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Coxeter Elements of the Symmetric Groups Whose Powers Afford the Longest Elements

Dedicated to Professor Ken-ichi SHINODA

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Abstract. The purpose of this paper is to present a condition for the power of a Coxeter element of $\mathfrak{S}_n$ to become the longest element. To be precise, given a product C of $n-1$ distinct adjacent transpositions of $\mathfrak{S}_n$ in any order, we describe a condition for C such that the $(n/2)$-th power $C^{n/2}$ of C becomes the longest element, in terms of the Amida diagrams.

Introduction

It is well known that the symmetric group $\mathfrak{S}_n$ is defined by the generators $S = \{s_i\}_{i=1}^{n-1}$ of transpositions. Consider a product of the distinct $n-1$ generators in any order $s_{i_1}s_{i_2}\cdots s_{i_{n-1}}$. Such an element is called a *Coxeter element*. All Coxeter elements are conjugate to each other and have the same cycle type (n), a single n-cycle (Remark 1), and accordingly $(s_{i_1}s_{i_2}\cdots s_{i_{n-1}})^n = 1$.

Now suppose that n is even and consider the power of a Coxeter element $(s_{i_1}s_{i_2}\cdots s_{i_{n-1}})^{n/2}$. If this is reduced, then it is the longest, since the longest element of $\mathfrak{S}_n$ is the unique one of length $n(n-1)/2$. A natural question arises: Does $(s_{i_1}s_{i_2}\cdots s_{i_{n-1}})^{n/2}$ afford the longest element in $\mathfrak{S}_n$ for any permutation of the generators? Actually, this does not hold. For example, $(s_1s_3s_2)^2$ affords the longest element in $\mathfrak{S}_4$ while $(s_1s_2s_3)^2$ does not.

In this article, we first show in case n is even which Coxeter elements in $\mathfrak{S}_n$ afford the longest element by taking its $(n/2)$-th power (THEOREM 2).

Then we also consider the case where n is odd, say $n = 2m-1$. In this case we cannot define $(s_{i_1}s_{i_2}\cdots s_{i_{2m-2}})^{n/2}$. Instead, we consider the following word

$$w_2(w_1w_2)^{m-1},$$

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where $w_1 = s_{i_1} s_{i_2} \cdots s_{i_{m-1}}$ and $w_2 = s_{i_m} s_{i_{m+1}} \cdots s_{i_{2m-2}}$. For some Coxeter elements, this expression also affords the longest element in $\mathfrak{S}_{2m-1}$. For example, a Coxeter element $s_1 s_3 s_2 s_4$ affords the longest element $s_2 s_4 (s_1 s_3 s_2 s_4)^2$ in $\mathfrak{S}_5$, while $s_3 s_4 (s_1 s_2 s_3 s_4)^2$ does not. We also show in case n is odd which Coxeter elements afford the longest element in $\mathfrak{S}_n$ (THEOREM 3).

In both cases, the proofs are described in terms of the Amida (Ghost legs) diagrams, ladder lotteries in Japan. By the Amida diagrams, we can geometrically understand what is going on. The reader may get interested in what would hold about the longest elements of other finite Coxeter groups. In types $B(C)_n$, D_n (n even), E_7, E_8, F_4, G_2, H_3, H_4 and $I_2(m)$, all Coxeter elements afford the longest elements by taking their powers. In types D_n (n odd) and E_6, some Coxeter elements afford the longest ones and others do not. For these types we also have had a description of which Coxeter elements afford the longest ones, by "folding" their root systems. This will be shown in a future work.

After posting the preliminary version of this article to the math arXiv, the author was informed of the existence of the paper [2]. In their paper, the same problem is solved for all finite Coxeter groups as a corollary of a word problem.

1. Preliminaries

The symmetric group $\mathfrak{S}_n$ is a Coxeter system of type A_{n-1} [1, 3], which is defined by the generators:

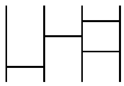
$$S = \{s_1, s_2, \ldots, s_{n-1}\}$$

and the relations:

$$\begin{aligned} s_i^2 &= 1 && (1 \le i \le n-1), \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} && (1 \le i \le n-2), \\ s_i s_j &= s_j s_i && (1 \le i, j \le n-1,\ |i-j| \ge 2). \end{aligned}$$

Each $w \in \mathfrak{S}_n$ can be written in the form $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ (not necessarily distinct) for $s_{i_j} \in S$ ($j = 1, 2, \ldots, r$). If r is as small as possible, then call it the *length* of w and written $\ell(w)$, and call any expression of w as a product of r elements of S a *reduced expression*. There may be more than one reduced expressions for an element $w \in \mathfrak{S}_n$.

Throughout this paper, we will describe elements of $\mathfrak{S}_n$ drawing the following pictures called the Amida (Ghost legs) diagrams.



(1)

An *Amida diagram* consists of n vertical lines and horizontal segments placed between adjacent vertical lines like ladders so that the end points of each horizontal segment meet the vertical lines and so that they do not meet any other horizontal segments' end points.

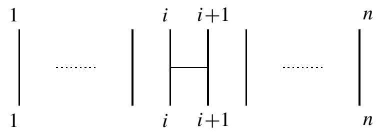
The n runners who start from the bottoms of the vertical lines go up along the lines. If they find horizontal segments on their right [resp. left], they turn right [resp. left] and go along the segments. They necessarily meet the adjacent vertical lines. Then again they go up the vertical lines and iterate this trip until they arrive at the tops of the vertical lines. If the i-th runner arrives at the σ_i-th position ($i = 1, 2, \ldots, n$), we consider the Amida diagram as one of the expressions of

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & i & \cdots & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_i & \cdots & \sigma_n \end{pmatrix} \in \mathfrak{S}_n.$$

For example the Amida diagram as in (1) corresponds to $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$.

We can also consider the product of Amida diagrams. If D_1 and D_2 are Amida diagrams of $\mathfrak{S}_n$ then the product $D_1 D_2$ is defined to be an Amida diagram obtained from D_1 and D_2 by putting the former on the latter.

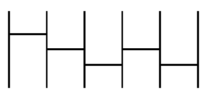
A generator $s_i \in \mathfrak{S}_n$ corresponds to an Amida diagram which consists of n vertical lines with only one horizontal segment between the i-th and the $(i+1)$-st vertical lines.



Since $s_1, s_2, \ldots, s_{n-1}$ generate $\mathfrak{S}_n$, any word in $\mathfrak{S}_n$ can be expressed as an Amida diagram. For example (1) denotes $s_3 s_2 s_3 s_1$.

A *Coxeter element* in $\mathfrak{S}_n$ is a product of distinct $n-1$ generators $\{s_1, \ldots, s_{n-1}\}$ in any order. By the definition we have $(n-1)!$ expressions of length $n-1$ for all Coxeter elements. However it may happen that different permutations of the distinct $n-1$ generators yield the same Coxeter element. For example, expressions $s_1 s_3 s_2 s_4$ and $s_3 s_1 s_4 s_2$ are the same element.

We want to count all distinct Coxeter elements. Amida diagrams give us a convenient tool for doing it. In order to express a Coxeter element by an Amida diagram, we have only to place the i-th horizontal segment (which corresponds to s_i) between the i-th and the $(i+1)$-st vertical lines: Place the first segment between the 1-st and the 2-nd vertical lines. The second segment is placed between the 2-nd and the 3-rd vertical lines so that it is not placed on the same height as the 1-st one's. The third segment is placed between the 3-rd and the 4-th vertical lines so that it is not placed on the same height as the 2-nd one's. Iterate this procedure until $(n-1)$-st segment is placed. We call the Amida diagrams obtained in this way *standard*. The following is a standard Amida diagram for a Coxeter element $s_1 s_2 s_4 s_3 s_5$ in $\mathfrak{S}_6$.



Since the i-th segment is not placed on the same height as the $(i-1)$-st one's, the former one must be placed higher or lower than the latter one's. For the fixed n vertical lines, the standard Amida diagram for an expression of a Coxeter element is uniquely defined up to graph isotopy. This graph isotopy also compatible with the commutativity among non-adjacent generators in S of $\mathfrak{S}_n$.

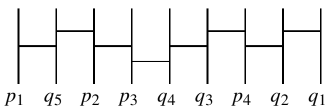
We label the i-th vertical line with a $+$ or $-$ sign ($i = 2, 3, \dots n-1$), according to the positions of the $(i-1)$-st and the i-th horizontal segments. If the i-th horizontal segment is placed higher [resp. lower] than the $(i-1)$-st one's we label the i-th vertical line with a $+$ [resp. $-$] sign. Then we have a sequence $[\varepsilon_2, \varepsilon_3, \dots, \varepsilon_{n-1}]$ of $+$ and $-$ signs of length $n-2$. Conversely, from a sequence of $+$ and $-$ signs of length $n-2$, we can obtain the corresponding Coxeter element: if ε_2 is positive [resp. negative] then multiply s_1 by s_2 from the left [resp. right], if ε_3 is positive [resp. negative] then multiply the previous one by s_3 from the left [resp. right] and repeat these multiplications until s_{n-1} is multiplied. Thus we can expect that the following theorem holds.

THEOREM 1. *There are 2^{n-2} Coxeter elements in $\mathfrak{S}_n$.*

To prove the theorem above, we have only to show that distinct sequences of signatures give distinct elements in $\mathfrak{S}_n$. This will be shown after Remark 1.

For C a Coxeter element in $\mathfrak{S}_n$, let $\varepsilon = [\varepsilon_2, \varepsilon_3, \dots, \varepsilon_{n-1}]$ be a sequence of plus and minus signs of length $n-2$ defined above (each sign is tagged to vertical lines except the left most and the right most ones). We call ε a *Coxeter path* of C and denote it by $p(C)$. Using the Coxeter path $p(C)$, we can define the *height* $ht(C)$ of C by $ht(C) = \sum_{i=2}^{n-1} \varepsilon_i$. Here ε_i takes the value $+1$ [resp. -1] if $+$ [resp. $-$] sign is assigned.

We also introduce the notion of *stanza*s and *co-stanza*s of a standard Amida diagram of a Coxeter element. Stanzas are ascending staircases from lower left to upper right and co-stanzas are ascending staircases from lower right to upper left. We label the beginning points of stanzas [resp. co-stanzas] as $p_1, p_2, \dots$ [resp. $q_1, q_2, \dots$] from left to right [resp. right to left] and call the stanza [resp. co-stanza] which starts at p_i [resp. q_i] the i-th stanza [resp. i-th co-stanza]. For example, the Amida diagram (2) of height 1 has 4 stanzas and 5 co-stanzas which start at p_1, p_2, p_3, p_4 and q_1, q_2, q_3, q_4, q_5 respectively.



(2)

As for the stanzas and co-stanzas, we note the following.

REMARK 1. Let $p_1 = 1, p_2, \dots, p_s$ and $q_1 = n, q_2, \dots, q_t$ be the beginning points of stanzas and co-stanzas respectively of the standard Amida diagram of a Coxeter element C.

(1) Let $p(C)$ be the Coxeter path of C. Then $p_2, p_3, \dots$ [resp. $q_2, q_3, \dots$] correspond to the coordinates of $p(C)$ which have $-$ [resp. $+$] signs.

(2) The height $ht(C)$ of C is equal to the number of co-stanzas minus the number of stanzas. Namely $ht(C) = t - s$, which is also equal to the number of $+$ signs minus the number of $-$ signs in $p(C)$.

(3) $C(p_1) = p_2, \ldots, C(p_{s-1}) = p_s, C(p_s) = q_1$, $C(q_1) = q_2, \ldots, C(q_{t-1}) = q_t, C(q_t) = p_1$. In particular, all Coxeter elements are conjugate[1]and their cycle type is (n) (a single n-cycle).

PROOF OF THEOREM 1. As we stated before Theorem 1, there exists a one to one correspondence between the standard Amida diagrams for Coxeter elements in $\mathfrak{S}_n$ and the sequences of $+$ and $-$ signs of length $n-2$. The sequences of signs determine the beginning points of stanzas and co-stanzas uniquely. By Remark 1(3), C, a Coxeter element in $\mathfrak{S}_n$ which (by the Amida diagram) yields stanzas starting at $p_1, p_2, \ldots$ and co-stanzas starting at $q_1, q_2, \ldots$ is a cyclic permutation $(p_1 = 1, p_2, \ldots, q_1, q_2, \ldots)$. This explains that Coxeter elements determined by the sequences of signs are all distinct. Hence we find the number of Coxeter elements in $\mathfrak{S}_n$ is 2^{n-2}. □

For σ an element of $\mathfrak{S}_n$, the *inversion number* $\iota(\sigma)$ is defined by

$$\iota(\sigma) = |\{(i, j)\ ;\ i < j,\ \sigma(i) > \sigma(j)\}|\,. \tag{3}$$

The inversion number $\iota(\sigma)$ coincides with the length $\ell(\sigma)$ and there is an Amida diagram for σ which has $\ell(\sigma)$ horizontal segments. The longest element $w_0 \in \mathfrak{S}_n$ maps i to $w_0(i) = n+1-i$, and $\ell(w_0) = n(n-1)/2$.

In terms of Amida diagrams, it is easy to show that Coxeter elements are characterized by the cycle type and the inversion number.

PROPOSITION 1. *Let σ be an element of $\mathfrak{S}_n$. If $\iota(\sigma) = n-1$ and the cycle type of σ is (n), that is if σ is an n-cycle, then σ is a Coxeter element of $\mathfrak{S}_n$.*

PROOF. Since $\iota(\sigma) = \ell(\sigma)$, we have an expression of σ whose Amida diagram has exactly $n-1$ horizontal segments. If there are more than one horizontal segments between an adjacent pair of vertical lines, then there is an adjacent pair of vertical lines which have no horizontal segments between them. A runner who starts at the bottom of one of them cannot move to the other. Such an Amida diagram does not represent an n-cycle. Thus, if $\iota(\sigma) = n-1$ and the cycle type of σ is (n), then its Amida diagram consists of $n-1$ horizontal segments, one for each pair of adjacent vertical lines. This implies σ is a Coxeter element in $\mathfrak{S}_n$. □

2. Coxeter elements whose powers afford the longest element in $\mathfrak{S}_{2m}$

In the previous section, we defined the standard Amida diagrams of Coxeter elements and showed that each of them is distinguished by a sequence of plus and minus signs of length $n-2$. In this section, we characterize the Coxeter elements which afford the longest element, when $n-1$, the number of Coxeter generators, is odd, say $n = 2m$.

[1]In any type of finite irreducible Coxeter groups, all Coxeter elements are conjugate [1, 3].

Let C be a Coxeter element in $\mathfrak{S}_{2m} = \langle s_1, s_2, \dots, s_{2m-1}\rangle$. Recall that the Coxeter number h (which is equal to the order of C) is $n = 2m$ (Remark 1(3)). In order that $C^{h/2} = C^{n/2} = C^m$ is the longest element, it should hold that

$$C^m(j) = n + 1 - j \quad \text{for } j = 1, 2, \dots, n\,. \tag{4}$$

Since C is a bijection, there exists $k \in \{1, 2, \dots, n\}$ such that $C(k) = j$. Hence (4) would be written as $C^m(C(k)) = n+1-C(k)$ for $k = 1, 2, \dots, n$. Again applying (4) for k, we obtain

$$C^m(k) = n + 1 - k$$

and we have

$$C^m(C(k)) = C(C^m(k)) = C(n + 1 - k) = n + 1 - C(k)\,.$$

This implies that in the standard Amida diagram of C, a runner who starts at the k-th position from the left arrives at the $C(k)$-th position from the left, while a runner who starts at the k-th position from the right arrives at the $C(k)$-th position from the right. Hence we have the following theorem.

THEOREM 2. *Let $n = 2m$ be an even integer and C a Coxeter element in $\mathfrak{S}_{2m}$. Then C^m is the longest element in $\mathfrak{S}_{2m}$ if and only if the corresponding standard Amida diagram of C is symmetric with respect to the vertical axis between the m-th and the $(m + 1)$-st vertical lines.*

The symmetric standard Amida diagrams as in the theorem above are obtained from the left half of the diagram by reflecting the image of it with respect to the vertical axis. Hence we have the following Corollary.

COROLLARY 1. *The number of distinct Coxeter elements which satisfy the above theorem is 2^{m-1}.*

3. Admissible Coxeter elements

Before we consider the case $n = 2m - 1$, we introduce the notion of admissible Coxeter elements. Admissible Coxeter elements are inductively defined from the ones in $\mathfrak{S}_n$ to the ones in $\mathfrak{S}_{n+2}$.

Let us consider the symmetric group $\mathfrak{S}_{n+2}$ as the permutation group of $n + 2$ letters $\{0, 1, 2, \dots, n, n + 1\}$ generated by the transpositions $s_0 = (0, 1)$, $s_1 = (1, 2)$, $s_2 = (2, 3)$, …, $s_{n-1} = (n - 1, n)$ and $s_n = (n, n + 1)$. For

$$w = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix} \in \mathfrak{S}_n$$

we define $\overline{w} \in \mathfrak{S}_{n+2}$ by

$$\overline{w} = \begin{pmatrix} 0 & 1 & 2 & \cdots & n & n+1 \\ 0 & i_1 & i_2 & \cdots & i_n & n+1 \end{pmatrix}.$$

If there is no confusion, we merely write w to refer to the image $\overline{w} \in \mathfrak{S}_{n+2}$.

DEFINITION 1. Let C be a Coxeter element in $\mathfrak{S}_n$. We identify C with $\overline{C} \in \mathfrak{S}_{n+2}$. Then $s_0 s_n C$, $C s_0 s_n$, $s_0 C s_n$ and $s_n C s_0$ are all Coxeter elements in $\mathfrak{S}_{n+2}$. We call these elements *extensions* of C.

REMARK 2. Every Coxeter element in $\mathfrak{S}_{n+2}$ is obtained from $\mathfrak{S}_n$ in the above way. In other words, a Coxeter element $C \in \mathfrak{S}_n$ has (exactly) one of the expressions of the form $s_1 s_{n-1} C'$, $C' s_1 s_{n-1}$, $s_1 C' s_{n-1}$ or $s_{n-1} C' s_1$, where C' is an expression of a Coxeter element in $\mathfrak{S}_{n-2} = \langle s_2, s_3, \dots, s_{n-2} \rangle$.

As for the heights of extensions, we have the following lemma.

LEMMA 1. *Let C be a Coxeter element in $\mathfrak{S}_n$ and $\eta = ht(C)$ its height. Then $ht(s_0 s_n C)$, $ht(C s_0 s_n)$, $ht(s_0 C s_n)$ and $ht(s_n C s_0)$ are η, η, $\eta - 2$ and $\eta + 2$ respectively.*

PROOF. In terms of the standard Amida diagrams, multiplying s_0 from the left corresponds to adding an s_0 segment at a position higher than that of the s_1 segment. Since the height of a Coxeter element is measured by the relative positions of the horizontal segments, this addition of s_0 lowers the height of C by 1. Similarly, multiplying s_0 from the right raises the height of C by 1. Multiplying s_n from the left [right] also raises [lowers] the height of C by 1. The result follows from these observations. □

With the above preparatory result at hand, the admissible Coxeter elements in $\mathfrak{S}_{2m-1}$ are defined as follows.

DEFINITION 2. There exists two Coxeter elements $s_1 s_2$ and $s_2 s_1$ in $\mathfrak{S}_3$. Both of them are by definition admissible. Let $C \in \mathfrak{S}_{2m-1} (m \geq 2)$ be an admissible Coxeter element and $\mathcal{E}(C)$ one of the extensions of C. If $|ht(\mathcal{E}(C))| \leq 1$ then the extension is called *admissible*. Otherwise the extension is *non-admissible*. An *admissible Coxeter element* in $\mathfrak{S}_{2m-1}$ is defined as a Coxeter element in $\mathfrak{S}_{2m-1}$ obtained from $s_1 s_2$ or $s_2 s_1$ in $\mathfrak{S}_3$ by the iterative application of the admissible extensions.

Since the height of $s_2 s_1$ [resp. $s_1 s_2$] in $\mathfrak{S}_3$ is 1 [resp. -1], by Lemma 1 the heights of admissible Coxeter elements are $+1$ or -1. So the definition above is rewritten as follows.

REMARK 3. Let C be an admissible Coxeter element in $\mathfrak{S}_{2m-1}$

(1) If C has an expression such that $ht(C) = 1$, then the expressions $s_0 s_{2m-1} C$, $C s_0 s_{2m-1}$ and $s_0 C s_{2m-1}$ are admissible in $\mathfrak{S}_{2m+1}$.
(2) If C has an expression such that $ht(C) = -1$, then the expressions $s_0 s_{2m-1} C$, $C s_0 s_{2m-1}$ and $s_{2m-1} C s_0$ are admissible in $\mathfrak{S}_{2m+1}$.

Note that $ht(C) = \pm 1$ does not mean C is admissible. For example $C = s_0 \overline{s_4 s_3 s_2 s_1} s_5 \in \mathfrak{S}_{5+2}$ has its height $ht(C) = 1$, but C is non-admissible, since $ht(s_4 s_3 s_2 s_1) = 3$.

From Definition 2 and Remark 3, we have the following corollary.

COROLLARY 2. *There are $2 \cdot 3^{m-2}$ admissible Coxeter elements in $\mathfrak{S}_{2m-1}$.*

4. Coxeter elements whose powers afford the longest element in $\mathfrak{S}_{2m-1}$

All Coxeter elements in $\mathfrak{S}_{2m-1}$ have the same order $h = 2m - 1$. In this case, the situation is rather complicated. Since $h/2 = (2m-1)/2$ is a half integer, we cannot define $h/2$-nd power of a Coxeter element. On the other hand, a Coxeter element $C \in \mathfrak{S}_{2m-1}$ has even length $\ell(C) = 2m - 2$. Hence putting $w_1 = s_{i_1} \cdots s_{i_{m-1}}$ and $w_2 = s_{i_m} \cdots s_{i_{2m-2}}$, we consider the following word

$$C^{h/2} = C_{w_2}^{h/2} = w_2(w_1w_2)^{m-1} = (s_{i_m}s_{i_{m+1}} \cdots s_{i_{2m-2}})C^{m-1} . \tag{5}$$

Note that the definition of $C_{w_2}^{h/2}$ above depends on the choice of w_2 (and w_1). For example, for $C = C_1 = s_1s_3s_2s_4$ a Coxeter element in $\mathfrak{S}_5$, another expression $C_2 = s_3s_4s_1s_2$ coincides with C. According to the equation (5), we have $C_1^{5/2} = s_3s_4C_1^2 = s_3s_4C^2$ and $C_2^{5/2} = s_1s_2C_2^2 = s_3s_4C^2$ which do not coincide. However we have the following lemma.

LEMMA 2. *Let $C = s_{i_1}s_{i_2} \cdots s_{i_{2m-2}} \in \mathfrak{S}_{2m-1}$ be a Coxeter element in $\mathfrak{S}_{2m-1}$ and $C = w_1w_2$ an expression of C such that $\ell(w_1) = \ell(w_2) = m - 1$. If $C_{w_2}^{h/2}$ affords the longest element in $\mathfrak{S}_{2m-1}$ for the expression, then such w_2 is uniquely determined.*

PROOF. Assume that w_1w_2 and $w_1'w_2'$ are both expressions of C. We further assume that both $C_{w_2}^{h/2} = w_2C^{m-1}$ and $C_{w_2'}^{h/2} = w_2'C^{m-1}$ are the longest element in $\mathfrak{S}_{2m-1}$. Since the longest element in $\mathfrak{S}_{2m-1}$ is unique, they coincide. Hence we have $w_2 = w_2'$. □

By the above lemma, we merely write $C^{h/2}$ for $C_{w_2}^{h/2}$ in the following lemma.

LEMMA 3. *Let $C = s_{i_1}s_{i_2} \cdots s_{i_{2m-2}} \in \mathfrak{S}_{2m-1}$ be an admissible Coxeter element which affords the longest element in $\mathfrak{S}_{2m-1}$ by $C^{h/2}$. Let $\mathcal{E}(C)$ be one of the admissible extensions of C. Then the following holds.*

(1) *If $ht(C) = 1$ and $\mathcal{E}(C)$ is written as $s_0s_{2m-1}C$, then $s_0(s_{i_m}s_{i_{m+1}} \cdots s_{i_{2m-2}})\mathcal{E}(C)^m$ is the longest element in $\mathfrak{S}_{2m+1}$.*
(2) *If $ht(C) = 1$ and $\mathcal{E}(C)$ is written as Cs_0s_{2m-1}, then $(s_{i_m}s_{i_{m+1}} \cdots s_{i_{2m-2}})s_0\mathcal{E}(C)^m$ is the longest element in $\mathfrak{S}_{2m+1}$.*
(3) *If $ht(C) = 1$ and $\mathcal{E}(C)$ is written as s_0Cs_{2m-1}, then $(s_{i_m}s_{i_{m+1}} \cdots s_{i_{2m-2}})s_{2m-1}\mathcal{E}(C)^m$ is the longest element in $\mathfrak{S}_{2m+1}$.*
(4) *If $ht(C) = -1$ and $\mathcal{E}(C)$ is written as $s_0s_{2m-1}C$, then $s_{2m-1}(s_{i_m}s_{i_{m+1}} \cdots s_{i_{2m-2}})\mathcal{E}(C)^m$ is the longest element in $\mathfrak{S}_{2m+1}$.*
(5) *If $ht(C) = -1$ and $\mathcal{E}(C)$ is written as Cs_0s_{2m-1}, then $(s_{i_m}s_{i_{m+1}} \cdots s_{i_{2m-2}})s_{2m-1}\mathcal{E}(C)^m$ is the longest element in $\mathfrak{S}_{2m+1}$.*
(6) *If $ht(C) = -1$ and $\mathcal{E}(C)$ is written as $s_{2m-1}Cs_0$, then $(s_{i_m}s_{i_{m+1}} \cdots s_{i_{2m-2}})s_0\mathcal{E}(C)^m$ is the longest element in $\mathfrak{S}_{2m+1}$.*

PROOF. We prove the theorem by induction on m. If $m = 2$, then there are two Coxeter elements s_2s_1 and s_1s_2 in $\mathfrak{S}_{2\cdot 2-1}$. Both of them are by definition admissible and $s_1(s_2s_1)^1 =$

$s_2(s_1s_2)^1$ is the longest element in $\mathfrak{S}_{2\cdot 2-1}$. So both s_2s_1 and s_1s_2 satisfy the hypothesis. Consider the case $C = s_2s_1 \in \mathfrak{S}_{2\cdot 2-1}$. Since $ht(s_2s_1) = 1$ we have only to consider the case (1)(2)(3). If $\mathcal{E}(C) = s_0s_3C = s_0s_3s_2s_1$, then we can check that $s_0(s_1)(s_0s_3s_2s_1)^2$ affords the longest element in $\mathfrak{S}_{2\cdot 2+1}$ by direct calculation. Similarly, if $\mathcal{E}(C) = Cs_0s_3 = s_2s_1s_0s_3$, then $(s_1)s_0(s_2s_1s_0s_3)^2$ affords the longest element and if $\mathcal{E}(C) = s_0Cs_3 = s_0s_2s_1s_3$, then $(s_1)s_3(s_0s_2s_1s_3)^2$ affords the longest element. The case $C = s_1s_2$ will be verified similarly.

Before moving on to the case $m \geq 3$, we rewrite the hypothesis. If $ht(C) = -1$, then taking the mirror image of the Amida diagram of C with respect to the vertical axis, we can attribute this case to the case $ht(C) = 1$. Hence we have only to consider the case (1)(2)(3).

In case $ht(C) = 1$ the numbers of stanzas and co-stanzas of C are $m - 1$ and m respectively. So we can put $C = (p_1, p_2, \ldots, p_{m-1}, q_1, q_2, \ldots, q_m)$ in the cyclic presentation (Remark 1). Then we have

$$C^{m-1} = \begin{pmatrix} p_1 & p_2 & \cdots & p_{m-1} & q_1 & q_2 & q_3 & \cdots & q_{m-1} & q_m \\ q_1 & q_2 & \cdots & q_{m-1} & q_m & p_1 & p_2 & \cdots & p_{m-2} & p_{m-1} \end{pmatrix}. \tag{6}$$

We note that $p_1 = 1$ and $q_1 = 2m - 1$. The hypothesis that the word $C^{h/2}$ is the longest element implies that $C^{h/2}(k) = 2m - k$ $(k = 1, 2, \ldots, 2m - 1)$. Hence for p_i $(i = 1, 2, \ldots, m - 1)$, q_j $(j = 2, 3 \ldots, m)$ and $q_1 = 2m - 1$ we have

$$2m - p_i = C^{h/2}(p_i) = s_{i_m}s_{i_{m+1}} \cdots s_{i_{2m-2}}C^{m-1}(p_i) = s_{i_m}s_{i_{m+1}} \cdots s_{i_{2m-2}}(q_i), \tag{7}$$

$$2m - q_j = C^{h/2}(q_j) = s_{i_m}s_{i_{m+1}} \cdots s_{i_{2m-2}}C^{m-1}(q_j) = s_{i_m}s_{i_{m+1}} \cdots s_{i_{2m-2}}(p_{j-1}) \tag{8}$$

and

$$\begin{aligned} 1 = 2m - q_1 &= C^{h/2}(q_1) \\ &= s_{i_m}s_{i_{m+1}} \cdots s_{i_{2m-2}}C^{m-1}(q_1) \\ &= s_{i_m}s_{i_{m+1}} \cdots s_{i_{2m-2}}(q_m) \end{aligned} \tag{9}$$

respectively.

Now we prove the case (1). In this case $\mathcal{E}(C) = s_0s_{2m-1}C$. Then we have

$$\mathcal{E}(C) = s_0s_{2m-1}C = (0, p_1, p_2, \ldots, p_{m-1}, 2m, q_1, q_2, \ldots, q_m)$$

and

$$\mathcal{E}(C)^m = \begin{pmatrix} 0 & p_1 & p_2 & \cdots & p_{m-1} & 2m & q_1 & q_2 & q_3 & \cdots & q_m \\ 2m & q_1 & q_2 & \cdots & q_{m-1} & q_m & 0 & p_1 & p_2 & \cdots & p_{m-1} \end{pmatrix}. \tag{10}$$

For p_i $(i = 1, 2, \ldots, m - 1)$, we have

$$\begin{aligned} s_0(s_{i_m}s_{i_{m+1}} \cdots s_{i_{2m-2}})\mathcal{E}(C)^m(p_i) &= s_0(s_{i_m}s_{i_{m+1}} \cdots s_{i_{2m-2}})(q_i) \\ &= s_0(2m - p_i) \quad (\because (7)) \end{aligned}$$

$$= 2m - p_i \quad (\because p_i < 2m-1)$$

and for q_j $(j = 2, 3, \ldots, m)$, we have

$$\begin{aligned} s_0(s_{i_m}s_{i_{m+1}}\cdots s_{i_{2m-2}})\mathcal{E}(C)^m(q_j) &= s_0(s_{i_m}s_{i_{m+1}}\cdots s_{i_{2m-2}})(p_{j-1}) \\ &= s_0(2m - q_j) \quad (\because (8)) \\ &= 2m - q_j \quad (\because q_j < q_1 = 2m-1)\,. \end{aligned}$$

For 0, $2m$ and $q_1 = 2m-1$, we have

$$s_0(s_{i_m}s_{i_{m+1}}\cdots s_{i_{2m-2}})\mathcal{E}(C)^m(0) = s_0(s_{i_m}s_{i_{m+1}}\cdots s_{i_{2m-2}})(2m) = 2m\,,$$
$$s_0(s_{i_m}s_{i_{m+1}}\cdots s_{i_{2m-2}})\mathcal{E}(C)^m(2m) = s_0(s_{i_m}s_{i_{m+1}}\cdots s_{i_{2m-2}})(q_m) = 0 \quad (\because (9))$$

and

$$s_0(s_{i_m}s_{i_{m+1}}\cdots s_{i_{2m-2}})\mathcal{E}(C)^m(q_1) = s_0(s_{i_m}s_{i_{m+1}}\cdots s_{i_{2m-2}})(0) = 1$$

respectively. Thus, we find that $s_0(s_{i_m}\cdots s_{i_{2m-2}})\mathcal{E}(C)^m$ is the longest element. In order to continue the induction, we further have to show that

$$\mathcal{E}(C)^{m+1/2} = s_0(s_{i_m}\cdots s_{i_{2m-2}})\mathcal{E}(C)^m\,.$$

In other words, we have to show that $\mathcal{E}(C)$ has an expression $\mathcal{E}(C) = w_1w_2$ such that $w_2 = s_0(s_{i_m}\cdots s_{i_{2m-2}})$ and $\ell(w_1) = \ell(w_2) = m$. Since we already know that $\mathcal{E}(C)$ has an expression $s_0s_{2m-1}(s_{i_1}\cdots s_{i_{m-1}})(s_{i_m}\cdots s_{i_{2m-2}})$, we have only to show that

$$s_0s_{2m-1}(s_{i_1}\cdots s_{i_{m-1}})(s_{i_m}\cdots s_{i_{2m-2}}) = s_{2m-1}(s_{i_1}\cdots s_{i_{m-1}})s_0(s_{i_m}\cdots s_{i_{2m-2}})\,. \tag{11}$$

By the equation (9) we find that $s_{i_m}\cdots s_{i_{2m-2}}$ involves s_1. This means $s_{2m-1}(s_{i_1}\cdots s_{i_{m-1}})$ does not involve s_1. Hence we can move the s_0 in the left hand side of the equation (11) rightward and we have the right hand side.

Next we prove the case (2). In this case $\mathcal{E}(C) = Cs_0s_{2m-1}$. Then we have

$$\mathcal{E}(C) = Cs_0s_{2m-1} = (0, p_2, p_3, \ldots, p_{m-1}, q_1, 2m, q_2, q_3, \ldots, q_m, p_1)$$

and

$$\mathcal{E}(C)^m = \begin{pmatrix} 0 & p_2 & p_3 & \cdots & p_{m-1} & q_1 & 2m & q_2 & q_3 & \cdots & q_m & p_1 \\ 2m & q_2 & q_3 & \cdots & q_{m-1} & q_m & p_1 & 0 & p_2 & \cdots & p_{m-1} & q_1 \end{pmatrix}. \tag{12}$$

Similar to the case (1), we have the following. For p_i $(i = 1, 2, \ldots, m-1)$, we have

$$(s_{i_m}s_{i_{m+1}}\cdots s_{i_{2m-2}})s_0\mathcal{E}(C)^m(p_i) = 2m - p_i\,.$$

For q_j $(j = 3, 4, \ldots, m)$ we have

$$(s_{i_m}s_{i_{m+1}}\cdots s_{i_{2m-2}})s_0\mathcal{E}(C)^m(q_j) = 2m - q_j\,.$$

For 0, q_1, $2m$ and q_2, we have

$$(s_{i_m} s_{i_{m+1}} \cdots s_{i_{2m-2}}) s_0 \mathcal{E}(C)^m (0) = 2m\,,$$

$$(s_{i_m} s_{i_{m+1}} \cdots s_{i_{2m-2}}) s_0 \mathcal{E}(C)^m (q_1) = 2m - q_1\,,$$

$$(s_{i_m} s_{i_{m+1}} \cdots s_{i_{2m-2}}) s_0 \mathcal{E}(C)^m (2m) = 0$$

and

$$(s_{i_m} s_{i_{m+1}} \cdots s_{i_{2m-2}}) s_0 (\mathcal{E}(C))^m (q_2) = 2m - q_2$$

respectively. Thus, we find that $(s_{i_m} \cdots s_{i_{2m-2}}) s_0 \mathcal{E}(C)^m$ is the longest element. In order to continue the induction, we have to check that $\mathcal{E}(C)$ can be written as $\mathcal{E}(C) = w_1 w_2$ such that $w_2 = (s_{i_m} s_{i_{m+1}} \cdots s_{i_{2m-2}}) s_0$ and $\ell(w_1) = \ell(w_2) = m$. Since $\mathcal{E}(C) = C s_0 s_{2m-1} = (s_{i_1} \cdots s_{i_{m-1}})(s_{i_m} \cdots s_{i_{2m-2}}) s_0 s_{2m-1}$, we have to show that

$$(s_{i_1} \cdots s_{i_{m-1}})(s_{i_m} \cdots s_{i_{2m-2}}) s_0 s_{2m-1} = (s_{i_1} \cdots s_{i_{m-1}}) s_{2m-1} (s_{i_m} \cdots s_{i_{2m-2}}) s_0\,. \tag{13}$$

By the equation (7), $s_{i_m} \cdots s_{i_{2m-2}}$ maps $q_1 = 2m - 1$ to $2m - p_1 = 2m - 1$. This implies $s_{i_m} \cdots s_{i_{2m-2}}$ does not involve s_{2m-2}. Hence we can move s_{2m-1} in the left hand side of the equation (13) leftward. Thus we obtain the right hand side of the equation.

Finally consider the case (3). In this case $\mathcal{E}(C) = s_0 C s_{2m-1}$. Then we have

$$\mathcal{E}(C) = s_0 C s_{2m-1} = (0, p_1, p_2, \ldots, p_{m-1}, q_1, 2m, q_2, q_3, \ldots, q_m)$$

and

$$\mathcal{E}(C)^m = \begin{pmatrix} 0 & p_1 & p_2 & \cdots & p_{m-1} & q_1 & 2m & q_2 & \cdots & q_{m-1} & q_m \\ q_1 & 2m & q_2 & \cdots & q_{m-1} & q_m & 0 & p_1 & \cdots & p_{m-2} & p_{m-1} \end{pmatrix}. \tag{14}$$

Similar to the case (1) and (2), we can check that $(s_{i_m} s_{i_{m+1}} \cdots s_{i_{2m-2}}) s_{2m-1} \mathcal{E}(C)^m$ is the longest element in $\mathfrak{S}_{2m+1}$. In this case $(s_{i_m} \cdots s_{i_{2m-2}}) s_{2m-1} \mathcal{E}(C)^m$ is already of the desired form. So we can continue the induction.

Thus we have completed the proof of the lemma. □

Finally, we obtain the following theorem.

THEOREM 3. *Let C be a Coxeter element in $\mathfrak{S}_{2m-1}$. If C is admissible, then there exists an expression $w_1 w_2$ of C such that $C_{w_2}^{h/2} = w_2 C^{m-1}$ affords the longest element. Conversely, if $C_{w_2}^{h/2} = w_2 C^{m-1}$ is the longest element in $\mathfrak{S}_{2m-1}$ for an expression $w_1 w_2$ of C, then C is admissible.*

PROOF. By Remark 3 and the previous lemma, we find that the first statement of the theorem holds. In order to prove the second statement we have to show that if $C \in \mathfrak{S}_{2m-1}$ is a non-admissible Coxeter element then $C^{h/2} = C_{w_2}^{h/2}$ in (5) is not the longest element in $\mathfrak{S}_{2m-1}$ for any choice of w_2.

First we show that if $ht(C) \geq 3$ then $C^{h/2} = C_{w_2}^{h/2}$ can not be the longest element. Suppose that $ht(C) = 2k+1$ ($k \geq 1$) and $m > k+1$. By Remark 1, the numbers of stanzas and co-stanzas of C are $m-k-1$ and $m+k$ respectively, and $C = (p_1, p_2, \dots, p_{m-k-1}, q_1, q_2, \dots, q_{m+k})$. Hence C^{m-1} becomes (in two-line form) as follows.

$$\begin{pmatrix} p_1 & p_2 & \cdots & p_{m-k-1} & q_1 & q_2 & \cdots & q_{k+1} & q_{k+2} & q_{k+3} & \cdots & q_m & q_{m+1} & q_{m+2} & \cdots & q_{m+k} \\ q_{k+1} & q_{k+2} & \cdots & q_{m-1} & q_m & q_{m+1} & \cdots & q_{m+k} & p_1 & p_2 & \cdots & p_{m-k-1} & q_1 & q_2 & \cdots & q_k \end{pmatrix}.$$

Note that q_{k+2} goes to $p_1 = 1$ and $q_1 = 2m-1$ goes to q_m. In order that w_2C^{m-1} is the longest element, w_2 has to map $1 = p_1 = C^{m-1}(q_{k+2})$ to $2m - q_{k+2}$ and $q_m = C^{m-1}(q_1) = C^{m-1}(2m-1)$ to 1. Since $1 = p_1 < q_{m+k} < q_{m+k-1} < \cdots < q_m$ and $2m-1 = q_1 > q_2 > \cdots > q_{k+2}$, we have $q_m \geq k+2 \geq 3$ and $2m - q_{k+2} \geq k+2 \geq 3$. Hence w_2 has to satisfy $w_2(1) \geq 3$ and $w_2(q_m) = 1$ ($q_m \geq 3$). Under the condition that w_2 consists of $m-1$ distinct generators from $\{s_1, \dots, s_{2m-2}\}$, the former implies w_2 involves a sequence $v_0s_2v_1s_1v_2$ and the latter implies it involves a sequence $v'_0s_1v'_1s_2v'_2$. Here each of $v_1, v_2, v_3, v'_1, v'_2, v'_3$ are sequences of the generators which have neither s_1 nor s_2 and which may be empty. This contradicts the assumption that w_2 is a distinct product of the generators. Hence if $ht(C) \geq 3$, then $C^{h/2} = C_{w_2}^{h/2}$ does not afford the longest element for any w_2. Similarly if $ht(C) \leq -3$, then we find $C^{h/2}$ does not afford the longest element either.

Next we show that if $|ht(C)| \geq 3$ then $\ell(C^{m-1}) < 2(m-1)^2$ holds. Suppose that C is a Coxeter element in $\mathfrak{S}_{2m-1}$ and $|ht(C)| \geq 3$. For $C^{m-1} \in \mathfrak{S}_{2m-1}$ there uniquely exist w' and w'' such that $w''C^{m-1}$ and $C^{m-1}w'$ are the longest element and such that $\ell(w'') + \ell(C^{m-1}) = \ell(C^{m-1}) + \ell(w') = (2m-1)(2m-2)/2$ [3]. Here we note that $\ell(w') = \ell(w'') \geq m-1$, since $\ell(C^{m-1}) \leq 2(m-1)^2$. Hence for the w' and w'' we have $C^{m-1}w'w''C^{m-1} = 1$. On the other hand, we already have $C^h = C^{m-1}CC^{m-1} = 1$. Hence we have $C = w'w''$. If $\ell(w') = \ell(w'') = m-1$, then w' and w'' must be distinct products of the $m-1$ generators. This contradicts the previous argument. Hence we have $\ell(w'') > m-1$. Since the length of the longest element in $\mathfrak{S}_{2m-1}$ is $(m-1)(2m-1)$, we have $\ell(C^{m-1}) < 2(m-1)^2$.

Finally we show that for any non-admissible Coxeter element C, we have $\ell(C^{m-1}) < 2(m-1)^2$ and accordingly $C^{h/2} = C_{w_2}^{h/2}$ does not afford the longest element for any w_2. By the previous argument if $|ht(C)| \geq 3$, then the claim holds. By the definition of admissibility, every non-admissible Coxeter element is obtained from iterative extensions of a Coxeter element of height ± 3. So we have only to consider the case $|ht(\tilde{C})| \leq 1$ where $\tilde{C}$ is an extension of a Coxeter element C of height ± 3 or ± 1. By the same argument as in Lemma 3, the case where $ht(C) < 0$ will be attributed to the case $ht(C) > 0$. Hence we find that if the following statements are verified, we obtain the desired inequality.

- For a Coxeter element $C \in \mathfrak{S}_{2m-1}$ of height 3 and $\mathcal{E}(C) = s_0Cs_{2m-1}$ of height 1, we have $\ell(\mathcal{E}(C)^m) < 2m^2$.
- For a non-admissible Coxeter element $C \in \mathfrak{S}_{2m-1}$ of height 1 and $\mathcal{E}(C) = Cs_0s_{2m-1}, s_0s_{2m-1}C$ of height 1, and $\mathcal{E}(C) = s_0Cs_{2m-1}$ of height -1, we have $\ell(\mathcal{E}(C)^m) < 2m^2$.

First consider the case $ht(C) = 3$ and $\mathcal{E}(C) = s_0Cs_{2m-1}$. In this case, we have $C = (p_1, p_2, \ldots, p_{m-2}, q_1, q_2, \ldots, q_{m+1})$,

$$C^{m-1} = \begin{pmatrix} p_1 & \cdots & p_{m-2} & q_1 & q_2 & q_3 & \cdots & q_m & q_{m+1} \\ q_2 & \cdots & q_{m-1} & q_m & q_{m+1} & p_1 & \cdots & p_{m-2} & q_1 \end{pmatrix}$$
$$= \begin{pmatrix} p_1 & < & \cdots & < & \boxed{\begin{matrix} q_2 \\ q_{m+1} \end{matrix}} & < & \cdots & < & q_1 \\ q_2 & & \cdots & & & & \cdots & & q_m \end{pmatrix}, \tag{15}$$

and $\mathcal{E}(C) = (0, p_1, p_2, \ldots, p_{m-2}, q_1, 2m, q_2, q_3, \ldots, q_{m+1})$,

$$\mathcal{E}(C)^m = \begin{pmatrix} 0 & p_1 & \cdots & p_{m-2} & q_1 & 2m & q_2 & q_3 & \cdots & q_m & q_{m+1} \\ 2m & q_2 & \cdots & q_{m-1} & q_m & q_{m+1} & 0 & p_1 & \cdots & p_{m-2} & q_1 \end{pmatrix}$$
$$= \begin{pmatrix} \boxed{\begin{matrix} 0 \\ 2m \end{matrix}} & \begin{matrix} p_1 \\ q_2 \end{matrix} & \begin{matrix} < \\ \end{matrix} & \begin{matrix} \cdots \\ \cdots \end{matrix} & < & \boxed{\begin{matrix} q_2 \\ 0 \end{matrix}} & < & \begin{matrix} \cdots \\ \cdots \end{matrix} & < & \begin{matrix} q_1 \\ q_m \end{matrix} & \boxed{\begin{matrix} 2m \\ q_{m+1} \end{matrix}} \end{pmatrix}. \tag{16}$$

Here the two-line forms (16) and (15) are obtained by sorting the columns so that the entries in the first rows are lined up in increasing order. Note that (16) differs from (15) only by the boxed columns. Note also that if there exist some p_is between q_2 and q_1 in the first rows of (16) and (15), then the corresponding entries q_{i+1}s in the second rows satisfy $q_{m+1} < q_{i+1}$.

For $\sigma \in \mathfrak{S}_n$, we consider the following set of "inverted pairs" (instead of inversion pairs):

$$I(\sigma) = \{(\sigma(i), \sigma(j)) \mid i < j,\ \sigma(i) > \sigma(j)\}.$$

The size of $I(\sigma)$ coincides with the inversion number defined in (3) and hence it coincides with the length $\ell(\sigma)$. Now we compare $I(\mathcal{E}(C)^m)$ with $I(C^{m-1})$ paying attention on the second rows of the two-line forms (16) and (15).

The leftmost column entry $2m$ in (16) forms $2m$ inverted pairs. If there exist t p_is between q_2 and q_1 in the first rows of (16) and (15), then the middle boxed column entry 0 in (16) forms $2m - 3 - t$ inverted pairs other than $(2m, 0)$. Further, the rightmost column entry q_{m+1} forms $t + 1$ new inverted pairs in $I(\mathcal{E}(C)^m)$ which are not in $I(C^{m-1})$. Hence we find $\ell(\mathcal{E}(C)^m) = \ell(C^{m-1}) + 4m - 2 < 2m^2$.

Next consider the case $ht(C) = \pm 1$. Let $\mathcal{E}(C)$ be $s_0s_{2m-1}C$ [resp. Cs_0s_{2m-1}, s_0Cs_{2m-1}], an extension of C. In this case we already have the two-line forms of C^{m-1} and $\mathcal{E}(C)^m$ by the equations (6) and (10) [resp. (12), (14)] (Note that these equations still hold even in the case C is non-admissible). Comparing the number of inverted pairs of (6) with that of (10) [resp. (12), (14)] as in the case where $ht(C) = 3$, we similarly obtain

$\ell(\mathcal{E}(C)^m) = \ell(C^{m-1}) + 4m - 2$. This implies that if $\ell(C^{m-1}) < 2(m-1)^2$ then we have $\ell(\mathcal{E}(C)^m) < 2m^2$. By inductive argument, we find that for C an arbitrary non-admissible Coxeter element in $\mathfrak{S}_{2m-1}$, $\ell(C^{m-1}) < 2(m-1)^2$ holds.

Thus we have completed the proof. □

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References

[1] N. BOURBAKI, *Groupes et algèbres de Lie IV, V, VI*, Hermann, Paris, 1968.

[2] C. CEBALLOS, J.-P. LABBÉ and C. STUMP, Subword complexes, cluster complexes, and generalized multi-associahedra, J. Algebraic Combin. **39** (2014), 17–51.

[3] J. E. HUMPHREYS, *Reflection Groups and Coxeter Groups*, Cambridge University Press, 1990.

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