 1$, then

$$\langle \pi_\nu(\gamma_a)\varphi, \varphi' \rangle_{\mathcal{U}} = \langle \varphi, \pi_{-\nu}(\gamma_{a^{-1}})\varphi' \rangle_{\mathcal{U}}\,, \quad \varphi \in I_\nu,\ \varphi' \in I_{-\nu},\ a \in E^\times\,, \tag{5.1}$$

$$\langle \varphi_w^{(\nu)}, \varphi_{w'}^{(-\nu)} \rangle_{\mathcal{U}} = \delta(w = w')\,\mathrm{vol}(\mathcal{C}w\mathcal{C}; \mathrm{d}g)\,, \quad w,\ w' \in W\,. \tag{5.2}$$

LEMMA 5.1. *There exists $l_0 \in \mathbb{N}$ such that the operator $\pi_\nu(\gamma_a)$ $(a \in E - \varpi^{-l_0}\mathfrak{o}_E)$ acting on $I_\nu^{\mathcal{C}}$ is diagonalizable with eigenvalues $|a|_E^{\nu-(m-1)/2}$, $|a|_E^{-\nu-(m-1)/2}$ and $|a|_E^{-1}$.*

PROOF. First, we compute the Jacquet module of $\pi_{-\nu}$ along P instead of $\bar{P}$. The set $W = \{e, w_0, w_1\}$ is a complete system of representatives of the double coset space $P\backslash G/P \cong W_M\backslash W_G/W_M$. The coset Pw_0P is Zariski open in G; the Zariski closure of the coset Pw_1P is $Pw_1P \cup P$. We have the obvious isomorphism $M \cong E^\times \times G_1$. From [4, §6], the Jacquet module $(I_{-\nu})_N$, viewed as an M-module, has a filtration $\{0\} \subset J_{w_0} \subset J_{w_1} \subset J_e = (I_{-\nu})_N$ with the successive quotients isomorphic to the (unnormalized) parabolic inductions

$$Ind(x^{-1}(\sigma_{M\cap xNx^{-1}})\delta^{1/2}|x^{-1}Px \cap M, M)\,, \qquad (x \in W)\,,$$

where σ is the quasi-character $\mathsf{m}[h]\,\mathsf{d}[t] \mapsto |t|_E^{-\nu}$ of M, δ is the modulus character whose restriction to T is $\prod_\alpha \alpha^{m(\alpha)} \prod_\beta \beta^{-m(\beta)}$ with $\alpha \in \Sigma^+ - \Sigma_M^+$, $x\alpha \in \Sigma^- - \Sigma_M^-$ and $\beta \notin \Sigma^+ - \Sigma_M^+$, $x\beta \in \Sigma^- - \Sigma_M^-$. By explicating these, we have $M(= E^\times \times G_1)$-isomorphisms

$$J_e/J_{w_1} \cong |\,|_E^{-\nu+(m-1)/2} \boxtimes \mathbf{1}_{G_1}\,, \quad J_{w_1}/J_{w_0} \cong |\,|_E \boxtimes \mathrm{Ind}_{P_1}^{G_1}(|\,|_E^{-\nu-1/2})\,,$$
$$J_{w_0} \cong |\,|_E^{\nu+(m-1)/2} \boxtimes \mathbf{1}_{G_1}\,,$$

where P_1 is the maximal F-parabolic subgroup of G_1 stabilizing the one dimensional subspace Ee_2 and $\mathrm{Ind}_{P_1}^{G_1}(|\,|_E^s)$ is the normalized parabolic induction from the character $h \mapsto |\mathbf{h}(he_2, e_{m-1})|_E^s$ of P_1. Since there is a perfect M-invariant pairing between $(I_{-\nu})_N$ and $(I_\nu)_{\bar{N}}$ ([4, Theorem 4.2.4]), we have an M-filtration $J_w^\perp$ $(w \in W)$ of $(I_\nu)_{\bar{N}}$ such that

$$(I_\nu)_{\bar{N}}/J_{w_0}^\perp \cong |\,|_E^{-\nu-(m-1)/2} \boxtimes \mathbf{1}_{G_1}\,, \quad J_{w_0}^\perp/J_{w_1}^\perp \cong |\,|_E^{-1} \boxtimes \mathrm{Ind}_{P_1}^{G_1}(|\,|_E^{\nu+1/2})\,,$$
$$J_{w_1}^\perp \cong |\,|_E^{\nu-(m-1)/2} \boxtimes \mathbf{1}_{G_1}\,.$$

By [4, Lemma 4.1.1], there exists $l_0 \in \mathbb{N}$ such that

$$j_{\bar{P}}(\pi_\nu(\gamma_a)v) = (\pi_\nu)_{\bar{P}}(\mathsf{d}[a])(j_{\bar{P}}(v))\,, \quad v \in I_\nu^{\mathcal{C},\mathrm{can}} \tag{5.3}$$

for all $a \in E - \varpi^{l_0}\mathfrak{o}_E$. From the above description of Jacquet modules, the $\mathcal{M}$-invariant parts $((I_\nu)_{\bar{N}}/J_{w_0}^\perp)^{\mathcal{M}}$, $(J_{w_0}^\perp/J_{w_1}^\perp)^{\mathcal{M}}$ and $(J_{w_1}^\perp)^{\mathcal{M}}$ are one dimensional with $(\pi_\nu)_{\bar{P}}(\mathsf{d}[a])$ acting on by the scalar $|a|_E^{\nu-(m-1)/2}$, $|a|_E^{-1}$ and $|a|_E^{-\nu-(m-1)/2}$, respectively. Hence $((I_\nu)_{\bar{N}})^{\mathcal{M}}$ is a 3-dimensional space. Since $I_\nu^{\mathcal{C}}$ is also 3-dimensional as we remarked above, the composite of two surjections

$$I_\nu^{\mathcal{C}} \xrightarrow{\pi_\nu(\gamma_a)} I_\nu^{\mathcal{C},\mathrm{can}} \xrightarrow{j_{\bar{P}}} ((I_\nu)_{\bar{N}})^{\mathcal{M}}$$

with $a \in E - \varpi^{l_0}\mathfrak{o}_E$ is a linear bijection. Hence, by (5.3), the operator $\pi_\nu(\gamma_a)$ with $a \in E - \varpi^{l_0}\mathfrak{o}_E$ on $(I_\nu)^{\mathcal{C}}$ has the eigenvalues $|a|_E^{-\nu-(m-1)/2}$, $|a|_E^{-1}$ and $|a|_E^{\nu-(m-1)/2}$ as desired. □

Let $T(\nu) : I_\nu \to I_{-\nu}$ be the standard G-intertwining operator, which is defined by an analytic continuation of the integral

$$[T(\nu)\varphi](g) = \int_{\bar{N}} \varphi(\bar{n}w_0 g)\,\mathrm{d}\bar{n}\,, \qquad g \in G$$

absolutely convergent for $\mathrm{Re}(\nu) \gg 0$, where $\mathrm{d}\bar{n}$ is the Haar measure on $\bar{N}$ such that $\mathrm{vol}(\mathcal{N}_1^-; \mathrm{d}\bar{n}) = 1$. We note that $\mathrm{vol}(\bar{N} \cap \mathcal{U}; \mathrm{d}\bar{n}) = q^{2m-3}$.

LEMMA 5.2. *Let $\varphi \in I_\nu^{\mathcal{C}}$ with $\mathrm{Re}(\nu) \gg 0$. Then*

$$\lim_{l\to\infty} q_E^{-l(\nu-(m-1)/2)}[\pi_\nu(\gamma_{\varpi^{-l}})\varphi)](e) = [T(\nu)\varphi](w_0)\,.$$

PROOF. From definition,

$$[\pi_\nu(\gamma_{\varpi^{-l}})\varphi](e) = \mathrm{vol}(\mathcal{C}\mathsf{d}[\varpi^{-l}]\mathcal{C})^{-1}\int_{g\in\mathcal{C}\mathsf{d}[\varpi^{-l}]\mathcal{C}} \varphi(g)\mathrm{d}g = \mathrm{vol}(\mathcal{C})^{-1}\int_{\mathcal{C}} \varphi(c\mathsf{d}[\varpi^{-l}])\mathrm{d}c\,.$$

We have the Iwahori factorization $c = umv$ with $m \in \mathcal{M}$, $v \in \mathcal{N}_0$ and $u \in \mathcal{N}_1^-$. The measure is decomposed to $\mathrm{d}c = \mathrm{vol}(\mathcal{C})\mathrm{d}u\,\mathrm{d}m\,\mathrm{d}v$ as before. Since $\mathsf{d}[\varpi^{-l}]^{-1}mv\mathsf{d}[\varpi^{-l}] \in \mathcal{C}$,

$$\begin{aligned}[\pi_\nu(\gamma_{\varpi^{-l}})\varphi](e) &= \int_{\mathcal{N}_1^-} \varphi(u\mathsf{d}[\varpi^{-l}])\mathrm{d}u = |\varpi^{-l}|_E^{\nu+(m-1)/2}\int_{\mathcal{N}_1^-} \varphi(\mathsf{d}[\varpi^{l}]u\mathsf{d}[\varpi^{-l}])\mathrm{d}u \\ &= |\varpi^{-l}|_E^{\nu-(m-1)/2}\int_{\bar{N}[l]} \varphi(\bar{n})\mathrm{d}\bar{n}\end{aligned}$$

with $\bar{N}[l] = \{\mathsf{n}(\varpi^{-l+1}X; \varpi^{-2l+1}b) \mid X \in \mathcal{L} \cap V_1,\ b \in \mathfrak{o}_F\}$. Hence

$$\lim_{l\to\infty} q_E^{-l(\nu-(m-1)/2)}[\pi_\nu(\gamma_{\varpi^{-l}})\varphi)](e) = \int_{\bar{N}} \varphi(\bar{n})\mathrm{d}\bar{n} = [T(\nu)\varphi](w_0)\,.$$

□

The C-function is defined by the relation $T(\nu)f_0^{(\nu)} = C(\nu)\,f_0^{(-\nu)}$. We have the explicit formula

$$C(\nu) = \mathrm{vol}(\bar{N} \cap \mathcal{U}; \mathrm{d}\bar{n})\,\frac{1 - q^{-(m-1)}q_E^{-\nu}}{1 - q^{m-3}q_E^{-\nu}}\,\frac{1 - (-1)^m q^{-1}q_E^{-\nu}}{1 - (-1)^m q_E^{-\nu}}\,. \tag{5.4}$$

In what follows, we use the phrase "for a generic ν" to mean "for all ν in an open dense subset of $\mathbb{X}_E$". For a generic ν, set

$$\begin{aligned}\psi_+^{(\nu)} &= \mathrm{vol}(\mathcal{C})^{-1}\varphi_e^{(\nu)}, \quad \psi_-^{(\nu)} = T(-\nu)(\psi_+^{(-\nu)}), \\ \psi_0^{(\nu)} &= f_0^{(\nu)} - \mathrm{vol}(\mathcal{C})\{C(\nu)\,\psi_+^{(\nu)} + \psi_-^{(\nu)}\}.\end{aligned} \tag{5.5}$$

LEMMA 5.3. *For all $l \in \mathbb{N}^*$,*

$$\pi_\nu(\gamma_{\varpi^{-l}})\psi_+^{(\nu)} = q_E^{l(\nu-(m-1)/2)}\psi_+^{(\nu)}, \quad \pi_\nu(\gamma_{\varpi^{-l}})\psi_-^{(\nu)} = q_E^{l(-\nu-(m-1)/2)}\psi_-^{(\nu)},$$
$$\pi_\nu(\gamma_{\varpi^{-l}})\psi_0^{(\nu)} = q_E^{-l}\psi_0^{(\nu)}.$$

PROOF. Consider the function $f(g) = \int_{\mathcal{C}} \varphi_w^{(-\nu)}(cg)\,\mathrm{d}c$ in $g \in G$. Since $f(\mathsf{d}[\varpi^l]) = f(g)$ for all $g \in \mathcal{C}\mathsf{d}[\varpi^l]\mathcal{C}$,

$$\begin{aligned}f(\mathsf{d}[\varpi^l]) &= \mathrm{vol}(\mathcal{C}\mathsf{d}[\varpi^l]\mathcal{C})^{-1}\int_G f(x) 1_{\mathcal{C}\mathsf{d}[\varpi^l]\mathcal{C}}(x)\,\mathrm{d}x \\ &= \mathrm{vol}(\mathcal{C}\mathsf{d}[\varpi^l]\mathcal{C})^{-1}\int_G\int_{\mathcal{C}} \varphi_w^{(-\nu)}(cg)\,1_{\mathcal{C}\mathsf{d}[\varpi^l]\mathcal{C}}(g)\,\mathrm{d}g\,\mathrm{d}c \\ &= \mathrm{vol}(\mathcal{C}\mathsf{d}[\varpi^l]\mathcal{C})^{-1}\int_G\int_{\mathcal{C}} \varphi_w^{(-\nu)}(g)\,1_{\mathcal{C}\mathsf{d}[\varpi^l]\mathcal{C}}(c^{-1}g)\,\mathrm{d}g\,\mathrm{d}c \\ &= \mathrm{vol}(\mathcal{C}\mathsf{d}[\varpi^l]\mathcal{C})^{-1}\mathrm{vol}(\mathcal{C})\int_G \varphi_w^{(-\nu)}(g)\,1_{\mathcal{C}\mathsf{d}[\varpi^l]\mathcal{C}}(g)\,\mathrm{d}g = \mathrm{vol}(\mathcal{C})\,[\pi_{-\nu}(\gamma_{\varpi^l})\varphi_w^{(-\nu)}](e).\end{aligned}$$

On the other hand, we compute $f(\mathsf{d}[\varpi^l])$ by the Iwahori factorization $\mathrm{d}c = \mathrm{vol}(\mathcal{C})\mathrm{d}v\mathrm{d}m\mathrm{d}u$. Since $\mathsf{d}[\varpi^l]^{-1}\mathcal{N}_1^-\mathsf{d}[\varpi^l] \subset \mathcal{C}$, we have

$$\begin{aligned}f(\mathsf{d}[\varpi^l]) &= \mathrm{vol}(\mathcal{C})\int_{\mathcal{N}} \varphi_w^{(-\nu)}(v\mathsf{d}[\varpi^l])\mathrm{d}v = \mathrm{vol}(\mathcal{C})|\varpi^l|_E^{-\nu+(m-1)/2}\mathrm{vol}(\mathcal{N})\,\varphi_w^{(-\nu)}(e) \\ &= \mathrm{vol}(\mathcal{C})q_E^{l(\nu-(m-1)/2)}\,\delta(w=e).\end{aligned}$$

Thus we obtain $[\pi_{-\nu}(\gamma_{\varpi^l})\varphi_w^{(-\nu)}](e) = q_E^{l(\nu-(m-1)/2)}\,\delta(w=e)$ for $w \in \{e, w_0, w_1\}$. From this, by (5.1),

$$\begin{aligned}\langle\pi_\nu(\gamma_{\varpi^{-l}})\varphi_e^{(\nu)}, \varphi_w^{(-\nu)}\rangle_{\mathcal{U}} &= \langle\varphi_e^{(\nu)}, \pi_{-\nu}(\gamma_{\varpi^l})\varphi_w^{(-\nu)}\rangle_{\mathcal{U}} = \mathrm{vol}(\mathcal{C})[\pi_{-\nu}(\gamma_l)\varphi_w^{(-\nu)}](e) \\ &= q_E^{l(\nu-(m-1)/2)}\mathrm{vol}(\mathcal{C})\,\delta(w=e).\end{aligned} \tag{5.6}$$

By (5.2), we obtain $\pi_\nu(\gamma_{\varpi^{-l}})\varphi_e^{(\nu)} = q_E^{l(\nu-(m-1)/2)}\varphi_e^{(\nu)}$ as desired. Then

$$\begin{aligned}\pi_\nu(\gamma_{\varpi^{-l}})\psi_-^{(\nu)} &= \pi_\nu(\gamma_{\varpi^{-l}}) \circ T(-\nu)\psi_+^{(-\nu)} = T(-\nu) \circ \pi_{-\nu}(\gamma_{\varpi^{-l}})\psi_+^{(-\nu)} \\ &= q_E^{l(-\nu-(m-1)/2)}T(-\nu)\psi_+^{(-\nu)} \\ &= q_E^{l(-\nu-(m-1)/2)}\psi_-^{(\nu)}.\end{aligned}$$

By Lemma 5.1, there exists a projector $\mathrm{Pr}_0^{(\nu)}$ from $I_\nu^{\mathcal{C}}$ (depending meromorphically on ν) onto the $|a|_E^{-1}$-eigenspace of $\pi_\nu(\gamma_a)$ ($a \in E - \varpi^{-l_0}\mathfrak{o}_E$). Set $\psi_0^{(\nu)} = \mathrm{Pr}_0^{(\nu)}(f_0^{(\nu)})$. Since $\psi_\pm^{(\nu)}$ (for generic ν) is $|a|_E^{\pm\nu-(m-1)/2}$-eigenvector of $\pi_\nu(\gamma_a)$ ($a \in E - \varpi^{l_0}\mathfrak{o}_E$) as shown above, from Lemma 5.1 the space $I_\nu^{\mathcal{C}}$ is a direct sum of $\mathbb{C}\psi_+^{(\nu)}$, $\mathbb{C}\psi_-^{(\nu)}$ and $\mathrm{Im}(\mathrm{Pr}_0^{(\nu)})$. We have the expression $f_0^{(\nu)} = b_+(\nu)\psi_+^{(\nu)} + b_-(\nu)\psi_-^{(\nu)} + \psi_0^{(\nu)}$ with some $b_\pm(\nu) \in \mathbb{C}$. Thus for $\mathrm{Re}(\nu) \gg 0$,

$$q_E^{-l(\nu-(m-1)/2)}[\pi_\nu(\gamma_{\varpi^{-l}}) f_0^{(\nu)}](e) = b_+(\nu)\psi_+^{(\nu)}(e) + q_E^{-2l\nu} b_-(\nu)\psi_-^{(\nu)}(e) + q_E^{-l(\nu-(m-1)/2)-l}\psi_0^{(\nu)}(e)\,,$$

which yields

$$[T(\nu) f_0^{(\nu)}](w_0) = b_+(\nu)\psi_+^{(\nu)}(e)$$

in the limit $l \to \infty$ by Lemma 5.2. The left-hand side equals $C(\nu)$ and the right-hand side $b_+(\nu)\mathrm{vol}(\mathcal{C})^{-1}$. Thus $b_+(\nu) = \mathrm{vol}(\mathcal{C})\, C(\nu)$ for $\mathrm{Re}(\nu) \gg 0$. The relation $T(\nu) f_0^{(\nu)} = C(\nu) f_0^{(-\nu)}$ yields

$$b_+(\nu)T(\nu)\psi_+^{(\nu)} + b_-(\nu)T(\nu)\psi_-^{(\nu)} + T(\nu)\psi_0^{(\nu)} = C(\nu)\big\{b_+(-\nu)\psi_+^{(-\nu)} + b_-(-\nu)\psi_-^{(-\nu)} + \psi_0^{(-\nu)}\big\}\,.$$

Since $T(\nu)$ preserves the eigenspace decomposition of $\pi_\nu(\gamma_a)$, we obtain $b_+(\nu)T(\nu)\psi_+^{(\nu)} = C(\nu)b_-(-\nu)\psi_-^{(-\nu)}$. Since $T(\nu)\psi_+^{(\nu)} = \psi_-^{(-\nu)}$, we have $b_-(-\nu) = C(\nu)^{-1}b_+(\nu) = \mathrm{vol}(\mathcal{C})$. Therefore, $\psi_0^{(\nu)} = f_0^{(\nu)} - b_+(\nu)\psi_+^{(\nu)} - \psi_-^{(\nu)} = f_0^{(\nu)} - \mathrm{vol}(\mathcal{C})\big\{C(\nu)\psi_+^{(\nu)} + \psi_-^{(\nu)}\big\}$. □

5.2. Poisson integrals on unitary hyperbolic spaces.

LEMMA 5.4. *For $\nu \in \mathbb{X}_E$ such that $q_E^{-\nu} \neq q_E^{-(m-3)/2}$, we have* $\dim_{\mathbb{C}} \mathrm{Hom}_H(I_\nu, \mathbb{C}) \leqslant 1$.

PROOF. We regard G as an $H \times P$-space by letting H and P act on G by the right translation and by the left translation, respectively. Let χ_ν be the character of $H \times P$ defined by $\chi_\nu(h, \mathsf{m}[g_1]\mathsf{d}[t]n) = |t|_E^{-\nu+(m-1)/2}$ ($h \in H$, $g_1 \in G_1$, $t \in E^\times$, $n \in N$). The space $\mathrm{Hom}_H(I_\nu, \mathbb{C})$ is isomorphic to $\mathrm{Hom}_{H\times P}(C_c^\infty(G), \chi_\nu)$, the space of distributions on G with an $H \times P$-equivariance. Set $O_1 = \{g \in G \,|\, \mathbf{h}(g^{-1}\ell_0, e_1) \neq 0\}$ and $O_2 = G - O_1$. Then we see that $O_1 = HP$ is an open $H \times P$-orbit in G and O_2 is a closed orbit. From the exact sequence of smooth $H \times P$-modules

$$0 \to C_c^\infty(O_1) \to C_c^\infty(G) \to C_c^\infty(O_2) \to 0\,,$$

we have an exact sequence of vector spaces

$$0 \to \mathrm{Hom}_{H\times P}(C_c^\infty(O_2), \chi_\nu) \to \mathrm{Hom}_{H\times P}(C_c^\infty(G), \chi_\nu) \to \mathrm{Hom}_{H\times P}(C_c^\infty(O_1), \chi_\nu)\,. \tag{5.7}$$

Let $O \subset G$ be the $H \times P$-orbit of a point $x_0 \in O$. Then the space $\mathrm{Hom}_{H\times P}(C_{\mathrm{c}}^{\infty}(O), \chi_\nu)$ is zero unless the character $\xi_{x_0} = (\chi_\nu | P \cap x_0 H x_0^{-1}) \cdot \delta_{P\cap x_0 H x_0^{-1}} \cdot (\delta_P | P \cap x_0 H x_0^{-1})^{-1}$ of the stabilizer $P \cap x_0 H x_0^{-1}$ is trivial, in which case the space is one dimensional ([3, Proposition 3.2 (p. 30)]). Here $\delta_{P\cap x_0 H x_0^{-1}}$ is the modulus character of the stabilizer. For $x_0 = e$, then $O = O_1$ and χ_ν is trivial on $P \cap H = \{\mathsf{m}[g_1] | g_1 \in G_1\}$. We may suppose $\ell_0 = ae_1 + e_m$ with $a + \bar{a} = 1$. Then, the element w_1 belongs to O_2 and $P \cap w_1 H w_1^{-1} = \{\mathsf{d}[t]\mathsf{m}[h]\mathsf{n}(X; b) | t \in E^\times,\ hw_1\ell_0 = w_1\ell_0,\ \mathbf{h}(X, w_1\ell_0) = 0,\ b \in F^\times\}$. We have $\xi_{w_1}(h, \mathsf{d}[t]\mathsf{m}[h]n) = |t|_E^{-\nu+(m-3)/2}$. For ν such that $q_E^{\nu-(m-3)/2} \neq 1$, the character ξ_{w_1} is non trivial, and thus $\mathrm{Hom}_{H\times P}(C_{\mathrm{c}}^{\infty}(O_2), \chi_\nu) = \{0\}$ for such ν. Hence, from (5.7), we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_{H\times P}(C_{\mathrm{c}}^{\infty}(G), \chi_\nu) \leqslant \dim_{\mathbb{C}} \mathrm{Hom}_{H\times P}(C_{\mathrm{c}}^{\infty}(O_1), \chi_\nu) = 1 \quad \text{if } q_E^{\nu-(m-3)/2} \neq 1\,.$$

□

Let Y_ν ($\nu \in \mathbb{X}_E$) be a function on G define by

$$Y_\nu(g) = \begin{cases} 0\,, & (g \in G - HP), \\ |\mathbf{h}(g^{-1}\ell_0, e_1)|_E^{\nu-(m-1)/2}\,, & (g \in HP)\,. \end{cases}$$

Since $HP = \{g \in G | \mathbf{h}(g^{-1}\ell_0, e_1) \neq 0\}$, the function Y_ν is continuous on G if $\mathrm{Re}(\nu) \geqslant \frac{m-1}{2}$. Therefore, we can define a $\mathbb{C}$-linear form $\Xi(\nu) : I_\nu \to \mathbb{C}$ ($\mathrm{Re}(\nu) \geqslant \frac{m-1}{2}$) by the Poisson integral

$$\langle \Xi(\nu), f \rangle = \int_{\mathcal{U}} Y_\nu(k) f(k)\, \mathrm{d}k\,, \qquad f \in I_\nu\,,$$

where $\mathrm{d}k$ is the Haar measure on $\mathcal{U}$ such that $\mathrm{vol}(\mathcal{U}) = 1$. For any $f \in I_0$, let $f^{(\nu)} \in I_\nu$ denote its flat extension. We define an action $\pi_\nu^0(g)$ of $g \in G$ on I_0 by setting $[\pi_\nu^0(g)f](k) = f^{(\nu)}(kg)$ for $f \in I_0$.

LEMMA 5.5. *There exists* $\boldsymbol{\Xi}_\nu \in I_0^* \otimes_{\mathbb{C}} \mathbb{C}(q_E^{-\nu})$ *such that, for all* $f \in I_0$, *we have* $\langle \boldsymbol{\Xi}_\nu, \pi_\nu^0(h) f \rangle = \langle \boldsymbol{\Xi}_\nu, f \rangle$ *for all* $h \in H$ *and for a generic* ν, *and* $\langle \boldsymbol{\Xi}_\nu, f \rangle = \langle \Xi(\nu), f^{(\nu)} \rangle$ *for* $\mathrm{Re}(\nu) \geqslant (m-1)/2$.

PROOF. This follows from Lemma 5.4 by Bernstein's theorem [7, §12.2 (p. 127)]. □

From this lemma, there exists a polynomial $R(z) \in \mathbb{C}[z]$ such that $\nu \mapsto R(q_E^{-\nu})\Xi(\nu)$ ($\mathrm{Re}(\nu) > (m-1)/2$) extends to an entire family of H-invariant functional on I_ν defined for all $\nu \in \mathbb{C}$.

Recall the $\mathcal{U}$-spherical vector $f_0^{(\nu)} \in I_\nu^{\mathcal{U}}$. Set

$$a(\nu) = \langle \Xi(\nu), f_0^{(\nu)} \rangle\,, \quad \mathrm{Re}(\nu) \geqslant \tfrac{m-1}{2}\,.$$

LEMMA 5.6. *We have*

$$a(\nu) = \frac{1-q^{-2}}{1-(-1)^m q^{-m}} \frac{1-(-1)^m q^{-2\nu-1}}{1-q^{-2\nu+m-3}}\,, \quad \mathrm{Re}(\nu) \geqslant \tfrac{m-1}{2}\,.$$

PROOF. Since the group $\mathcal{U}$ acts transitively on $\mathcal{L}_{\mathrm{prim}}$, we have $\mathcal{U}/(P^1\cap\mathcal{U}) \cong \mathbf{\Sigma}(V,0)\cap\mathcal{L}_{\mathrm{prim}}$ by mapping a class $k(P^1\cap\mathcal{U})$ to the vector ke_1. Thus

$$a(\nu) = \int_{\mathcal{U}/(P^1\cap\mathcal{U})} |\mathbf{h}(\ell_0, ke_1)|_E^{\nu-(m-1)/2}\mathrm{d}k = \int_{\mathbf{\Sigma}(V,0)\cap\mathcal{L}_{\mathrm{prim}}} |\mathbf{h}(\ell_0, Z)|_E^{\nu-(m-1)/2}\,\mathrm{d}_0 Z\,,$$

where $\mathrm{d}_0 Z$ is the $\mathcal{U}$-invariant measure on $\mathbf{\Sigma}(V,0)$ such that $\mathrm{vol}(\mathbf{\Sigma}(V,0)\cap\mathcal{L}_{\mathrm{prim}}) = 1$, which should be proportional to the restriction of the measure $|\omega_{V,0}|_F$ to $\mathbf{\Sigma}(V,0)\cap\mathcal{L}_{\mathrm{prim}}$; let C denote the proportionality constant. Let $a_1(\nu)$ be the integral of $|\mathbf{h}(\ell_0,Z)|_E^{\nu-(m-1)/2}$ over $\mathbf{\Sigma}(V,0)\cap\mathcal{L}$ with respect to the measure $|\omega_{V,0}|_F$. Since $\mathcal{L}_{\mathrm{prim}} = \mathcal{L} - \varpi\mathcal{L}$, Lemma 4.1 yields the relation

$$a(\nu) = C\,a_1(\nu)(1-|\varpi|_E^{m-1}|\varpi|_E^{\nu-(m-1)/2}) = C\,a_1(\nu)(1-q^{-2\nu-m+1})\,. \tag{5.8}$$

Recall $V_1 = (Ee_1+Ee_m)^{\perp}$. Set $\mathcal{L}_1 = \mathcal{L}\cap V_1$. If we write a general point $Z\in\mathcal{L}$ in the form $Z = z_1e_1 + X + z_2e_m$ ($z_1, z_2 \in \mathfrak{o}_E$, $X\in\mathcal{L}_1$), then $\mathbf{h}(\ell_0, Z) = a\bar{z}_2 + \bar{z}_1$. Thus,

$$\begin{aligned} a_1(\nu) &= \int_{z_1,z_2\in\mathfrak{o}_E} |a\bar{z}_2+\bar{z}_1|_E^{\nu-(m-1)/2}\, v_1\left(-\mathrm{tr}_{E/F}(z_1\bar{z}_2)\right)\mathrm{d}\mu(z_1)\,\mathrm{d}\mu(z_2) \\ &= \sum_{l\in\mathbb{N}} v_1(\varpi^l)\,A(l)\,, \end{aligned} \tag{5.9}$$

where

$$A(l) = \int_{\substack{z_1,z_2\in\mathfrak{o}_E \\ \mathrm{tr}_{E/F}(z_1\bar{z}_2)\in\varpi^l\mathfrak{o}_F^\times}} |a\bar{z}_1+\bar{z}_2|_E^{\nu-(m-1)/2}\mathrm{d}\mu(z_1)\mathrm{d}\mu(z_2)\,,$$

$$v_1(t) = \mathrm{vol}(\mathbf{\Sigma}(V_1,t)\cap\mathcal{L}_1;\,|\omega_{V_1,t}|_F)\,.$$

From Lemma 4.2,

$$v_1(\varpi^l) = \frac{1-(-1)^{m-1}q^{-(m-1)}}{1-(-1)^{m-1}q^{-(m-2)}}(1-(-1)^{m(l+1)}q^{-(m-2)(l+1)})\,. \tag{5.10}$$

To compute the integral $A(l)$, we further set

$$A_{ij}(l) = \int_{\substack{u_1,u_2\in\mathfrak{o}_E^\times \\ \mathrm{tr}_{E/F}(u_1\bar{u}_2)\in\varpi^{l-i-j}\mathfrak{o}_F^\times}} |a\varpi^i\bar{u}_1+\varpi^j\bar{u}_2|_E^{\nu-(m-1)/2}\mathrm{d}\mu(u_1)\mathrm{d}\mu(u_2)\,.$$

The evaluation of these integrals is given in Lemma 5.7 below. Plugging (5.10) and (5.11) to (5.9) and noting (5.8), by a direct computation, we obtain

$$a(\nu) = C(1-q^{-2(m-2)}T)\sum_{l=0}^{\infty} v_1(\varpi^l)A(l)$$

$$= C\,(1-q^{-2})(1-(-1)^{m-1}q^{-(m-1)})\frac{1-(-1)^m q^{-(m-2)}T}{1-T}\,,$$

with $T = q^{-2\nu+m-3}$ if $\mathrm{Re}\,\nu > \frac{m-3}{2}$. The constant C can be determined by setting $\nu = (m-1)/2$ and using the relation $a((m-1)/2) = 1$. $\square$

LEMMA 5.7. *We have $A_{ij}(l) = 0$ unless $i + j \geqslant l$. Set $T = q^{-2\nu+m-3}$. Then*

$$A_{i,j}(l) = (q^2T)^{\inf(i,j)}(1-q^{-2})$$
$$\times \begin{cases} q^{i+j-l}(1-q^{-1})^{1+\delta(i+j>l)}\,, & (l \geqslant i+j, i \neq j)\,,\\ q^{2i-l}(1-q^{-1})^2\,, & (l > 2i, i = j)\,,\\ (1-q^{-1}-q^{-2}+q^{-1}T)(1-T)^{-1}\,, & (l = 2i, i = j)\,. \end{cases}$$

We have

$$A(l) = \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} q^{-2(i+j)}A_{ij}(l) \tag{5.11}$$
$$= \frac{q^{-l}(1-q^{-2})}{1-T}\begin{cases}(1-q^{-2})(1-T^{(l+1)/2})\,, & (l \equiv 1 \pmod 2)\,,\\ 1-q^{-2}-q^{-1}T^{l/2}(1-T)\,, & (l \equiv 0 \pmod 2)\,.\end{cases}$$

PROOF. A direct computation. $\square$

5.3. Spherical functions. We define the normalized Poisson integral by $\Xi^0(\nu) = a(\nu)^{-1}\Xi(\nu)$ and set

$$\Omega_\nu(g) = \langle \Xi^0(\nu), \pi_\nu(g)f_0^{(\nu)}\rangle\,, \quad g \in G\,.$$

LEMMA 5.8. *For a generic ν,*

$$\Xi^0(-\nu)\circ T(\nu) = C(\nu)\,\Xi^0(\nu)\,.$$

PROOF. For a generic $\nu \in \mathbb{X}_E$, the functional $\Xi^0(-\nu)\circ T(\nu)$ belongs to the space $\mathrm{Hom}_H(I_\nu, \mathbb{C})$, which is one dimensional with the basis $\Xi^0(\nu)$ by Lemma 5.4. Thus there exists $b(\nu) \in \mathbb{C}$ such that $\Xi^0(-\nu)\circ T(-\nu) = b(\nu)\Xi^0(\nu)$. Apply this to the vector $f_0^{(\nu)} \in I_\nu$ and use the relation $T(\nu)f_0^{(\nu)} = C(\nu)f_0^{(-\nu)}$. Then $b(\nu) = C(\nu)$ is obtained. $\square$

LEMMA 5.9. *We have*

$$\Omega_\nu(\mathsf{d}[\varpi^{-l}]) = \langle \Xi^0(\nu), \pi_\nu(\gamma_{\varpi^{-l}})f_0^{(\nu)}\rangle\,, \quad l \in \mathbb{N}\,.$$

PROOF. It suffices to show the inclusion $\mathcal{C}\mathsf{d}[\varpi^{-l}]\mathcal{C} \subset H_0\mathsf{d}[\varpi^{-l}]\mathcal{U}$. Let $c, c_1 \in \mathcal{C}$. If we set $c_1^{-1}\ell_0 = \sum_{j=1}^m a_je_j$ with $a_j \in \mathfrak{o}_E$, then, for any $\xi \in \mathcal{L}$, we have

$$\begin{aligned}&\mathbf{h}(\varpi^l(c_1\mathsf{d}[\varpi^{-l}]c)^{-1}\ell_0,\xi)\\&\quad=\varpi^l\mathbf{h}(\mathsf{d}[\varpi^l]c_1^{-1}\ell_0,c(\xi))=\sum_{j=1}^m\varpi^l a_j\,\mathbf{h}(\mathsf{d}[\varpi^l]e_j,c(\xi))\\&\quad=\varpi^{2l}a_1\,\mathbf{h}(e_1,c(\xi))+\sum_{j=2}^{m-1}\varpi^l a_j\,\mathbf{h}(e_j,c(\xi))+a_m\,\mathbf{h}(e_m,c(\xi))\in\mathcal{L}\,.\end{aligned}$$

Hence $\varpi^l(c_1\mathsf{d}[\varpi^{-l}]c)^{-1}\ell_0\in\mathcal{L}$. Since there exist $t\in\mathfrak{o}_E^\times$ and $e\in\mathcal{L}$ such that $c_1(e_1)=te_1+\varpi e$,

$$a_m=\mathbf{h}(c_1^{-1}\ell_0,e_1)=\mathbf{h}(\ell_0,c_1(e_1))=\mathbf{h}(\ell_0,te_1+\varpi e)=t\mathbf{h}(\ell_0,e_1)+\varpi\mathbf{h}(\ell_0,e)\,.$$

By $\mathbf{h}(\ell_0,e_1)=1$ and $\mathbf{h}(\ell_0,e)\in\mathfrak{o}_E$, we have $a_m\in\mathfrak{o}_E^\times$. This shows $(c_1\mathsf{d}[\varpi^{-l}]c)^{-1}\ell_0\in\varpi^{-l}\mathcal{L}_{\mathrm{prim}}$, or equivalently $c_1\mathsf{d}[\varpi^{-l}]c\in H_0\mathsf{d}[\varpi^{-l}]\mathcal{U}$ as desired. □

LEMMA 5.10. *For a generic ν, we have*

$$\big\langle\Xi^0(\nu),\psi_+^{(\nu)}\big\rangle=a(\nu)^{-1}\,,\quad\big\langle\Xi^0(\nu),\psi_-^{(\nu)}\big\rangle=C(-\nu)a(-\nu)^{-1}\,,\quad\big\langle\Xi^0(\nu),\psi_0^{(\nu)}\big\rangle=0\,.$$

PROOF. Let $c\in\mathcal{C}$. By definition, we have $c^{-1}\ell_0=t\ell_0+\varpi e$ with some $t\in\mathfrak{o}_E^\times$ and $e\in\mathcal{L}$. Hence $Y_\nu(c)=|\mathbf{h}(c^{-1}\ell_0,e_1)|_E^{\nu-(m-1)/2}=|t+\varpi\mathbf{h}(e,e_1)|_E^{\nu-(m-1)/2}=1$. Thus

$$\langle\Xi^0(\nu),\psi_+^{(\nu)}\rangle=a(\nu)^{-1}\int_{\mathcal{U}}Y_\nu(k)\psi_+^{(\nu)}(k)\mathrm{d}k=a(\nu)^{-1}\mathrm{vol}(\mathcal{C})^{-1}\int_{\mathcal{C}}Y_\nu(c)\mathrm{d}c=a(\nu)^{-1}\,.$$

This proves the first formula. To have the second one, we start with Lemma 5.8. Since $\psi_-^{(-\nu)}=T(\nu)\psi_+^{(\nu)}$, we have

$$\big\langle\Xi^0(-\nu),\psi_-^{(-\nu)}\big\rangle=C(\nu)\big\langle\Xi^0(\nu),\psi_+^{(\nu)}\big\rangle=C(\nu)a(\nu)^{-1}\,.$$

To prove the assertion for $\psi_0^{(\nu)}$ we use the theory developed in [14], which asserts the existence of a linear map

$$r_{\bar P}:\mathrm{Hom}_H(I_\nu,\mathbb{C})\longrightarrow\mathrm{Hom}_{H\cap M}((I_\nu)_{\bar N},\mathbb{C})$$

such that

$$\langle\Xi,f\rangle=\langle r_{\bar P}(\Xi),\bar f\rangle \tag{5.12}$$

for any $\bar f\in((I_\nu)_{\bar N})^{\mathcal{M}}$ and its canonical lifting $f\in I_\nu^{\mathcal{C}}$. For $a\in E^\times$, let $\tau_a^*\in\mathrm{Hom}((I_\nu)_{\bar N},\mathbb{C})$ be the dual of the operator $(\pi_\nu)_{\bar P}(\mathsf{d}[a])$ on $(I_\nu)_{\bar N}$. Since $H\cap M=\{\mathsf{m}[h]|h\in G_1\}$, the operators τ_a^* preserve the space of $H\cap M$-invariant linear functionals $\mathrm{Hom}_{H\cap M}((I_\nu)_{\bar N},\mathbb{C})$. Recall the M-filtration $J_w^\perp$ ($w\in\{e,w_1,w_0\}$) of $(I_\nu)_{\bar N}$ constructed in the proof of Lemma 5.1. From what we had shown there, for $a\in E^\times$, $\mathrm{Hom}_{H\cap M}(I_\nu)_{\bar N}/J_{w_0}^\perp,\mathbb{C})\cong\mathbb{C}$ with τ_a^* acting by the scalar $|a|_E^{\nu-(m-1)/2}$, $\mathrm{Hom}_{H\cap M}(J_{w_1}^{\perp}, \mathbb{C}) \cong \mathbb{C}$ with τ_a^* acting by the scalar $|a|_E^{-\nu-(m-1)/2}$ and $\mathrm{Hom}_{H\cap M}(J_{w_0}^{\perp}/J_{w_1}^{\perp}, \mathbb{C}) \cong \mathrm{Hom}_{G_1}\big(\mathrm{Ind}_{P_1}^{G_1}(|\,|_E^{\nu+1/2}), \mathbb{C}\big)$ with τ_a^* acting by the scalar $|a|_E^{-1}$. For a generic value of ν, we have $\mathrm{Hom}_{G_1}\big(\mathrm{Ind}_{P_1}^{G_1}(|\,|_E^{\nu+1/2}), \mathbb{C}\big) = \{0\}$. Hence, for such ν, $\mathrm{Hom}_{H\cap M}\big((I_\nu)_{\bar{N}}, \mathbb{C}\big)$ is an at most two dimensional space on which the operators τ_a^* ($a \in E^\times$) has a multiplicity free spectrum contained in the set $\big\{|a|_E^{\nu-(m-1)/2}, |a|_E^{-\nu-(m-1)/2}\big\}$. Let $r_{\bar{P}}(\varXi(\nu)) = \xi_+ + \xi_-$ with $\xi_\pm$ such that $\tau_a^*(\xi_\pm) = |a|_E^{\pm\nu-(m-1)/2}\xi_\pm$ for all $a \in E^\times$. From Lemma 5.3, $\psi_0^{(\nu)}$ is an eigenvector of $\pi_\nu(\gamma_a)$ for all $a \in E - \mathfrak{o}_E$ with non-zero eigenvalues. Hence, by [4, Lemma 4.3.2], $\psi_0^{(\nu)} \in I_\nu^{\mathcal{C}}$ is the canonical lifting of $j_{\bar{P}}(\psi_0^{(\nu)}) \in \big((I_\nu)_{\bar{N}}\big)^{\mathcal{M}}$ in the sense of [4, §4.1.6]. Applying (5.12), we have

$$\big\langle \varXi^0(\nu), \psi_0^{(\nu)}\big\rangle = \big\langle r_{\bar{P}}(\varXi^0(\nu)), j_{\bar{P}}(\psi_0^{(\nu)})\big\rangle = \big\langle \xi_+, j_{\bar{P}}(\psi_0^{(\nu)})\big\rangle + \big\langle \xi_-, j_{\bar{P}}(\psi_0^{(\nu)})\big\rangle.$$

The two summands in the right-hand side both vanish for a generic ν, due to the relation $\big(|a|_E^{\pm\nu-(m-1)/2} - |a|_E^{-1}\big)\big\langle \xi_\pm, j_{\bar{P}}(\psi_0^{(\nu)})\big\rangle = 0$ ($a \in E - \varpi^{l_0}\mathfrak{o}_E$) obtained from the computation

$$\begin{aligned} |a|_E^{\pm\nu-(m-1)/2}\big\langle \xi_\pm, j_{\bar{P}}(\psi_0^{(\nu)})\big\rangle &= \big\langle \tau_a^*(\xi_\pm), j_{\bar{P}}(\psi_0^{(\nu)})\big\rangle \\ &= \big\langle \xi_\pm, \pi_{\bar{P}}(\mathsf{d}[a])\, j_{\bar{P}}(\psi_0^{(\nu)})\big\rangle \\ &= \big\langle \xi_\pm, j_{\bar{P}}(\pi_\nu(\gamma_a)\psi_0^{(\nu)})\big\rangle = |a|_E^{-1}\big\langle \xi_\pm, j_{\bar{P}}(\psi_0^{(\nu)})\big\rangle. \end{aligned}$$

We note that the third equality is due to (5.3) and the last one is from Lemma 5.3. □

LEMMA 5.11. *For a generic* ν,

$$\varOmega_\nu(\mathsf{d}[\varpi^{-l}]) = \frac{C(\nu)a(-\nu)\, q_E^{l(\nu-(m-1)/2)} + C(-\nu)a(\nu)\, q_E^{l(-\nu-(m-1)/2)}}{C(\nu)a(-\nu) + C(-\nu)a(\nu)}, \quad l \in \mathbb{N}.$$

PROOF. From Lemmas 5.9, 5.3, (5.5) and Lemma 5.10, we have

$$\begin{aligned} \varOmega_\nu(\mathsf{d}[\varpi^{-l}]) &= \mathrm{vol}(\mathcal{C})\big\{C(\nu)q_E^{l(\nu-(m-1)/2)}\,\langle \varXi^0(\nu), \psi_+^{(\nu)}\rangle + q_E^{l(-\nu-(m-1)/2)}\,\langle \varXi^0(\nu), \psi_-^{(\nu)}\rangle\big\} \\ &\quad + q_E^{-l}\,\big\langle \varXi^0(\nu), \psi_0^{(\nu)}\big\rangle \\ &= \mathrm{vol}(\mathcal{C})a(\nu)^{-1}a(-\nu)^{-1}\big\{C(\nu)a(-\nu)q_E^{l(\nu-(m-1)/2)} + C(-\nu)a(\nu)q_E^{l(-\nu-(m-1)/2)}\big\}. \end{aligned}$$

Since $\varOmega_\nu(e) = 1$, we have the desired formula from this. □

From (5.4) and Lemma 5.6, we have that $C(\nu)a(-\nu) + C(-\nu)a(\nu)$ is equal to

$$\mathrm{vol}(\bar{N} \cap \mathcal{U}; d\bar{n})\frac{1 - q^{-2}}{1 - (-1)^m q^{-m}}\, Q_{E,m}\frac{(1 - (-1)^m q^{-2\nu-1})(1 - (-1)^m q^{2\nu-1})}{(1 - q^{-2\nu+m-3})(1 - q^{2\nu+m-3})}.$$

By using this, we have the first formula of Theorem 4.5 from Lemma 5.11.

REMARK: Let $E \cong F \oplus F$. In this case, the explicit formula of $\varOmega_\nu$ may be deducible from [19, Theorem 1.2.1]; although it is a far reaching result, its proof requires a bunch of sophisticated techniques. Our direct argument described above works with some modifications in this case also. All objects we introduced above make sense. We can identify $G \cong \mathrm{GL}_m(F)$ so that $\mathcal{U}$ corresponds to $\mathrm{GL}_m(\mathfrak{o}_F)$ and P to a standard parabolic subgroup with Levi subgroup $\mathrm{GL}_1 \times \mathrm{GL}_{m-2} \times \mathrm{GL}_1$. Let w_0, w_1 and w_2 denote the transpositions $(1\,m)$, (12), $(m-1\,m)$ belonging to S_m, the symmetric group of degree m; then 7 elements $e, w_0, w_1, w_2, w_1w_0, w_2w_0, w_1w_2$ form a complete system of representatives from the double cosets in $S_{m-2}\backslash S_m/S_{m-2} \cong \mathcal{C}\backslash\mathcal{U}/\mathcal{C}$. Instead of Lemma 5.1, we have that the operator $\pi_\nu(\gamma_{(\varpi^{-l_1},\varpi^{-l_2})})$ $(l_1, l_2 \in \mathbb{N})$ acting on the 7-dimensional space $I_\nu^{\mathcal{C}}$ has eigenvalues $q^{(l_1+l_2)(\nu-\frac{m-1}{2})}$, $q^{(l_1+l_2)(-\nu-\frac{m-1}{2})}$, $q^{-l_1+l_2(\nu-\frac{m-1}{2})}$, $q^{l_1(\nu-\frac{m-1}{2})-l_2}$, $q^{-(l_1+l_2)}$, $q^{l_1(-\nu-\frac{\nu-1}{2})-l_2}$, $q^{-l_1+l_2(-\nu-\frac{m-1}{2})}$. Lemma 5.2 and the formula (5.4) hold true as they are with ϖ replaced with ϖ_E. With $\psi_\pm^{(\nu)}$ defined as in (5.5), we have the first two formulas in Lemma 5.3 as they are with ϖ replaced with ϖ_E; instead of the third one, we claim that there exists a projector $\mathrm{Pr}^{(\nu)}$ from $I_\nu^{\mathcal{C}}$ onto the annihilator of $\mathbb{C}\psi_+^{(-\nu)} + \mathbb{C}\psi_-^{(-\nu)}$ with respect to the pairing $\langle\,,\rangle_{\mathcal{U}}$, depending meromorphically on ν. Lemma 5.4 holds true with larger exceptions of ν, because the complement of the unique open double coset HP has 3 low dimensional ones. The statement of Lemma 5.6 should be modified as

$$a(\nu) = \frac{(1-q^{-1})^2}{1-q^{-m}} \frac{1-q^{-2\nu-1}}{(1-q^{-\nu+(m-3)/2})^2}\,.$$

The proof of Lemma 5.6 should be changed as follows. Let $a_1(\nu)$ be as in the proof; then in place of (5.8), we have $a(\nu) = C(1-q^{-\nu-(m-1)/2})^2 a_1(\nu)$. Writing a general point $Z \in \mathcal{L}$ as $Z = z_1e_1 + X + z_2e_m$ $(z_1, z_2 \in \mathfrak{o}_E, X \in \mathcal{L}_1)$, we have the formula (5.9) as it is. Set $a = (a', a'')$ and $z_j = (z'_j, z''_j)$ with $a', a'', z'_j, z''_j \in \mathfrak{o}_F$. Making the variable change $u' = z'_1 - a''z'_2$, $u'' = z''_1 - a'z''_2$ to get rid of z'_1, z''_1, we see

$$A(l) = \int_{\substack{(u',u'',z'_2,z''_2)\in\mathfrak{o}_F^4,\\ u''z_2+u'z''_2+\mathrm{tr}_{E/F}(a)\,z'_2z''_2\in\varpi^l\mathfrak{o}_F^\times}} |u'|_F^{\nu-(m-1)/2}|u''|_F^{\nu-(m-1)/2}\mathrm{d}u'\mathrm{d}u''\,\mathrm{d}z'_2\mathrm{d}z''_2$$

By the variable change $z'_2 = -\mathrm{tr}_{E/F}(a)\,u' + v'$, $z''_2 = -\mathrm{tr}_{E/F}(a)\,u'' - v''$, noting $\mathrm{tr}_{E/F}(a) \in \mathfrak{o}_F^\times$, we get

$$\begin{aligned} A(l) &= \int_{\substack{(u',u'',v',v'')\in\mathfrak{o}_F^4\\ u'u''+v'v''\in\varpi^l\mathfrak{o}_F^\times}} |u'|_F^{\nu-(m-1)/2}|u''|_F^{\nu-(m-1)/2}\mathrm{d}u'\mathrm{d}u''\,\mathrm{d}v'\mathrm{d}v'' \\ &= \int_{t\in\mathfrak{o}_F} |t|_F^{\nu-(m-1)/2} v_1(t)v_2(t)\mathrm{d}t \end{aligned}$$

with $v_2(t) = \mathrm{vol}\big\{(v', v'') \in \mathfrak{o}_F^2 | v'v'' \in -t + \varpi^l\mathfrak{o}_F^\times\big\}$ and $v_1(t) = \mathrm{vol}\big(\big\{(u', u'') \in \mathfrak{o}_F^2 | u'u'' = t\big\}; \mathrm{d}_t u\big)$, where $\mathrm{d}_t u$ is the fiber measure on $\gamma^{-1}(t)$ of the mapping $\gamma : (u', u'') \mapsto t = u'u''$.

Since

$$\int_F v_1(t)\psi(t\tau)\mathrm{d}t = \int_{u',u''\in\mathfrak{o}_F} \psi(u'u''\tau)\mathrm{d}u'\mathrm{d}u'' = \inf\left(1, |\tau|_F^{-1}\right),$$

we easily have $v_1(t) = (1-q^{-1})(1+\mathrm{ord}_F(t))$ $(t \in \mathfrak{o}_F)$ by the Fourier inversion formula. By using this,

$$v_2(t) = \int_{\substack{y\in\mathfrak{o}_F \\ v\in -t+\varpi^l\mathfrak{o}_F^\times}} v_1(y)\mathrm{d}y = \sum_{k=0}^{\infty}(1-q^{-1})(1+k)\int_{\substack{y\in\varpi^k\mathfrak{o}_F^\times \\ v\in -t+\varpi^l\mathfrak{o}_F^\times}} \mathrm{d}y\,.$$

By evaluating the y-integral, we obtain

$$v_2(t) = (1-q^{-1})^2 q^{-l}\begin{cases} 1+\mathrm{ord}_F(t)\,, & (l > \mathrm{ord}_F(t))\,, \\ (l+1) + q^{-1}(1-q^{-1})^{-2}\,, & (l = \mathrm{ord}_F(t))\,, \\ 1+l\,, & (l < \mathrm{ord}_F(t))\,. \end{cases}$$

Set $T = q^{-(\nu-(m-1)/2+1)}$. By plugging the values $v_1(t)$ and $v_2(t)$ computed above,

$$\begin{aligned} A(l) &= \sum_{k=0}^{\infty} q^{-k(s-(m-1)/2)-k}(1-q^{-1})v_1(\varpi^k)v_2(\varpi^k) \\ &= (1-q^{-1})^4 q^{-l}\left\{\sum_{k=0}^{\infty}(k+1)^2T^k + \sum_{k=l}^{\infty}((1+k)(1+l)-(1+k)^2)T^k \right. \\ &\qquad \left. + (1+l)(1-q^{-1})^{-2}q^{-1}T^l\right\} \\ &= (1-q^{-1})^4 q^{-l}\left\{\tfrac{1+T}{(1-T)^3} - (l+2)\tfrac{T^{l+1}}{(1-T)^2} - \tfrac{2T^{l+2}}{(1-T)^3} + \tfrac{q^{-1}(l+1)T^l}{(1-q^{-1})^2}\right\}. \end{aligned}$$

A computation reveals

$$\begin{aligned} a_1(\nu) &= \sum_{l=0}^{\infty}\frac{1-q^{-(m-1)}}{1-q^{-(m-2)}}(1-q^{-(l+1)(m-2)})A(l) \\ &= (1-q^{-1})^2(1-q^{-(m-1)})\frac{1-q^{-(m-2)}T^2}{(1-T)^2(1-q^{-(m-2)}T)^2}\,. \end{aligned}$$

The remaining part of the proof is similar. Lemmas 5.8, 5.9 and 5.10 hold true as they are. Then we complete the evaluation of $\Omega_\nu(\mathsf{d}[\varpi_E^{-l}])$ as in Lemma 5.11.

5.4. The proof of the Fourier inversion formulas. The aim of this subsection is to provide a proof of Theorems 2.1, 3.1 and 4.6. Our argument relies on the explicit formulas of spherical functions recalled in §2.2, §3.1 for GL(2) case and given in Theorem 4.5 for the unitary case. A strong resemblance of the formulas for these cases is evident. We describe the argument only for the unitary group over a quadratic field E for clarity of argument leaving the remaining cases for the readers, because the necessary modifications are immediate.

THEOREM 5.12. *For any* $f \in C_c(H\backslash G/\mathcal{U})$, *we have the inversion formula*

$$[\mathcal{F}_{\mathbf{h}}^*\mathcal{F}_{\mathbf{h}} f](g) = f(g), \quad g \in G.$$

PROOF. Let $l \in \mathbb{N}$ and f_l the characteristic function of $H\mathsf{d}[\varpi^{-l}]\mathcal{U}$. Since f_l's form a $\mathbb{C}$-basis of $C_c(H\backslash G/\mathcal{U})$, it suffices to show the formula for those functions. As in the proof of Theorem 4.8, we have

$$\begin{aligned}
[\mathcal{F}_{\mathbf{h}}^*\mathcal{F}_{\mathbf{h}} f_l](\mathsf{d}[\varpi^{-k}]) &= \int_{\mathbb{X}_E^0} [\mathcal{F}_{\mathbf{h}} f_l](\nu)\, \Omega_\nu(\mathsf{d}[\varpi^{-k}])\, \mathrm{d}\Lambda_{\mathbf{h}}(\nu) \\
&= \frac{1}{\pi\sqrt{-1}} \int_{\mathbb{X}_E^0} \left\{ \int_{H\backslash G} f_l(g)\Psi_\nu(g)\mathrm{d}g \right\} \Omega_\nu(\mathsf{d}[\varpi^{-k}])\mathrm{d}\mu_m(\nu) \\
&= \frac{1}{\pi\sqrt{-1}} \int_{\mathbb{X}_E^0} v(\varpi^{-l})\Psi_\nu(\mathsf{d}[\varpi^{-l}])\Omega_\nu(\mathsf{d}[\varpi^{-k}])\, \mathrm{d}\mu_m(\nu)\,, \qquad (5.13)
\end{aligned}$$

where we set $v(\varpi^{-l}) = \mathrm{vol}(H\backslash H\mathsf{d}[\varpi^{-l}]\mathcal{U})$. By plugging the formulas (4.6) and the first formula in Theorem 4.5, we have the following expression for (5.13):

$$\frac{v(\varpi^{-l})Q_{E,m}^{-1}\, q_E^{-(k+l)(m-1)/2}}{\pi\sqrt{-1}} \int_{\mathbb{X}_E^0} \left\{ q_E^{-\nu(l-k)} + (-1)^{m-1} \frac{1 - q_E^{\nu-(m-1)/2}}{1 - q_E^{-(\nu+(m-1)/2)}} q_E^{-\nu(l+k+1)} \right\} \log q\, \mathrm{d}\nu.$$

By the change of variables $z = q_E^{-\nu} = q^{-2\nu}$, this becomes

$$\begin{aligned}
&\frac{v(\varpi^{-l})Q_{E,m}^{-1}\, q_E^{-(k+l)(m-1)/2}}{2\pi\sqrt{-1}} \oint_{|z|=1} \left\{ z^{l-k} + (-1)^{m-1} \frac{1 - q_E^{-(m-1)/2} z^{-1}}{1 - q_E^{-(m-1)/2} z} z^{k+l+1} \right\} \frac{\mathrm{d}z}{z} \\
&= v(\varpi^{-l})Q_{E,m}^{-1}\, q_E^{-(k+l)(m-1)/2} \{\delta(k=l) - (-1)^{m-1} q_E^{-(m-1)/2} \delta(k=l=0)\} \\
&= v(\varpi^{-l})Q_{E,m}^{-1}\, q_E^{-l(m-1)} \delta(k=l) \{1 + (-1)^m q_E^{-(m-1)/2}\}^{\delta(l=0)}.
\end{aligned}$$

The last expression is equal to $\delta(l = k)$ by the relation $v(\varpi^{-l}) = q_E^{l(m-1)}(1 + (-1)^m q_E^{-(m-1)/2})^{\delta(l>0)}$ from [8, Lemma 1.12]. Thus

$$[\mathcal{F}_{\mathbf{h}}^*\mathcal{F}_{\mathbf{h}} f_l](\mathsf{d}[\varpi^{-k}]) = \delta(k = l) = f_l(\mathsf{d}[\varpi^{-k}])$$

as desired. □

For $1 \leqslant j \leqslant [\frac{m}{2}]$, let $\check{\varepsilon}_j$ be a cocharacter of T defined as $\check{\varepsilon}_j(t) = \mathsf{d}(t_1, \dots, t_m)$ with $t_j = t$, $t_{m-j+1} = t^{-1}$ and $t_i = 1$ $(i \neq j, m-j+1)$. If we identify the dual torus $\hat{T}(\mathbb{C}) = \mathrm{Hom}(X_*(T), \mathbb{C}^\times)$ with $(\mathbb{C}^\times)^{[\frac{m}{2}]}$ by the map $\hat{T}(\mathbb{C}) \ni \lambda \mapsto (\lambda(\check{\varepsilon}_j))_{j=1}^{[\frac{m}{2}]} \in (\mathbb{C}^\times)^{[\frac{m}{2}]}$, then the maximal spectrum of the Hecke algebra $\mathcal{H}(G, \mathcal{U})$ for the pair $(G, \mathcal{U})$ is parametrized by the orbit space $(\mathbb{C}^\times)^{[\frac{m}{2}]}/W_G$ through the Satake isomorphism. By definition, the spherical function Ω_ν satisfies the Hecke eigenequation

$$\Omega_\nu * \phi = \Omega_\nu * \check{\phi} = \Omega_\nu \hat{\phi}(\nu)\,, \quad \phi \in \mathcal{H}(G,\mathcal{U})\,, \tag{5.14}$$

where $\check{\phi}(g) = \phi(g^{-1})$, and $\hat{\phi}(\nu)$ denote the image of ϕ under the $\mathbb{C}$-algebra homomorphism $\mathcal{H}(G,\mathcal{U}) \to \mathbb{C}$ with the Satake parameter $\big(q_E^{-\nu}, \{q_E^{-\frac{m-2j-1}{2}}\}_{j=1}^{m-2}\big)\, W_G$. Let $\mathcal{P}_H : \mathcal{H}(G,\mathcal{U}) \to C_{\mathrm{c}}(H\backslash G/\mathcal{U})$ be the linear map defined by

$$[\mathcal{P}_H(\phi)](g) = \int_H \phi(hg)\,\mathrm{d}h\,, \quad \phi \in \mathcal{H}(G,\mathcal{U})\,,$$

where $\mathrm{d}h$ is the Haar measure on H such that $\mathrm{vol}(H \cap \mathcal{U}) = 1$.

LEMMA 5.13. (1) *For $\phi \in \mathcal{H}(G,\mathcal{U})$, $[\mathcal{F}_{\mathbf{h}}(\mathcal{P}_H(\phi))](\nu) = \hat{\phi}(\nu)$.*

(2) *For $f \in C_{\mathrm{c}}(H\backslash G/\mathcal{U})$, we have*

$$[\mathcal{F}_{\mathbf{h}}(f * \phi)](\nu) = [\mathcal{F}_{\mathbf{h}} f](\nu)\,\hat{\phi}(\nu)\,, \quad \phi \in \mathcal{H}(G,\mathcal{U})\,. \tag{5.15}$$

PROOF. This follows from (5.14) by a straightforward computation. □

The sign change $\nu \to -\nu$ on $\mathbb{X}_E^0$ defines an action of the group $S = \{\pm 1\}$ on the space $\mathbb{X}_E^0$. The space of S-invariant Laurent polynomials $\mathbb{C}[z, z^{-1}]^S$ is embedded to the space of continuous functions $C(\mathbb{X}_E^0/S)$ by $z \mapsto q_E^{-\nu}$.

LEMMA 5.14. *For any $f \in C_{\mathrm{c}}(H\backslash G/\mathcal{U})$, we have $\mathcal{F}_{\mathbf{h}} f \in \mathbb{C}[z, z^{-1}]^S$.*

PROOF. It suffices to show that $\mathcal{F}_{\mathbf{h}} f_l \in \mathbb{C}[z, z^{-1}]^S$ for any $l \in \mathbb{N}$, where f_l denotes the characteristic function of the double coset $H\mathsf{d}[\varpi^{-l}]\mathcal{U}$ on G. We have

$$\mathcal{F}_{\mathbf{h}} f_l(\nu) = \Omega_\nu(\mathsf{d}[\varpi^{-l}])\,\mathrm{vol}(H\mathsf{d}[\varpi^{-l}]\mathcal{U})\,,$$

which is seen to belong to $\mathbb{C}[z, z^{-1}]^S$ from the formula in Theorem 4.5. □

LEMMA 5.15. *The linear map $\mathcal{P}_H : \mathcal{H}(G,\mathcal{U}) \to C_{\mathrm{c}}(H\backslash G/\mathcal{U})$ is surjective.*

PROOF. Let $f \in C_{\mathrm{c}}(H\backslash G/\mathcal{U})$. Then $\mathcal{F}_{\mathbf{h}} f(\nu) \in \mathbb{C}[z, z^{-1}]^S$ from Lemma 5.14. By using the Satake isomorphism, we can find a function $\phi \in \mathcal{H}(G,\mathcal{U})$ such that $\mathcal{F}_{\mathbf{h}} f(\nu) = \hat{\phi}(\nu)$. For such ϕ, from Lemma 5.13 (1) and Theorem 5.12,

$$f = \mathcal{F}_{\mathbf{h}}^* \mathcal{F}_{\mathbf{h}} f = \mathcal{F}_{\mathbf{h}}^*(\hat{\phi}) = \mathcal{F}_{\mathbf{h}}^* \mathcal{F}_{\mathbf{h}}(\mathcal{P}_H(\phi)) = \mathcal{P}_H(\phi)$$

as desired. □

LEMMA 5.16. *Let $\mathcal{A}$ be the image of $C_{\mathrm{c}}(H\backslash G/\mathcal{U})$ by the transform $\mathcal{F}_{\mathbf{h}}$. The subspace $\mathcal{A}$ is everywhere dense in $C(\mathbb{X}_E^0/S)$ with respect to the topology of uniform convergence.*

PROOF. Obviously $\mathcal{A}$ is a $\mathbb{C}$-subspace of $C(\mathbb{X}_E^0/S)$. Moreover, Lemmas 5.13 and 5.15 combined show that $\mathcal{A}$ is closed under the pointwise product of functions. Since $\mathcal{F}_{\mathbf{h}}$ maps the characteristic function of $H\backslash H\mathcal{U}$ to 1, $\mathcal{A}$ contains the unit element of $C(\mathbb{X}_E^0/S)$. Since $\overline{\Omega_\nu(g)} = \Omega_{-\nu}(g) = \Omega_\nu(g)$ for $\nu \in \mathbb{X}_E^0$, the space $\mathcal{A}$ is stable under the complex conjugation. We shall show that $\mathcal{A}$ separates the points of $\mathbb{X}_E^0/S$, i.e., for any two different points ν and ν' in $\mathbb{X}_E^0/S$, there exists some $f \in C_c(H\backslash G/\mathcal{U})$ such that $[\mathcal{F}_{\mathbf{h}}f](\nu) \neq [\mathcal{F}_{\mathbf{h}}f](\nu')$. Suppose $[\mathcal{F}_{\mathbf{h}}f](\nu) = [\mathcal{F}_{\mathbf{h}}f](\nu')$ for all f contrarily. Then $\Omega_\nu(g) = \Omega_{\nu'}(g)$ on G. From the Hecke eigenequation (5.14) , we have $\hat{\phi}(\nu) = \hat{\phi}(\nu')$ for all $\phi \in \mathcal{H}(G,\mathcal{U})$, which implies $(q_E^{-\nu}, \{q_E^{-\frac{m-2j-1}{2}}\}_{j=1}^{m-2})W_G = (q_E^{-\nu'}, \{q_E^{-\frac{m-2j-1}{2}}\}_{j=1}^{m-2})W_G$. Since ν, ν' are purely imaginary, this forces us to have $\nu = \pm\nu'$, a contradiction. Now we apply the Stone-Weierstrass theorem [17, Theorem 7.31] to conclude that $\mathcal{A}$ is everywhere dense in $C(\mathbb{X}_E^0/S)$. □

We endow the spaces $C_c(H\backslash G/\mathcal{U})$ and $C(\mathbb{X}_E^0/S)$ with hermitian inner-products defined by

$$(f|f_1)_{H\backslash G} = \int_{H\backslash G} f(g)\overline{f_1(g)}\,\mathrm{d}g\,, \quad (\alpha|\alpha_1)_{\mathbb{X}_E^0} = \int_{\mathbb{X}_E^0} \alpha(\nu)\overline{\alpha_1(\nu)}\,\mathrm{d}\Lambda_{\mathbf{h}}(\nu)\,,$$

respectively. We consider the first inner-product for functions outside compactly supported ones as long as it makes sense.

LEMMA 5.17. *Let $\alpha \in \mathbb{C}[z, z^{-1}]^S$. Then $\mathcal{F}_{\mathbf{h}}^*\alpha \in C_c(H\backslash G/\mathcal{U})$.*

PROOF. We use the notation introduced in the proof of Theorem 5.12. In the same way as there, we have the contour integral expression

$$[\mathcal{F}_{\mathbf{h}}^*\alpha](\mathsf{d}[\varpi^{-l}]) = \frac{1}{\pi\sqrt{-1}} \int_{\mathbb{X}_E^\delta} \alpha(\nu)\Psi_\nu(\mathsf{d}[\varpi^{-l}])\,\mathrm{d}\mu_m(\nu)\,.$$

For $l \in \mathbb{N}$, set $\alpha_l(\nu) = z^l + z^{-l}$ with $z = q_E^{-\nu}$. Then α_l's form a linear basis of $\mathbb{C}[z, z^{-1}]^S$. We have

$$\begin{aligned}[\mathcal{F}_{\mathbf{h}}^*\alpha_l](\mathsf{d}[\varpi^{-l'}]) &= \frac{1}{\pi\sqrt{-1}} \int_{\mathbb{X}_E^\delta} \alpha_l(\nu)\Psi_\nu(\mathsf{d}[\varpi^{-l'}])\,\mathrm{d}\mu_m(\nu)\\ &= \frac{1}{2\pi\sqrt{-1}} \oint_{|z|=q_E^{-\delta}} q^{-l'(m-1)} \frac{z^{l'-l-1}(z^{2l}+1)(1-(-1)^m z)}{1-q^{-(m-1)}z}\,\mathrm{d}z\,.\end{aligned}$$

If $l' \geqslant l+1$, then the integrand is evidently holomorphic in z on the disc $|z| < q_E^{-\delta}$. Thus the integral vanishes by Cauchy's theorem. Hence the support of the function $\mathcal{F}_{\mathbf{h}}^*\alpha_l$ is contained in the union of finite number of cosets $H\mathsf{d}[\varpi^{-l'}]\mathcal{U}$ $(0 \leqslant l' \leqslant l)$. □

LEMMA 5.18. *The mapping $f \mapsto \mathcal{F}_{\mathbf{h}}f$ from $C_c(H\backslash G/\mathcal{U})$ to $C(\mathbb{X}_E^0/S)$ preserves the inner-products. The mapping $\alpha \mapsto \mathcal{F}_{\mathbf{h}}^*\alpha$ from $\mathbb{C}[z, z^{-1}]^S$ to $C_c(H\backslash G/\mathcal{U})$ preserves the inner-*

products. Moreover,

$$(\mathcal{F}_{\mathbf{h}} f|\alpha)_{\mathbb{X}_E^0} = (f|\mathcal{F}_{\mathbf{h}}^*\alpha)_{H\backslash G} \tag{5.16}$$

for any $f \in C_{\mathrm{c}}(H\backslash G/\mathcal{U})$ *and* $\alpha \in C(\mathbb{X}_E^0/S)$.

PROOF. The formula (5.16) is proved by a direct application of Fubini's theorem. Then by applying the formula (5.16) and using Theorem 5.12, we have

$$(\mathcal{F}_{\mathbf{h}} f_1|\mathcal{F}_{\mathbf{h}} f)_{\mathbb{X}_E^0} = (f_1|\mathcal{F}_{\mathbf{h}}^*\mathcal{F} f)_{H\backslash G} = (f_1|f)_{H\backslash G} \quad \text{for all } f_1,\ f \in C_{\mathrm{c}}(H\backslash G/\mathcal{U}).$$

This shows the first assertion. It remains to prove the second assertion. To argue, let f, $f_1 \in C_{\mathrm{c}}(H\backslash G/\mathcal{U})$ and set $\alpha = \mathcal{F}_{\mathbf{h}} f$, $\alpha_1 = \mathcal{F}_{\mathbf{h}} f_1$. Then by Theorem 5.12 and by the first assertion we just established,

$$(\mathcal{F}_{\mathbf{h}}^*\alpha|\mathcal{F}_{\mathbf{h}}^*\alpha_1)_{H\backslash G} = (\mathcal{F}_{\mathbf{h}}^*\mathcal{F}_{\mathbf{h}} f|\mathcal{F}_{\mathbf{h}}^*\mathcal{F}_{\mathbf{h}}\alpha_1)_{H\backslash G} = (f|f_1)_{H\backslash G} = (\alpha|\alpha_1)_{\mathbb{X}_E^0}\,.$$

This shows $(\mathcal{F}_{\mathbf{h}}^*\alpha|\mathcal{F}_{\mathbf{h}}^*\alpha_1)_{H\backslash G} = (\alpha|\alpha_1)_{\mathbb{X}_E^0}$ for any α, α_1 belonging to the space $\mathcal{A}$ defined in Lemma 5.16. By Lemma 5.16 (i), the same formula remains valid for all elements of $L^2(\mathbb{X}_E^0/S;\, \mathrm{d}\Lambda_{\mathbf{h}})$, in particular for those from $\mathbb{C}[z, z^{-1}]^S$. □

The following lemma shows the space $\mathcal{A}$ in Lemma 5.16 coincides with $\mathbb{C}[z, z^{-1}]^S$.

THEOREM 5.19. *For any* $\alpha \in \mathbb{C}[z, z^{-1}]^S$, *the integral* $\mathcal{F}_{\mathbf{h}}^*\alpha$ *belongs to* $C_{\mathrm{c}}(H\backslash G/\mathcal{U})$, *and*

$$[\mathcal{F}_{\mathbf{h}}\mathcal{F}_{\mathbf{h}}^*\alpha](\nu) = \alpha(\nu)\,, \quad \nu \in \mathbb{X}_E^0\,. \tag{5.17}$$

PROOF. Let α, $\beta \in \mathbb{C}[z, z^{-1}]^S$. Then by (5.16) and by the second assertion of Lemma 5.18, we have $(\mathcal{F}_{\mathbf{h}}\mathcal{F}_{\mathbf{h}}^*\alpha|\beta)_{\mathbb{X}_E^0} = (\mathcal{F}_{\mathbf{h}}^*\alpha|\mathcal{F}_{\mathbf{h}}^*\beta)_{H\backslash G} = (\alpha|\beta)_{\mathbb{X}_E^0}$. Lemma 5.16 tells that $\mathbb{C}[z, z^{-1}]^S$ is dense in $L^2(\mathbb{X}_E^0/S;\, \mathrm{d}\Lambda_{\mathbf{h}})$. Thus we should have the identity $\mathcal{F}_{\mathbf{h}}\mathcal{F}_{\mathbf{h}}^*\alpha = \alpha$ in the L^2-space. Since both sides of the equality are continuous (for the left-hand side, use Lemmas 5.16 and 5.17), we have the point-wise equality (5.17). □

6. Appendix 2: A Cartan type decomposition for unitary groups (the split case)

We continue to hold all the notation and assumptions in §1.2.2. The aim of this subsection is to give a proof of the decomposition (4.2) for the case $E = F \oplus F$. A vector $\ell \in \mathcal{L}$ is called to be primitive in $\mathcal{L}$ if $\ell \notin \mathfrak{m}\mathcal{L}$ for any maximal ideal $\mathfrak{m} \subset \mathfrak{o}_E$; the set of all such vectors is denoted by $\mathcal{L}_{\mathrm{prim}}$. If we set $\varpi_1 = (\varpi, 1)$ and $\varpi_2 = (1, \varpi)$, then $\mathcal{L}_{\mathrm{prim}} = \mathcal{L} - (\varpi_1\mathcal{L} \cup \varpi_2\mathcal{L})$.

LEMMA 6.1. *For any primitive vector* $\ell \in \mathcal{L}$, *there exists* $k \in \mathcal{U}$ *such that* $\mathbf{h}(k(\ell), e_1) = 1$.

PROOF. By the $\mathfrak{o}_E$-basis of $\mathcal{L}$ fixed in §4, we identify $\mathcal{L} = \mathfrak{o}_E^m = \mathfrak{o}_F^m \oplus \mathfrak{o}_F^m$ and realize G as a matrix group

$$G = \{(g, T^{-1t}g^{-1}T) |\, g \in \mathrm{GL}_m(F)\}\,,$$

where $T = (\mathbf{h}(e_i, e_j))_{1\leqslant i,j\leqslant m}$. Then $\mathcal{U} = \{(g, T^{-1t}g^{-1}T) |\, g \in \mathrm{GL}_m(\mathfrak{o}_F)\}$. Set $\ell = (x, y)$ with $x, y \in \mathfrak{o}_F^m$. Since x is a primitive vector of $\mathfrak{o}_F^m$, by transforming by a suitable element in $\mathrm{GL}_m(\mathfrak{o}_F)$, we may suppose $x = \begin{bmatrix} 0 \\ 0_{m-2} \\ 1 \end{bmatrix}$. Set $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ with $y_1, y_3 \in \mathfrak{o}_F$ and $y_2 \in \mathfrak{o}_F^{m-2}$. If $y_3 \in \mathfrak{o}_F^\times$, then the element $k = (\mathsf{d}(y_3, 1, \ldots, 1), T^{-1}\mathsf{d}(y_3^{-1}, 1, \ldots, 1)T)$ belongs to $\mathcal{U}$ and

$$k(\ell) = \left(\begin{bmatrix} 0 \\ 0_{m-2} \\ 1 \end{bmatrix}, \begin{bmatrix} y_1' \\ y_2' \\ 1 \end{bmatrix} \right) \quad \text{with some } y_1' \in \mathfrak{o}_F \text{ and } y_2' \in \mathfrak{o}_F^{m-2}\,.$$

Hence $\mathbf{h}(k(\ell), e_1) = 1$. If $y_3 \notin \mathfrak{o}_F^\times$, then (y_2, y_3) is a primitive vector of $\mathfrak{o}_F^{m-1}$. Therefore, there exists $A_{12} \in \mathrm{M}_{1,m-1}(\mathfrak{o}_F)$ and $a_{13} \in \mathfrak{o}_F$ such that $A_{12}T_0y_2 + a_{13}y_1 = 1 - y_3$, where $T_0 = (\mathbf{h}(e_i, e_j))_{2\leqslant i,j\leqslant m-1}$. Set $u = \begin{bmatrix} 1 & A_{12} & a_{13} \\ 0 & 1_{m-2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $k = ({}^tu^{-1}, T^{-1}uT)$. Then $k \in \mathcal{U}$ and $k(\ell) = \left(\begin{bmatrix} 0 \\ 0_{m-2} \\ 1 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ 1 \end{bmatrix} \right)$. Thus $\mathbf{h}(k(\ell), e_1) = 1$. $\square$

LEMMA 6.2. *Suppose $\ell_0 = ae_1 + e_m$ with $\mathbf{h}[\ell_0] = a + \bar{a} = 1$. Let ℓ_1 be a primitive vector of $\mathcal{L}$ such that $\mathbf{h}[\ell_1] = \varpi^l \mathbf{h}[\ell_0]$ with $l \in \mathbb{N}$. Then, there exists $k \in \mathcal{U}$ such that $k\ell_1 = a\varpi^l e_1 + e_m$.*

PROOF. By Lemma 6.1, we may assume $\mathbf{h}(\ell_1, e_1) = 1$. Then there exist $a_j \in \mathfrak{o}_E$ $(1 \leqslant j \leqslant m-1)$ such that $\ell_1 = \sum_{j=1}^{m-1} a_j e_j + e_m$. Set $\xi = \sum_{j=2}^{m-1} a_j e_j$ and $b = 2^{-1}\mathrm{tr}_{E/F}\left(\frac{\varpi^l a + \bar{a}_1}{\sqrt{\theta}}\right)$. The element $k = \mathsf{n}(-\xi; b) \in G$ belongs to $\mathcal{U}$, and $k(\ell_1) = \left(a_1 + 2^{-1}\mathbf{h}[\xi] + b\sqrt{\theta}\right) e_1 + e_m$. From $\mathbf{h}[\ell_1] = \varpi^l \mathbf{h}[\ell_0] = \varpi^l(a + \bar{a})$, we have $a_1 + 2^{-1}\mathbf{h}[\xi] + b\sqrt{\theta} = \varpi^l a$. $\square$

For any $(l_1, l_2) \in \mathbb{Z}^2$, set

$$\mathcal{G}_{l_1,l_2} = \{g \in G |\, g^{-1}\ell_0 \in \varpi_1^{-l_1}\varpi_2^{-l_2}\mathcal{L}_{\mathrm{prim}}\}\,.$$

If $g \in \mathcal{G}_{l_1,l_2}$, then $\varpi_1^{l_1}\varpi_2^{l_2}g^{-1}\ell_0 \in \mathcal{L}$; since $\mathbf{h}[\mathcal{L}] \subset \mathfrak{o}_F$, we have $\varpi^{l_1+l_2} = \mathbf{h}[\varpi_1^{l_1}\varpi_2^{l_2}g^{-1}\ell_0] \in \mathfrak{o}_F$ and thus $l_1 + l_2 \geqslant 0$. Hence G is a disjoint union of $\mathcal{G}_{l_1,l_2}$ with $(l_1, l_2) \in \mathbb{Z}^2$, $l_1 + l_2 \geqslant 0$. Let us show $\mathcal{G}_{l_1,l_2} = H_0\mathsf{d}[\varpi_1^{-l_1}\varpi_2^{-l_2}]\mathcal{U}$ for any $(l_1, l_2) \in \mathbb{Z}^2$, $l_1 + l_2 \geqslant 0$. We have $\mathsf{d}[\varpi_1^{-l_1}\varpi_2^{-l_2}] \in \mathcal{G}_{l_1,l_2}$ because

$$\varpi_1^{l_1}\varpi_2^{l_2}\mathsf{d}[\varpi_1^{-l_1}\varpi_2^{-l_2}]\ell_0 = \varpi^{l_1+l_2}ae_1 + e_m \in \mathcal{L}_{\mathrm{prim}}\,.$$

Since $H_0\mathcal{G}_{l_1,l_2}\mathcal{U} \subset \mathcal{G}_{l_1,l_2}$, we obtain the inclusion $H_0\mathsf{d}[\varpi_1^{-l_1}\varpi_2^{-l_2}]\mathcal{U} \subset \mathcal{G}_{l_1,l_2}$. To have the converse inclusion, we let g be any element from $\mathcal{G}_{l_1,l_2}$. Then $\ell_1 = \varpi_1^{l_1}\varpi_2^{l_2}g^{-1}\ell_0$ belongs to $\mathcal{L}_{\rm prim}$ and $\mathbf{h}[\ell_1] = \varpi^{l_1+l_2}\mathbf{h}[\ell_0]$. By Lemma 6.2, there exists $k \in \mathcal{U}$ such that $k\ell_1 = \varpi^{l_1+l_2}ae_1 + e_m = \varpi_1^{l_1}\varpi_2^{l_2}\mathsf{d}[\varpi_1^{-l_1}\varpi_2^{-l_2}]\ell_0$. Thus $g^{-1}\ell_0 = \mathsf{d}[\varpi_1^{-l_1}\varpi_2^{-l_2}]\ell_0$, or equivalently $\mathsf{d}[\varpi_1^{-l_1}\varpi_2^{-l_2}]^{-1}g \in H_0$. This shows $g \in H_0\mathsf{d}[\varpi_1^{-l_1}\varpi_2^{-l_2}]\ell_0]\mathcal{U}$. We proved the disjoint decomposition

$$G = \bigcup_{l_1,l_2\in\mathbb{Z},l_1+l_2\geqslant 0} H_0\mathsf{d}[\varpi_1^{-l_1}\varpi_2^{-l_2}]\mathcal{U}.$$

To deduce (4.2) from this, it remains to note the equality $\bigcup_{l_1+l_2=l} H_0\mathsf{d}[\varpi_1^{-l_1}\varpi_2^{-l_2}]\mathcal{U} = H\mathsf{d}[\varpi_1^{-l}]\mathcal{U}$ for any $l \in \mathbb{N}$, which follows from the relation $\mathsf{d}[(\varpi_1\varpi_2^{-1})^{l_2}] \in H$.

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