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Toward Noether's Problem for the Fields of Cross-ratios

Dedicated to Professor Ken-ichi SHINODA

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Abstract. In this article, we consider an analogue of Noether's problem for the fields of cross-ratios, and discuss on a rationality problem which connects this with Noether's problem. We show that the affirmative answer of the analogue implies the affirmative answer for Noether's Problem for any permutation group with odd degree. We also obtain some negative results for various permutation groups with even degree.

1. Introduction

Let k be a field and consider the action of the symmetric group $\mathfrak{S}_n$ on the rational function field $L_n := k(x_1, \dots, x_n)$ of n variables over k by permutation; $\sigma(x_i) := x_{\sigma(i)}$.

E. Noether [13, 14] proposed the following problem as a basic strategy for the inverse Galois problem.

PROBLEM 1 (Noether's Problem). *For a subgroup H of $\mathfrak{S}_n$, is the fixed subfield L_n^H of L_n rational (i.e. purely transcendental) over k? That is, whether there exist $t_1, \dots, t_n \in L_n$ with $L_n^H = k(t_1, \dots, t_n)$?*

This problem is highly non-trivial even in the case of cyclic groups. The affirmative answer for H implies the existence of a generic polynomial over k for H if k is infinite. This problem is generalized to the following *Rationality Problem*, which is sometimes also called *General Noether Problem* in this context:

PROBLEM 2 (Rationality Problem). *For a finite subgroup H of $\mathrm{Aut}_k(L_n)$, is the fixed subfield L_n^H of L_n rational over k?*

In this article we consider the subfield of L_n generated by cross-ratios of variables and the action of $\mathfrak{S}_n$ on it. The projective general linear group $\mathrm{PGL}(2, k)$ of degree two acts on L_n from the left by diagonal linear transformation: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x_i := \dfrac{ax_i + b}{cx_i + d}$. For $n \geq 3$, let

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$K_n := L_n^{\mathrm{PGL}(2,k)}$ be the fixed field of L_n under the action of $\mathrm{PGL}(2,k)$. Then K_n is generated by cross-ratios among x_i's, and is rational over k of transcendental degree $n-3$. Since the actions of $\mathrm{PGL}(2,k)$ and $\mathfrak{S}_n$ on L_n commute with each other, $\mathfrak{S}_n$ acts also on K_n, faithfully for $n \geq 5$. We assume $n \geq 5$ throughout this article. Our interest is to consider an analogue of Noether's Problem for K_n.

PROBLEM 3 (Cross-Ratio Noether's Problem). *For a subgroup H of $\mathfrak{S}_n$, is the fixed field K_n^H rational over k? That is, whether there exist $t_1, \dots, t_{n-3} \in K_n$ with $K_n^H = k(t_1, \dots, t_{n-3})$?*

It has various importance to consider Cross-Ratio Noether's Problem. First, it has a geometric background, that is, K_n is the function field of the moduli space $\mathcal{M}_{0,n}$ of projective lines with ordered n marked points. Secondly, an affirmative answer of this problem yields a generic polynomial. By the theorem of Kemper-Mattig [10], if Cross-Ratio Noether's Problem over k for H is affirmative, we have a generic polynomial over k for H, while the rationality of the fixed field under a general action of a finite group H does not necessarily imply the existence of a generic polynomial over k for H. Thirdly, since the transcendental degree of K_n over k is smaller than that of L_n, the actual calculation for Cross-Ratio Noether's Problem turns to be easier than that of Noether's Problem, especially for small n. In fact, in preceding works [7, 8] of Hashimoto and the author, they gave affirmative answers for all transitive groups of degree 5 and for that of degree 6 except two cases. And last of all, it has natural relationship with Noether's Problem as follows. To connect Noether's Problem and Cross-Ratio Noether's Problem, we consider the rationality of L_n^H over K_n^H.

PROBLEM 4. *For a subgroup H of $\mathfrak{S}_n$, is the fixed field L_n^H rational over K_n^H? That is, whether there exist $t_1, t_2, t_3 \in L_n^H$ with $L_n^H = K_n^H(t_1, t_2, t_3)$?*

If this is true, then the affirmative answer of Cross-Ratio Noether's Problem for H implies the affirmative answer of Noether's Problem for H. We notice that, if this is true for $\mathfrak{S}_n$, then it is true also for any subgroup H of $\mathfrak{S}_n$, since the same generators $t_1, t_2, t_3 \in L_n^{\mathfrak{S}_n}$ work also for H.

To investigate this problem, we further divide it into three steps according to the sequence of subgroups of $\mathrm{PGL}(2,k)$:

$$\mathrm{PGL}(2,k) \supset B := \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \rhd U := \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \rhd \{1\}\,. \tag{1}$$

Let us consider the fixed fields L_n^U, L_n^B under the action of U, B respectively. Then each step of the extension

$$L_n \supset L_n^U \supset L_n^B \supset K_n$$

is purely transcendental of transcendental degree 1. We consider the extension of the fixed fields

$$L_n^{\mathfrak{S}_n} \supset (L_n^U)^{\mathfrak{S}_n} \supset (L_n^B)^{\mathfrak{S}_n} \supset K_n^{\mathfrak{S}_n}$$

by $\mathfrak{S}_n$. As we shall see in Section 2, provided the characteristic of k does not divide n, the extensions $(L_n)^{\mathfrak{S}_n}/(L_n^U)^{\mathfrak{S}_n}$ and $(L_n^U)^{\mathfrak{S}_n}/(L_n^B)^{\mathfrak{S}_n}$ are purely transcendental of transcendental degree 1. Hence it is the remaining problem to see the extension $(L_n^B)^{\mathfrak{S}_n}/K_n^{\mathfrak{S}_n}$. We put $\widetilde{K}_n := L_n^B$. Our main problem in this article is the following:

PROBLEM 5. *For a subgroup H of $\mathfrak{S}_n$, is the fixed field $\widetilde{K}_n^H$ rational over K_n^H? That is, whether there exists $t \in \widetilde{K}_n^H$ with $\widetilde{K}_n^H = K_n^H(t)$?*

For the most important case of $H = \mathfrak{S}_n$, we show the following theorem:

THEOREM 1. *Let n be an integer with $n \geq 5$. Assume that a base field k is infinite.*

(i) *When n is odd, for any base field k (in particular, for $k = \mathbb{Q}$), $\widetilde{K}_n^{\mathfrak{S}_n}$ is rational over $K_n^{\mathfrak{S}_n}$.*
(ii) *Assume that the characteristic of k is not two. When n is even, for any base field k (in particular, even when $k = \overline{k}$), $\widetilde{K}_n^{\mathfrak{S}_n}$ is not rational over $K_n^{\mathfrak{S}_n}$.*

The statements (i) and (ii) will be proved in Section 5 and 6 respectively.

Our theorem implies that $\widetilde{K}_n^H$ is rational over K_n^H for any subgroup H of $\mathfrak{S}_n$ when n is odd and $n \geq 5$. Hence we have

COROLLARY 1. *Let k be an infinite field whose characteristic does not divide n. For an odd integer n with $n \geq 5$ and a subgroup H of $\mathfrak{S}_n$, the affirmative answer for Cross-Ratio Noether's Problem for H implies that for Noether's Problem for H.*

Combining this with [7], we reprove the following:

COROLLARY 2. *Let k be a field of characteristic 0. For all transitive subgroup H of $\mathfrak{S}_5$, Noether's Problem for H is affirmative, that is, L_5^H is rational over k.*

As the case of $H = \mathfrak{A}_5$, this includes the characteristic zero case of Maeda's outstanding theorem [11], the affirmative answer of Noether's Problem for $\mathfrak{A}_5$.

It is a remaining problem that, for which subgroup H of $\mathfrak{S}_n$, $\widetilde{K}_n^H$ is rational over K_n^H when n is even. We discuss some cases in Section 7.

THEOREM 2. *Assume that k is infinite and of characteristic different from two. For any transitive subgroup H of degree $n = 2^e$ ($e \geq 3$), $\widetilde{K}_n^H$ is not rational over K_n^H.*

THEOREM 3. *Assume that k is infinite and of characteristic different from two. For any transitive subgroup T of $\mathfrak{S}_6$, $\widetilde{K}_6^T$ is not rational over K_6^T.*

We must notice that the non-rationality of $\widetilde{K}_n^H/K_n^H$ necessarily imply *neither* the non-rationality of L_n^H/K_n^H *nor* the negative answer for Noether's Problem for H.

2. Preliminaries

Throughout this article, we assume that n is an integer with $n \geq 5$. The projective general linear group $G := \mathrm{PGL}(2, k)$ acts on $L_n := k(x_1, \dots, x_n)$ from the left as linear fractional transformation diagonally:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x_i := \frac{ax_i + b}{cx_i + d}.$$

The fixed field $K_n := L_n^G$ is generated by cross-ratios of the variables x_i's, and is a purely transcendental extension (a rational function field) over k of transcendental degree $(n-3)$. In fact, we have

$$K_n = k\left(\frac{x_i - x_1}{x_i - x_2} \Big/ \frac{x_3 - x_1}{x_3 - x_2} \;\middle|\; i = 4, \dots, n\right), \tag{2}$$

and L_n is purely transcendental over K_n of transcendental degree 3. We shall call K_n *the field of cross-ratios*.

Since the actions of G and $\mathfrak{S}_n$ on L_n commute with each other, $\mathfrak{S}_n$ acts also on K_n. When $n \geq 5$, this action is faithful. Then *Cross-Ratio Noether's Problem* asks, for a subgroup H of $\mathfrak{S}_n$, whether the fixed field K_n^H is rational over k or not (Problem 3).

To see the relationship between Noether's Problem and Cross-Ratio Noether's Problem, we ask the rationality of L_n^H over K_n^H (Problem 4).

If both Problems 3 and 4 for H is affirmative (that is, K_n^H is rational over k and L_n^H is rational over K_n^H), then L_n^H is rational over k, hence also Noether's Problem for H is affirmative. We do not know whether the converse is true or not.

If Problem 4 for $\mathfrak{S}_n$ is true, then there exists algebraically independent generators (minimal bases) $t_1, t_2, t_3 \in L_n^{\mathfrak{S}_n}$. For any subgroup H of $\mathfrak{S}_n$, we have $L_n^H = K_n^H(t_1, t_2, t_3)$. Hence Problem 4 for $\mathfrak{S}_n$ implies Problem 4 for H.

The extension L_n^H / K_n^H is of transcendental degree 3. We can subdivide this into three steps as follows. Consider a sequence

$$G = \mathrm{PGL}(2, k) \supset B := \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\} \rhd U := \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\} \rhd \{1\} \tag{3}$$

of subgroups of $G = \mathrm{PGL}(2, k)$ and the fixed subfields of the action of them. Each step of the sequence $L_n \supset L_n^U \supset L_n^B \supset L_n^G = K_n$ of the fixed fields is a purely transcendental extension of transcendental degree 1. In fact, we have

$$L_n^B = k\left(\frac{x_i - x_1}{x_i - x_2} \;\middle|\; i = 3, \dots, n\right) = K_n\left(\frac{x_3 - x_1}{x_3 - x_2}\right), \tag{4}$$

$$L_n^U = k\,(x_i - x_1 \mid i = 2, \dots, n) = L_n^B(x_2 - x_1), \tag{5}$$

$$L_n = L_n^U(x_1). \tag{6}$$

We shall call L_n^U *the field of differences* and L_n^B *the field of ratios of differences*. In the following we denote L_n^B also by $\widetilde{K}_n$. Our problem is whether the rationality of these extensions descend to the fixed fields under permutation groups.

As mentioned below, for the upper two steps $L_n \supset L_n^U \supset \widetilde{K}_n$, this is always true provided the characteristic of k does not divide n: each step of $L_n^{\mathfrak{S}_n} \supset (L_n^U)^{\mathfrak{S}_n} \supset \widetilde{K}_n^{\mathfrak{S}_n}$ is purely transcendental.

Indeed, let s_i $(1 \leq i \leq n)$ be the i-th fundamental symmetric polynomials of $x_1, x_2, \ldots, x_n$, that is, the polynomials defined by

$$F(X) := \prod_{i=1}^{n}(X + x_i) =: X^n + \sum_{i=1}^{n} s_i X^{n-i} \,. \tag{7}$$

Then we have $L_n^{\mathfrak{S}_n} = k(s_1, \ldots, s_n)$. For $(L_n^U)^{\mathfrak{S}_n}$ and $\widetilde{K}_n^{\mathfrak{S}_n}$, we have the following[1]

PROPOSITION 1. *Assume that the characteristic of k does not divide n. Define the polynomials t_i $(2 \leq i \leq n)$ in $x_1, x_2, \ldots, x_n$ by*

$$X^n + \sum_{i=2}^{n} t_i X^{n-i} := F\left(X - \frac{s_1}{n}\right) . \tag{8}$$

Then

(i) $(L_n^U)^{\mathfrak{S}_n} = k(t_2, \ldots, t_n)$ *and* $L_n^{\mathfrak{S}_n} = (L_n^U)^{\mathfrak{S}_n}(s_1)$.

(ii) $\widetilde{K}_n^{\mathfrak{S}_n} = k\left(t_i\left(\frac{t_2}{t_3}\right)^i \,\middle|\, 3 \leq i \leq n\right)$ *and* $(L_n^U)^{\mathfrak{S}_n} = \widetilde{K}_n^{\mathfrak{S}_n}\left(\frac{t_2}{t_3}\right)$.

PROOF. (i) When we substitute x_i by $x_i - a$ $(a \in k)$, $F(X)$ changes to

$$\prod_{i=1}^{n}\bigl(X + (x_i - a)\bigr) = \prod_{i=1}^{n}\bigl((X - a) + x_i\bigr) = F(X - a)\,,$$

and $\frac{s_1}{n}$ changes to $\frac{s_1}{n} - a$. Hence $F\left(X - \frac{s_1}{n}\right)$ is invariant under this translation, so are t_j $(2 \leq i \leq n)$. Since $k(s_1, \ldots, s_n) = k(s_1, t_2, \ldots, t_n) = k(t_2, \ldots, t_n)(s_1)$, we have $k(s_1, \ldots, s_n)^U = k(t_2, \ldots, t_n)(s_1)^U = k(t_2, \ldots, t_n)$.

(ii) When we substitute x_i by cx_i $(c \in k^\times)$, t_j changes to $c^j t_j$, and $u := t_2/t_3$ changes to $c^{-1}u$. Hence $u^j t_j$ is invariant under this. Since

$$k(t_2, \ldots, t_n) = k(u, t_3, \ldots, t_n) = k(u, u^3 t_3, \ldots, u^n t_n)$$

we have

[1] This fact was provided by K.Hashimoto to the author during their joint works [7, 8].

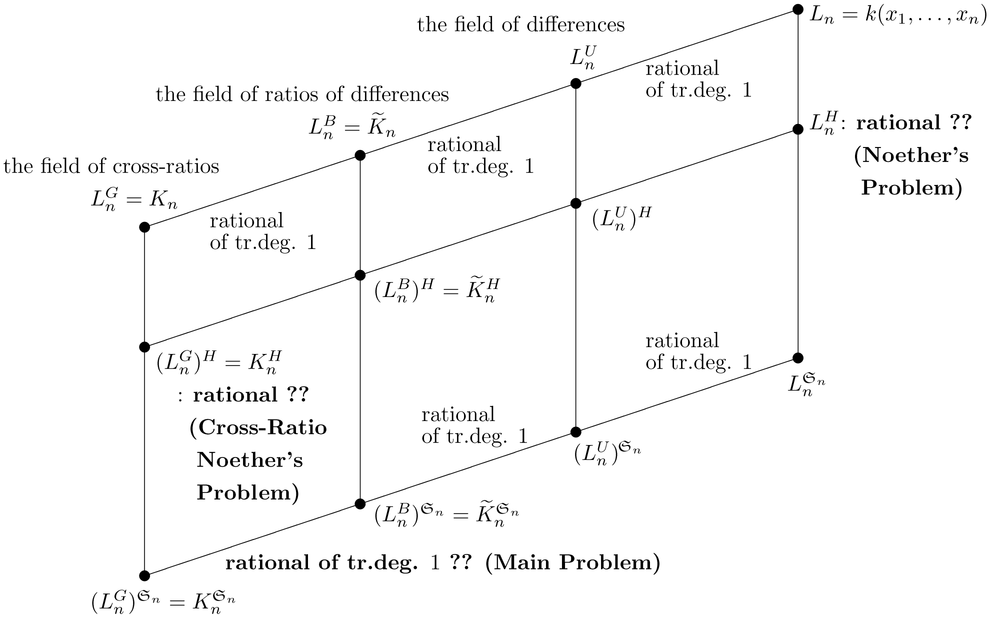

FIGURE 1. The total picture

$$k(t_2, \dots, t_n)^B = k(u^3 t_3, \dots, u^n t_n)(u)^B = k(u^3 t_3, \dots, u^n t_n)\,.$$

□

Hence our remaining problem is the lowest step: for a subgroup H of $\mathfrak{S}_n$, whether $\widetilde{K}_n^H$ is rational over K_n^H or not (Problem 5). In Sections 5 and 6, we shall show the following:

THEOREM 1. *Let n be an integer with $n \geq 5$. Assume that a base field k is infinite.*

(i) *When n is odd, for any base field k, $\widetilde{K}_n^{\mathfrak{S}_n}$ is rational over $K_n^{\mathfrak{S}_n}$.*
(ii) *Assume that the characteristic of k is not two. When n is even, for any base field k, $\widetilde{K}_n^{\mathfrak{S}_n}$ is not rational over $K_n^{\mathfrak{S}_n}$.*

3. The action of permutations on cross-ratios

In this preparatory section, we introduce a concise way to choose suitable generators of the field K_n of cross-ratios and the field $\widetilde{K}_n$ of ratios of differences and to calculate the action of $\mathfrak{S}_n$ on them (cf. Hashimoto-Tsunogai [7, 8]).

Let $\mathcal{M}_{0,n}$ be the moduli space of projective lines with ordered n marked points:

$$\begin{aligned}\mathcal{M}_{0,n} &= \big((\mathbb{P}^1)^n \smallsetminus (\text{weak diagonal})\big)/\mathrm{PGL}(2) \\ &= \{(x_1, \dots, x_n) \mid x_i \in \mathbb{P}^1, x_i \neq x_j (i \neq j)\}/\mathrm{PGL}(2),\end{aligned} \tag{9}$$

where $\mathrm{PGL}(2) = \mathrm{Aut}(\mathbb{P}^1)$ acts on $(\mathbb{P}^1)^n$ diagonally. We denote the class of $(x_1, \dots, x_n)$ by $[x_1, \dots, x_n]$. The function field of $\mathcal{M}_{0,n}$ is K_n:

$$k(\mathcal{M}_{0,n}) = k(x_1, \dots, x_n)^{\mathrm{PGL}(2)} = K_n\,. \tag{10}$$

The symmetric group $\mathfrak{S}_n$ of degree n acts on $\mathcal{M}_{0,n}$ from the left by permutation of components:

$$\sigma \cdot [x_1, \dots, x_n] := [x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)}] \qquad (\sigma \in \mathfrak{S}_n)\,. \tag{11}$$

The action of $\mathfrak{S}_n$ on K_n coincides with the action on $k(\mathcal{M}_{0,n})$ induced by the pull-back of the above action:

$$\sigma \cdot \varphi := \varphi \circ \sigma^{-1} \qquad (\sigma \in \mathfrak{S}_n, \varphi \in K)\,. \tag{12}$$

A point $P = [x_1, \dots, x_n]$ of $\mathcal{M}_{0,n}$ can be represented uniquely in the form, for example, $[y_1, \dots, y_{n-3}, 0, 1, \infty]$ by normalizing with PGL(2)-action. We consider

$$y_i(P) = y_i = \frac{x_i - x_{n-2}}{x_i - x_n} \Bigg/ \frac{x_{n-1} - x_{n-2}}{x_{n-1} - x_n} \tag{13}$$

as a function on $\mathcal{M}_{0,n}$. Then $y_1, \dots, y_{n-3}$ generate $k(\mathcal{M}_{0,n})$ and we have $K_n = k(\mathcal{M}_{0,n}) = k(y_1, \dots, y_{n-3})$. The action of $\mathfrak{S}_n$ on these generators is described as in the following example.

EXAMPLE 1. For simplicity, we introduce an example in the case of $n = 5$. Let us calculate the action of $\alpha = (1\ 2\ 3\ 4\ 5)$ on y_1, y_2. For $P = [x_1, \dots, x_5] = [y_1, y_2, 0, 1, \infty]$, we have

$$\alpha^{-1}(P) = [x_2, x_3, x_4, x_5, x_1] = [y_2, 0, 1, \infty, y_1] = \left[\frac{y_2 - 1}{y_2 - y_1}, \frac{1}{y_1}, 0, 1, \infty\right], \tag{14}$$

where we renormalize it by $\xi \mapsto \dfrac{\xi - 1}{\xi - y_1}$. Hence it follows that

$$\alpha : y_1 \longmapsto \frac{y_2 - 1}{y_2 - y_1}\,, \qquad y_2 \longmapsto \frac{1}{y_1}\,. \tag{15}$$

Next we describe the action of $\mathfrak{S}_n$ on $L_n^B = \widetilde{K}_n$ in the similar way. We take an element $z := \dfrac{x_{n-1} - x_n}{x_{n-1} - x_{n-2}} \in \widetilde{K}_n$ as a generator over K_n; $\widetilde{K}_n = K_n(z)$. Regarding z as a formal limit

$$z = \lim_{x_{n+1} \to \infty} \frac{x_{n+1} - x_{n-2}}{x_{n+1} - x_n} \Bigg/ \frac{x_{n-1} - x_{n-2}}{x_{n-1} - x_n}\,, \tag{16}$$

we can calculate the action on z simultaneously by putting z at the $(n+1)$-th component, since the operation taking a limit formally and the action of $\mathfrak{S}_n$ commute with each other.

EXAMPLE 2. In the previous example, we have also

$$\alpha^{-1}([y_1, y_2, 0, 1, \infty; z]) = [y_2, 0, 1, \infty, y_1; z] = \left[\frac{y_2 - 1}{y_2 - y_1}, \frac{1}{y_1}, 0, 1, \infty; \frac{z-1}{z-y_1}\right]. \quad (17)$$

Hence we obtain

$$\alpha(z) = \frac{z-1}{z-y_1}. \quad (18)$$

REMARK 1. We can take a generating system by considering a normalization other than $[y_1, \dots, y_{n-3}, 0, 1, \infty]$. In fact, we take various ways in the following sections. This geometric view is useful to choose a good generating system which is suitable for calculation we are in face of.

4. Conditions for the descent of rationality

In this section, to consider the condition to descent the rationality of $\widetilde{K}_n/K_n$ to $\widetilde{K}_n^H/K_n^H$, we discuss on the descent condition in more general situation.

Let K be an arbitrary field and $\widetilde{K} = K(X)$ be a rational function field of one variable over K. Let H be a finite subgroup of $\mathrm{Aut}(\widetilde{K})$ (hence H acts on $\widetilde{K}$ faithfully), and assume that

(i) H stabilizes K (that is, $\sigma(K) = K$ for any $\sigma \in H$),
(ii) H acts on K faithfully.

Then $\widetilde{K}^H$ is a rational function field of genus zero over K^H, and $K\widetilde{K}^H = \widetilde{K}$. We ask the rationality of $\widetilde{K}^H$ over K^H.

4.1. Descent to 2-Sylow subgroups. First we observe that

LEMMA 1. *If $\widetilde{K}^H$ is rational over K^H, then, for any subgroup H_1 of H, $\widetilde{K}^{H_1}$ is rational over K^{H_1}.*

PROOF. If we take an element $z \in \widetilde{K}^H$ satisfying $\widetilde{K}^H = K^H(z)$, then we have $\widetilde{K}^{H_1} = K^{H_1}(z)$. □

As an intermediate step, it is useful to consider a 2-Sylow subgroup of H. Let S be a 2-Sylow subgroup of H. We subdivide the descent from $\widetilde{K}/K$ to $\widetilde{K}^H/K^H$ into the following two steps:

(i) Is $\widetilde{K}^S/K^S$ rational? (2-Sylow descent)
(ii) If $\widetilde{K}^S/K^S$ is rational, is $\widetilde{K}^H/K^H$ rational? (odd degree descent)

The odd degree descent always holds from the following lemma:

LEMMA 2. *Let k be any field, K/k be an algebraic function field of one variable over k and $\widetilde{k}/k$ be a finite extension of odd degree* (*say*, $2m+1$). *Then if the function field $K\widetilde{k}/\widetilde{k}$ obtained by the extension of a constant field is rational* (*that is, there exists an element $z \in K\widetilde{k}$ satisfying $K\widetilde{k} = \widetilde{k}(z)$*)*, K/k is rational* (*that is, one can choose $z \in K$*).

PROOF. The divisor $D := N_{K\widetilde{k}/K} P + mD_0$, where P is a prime divisor of a rational function field $K\widetilde{k}/\widetilde{k}$ of degree one, $N_{K\widetilde{k}/K}$ is the norm from $K\widetilde{k}$ to K, and D_0 is a canonical divisor[2] of K/k, is a k-rational divisor of K of degree one. By Riemann-Roch Theorem, there exists a k-rational function $f \in K$ such that $f \in L(D)$. Since the divisor $D-(f)$ is a effective divisor of degree one, it is a k-rational prime divisor of degree one. Hence K/k is rational. □

Gathering the above two lemmata, we have the following proposition (see also [18] Theorem 5):

PROPOSITION 2. *Let S be a 2-Sylow subgroup of H. Then $\widetilde{K}^H$ is rational over K^H if and only if $\widetilde{K}^S$ is rational over K^S.*

Thus what we must do is to distinguish, for a 2-group S in $\mathrm{Aut}(\widetilde{K})$ which acts on K faithfully, whether the conic $\widetilde{K}^S/K^S$ is rational or not. Contrary to the odd degree descent, the validity of 2-Sylow descent depends on a case.

4.2. The case of "semi-affine" action. Let $\widetilde{K}/K$ be as above, and S be a finite 2-subgroup of $\mathrm{Aut}(\widetilde{K})$. If the action of S on $\widetilde{K}$ is "semi-affine" over K, we have an affirmative answer on the rationality of $\widetilde{K}^S$ over K^S. Although this is a special case of known results ([12] Lemma, [1] Theorem 3.1), here we give a more constructive proof for our case.

PROPOSITION 3. *Let K be a field and $\widetilde{K} = K(X)$ a rational function field over K. Let S be a finite 2-subgroup of* $\mathrm{Aut}(\widetilde{K})$, *and assume that the following conditions are satisfied*:

(i) *S stabilizes K and acts on K faithfully*,
(ii) *for any $\sigma \in S$, there exists $c_\sigma, d_\sigma \in K$ such that $\sigma(X) = c_\sigma X + d_\sigma$.*

Then the fixed field $\widetilde{K}^S$ is again rational over K^S, that is, there exists $Z \in \widetilde{K}^S$ such that $\widetilde{K}^S = K^S(Z)$.

PROOF. Since S is a 2-group, there is a central sequence $S = Z_0 \rhd Z_1 \rhd \cdots \rhd Z_{l-1} \rhd Z_l = \{1\}$ with $(Z_i : Z_{i+1}) = 2$. The unique non-trivial element $\tau \in Z_{l-1}$ is central in S and of order 2.

We prove the proposition by induction on the order of S, or l. The induction step is the following lemma:

LEMMA 3. *Let $\widetilde{K} = K(X)$ and S be as in the proposition. Let $\tau \in S$ be a central element of S of order 2. Then there exists an element $Z \in \widetilde{K}$ satisfying the following conditions*:

[2] Usually it is denoted by K, which causes the collision of notation here.

(i) $\tau(Z) = Z$,
(ii) $\widetilde{K} = K(Z)$ (*hence we have* $\widetilde{K}^{\langle\tau\rangle} = K^{\langle\tau\rangle}(Z)$),
(iii) *for any* $\sigma \in S$, *there exist* $c_\sigma, d_\sigma \in L^{\langle\tau\rangle}$ *such that* $\sigma(Z) = c_\sigma Z + d_\sigma$.

PROOF. By the assumption (ii) of the proposition, there exist $c_\tau, d_\tau \in K$ such that $\tau(X) = c_\tau X + d_\tau$. If $c_\tau \neq -1$, put $Z := X + \tau(X) = (c_\tau + 1)X + d_\tau$. If $c_\tau = -1$, take an element $a \in K$ with $\tau(a) \neq a$ and put $Z := aX + \tau(aX) = (a - \tau(a))X + d_\tau \tau(a)$. Then we have $\tau(Z) = Z$. Since the change of variables from X to Z is affine over K, it holds that $\widetilde{K} = K(Z)$ and also that for any $\sigma \in S$ there exist unique elements $c_\sigma, d_\sigma \in K$ such that $\sigma(Z) = c_\sigma Z + d_\sigma$. Since τ is central in S, $\sigma(Z) \in K^{\langle\tau\rangle}$. The uniqueness of c_σ and d_σ deduces that $c_\sigma, d_\sigma \in K^{\langle\tau\rangle}$. □

Applying this lemma for ($\widetilde{K} = K(X)/K$, $S \supset \langle\tau\rangle$), we obtain an element $Z \in \widetilde{K}^{\langle\tau\rangle}$ such that $\widetilde{K} = K(Z)$. Then we have $\widetilde{K}^{\langle\tau\rangle} = K^{\langle\tau\rangle}(Z)$ and the induced action of $S/\langle\tau\rangle = S/Z_{l-1}$ on $K^{\langle\tau\rangle}(Z)$ is semi-affine over $K^{\langle\tau\rangle}$ and faithful on $K^{\langle\tau\rangle}$. By the assumption of induction for ($\widetilde{K}^{\langle\tau\rangle} = K^{\langle\tau\rangle}(X)/K^{\langle\tau\rangle}$, $S/\langle\tau\rangle$), $(\widetilde{K}^{\langle\tau\rangle})^{S/\langle\tau\rangle} = \widetilde{K}^S$ is rational over $(K^{\langle\tau\rangle})^{S/\langle\tau\rangle} = K^S$. □

4.3. A recipe for a group S of order two. For the cases not covered by the argument of the previous subsection, we need the concrete determination of the fixed fields $\widetilde{K}^S$ and K^S, and an explicit description of the conic $\widetilde{K}^S/K^S$. Here we shall give a recipe which will be used in the proof of our theorems for the case S is of order two.

We consider the situation that $S = \langle\sigma\rangle \subset \mathrm{Aut}(\widetilde{K})$ satisfies the following conditions:

(i) $\#S = 2$ (i.e. $\sigma^2 = \mathrm{id}$),
(ii) S stabilizes K (i.e. $\sigma(K) = K$), and
(iii) S acts on K faithfully (i.e. $\sigma|_K \neq \mathrm{id}$).

We want to know whether the conic $\widetilde{K}^S/K^S$ is rational or not.

Since also $\sigma(X)$ generates $\widetilde{K} = K(X)$ over K, the action of σ on $\widetilde{K} = K(X)$ is "semi-linear fractional" over K:

$$\sigma(X) = \frac{aX + b}{cX + d} \qquad (a, b, c, d \in K, ad - bc \neq 0)\,. \tag{19}$$

Since

$$\sigma^2(X) = \sigma\left(\frac{aX+b}{cX+d}\right) = \frac{\sigma(a)\dfrac{aX+b}{cX+d} + \sigma(b)}{\sigma(c)\dfrac{aX+b}{cX+d} + \sigma(d)} = \frac{(\sigma(a)a + \sigma(b)c)X + (\sigma(a)b + \sigma(b)d)}{(\sigma(c)a + \sigma(d)c)X + (\sigma(c)b + \sigma(d)d)}$$

and $\sigma^2 = \mathrm{id}$, we have

$$\sigma(a)a + \sigma(b)c = \sigma(c)b + \sigma(d)d$$

and

$$\sigma(a)b + \sigma(b)d = \sigma(c)a + \sigma(d)c = 0\,.$$

The following lemma is tactically useful:

LEMMA 4. *There exists $Z \in \widetilde{K}$ such that $\widetilde{K} = K(Z)$ with $Z\sigma(Z) \in K^S$.*

PROOF. Let us take $Z := cX + d$. Then we have $\widetilde{K} = K(Z)$ and

$$\begin{aligned} Z\sigma(Z) &= (cX+d)\left(\sigma(c)\frac{aX+b}{cX+d} + \sigma(d)\right) = \sigma(c)(aX+b) + \sigma(d)(cX+d) \\ &= (\sigma(c)a + \sigma(d)c)X + (\sigma(c)b + \sigma(d)d) = \sigma(c)b + \sigma(d)d \in K\,. \end{aligned}$$

Since $Z\sigma(Z)$ is σ-invariant, we have $Z\sigma(Z) \in K^S$. □

REMARK 2. As another choice, we can take also $X/(aX+b)$ as a suitable choice of a generator of $\widetilde{K}$ over K, which may be useful for some calculation in other cases.

We take an element $Z \in \widetilde{K}$ as in the above lemma, and put $s := Z\sigma(Z) \in K^S$ and $U := Z + \sigma(Z) = Z + s/Z \in \widetilde{K}^S$. We define $\tau \in \mathrm{Aut}(\widetilde{K}/K)$ by $\tau(Z) := \sigma(Z)$.

CLAIM 1. *The group $G := \langle \sigma, \tau \rangle$ generated by σ and τ is isomorphic to the Klein's four group, that is, σ and τ satisfy the relations $\sigma^2 = \tau^2 = 1$ and $\sigma\tau = \tau\sigma$.*

PROOF. We can see easily that $\tau\sigma|_K = \sigma\tau|_K = \sigma|_K$ and that $\tau^2(Z) = \tau\sigma(Z) = \sigma\tau(Z) = Z$. The assertion follows from this. □

We determine the fixed field $\widetilde{K}^S$ as an intermediate field of $\widetilde{K}/\widetilde{K}^G$, because the other two intermediate fields $\widetilde{K}^{\langle\tau\rangle}$ and $\widetilde{K}^{\langle\sigma\tau\rangle}$ can be easily determined.

Since U is G-invariant and $\tau|_K = \mathrm{id}$, we have $\widetilde{K}^{\langle\tau\rangle} = K(Z)^{\langle\tau\rangle} = K(U)$ and $\widetilde{K}^G = (\widetilde{K}^{\langle\tau\rangle})^G = K(U)^G = K^G(U) = K^S(U)$. On the other hand, since Z is $\sigma\tau$-invariant, we have $\widetilde{K}^{\langle\sigma\tau\rangle} = K^S(Z) = \widetilde{K}^G(Z)$.

Here we treat the case that the characteristic is other than 2. Since $U = Z + s/Z \in \widetilde{K}^G$, we can take $Z - s/Z$ as a generator of $\widetilde{K}^{\langle\sigma\tau\rangle}$ over $\widetilde{K}^G$ instead of Z. Choose an element $a \in K \setminus K^S$ such that $\sigma(a) = -a$ and put $c := a^2$. Then we have $K = K^S(a)$ and $c \in K^S$. Hence $\widetilde{K}^{\langle\tau\rangle} = K(U) = \widetilde{K}^G(a)$. From these, we can take $V := \dfrac{Z - s/Z}{a}$ as a generator of $\widetilde{K}^S$ over $\widetilde{K}^G$; $\widetilde{K}^S = \widetilde{K}^G(V)$. Thus we have $\widetilde{K}^S = K(U, V)$ with one relation

$$U^2 - cV^2 = 4s\,, \tag{20}$$

since $V^2 = \dfrac{(Z - s/Z)^2}{a^2} = \dfrac{U^2 - 4s}{c}$. Hence $\widetilde{K}^S$ is rational over K^S if and only if the conic $U^2 - cV^2 = 4s$ over K^S has a K^S-rational point (U, V).

REMARK 3. The case of characteristic 2 is similar except the use of Artin-Schreier theory instead of Kummer theory.

Since the extension K/K^S is of Artin-Schreier type, there exists an element $a \in K \setminus K^S$ such that $\sigma(a) = a + 1$. Put $c := a(a+1)$, then we have $K = K^S(a)$ and $c \in K^S$. Hence

$\widetilde{K}^{\langle\tau\rangle} = K(U) = \widetilde{K}^G(a)$. Since $Z - \sigma(Z) = Z + \sigma(Z) = U$ and $\sigma(a) - a = 1$, we can take $V := Z - aU$ as a generator of $\widetilde{K}^S$ over $\widetilde{K}^G$; $\widetilde{K}^S = \widetilde{K}^G(V)$. Thus we have $\widetilde{K}^S = K(U, V)$ with one relation

$$V^2 - UV - cU^2 = s\,, \tag{21}$$

because

$$V^2 - UV = (Z^2 + a^2U^2) - (UZ - aU^2) = Z(U - Z) + a(a+1)U^2 = s + cU^2\,.$$

Hence $\widetilde{K}^S$ is rational over K^S if and only if the conic $V^2 - UV - cU^2 = s$ over K^S has a K^S-rational point (U, V).

4.4. A remark on an interpretation via Galois cohomology. One can find that in the both cases above the left hand side of the conic (20), (21) we obtained is the norm form of K/K^S, that is, $\widetilde{K}^S$ is rational over K^S if and only if $s \in N_{K/K^S}K^\times$. We can interpret this via Galois cohomology ([16, 17, 18]).

The extension $\widetilde{K}^S/K^S$ between the fixed fields can be parametrized by the Galois cohomology group $H^1(S, \mathrm{PGL}(2, K))$. Denote the set of the K^S-isomorphism classes of function fields L/K^S of one variable of genus 0 which split in K (that is, $KL \simeq K(X)$) by $E(S, K)$. The bijection between $E(S, K)$ and $H^1(S, \mathrm{PGL}(2, K))$ is obtained as follows: the correspondence $\sigma \mapsto \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, where $\sigma(X) = \dfrac{aX + b}{cX + d}$, gives a 1-cocycle $G \to \mathrm{PGL}(2, K)$, whose cohomology class does not depend on the choice of a generator X. Taking the transposition is to avoid to get an anti-cocycle.

Since $H^1(S, \mathrm{GL}(2, K)) = \{1\}$, from the central exact sequence

$$1 \to K^\times \to \mathrm{GL}(2, K) \to \mathrm{PGL}(2, K) \to 1\,,$$

we obtain the injective connecting homomorphism $H^1(S, \mathrm{PGL}(2, K)) \to H^2(S, K^\times)$. Moreover $H^2(S, K^\times)$ is isomorphic to $(K^S)^\times/N_{K/K^S}K^\times$, where $s \in (K^S)^\times$ corresponds to the class of a 2-cocycle determined by $(\sigma, \sigma) \mapsto s$.

The following proposition is essentially a version of [4][3] Theorem 2, and is simplified without loss of generality by using Lemma 4.

PROPOSITION 4. *Let $\widetilde{K} = K(Z)$ be a rational function field over a field K. Assume that $S = \langle\sigma\rangle \subset \mathrm{Aut}(\widetilde{K})$ satisfies the following conditions*:

(i) $\#S = 2$ (*i.e.* $\sigma^2 = \mathrm{id}$),
(ii) *S stabilizes K* (*i.e.* $\sigma(K) = K$),
(iii) *S acts on K faithfully* (*i.e.* $\sigma|_K \neq \mathrm{id}$), *and*
(iv) $\sigma(Z) = s/Z$ *with* $s \in K^S$.

[3]In the calculation in [4], there is a (non-serious) mistake. In p.46, $\alpha(U) = \dfrac{(W^2 - Y^2)U - 4WY}{WYU + (W^2 - Y^2)}$ is correct.

Then the image of the isomorphism class $[\widetilde{K}^S/K^S]$ under the composite

$$E(S,K) \simeq H^1(S,\mathrm{PGL}(2,K)) \longrightarrow H^2(S,K^\times) \longrightarrow (K^S)^\times/N_{K/K^S}K^\times \tag{22}$$

is given by s mod $N_{K/K^S}K^\times$. Hence $\widetilde{K}^S$ is rational over K^S if and only if $s \in N_{K/K^S}K^\times$.

PROOF. The 1-cocycle $f \in Z^1(S,\mathrm{PGL}(2,K))$ determined by the extension $\widetilde{K}^S/K^S$ is given by $f(\sigma) = \begin{pmatrix} 0 & s \\ 1 & 0 \end{pmatrix}$. Since $f(\sigma)\sigma(f(\sigma))f(\sigma^2)^{-1} = sI_2$, $[f] \in H^1(S,\mathrm{PGL}(2,K))$ is mapped to the 2-cocycle determined by $(\sigma,\sigma) \mapsto s$ by the definition of the connecting map. The isomorphism $H^2(S,K^\times) \longrightarrow (K^S)^\times/N_{K/K^S}K^\times$ maps this 2-cocycle to s mod $N_{K/K^S}K^\times$. □

In actual examples which we want to investigate, to determine whether s is a norm or not, we must know more precise information of K/K^S (such as explicit generators of K^S), so we need concrete calculation as in the following sections.

5. Rationality for odd n's

Now we return to our situation; K_n is the field of cross-ratios of n variables, $\widetilde{K}_n$ is the field of ratios of differences, and $\mathfrak{S}_n$ acts on them by permutation of indices of variables.

In this section, we assume that n is an odd integer with $n \geq 5$, and show Theorem 1 (i), that is, $\widetilde{K}_n^{\mathfrak{S}_n}$ is rational over $K_n^{\mathfrak{S}_n}$. We need 2-Sylow descent to obtain an affirmative answer for our main problem.

Let S be a 2-Sylow subgroup of $\mathfrak{S}_n$. First we consider the action of S on the set $\{1,\ldots,n\}$. Since n is odd, there exists an orbit consisting of a single element, say $\{n\}$, that is, $\sigma(n) = n$ for all $\sigma \in S$.

Owing to Proposition 3, to obtain Theorem 1 (i), it is enough to show the following:

LEMMA 5. *There exists an element $z \in \widetilde{K}_n$ such that $\widetilde{K}_n = K_n(z)$ and that $\sigma(z) = c_\sigma z + d_\sigma$ $(c_\sigma, d_\sigma \in K_n)$ for any $\sigma \in S$.*

PROOF. We shall show that the choice $z := \dfrac{x_{n-1}-x_n}{x_{n-1}-x_{n-2}} \in \widetilde{K}_n$ is suitable for this. Let $\sigma \in S$. Since $\sigma(n) = n$,

$$\sigma(z) = \frac{x_{\sigma(n-1)}-x_n}{x_{\sigma(n-1)}-x_{\sigma(n-2)}} = \frac{z - y_{\sigma(n-2)}}{y_{\sigma(n-1)}-y_{\sigma(n-2)}}, \tag{23}$$

where we put $y_i = \dfrac{x_i - x_{n-2}}{x_i - x_n} \Big/ \dfrac{x_{n-1}-x_{n-2}}{x_{n-1}-x_n} \in K_n$ $(i = 1,\ldots,n-3)$, $y_{n-2} = 0$ and $y_{n-1} = 1$. □

REMARK 4. We can find this choice of z by normalizing the n-th coordinate of points of $\mathcal{M}_{0,n}$ to ∞. Concretely, we can see this by considering

$$\begin{aligned}\sigma^{-1}([y_1,\dots,y_{n-3},0,1,\infty;z]) &= [*,\dots,*,y_{\sigma(n-2)},y_{\sigma(n-1)},\infty;z] \\ &= \left[*,\dots,*,0,1,\infty;\frac{z-y_{\sigma(n-2)}}{y_{\sigma(n-1)}-y_{\sigma(n-2)}}\right],\end{aligned} \tag{24}$$

where we employ a renormalization $\xi \longmapsto \dfrac{\xi - y_{\sigma(n-2)}}{y_{\sigma(n-1)}-y_{\sigma(n-2)}}$.

Thus, the assertion of Theorem 1 (i) follows from Proposition 3.

6. Non-rationality for even n's

In this section, we assume that n is an even integer with $n \geq 6$, and show Theorem 1 (ii), that is, $\widetilde{K}_n^{\mathfrak{S}_n}$ is not rational over $K_n^{\mathfrak{S}_n}$ provided the characteristic of k is not two. To show this, it suffices to find a (un)suitable subgroup $H \subset \mathfrak{S}_n$ such that $\widetilde{K}_n^H$ is not rational over K_n^H. This group H should be a 2-group. In this case, we can choose H as in the following proposition. Although the results in this and the next sections are concrete examples of known results (e.g. [6, 9]), we give a proof based on explicit computation since we need it finally for actual determination of (non-)rationality.

PROPOSITION 5. *Let $n = 2m+4 \geq 6$ $(m \geq 1)$ and put $\sigma := (1\ 2)\cdots(2m-1\ 2m)(2m+1\ 2m+2)(2m+3\ 2m+4)$. Then $\widetilde{K}_n^{\langle\sigma\rangle}$ is not rational over $K_n^{\langle\sigma\rangle}$.*

PROOF. Take a normalization for a point of $\mathcal{M}_{0,n}$ as $[y_1,\dots,y_{2m},y_0,1,0,\infty;z]$. Then we have

$$K_n = k(y_0,y_1,\dots,y_{2m}), \qquad \widetilde{K}_n = K_n(z). \tag{25}$$

The actions of σ on K_n and $\widetilde{K}_n$ are calculated as in Section 3:

$$\sigma : \begin{cases} y_0 \longmapsto y_0, \\ y_{2i-1} \longmapsto \dfrac{y_0}{y_{2i}} \quad (1 \leq i \leq m), \\ y_{2i} \longmapsto \dfrac{y_0}{y_{2i-1}} \quad (1 \leq i \leq m), \\ z \longmapsto \dfrac{y_0}{z}. \end{cases} \tag{26}$$

Put $\eta_i := y_{2i-1} + \sigma(y_{2i-1}) = y_{2i-1} + \dfrac{y_0}{y_{2i}}$ and $\eta_i' := y_{2i-1} - \sigma(y_{2i-1}) = y_{2i-1} - \dfrac{y_0}{y_{2i}}$ for $1 \leq i \leq m$. Then we have $K_n = k(y_0,\eta_1,\dots,\eta_m,\eta_1',\dots,\eta_m')$ and $\sigma(\eta_i) = \eta_i, \sigma(\eta_i') = -\eta_i'$. Hence, putting $\eta_{m+i} := \eta_1'\eta_i'$, we have $K_n^{\langle\sigma\rangle} = k(y_0,\eta_1,\dots,\eta_m,\eta_{m+1},\dots,\eta_{2m})$.

Furthermore, if we put

$$U := z + \sigma(z) = z + \frac{y_0}{z}\,, \qquad V := \frac{z - \sigma(z)}{\eta_1'} = \frac{z - \frac{y_0}{z}}{y_1 - \frac{y_0}{y_2}}\,, \tag{27}$$

then it holds that $\widetilde{K}_n^{\langle\sigma\rangle} = K_n^{\langle\sigma\rangle}(U, V)$ with

$$U^2 - \eta_{m+1} V^2 = 4y_0\,. \tag{28}$$

This is a conic over $K_n^{\langle\sigma\rangle}$. Thus the non-rationality of $\widetilde{K}_n^{\langle\sigma\rangle}/K_n^{\langle\sigma\rangle}$ is reduced to the following claim. □

CLAIM 2. *The conic $U^2 - \eta_{m+1}V^2 = 4y_0$ over $K_n^{\langle\sigma\rangle}$ has no $K_n^{\langle\sigma\rangle}$-rational points.*

PROOF. Since $K_n^{\langle\sigma\rangle} = k(\eta_1, \dots, \eta_m, \eta_{m+1}, \dots, \eta_{2m})(y_0)$, it suffices to show that the equation

$$U_0^2 - \eta_{m+1}V_0^2 = 4y_0 W_0^2$$

has no non-trivial solution (U_0, V_0, W_0) in the polynomial ring $k(\eta_1, \dots, \eta_m, \eta_{m+1}, \dots, \eta_{2m})[y_0]$. The both terms $U_0^2, \eta_{m+1}V_0^2$ in LHS are of even degree in y_0, while RHS is of odd degree in y_0. Hence the leading terms of $U_0^2, \eta_{m+1}V_0^2$ must be equal and cancelled in LHS. But it is impossible because η_{m+1} is not a square in $k(\eta_1, \dots, \eta_m, \eta_{m+1}, \dots, \eta_{2m})$. □

From Proposition 5 together with Lemma 1 in the previous section, we obtain the assertion of Theorem 1 (ii).

7. Non-rationality for transitive subgroups of even degree

We continue to keep the assumption on the base field k to be infinite and of characteristic different from two. In this section we shall discuss, for an even n and for a transitive group H in $\mathfrak{S}_n$ whether $\widetilde{K}_n^H$ is not rational over K_n^H. By Proposition 5, if a permutation group H in $\mathfrak{S}_{2m+4}$ has an element of cycle type 2^{m+2} (that is, conjugate to $(1\ 2)\cdots(n-1\ n)$), then $\widetilde{K}_n^H$ is not rational over K_n^H.

7.1. The case $n = 2^e$ $(e \geq 3)$. When the degree n is a power of 2, we can give a uniform answer for all transitive subgroups of $\mathfrak{S}_n$ owing to the following group-theoretical lemma:

LEMMA 6. *Let $n = p^e$ be a prime power. Then any transitive subgroup H in $\mathfrak{S}_n$ contains an element of cycle type $p^{n/p}$.*

PROOF. First we shall show that a p-Sylow subgroup S_p of H is transitive. Let H_1 (resp. S_1) be the stabilizer of the symbol 1 under the standard permutation action of H (resp.

S_p). Then $(H : H_1) = n = p^e$ follows from the transitivity of H. Hence $(H : S_1) = (H : H_1)(H_1 : S_1)$ is a multiple of p^e. On the other hand, we have $(H : S_1) = (H : S_p)(S_p : S_1)$ and $(H : S_p)$ is prime to p because S_p is a p-Sylow subgroup of H. Hence $(S_p : S_1)$ is divided by p^e. This shows the transitivity of S_p.

Since the center $Z = Z(S_p)$ is a non-trivial abelian p-group, Z contains an element σ of order p. Then σ must be of cycle type $p^{n/p}$. To show this, suppose that σ is of cycle type of p^k with $k < e/p$. Without loss of generality, we suppose $\sigma = (1 \cdots p) \cdots ((k-1)p+1 \cdots kp)$. Since S_p is transitive, there exists an element $\rho \in S_p$ such that $\rho(1) = kp + 1$. Then we have $\rho^{-1}\sigma\rho(1) = 1 \neq 2 = \sigma(1)$. This contradicts that σ is central in S_p. □

THEOREM 2. *Assume that k is infinite and of characteristic different from two. For any transitive subgroup H of degree $n = 2^e$ $(e \geq 3)$, $\widetilde{K}_n^H$ is not rational over K_n^H.*

PROOF. By applying the lemma above for $p = 2$, we know that any transitive group H of degree 2^e contains an element of cycle type $2^{n/2}$. Then the assertion follows from Proposition 5. □

When n is not a power of 2, there is a transitive subgroup H of $\mathfrak{S}_n$ such that H has no element of cycle type $2^{n/2}$. For such cases we need individual treatment.

7.2. The case $n = 6$. There are 16 conjugacy classes of transitive subgroups in $\mathfrak{S}_6$, listed in Butler-McKay [3] (see the table cited from Hashimoto-Tsunogai [8], where the left-most column is the symbol numbered in [3]). In [8], Cross-Ratio Noether's Problem for these groups is settled affirmatively except for ${}_6T_{12}$ and ${}_6T_{15}$. For these groups we shall show the following:

THEOREM 3. *Assume that k is infinite and of characteristic different from two. For any transitive subgroup T of $\mathfrak{S}_6$, $\widetilde{K}_6^T$ is not rational over K_6^T.*

We shall prove this theorem by showing that any transitive group T includes a 2-group H such that $\widetilde{K}_6^H$ is not rational over K_6^H. Consulting the table of the transitive groups of degree 6 and checking with a computer algebra system GAP [5], we can see the following:

LEMMA 7. *Any transitive group T of degree 6 includes a subgroup conjugate to one of the following*:

$$\begin{aligned} H_1 &= \langle (1\ 2)(3\ 4)(5\ 6) \rangle\,, \\ H_2 &= \langle (1\ 2)(3\ 4), (1\ 2)(5\ 6) \rangle\,, \\ \textit{or} \quad H_3 &= \langle (1\ 2\ 3\ 4)(5\ 6) \rangle\,. \end{aligned}$$

In particular, a transitive group T of degree 6 includes a conjugate of H_1 if and only if T is odd.

TABLE 1. The transitive groups of degree 6 (cf. [3, 8])

	order	sign	structure	generators	includes		
${}_6T_1$	6		C_6	α	H_1		
${}_6T_2$	6		$\mathfrak{S}_3(6)$	α^2, β	H_1		
${}_6T_3$	12		D_6	α, β	H_1		
${}_6T_4$	12	$+$	$\mathfrak{A}_4$	α^2, τ_1, τ_2		H_2	
${}_6T_5$	18		$\mathfrak{S}_3 \times C_3$	${}_6T_2, \gamma_1$	H_1		
${}_6T_6$	24		$\mathfrak{A}_4 \times C_2$	${}_6T_4, \theta$	H_1	H_2	
${}_6T_7$	24	$+$	$\mathfrak{S}_4{}^{(+)}$	${}_6T_4, \beta\theta$		H_2	H_3
${}_6T_8$	24		$\mathfrak{S}_4{}^{(-)}$	${}_6T_4, \beta$	H_1	H_2	
${}_6T_9$	36		$V_4 \ltimes (C_3 \times C_3)$	${}_6T_3, \gamma_1$	H_1		
${}_6T_{10}$	36	$+$	$C_4 \ltimes (C_3 \times C_3)$	$\alpha^2, \alpha\beta, \gamma_1, \delta$			H_3
${}_6T_{11}$	48		$\mathfrak{S}_4 \times C_2$	${}_6T_4, \beta, \theta$	H_1	H_2	H_3
${}_6T_{12}$	60	$+$	$\mathfrak{A}_5(6)$	${}_6T_4, \varphi$		H_2	
${}_6T_{13}$	72		$D_4 \ltimes (C_3 \times C_3)$	${}_6T_9, \delta$	H_1		H_3
${}_6T_{14}$	120		$\mathfrak{S}_5(6)$	${}_6T_8, \varphi$	H_1	H_2	
${}_6T_{15}$	360	$+$	$\mathfrak{A}_6$	${}_6T_7, \varphi$		H_2	H_3
${}_6T_{16}$	720		$\mathfrak{S}_6$	${}_6T_{15}, \beta$	H_1	H_2	H_3

$$\alpha = (123456), \beta = (14)(23)(56), \theta = \alpha^3 = (14)(25)(36),$$
$$\gamma_1 = (135), \gamma_2 = (246), \delta = (14)(2563),$$
$$\tau_1 = (14)(25), \tau_2 = (14)(36), \varphi = (15243).$$

REMARK 5. Since the permutation $(1\ 2)(3\ 4)(5\ 6)$ is odd, the even transitive subgroups cannot include a conjugate of H_1, from which the "only-if" part follows. Conversely, to show "if" part, it seems to need to consult the table of the transitive subgroups of $\mathfrak{S}_6$.

For the case of H_1, in Proposition 5, we have already shown that $\widetilde{K}_6^{H_1}$ is not rational over $K_6^{H_1}$. Hence for any *odd* transitive group T, $\widetilde{K}_6^T$ is not rational K_6^T.

In the following propositions, we shall treat the remaining two cases.

PROPOSITION 6. *For the group* $H := H_2 = \langle (1\ 2)(3\ 4), (1\ 2)(5\ 6)\rangle$, $\widetilde{K}_6^H$ *is not rational over* K_6^H.

PROOF. Put $\sigma := (1\ 2)(3\ 4)$, $\tau := (1\ 2)(5\ 6)$ and $H = \langle \sigma, \tau\rangle \supset S := \langle \sigma \rangle \supset \{1\}$. Choose generators of $K := K_6$ and $\widetilde{K} := \widetilde{K}_6$ according to a normalization $[-1-y_1, 1-y_1, \infty, 0, y_2, y_3; z]$, that is, first we take as $[-1, 1, \infty, y_1, *, *; *]$ then translate it by $\xi \mapsto \xi - y_1$, and put the fifth (resp. the sixth) coordinate to y_2 (resp. y_3). Then $K = k(y_1, y_2, y_3)$ and $\widetilde{K} = K(z)$.

The action of H is given as follows:

$$\sigma:\begin{cases} y_1 \longmapsto y_1 \\ y_2 \longmapsto \dfrac{Y}{y_2} \\ y_3 \longmapsto \dfrac{Y}{y_3} \\ z \longmapsto \dfrac{Y}{z}, \end{cases} \qquad \tau:\begin{cases} y_1 \longmapsto -y_1 \\ y_2 \longmapsto -y_3 \\ y_3 \longmapsto -y_2 \\ z \longmapsto -z, \end{cases} \tag{29}$$

where we put $Y := y_1^2 - 1$. To determine K^S and $\widetilde{K}^S$, we observe the action of σ on some typical elements in K and $\widetilde{K}$:

$$\begin{aligned} y_2 + y_3 &\longmapsto \frac{Y}{y_2y_3}(y_2+y_3)\,, & y_2 - y_3 &\longmapsto -\frac{Y}{y_2y_3}(y_2-y_3)\,, \\ y_2y_3 + Y &\longmapsto \frac{Y}{y_2y_3}(y_2y_3+Y)\,, & y_2y_3 - Y &\longmapsto -\frac{Y}{y_2y_3}(y_2y_3-Y)\,, \\ z + y_2 &\longmapsto \frac{Y}{y_2z}(z+y_2)\,, & z - y_2 &\longmapsto -\frac{Y}{y_2z}(z-y_2)\,. \end{aligned} \tag{30}$$

From this, we obtain σ-invariant elements $v_1 := \dfrac{y_2y_3+Y}{y_2+y_3}$, $v := \dfrac{y_2y_3-Y}{y_2-y_3} \in K^S$ and $z_1 := \dfrac{z+y_2}{z-y_2}\dfrac{y_2-y_3}{y_2+y_3} \in \widetilde{K}^S$. Since $K = k(y_1, y_2, y_3) = k(y_1, v_1, v)(y_3)$ and y_3 satisfies the quadratic equation $y_3^2 - 2\dfrac{(v_1v+Y)}{(v_1+v)}y_3 + Y = 0$ over $k(y_1, v_1, v)$, we have $[K : k(y_1, v_1, v)] \leq 2$, which implies $K^S = k(y_1, v_1, v)$. It also holds that $\widetilde{K}^S = K^S(z_1)$ since $\widetilde{K} = K(z_1)$. We also notice that $\sigma\left(\dfrac{y_2-y_3}{y_2+y_3}\right) = -\dfrac{y_2-y_3}{y_2+y_3}$ and hence $\left(\dfrac{y_2-y_3}{y_2+y_3}\right)^2$ is σ-invariant. If fact, we have $\left(\dfrac{y_2-y_3}{y_2+y_3}\right)^2 = \dfrac{Y-v_1^2}{Y-v^2}$.

The action of τ on the generators of $K^S = k(y_1, v_1, v)$ and $\widetilde{K}^S = K^S(z_1)$ is as follows:

$$\tau:\begin{cases} y_1 \longmapsto -y_1 \\ v_1 \longmapsto -v_1 \\ v \longmapsto v \\ z_1 \longmapsto -\dfrac{z_1 + \frac{Y-v_1^2}{Y-v^2}}{z_1+1}\,. \end{cases} \tag{31}$$

Hence $K^H = K(u, v, w)$, where we put $u := y_1v_1$, $w := v_1^2$. Note that $Y = y_1^2 - 1 =$

$\dfrac{u^2}{v_1^2} - 1 = \dfrac{u^2 - w}{w}$ is also τ-invariant. To determine $\widetilde{K}^H$, we take $Z := \dfrac{1}{z_1 + 1}$ as a generator of $\widetilde{K}^S$ over K^S to make the computation simpler, while Lemma 4 suggests us to consider $z_1 + 1$. Then $Z\tau(Z) = \dfrac{1}{1 - \frac{Y-w}{Y-v^2}} = \dfrac{Y - v^2}{w - v^2} \in K^H$. Put $U := (w - v^2)(Z + \tau(Z))$, $V := w(w - v^2)\dfrac{Z - \tau(Z)}{v_1}$, then we obtain a conic

$$V^2 - wU^2 = 4(w - v^2)(w(v^2 + 1) - u^2) \tag{32}$$

attaching to the extension $\widetilde{K}^H/K^H$. Thus the non-rationality of $\widetilde{K}^H/K^H$ is reduced to the following claim. □

CLAIM 3. *The conic $V^2 - wU^2 = 4(w - v^2)(w(v^2 + 1) - u^2)$ over K^H has no K^H-rational points.*

PROOF. Since $K^H = k(u, v)(w)$, it suffices to show that the equation

$$V_0^2 - wU_0^2 = 4(w - v^2)(w(v^2 + 1) - u^2)W_0^2$$

has no non-trivial solution (U_0, V_0, W_0) in the polynomial ring $k(u, v)[w]$. Since RHS (resp. V_0^2, wU_0^2) is of even (resp. even, odd) degree in w, the leading terms of V_0^2 and RHS must be equal. But it is impossible because $v^2 + 1$ is not a square in $k(u, v)$. □

PROPOSITION 7. *For the group $H := H_3 = \langle(1\ 2\ 3\ 4)(5\ 6)\rangle$, $\widetilde{K}_6^H$ is not rational over K_6^H.*

PROOF. Put $\sigma = (1\ 2\ 3\ 4)(5\ 6)$ and $H = \langle\sigma\rangle \supset S := \langle\sigma^2\rangle \supset \{1\}$. Choose generators of $K := K_6$ and $\widetilde{K} := \widetilde{K}_6$ according to a normalization $[a_0a_1a_2, a_1, a_0a_1, 1, 0, \infty; b]$. Then $K = k(a_0a_1a_2, a_1, a_0a_1) = k(a_0, a_1, a_2)$ and $\widetilde{K} = K(b)$.

The action of σ is as follows:

$$\sigma : \begin{cases} a_1 \longmapsto a_2\,, \\ a_2 \longmapsto 1/a_1\,, \\ a_0 \longmapsto a_0a_1\,, \\ b \longmapsto a_0a_1a_2/b\,, \end{cases} \qquad \sigma^2 : \begin{cases} a_1 \longmapsto 1/a_1\,, \\ a_2 \longmapsto 1/a_2\,, \\ a_0 \longmapsto a_0a_1a_2\,, \\ b \longmapsto b/a_1\,. \end{cases} \tag{33}$$

Put $K_0 := k(a_1, a_2)$. Then $K_0 \subset K = K_0(a_0) \subset \widetilde{K} = K(b)$ is a σ-stable tower of successively rational extensions. We can take another choice $a := \mathrm{tr}_H(a_0) = (1 + a_1)(1 + a_2)a_0$ of a generator of K/K_0. Since a is σ-invariant, we have $K^H = K_0(a)^H = K_0^H(a)$. Next put $b' := \mathrm{tr}_S(b) = \left(1 + \dfrac{1}{a_1}\right)b$, then $\widetilde{K} = K(b')$ and $\widetilde{K}^S = K(b')^S = K^S(b')$. Hence $\widetilde{K}^S$

is rational over K^S. (Although we can see this from the discussion in Section 5 considering that S has fixed points 5, 6 in the set $\{1, \dots, 6\}$, we need more concrete description to dig up $\widetilde{K}^H/K^H$.)

Since $N_{H/S}(b') = \Big(1+\dfrac{1}{a_1}\Big)b\Big(1+\dfrac{1}{a_2}\Big)\dfrac{a_0a_1a_2}{b} = a \in K^H$, b' satisfies the condition of Lemma 4. Take an element $a_- \in K_0^S \setminus K_0^H$ satisfying $\sigma(a_-) = -a_-$, and put $c := a_-^2 \in K_0^H \setminus (K_0^H)^2$. Then, by putting $U := \mathrm{tr}_{H/S}(b') = b' + \dfrac{a}{b'}$, $V := \dfrac{b'-\sigma(b')}{a_-}$, we obtain a conic

$$U^2 - cV^2 = 4a$$

attaching to the extension $\widetilde{K}^H/K^H$. Thus the non-rationality of $\widetilde{K}^H/K^H$ is reduced to the following claim. □

CLAIM 4. *The conic $U^2 - cV^2 = 4a$ over K^H has no K^H-rational points.*

PROOF. Since $K^H = K_0^H(a)$, it suffices to show that the equation

$$U_0^2 - cV_0^2 = 4aW_0^2$$

has no non-trivial solution (U_0, V_0, W_0) in the polynomial ring $K_0^H[a]$. Since RHS is of odd degree in a, the degrees of U_0 and V_0 in a are equal and their leading terms must be cancelled in LHS. But it is impossible because c is not a square in K_0^H. □

7.3. Some remarks for other n's. For small individual n's, we can say something by consulting the table of transitive groups of Butler-McKay [3], Butler [2] and Royle [15], and using GAP to check individual cases.

REMARK 6. When $n = 4m+2 \equiv 2 \pmod 4$, $\sigma = (1\ 2)\cdots(n-1\ n)$ of cycle type 2^{2m+1} is an odd permutation, hence σ is not contained in any even subgroup. Moreover, there are odd transitive subgroups which do not contain any conjugate of σ in general. When $n = 4m \equiv 0 \pmod 4$, $\sigma = (1\ 2)\cdots(n-1\ n)$ of cycle type 2^{2m} is an even permutation, hence even subgroups may contain σ, but in fact there are some transitive subgroups which do not contain any conjugate of σ.

EXAMPLE 3. In the case $n = 10$, according to the list of Butler-McKay [3], not as in the case of $n = 6$, there are some odd transitive groups which do not contain any element of cycle type 2^5.

EXAMPLE 4. In the case $n = 12$, there are 17 minimal transitive groups as listed in Royle [15] Section 4.3. Among them, the groups numbered 9, 12, 13, 15 do not contain any element of cycle type 2^6.

EXAMPLE 5. In the case $n = 14$, By checking the list of Butler [2] using GAP, we know that all odd transitive subgroups has an element of cycle type 2^7 as in the case of $n = 6$. Hence we have the following result.

THEOREM 4. *Assume that k is infinite and of characteristic different from two. Then, for any* odd *transitive subgroup T of $\mathfrak{S}_{14}$, $\widetilde{K}_{14}^T$ is not rational over K_{14}^T.*

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