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Construction of Double Grothendieck Polynomials of Classical Types using IdCoxeter Algebras

Dedicated to Professor Ken-ichi SHINODA

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Abstract. We construct double Grothendieck polynomials of classical types which are essentially equivalent to but simpler than the polynomials defined by A. N. Kirillov in arXiv:1504.01469 and identify them with the polynomials defined by T. Ikeda and H. Naruse in Adv. Math. (2013) for the case of maximal Grassmannian permutations. We also give geometric interpretation of them in terms of algebraic localization map and give explicit combinatorial formulas.

1. Introduction

Let G be a semisimple complex Lie group, $B \subset G$ be a fixed Borel subgroup of G, $T \subset B$ be a maximal torus in B, $\mathcal{F} := G/B$ and $W := N_G(T)/T$ be the corresponding flag variety and the Weyl group. Let ℓ be the rank of G. According to the famous Borel's theorem [4], the cohomology ring $H^*(G/B, \mathbb{Q})$ is isomorphic to the quotient $\mathbb{Q}[z_1, \ldots, z_\ell]/J_\ell$, where $z_i := c_1(L_i) \in H^2(G/B, \mathbb{Q})$, $i = 1, \ldots, \ell$, and $c_1(L_i)$ denotes the first Chern class of the standard line bundle L_i corresponding to the i-th fundamental weight ω_i over the complete flag variety $\mathcal{F} = G/B$ in question, J_ℓ stands for the ideal generated by the fundamental invariants associated with the Weyl group W .

To our best knowledge the first systematic and complete treatment of the Schubert Calculus has been done by I. N. Bernstein, I. M. Gelfand and S. I. Gelfand [2] and independently, by M. Demazure [6] in the beginning of 70's of the last century. A Schubert polynomial $S_w(Z_\ell)$, with $\ell = \mathrm{rk}(G)$, $Z_\ell = (z_1, z_2, \ldots, z_\ell)$, corresponding to an element w of the Weyl group W, by definition is a polynomial which expresses the Poincaré dual class $[X_{w_0 w}] \in H^*(G/B)$, where w_0 is the longest element in W, of the homology class of the Schubert variety $X_w := \overline{BwB/B} \subset G/B$ in terms of the Borel generators z_i, $1 \le i \le \ell$,

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in the cohomology ring of the flag variety $\mathcal{F}$. Therefore by the very definition, a Schubert polynomial $S_w(Z_\ell)$ is defined only modulo the ideal J_ℓ.

Hence it is an interesting problem to ask if there exists the “natural representative” of a Schubert polynomial $S_w(Z_\ell)$ in the ring $\mathbb{Q}[z_1, \dots, z_\ell]$ with “nice” combinatorial, algebraic and geometric properties.

For the type A_{n-1} flag varieties, A. Lascoux and M.-P. Schützenberger [24] constructed a family of Schubert polynomials[1] $\mathfrak{S}_w(X_n) \in \mathbb{Z}[X_n]$ with $w \in S_n$ where $X_n = (x_1, x_2, \dots, x_n)$ are indeterminates, and S_n is the symmetric group on the set of n letters $\{1, 2, \dots, n\}$. We will write the transposition $s_i = (i, i+1)$. Then S_n is a Coxeter group with distinguished set $I = \{s_1, s_2, \dots, s_{n-1}\}$ of generators. We list some of nice properties of the Schubert polynomials $\mathfrak{S}_w(X_n)$ according to [8].

(0) $\mathfrak{S}_w(X_n)$ is homogeneous of degree $\ell(w)$, $\mathfrak{S}_e(X_n) = 1$.

(1) (Compatibility conditions)

$$\partial_i^{(x)} \mathfrak{S}_w(X_n) = \begin{cases} \mathfrak{S}_{ws_i}(X_n) & \text{if } \ell(ws_i) = \ell(w) - 1\,, \\ 0 & \text{otherwise} \end{cases}$$

where $\partial_i^{(x)} f = \dfrac{f - s_i(f)}{x_i - x_{i+1}}$ is the divided difference operator with respect to x_i and x_{i+1}.

(2) the structural constants for the multiplication of Schubert polynomials $\mathfrak{S}_w(X_n)$, $w \in S_n$, coincide with the triple intersection numbers of Schubert varieties,

(3) $\mathfrak{S}_w(X_n)$ has nonnegative integer coefficients,

(4_w), (4_s) $\mathfrak{S}_w(X_n)$ is weakly and strongly stable i.e. for all $m > n$, we have

$$\mathfrak{S}_w(X_m) = \mathfrak{S}_w(X_n)\,, \text{ where } w \in S_n \subset S_m\,,$$

see Definition 8 in Section 5 below.

A new approach to the theory of type A Schubert polynomials which is based on the study of the type A nilCoxeter algebras, has been initiated by S. Fomin and R. Stanley [10]. The basic idea of that approach is to consider and study the generating function of all Schubert polynomials simultaneously, namely, to treat the following generating function

$$\mathfrak{S}(X_n) = \sum_{w \in S_n} \mathfrak{S}_w(X_n) u_w\,,$$

where u_w denotes the standard linear basis in the type A nilCoxeter algebra $\mathrm{NC_n}$ which is a

[1] We refer the reader to nicely written book [27] for comprehensive exposition of the Schubert polynomials.

$\mathbb{Z}$-algebra with generators $u_1, u_2, \ldots, u_{n-1}$ and relations

$$u_i^2 = 0 (1 \le i \le n-1), u_i u_j = u_j u_i (|i-j| > 1), u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1} (1 \le i \le n-2)\,.$$

We define $u_w = u_{i_1} \cdots u_{i_\ell}$ when $w = s_{i_1} \cdots s_{i_\ell} \in S_n$ is a reduced expression by the transpositions $s_i = (i, i+1)$. An unexpected and deep result discovered in [10] is that in the algebra $\mathrm{NC}_n[x_1, \ldots, x_n] = \mathrm{NC}_n \otimes \mathbb{Z}[x_1, \ldots, x_n]$ the polynomial $\mathfrak{S}(X_n)$ is completely factorizable in the product of linear factors. The basic tool to prove the factorizability property is the usage of the Yang–Baxter relation among the elements $h_i(x) = 1 + x u_i$ in the algebra $\mathrm{NC}_n[x, y]$, namely

$$(1 + x u_i)(1 + (x+y) u_{i+1})(1 + y u_i) = (1 + y u_{i+1})(1 + (x+y) u_i)(1 + x u_{i+1})\,. \quad (1)$$

The main consequence of the Yang–Baxter relation (1) is that the polynomials $A_k(x) = h_{n-1}(x) h_{n-2}(x) \ldots h_k(x)$, commute, namely

$$[A_k(x), A_k(y)] = 0\,.$$

It has been proved in [9] , [10] that

$$\mathfrak{S}(X_n) = \sum_{w \in S_n} \mathfrak{S}_w(X_n) u_w = A_1(x_1) A_2(x_2) \cdots A_{n-1}(x_{n-1})\,.$$

The double Schubert polynomials $\mathfrak{S}_w(X_n, Y_n)$ of type A, which were originally defined by A. Lascoux in [22], are combinatorially defined as follows. For the longest element $w_0 = [n, n-1, \ldots, 1] \in S_n$, it is defined by

$$\mathfrak{S}_{w_0}(X_n, Y_n) := \prod_{i+j \le n} (x_i + y_j)\,.$$

For general $w \in S_n$, it is define using divided difference operator as

$$\mathfrak{S}_w(X_n, Y_n) := \partial^{(x)}_{w^{-1} w_0} \mathfrak{S}_{w_0}(X_n, Y_n)\,.$$

Using nilCoxeter algebra NC_n the generating function $\mathfrak{S}(X_n, Y_n) = \sum_{w \in S_n} \mathfrak{S}_w(X_n, Y_n) u_w$ of double Schubert polynomials can be factored as follows.

$$\mathfrak{S}(X_n, Y_n) = A_{n-1}^{-1}(-y_{n-1}) A_{n-2}^{-1}(-y_{n-2}) \cdots A_1^{-1}(-y_1) A_1(x_1) A_2(x_2) \cdots A_{n-1}(x_{n-1})\,.$$

Later it was noticed by R. Goldin [11] that the double Schubert polynomials represent torus equivariant Schubert classes, cf. Theorem 2.4 in [11]. When $y_1 = y_2 = \cdots = y_n = 0$, the double Schubert polynomial $\mathfrak{S}_w(X_n, Y_n)$ becomes the single Schubert polynomial $\mathfrak{S}_w(X_n)$.

Construction of "good" representatives for the Schubert polynomials corresponding to the flag varieties of classical types B, C, D was initiated by S. Billey and M. Haiman [3] and independently by S. Fomin and A. N. Kirillov [8]. In [8] the authors extended an algebro-combinatorial approach (i.e. using nilCoxeter algebra and Yang-Baxter equations) to a definition and study extending the type A Schubert polynomials to the case of those of types B and

C. But it also works for type D as well. The key tool in a construction of the aforementioned polynomials is a unitary exponential solution to the quantum Yang–Baxter equations ([29]) with values in the nilCoxeter algebras of types B, C, D correspondingly. The exponential solution to the quantum Yang–Baxter equation associated with nilCoxeter algebra $\mathrm{NC}(W)$, (which is a specialization $\beta = 0$ of IdCoxeter algebra $\mathrm{Id}_\beta(W)$ in Definition 1) of Weyl group $W = W(X)$ of root system of type $X := A_{n-1}, B_n, C_n, D_n$, allows to construct a family of elements $R_i(x) \in \mathrm{NC}(R)[x]$ with $i = 1, \ldots, \mathrm{rk}(R)$ such that

$$R_i(x)R_i(y) = R_i(y)R_i(x), i = 1, \ldots, \mathrm{rk}(R)\,.$$

The elements $R_i(x_1), \ldots, R_i(x_\ell)$ with $i = 1, \ldots, \ell := \mathrm{rk}(R)$ are building blocks in the construction of the generating function for all Schubert polynomials corresponding to the flag variety associated with the root system R.

Now in order to ensure the compatibility conditions one needs to specify the action of simple transpositions of the corresponding Weyl group on the ring of polynomials $\mathbb{Q}[x_1, \ldots, x_\ell]$. In [8] and [18] the authors have chosen the natural or *standard* action of the Weyl group on the cohomology ring of the corresponding flag variety G/B. Namely,

$$\begin{aligned}
&s_i(x_i) = x_{i+1}\,, s_i(x_{i+1}) = x_i\,, s_i(x_j) = x_j \text{ if } j \neq i,\ i+1 \text{ (type } A)\,,\\
&s_0(x_1) = -x_1\,, s_0(x_i) = x_i \text{ if } i > 1 \text{ (types } B, C)\,,\\
&s_{\hat{1}}(x_1) = -x_2\,, s_{\hat{1}}(x_2) = -x_1\,, s_{\hat{1}}(x_i) = x_i \text{ if } i > 2 \text{ (type } D)\,.
\end{aligned}$$

Based on these choice of the action of the simple transpositions, the divided difference operators are defined uniquely. As was remarked in [8], it is easy to see that for root systems of types B, C (and D) it is impossible to find "good" representatives for the Schubert classes which satisfy the properties (0), (1), (2), (3) listed above. Nevertheless in [8] the authors introduce the so called Schubert polynomials of the first kind with nice combinatorial properties including those (0), (2), (3), (4_w), and therefore suitable for computation of the triple intersection numbers for Schubert varieties of classical type, the main Problem of the Schubert Calculus, see [8] for details.

In [3] the authors defined certain action of Weyl group on the ring of supersymmetric functions of infinite number of variables $\Gamma = (\mathbb{Z}[x_1, x_2, \ldots])^{SS}$ and define another family of Schubert polynomials, where SS means supersymmetric (for detail see § 4).

In [15] Ikeda, Mihalcea and the second author defined and studied the double Schubert polynomials of type B, C, D using localization map of equivariant cohomology. For K-theory there is analogous map and the image has the so called Goresky-Kottwitz-MacPherson property [12]. As mentioned for the case of Grassmannians in [17], the Schubert classes can be characterized by recurrence relations. These are essentially done already by Kostant-Kumar [20] and used in [21], see § 6 for more details.

In conclusion in the present paper we used an algebro-combinatorial construction of [7] to extend the algebro-geometric [17] constructions of the double Schubert polynomials of types B, C, D, to get double Grothendieck polynomials which represent the Schubert classes in the K-theory rings of the types B, C and D full flag varieties. Some of these polynomials also appear in more geometric context of connective K-theory of (non-maximal) Grassmannians in [14] , where the parameter $\beta = a_{1,1}$ has its meaning.

The formulas (3) and (6) obtained in section 8 lead to combinatorial descriptions of polynomials in questions in terms of either EYD, or compatible sequences, or set-valued tableaux [5]. We expect that after a certain change of idCoxeter algebra and replacing $A \oplus B$ in our formulas (3) in Lemma 9 and (6) in Lemma 10 by $F(A, B)$, where $F(x, y)$ stands for the universal formal group law, we come to formal power series which have a suitable interpretations in the theory of algebraic cobordism [26] of flag varieties.

1.1. Organization. In Section 2 we summarize the notations and definitions needed. In Section 3 we describe some basic properties of idCoxeter algebra $\mathrm{Id}_\beta(W)$. In Section 4 we define β-supersymmetric functions and K-theoretic Schur P- Q-functions and K-theoretic Stanley symmetric functions. In Section 5 we introduce the double Grothendieck polynomials of classical types and some fundamental properties. In Section 6 we give a geometric interpretation of the double Grothendieck polynomials using (algebraic) localization map. In Section 7 we introduce adjoint polynomials which are dual to the double Grothendieck polynomials. Finally in Section 8 we give two types of combinatorial formula for double Grothendieck polynomials using compatible sequences and excited Young diagrams.

2. Definitions and Notations

In this paper $W = W(X)$ is a Weyl group of type $X = A, B, C, D$. $I = I^X$ is the set of simple reflections in $W(X)$. We index the simple reflections by the same notation as in [15] §3.2. In particular, for type B and C, s_0 corresponds to the left most node of the Dynkin diagram with the relations $(s_0 s_1)^4 = 1$ and $(s_0 s_i)^2 = 1$ for $i \geq 2$. For type D, $s_{\hat{1}} := s_0 s_1 s_0$ and we consider $W(D)$ as the subgroup of $W(B)$ generated by $s_{\hat{1}}, s_1, \ldots$. For $X = B$ and C, the Weyl group $W(X_n) = \langle s_0, s_1, \ldots, s_{n-1}\rangle$ is the hyperoctahedral group and the elements are realized as signed permutations. (cf. [15] §3.3.) (Maximal) Grassmannian elements of type B_n and C_n are minimal length coset representatives of $W(B_n)/S_n = W(C_n)/S_n$ where $S_n = \langle s_1, \ldots, s_{n-1}\rangle$ is the parabolic subgroup corresponding to the index 0. For a Grassmannian element $w = [\bar{i}_1, \ldots, \bar{i}_\ell, i_{\ell+1}, \ldots, i_n]$ of type $X = B, C$, where $1 \leq i_1, \ldots, i_n \leq n$ are distinct integers with $i_1 > \cdots > i_\ell$ and $i_{\ell+1} < \cdots < i_n$, we associate strict partition $\lambda_X(w) = (i_1, \ldots, i_\ell)$. (Maximal) Grassmannian elements of type D_n are minimal length coset representatives of $W(D_n)/S_n$ where $S_n = \langle s_1, \ldots, s_{n-1}\rangle$ is the parabolic subgroup corresponding to the index $\hat{1}$. For a Grassmannian element $w = [\bar{i}_1, \ldots, \bar{i}_\ell, i_{\ell+1}, \ldots, i_n]$ of type D, where $1 \leq i_1, \ldots, i_n \leq n$ are distinct integers with $i_1 > \cdots > i_\ell$ and $i_{\ell+1} < \cdots < i_n$, we associate strict partition $\lambda_D(w) = (i_1 - 1, \ldots, i_\ell - 1)$. Note that for type D case ℓ is always even and we can omit $i_\ell - 1 = 0$ when $i_\ell = 1$.

We use Bruhat order $w \leq v$ on $W(X)$. Then it is known that for (maximal) Grassmannian elements $w, v \in W(X)$, we have

$$w \leq v \iff \lambda_X(w) \subset \lambda_X(v)\,.$$

The set of roots Δ_X is the set of orbits of simple roots.

Following [7], we prepare some notations. Let β be an indeterminate. We define operations $\oplus$ and $\ominus$ as follows.

$$x \oplus y := x + y + \beta xy, x \ominus y := (x - y)/(1 + \beta y) .$$

We also use the convention that

$$\bar{x} := \ominus x = -\frac{x}{1 + \beta x} .$$

Then we have $x \oplus \bar{x} = 0$. For a Weyl group W with the set I of Coxeter generators, we define idCoxeter algebra as follows.

DEFINITION 1 (IdCoxeter algebra). IdCoxeter algebra $\mathrm{Id}_\beta(W)$ for W is a $\mathbb{Z}[\beta]$ algebra with generators u_i for each $s_i \in I$ and relations as follows.

$$u_i^2 = \beta u_i ,$$

$$\underbrace{u_i u_j u_i \cdots}_{m_{i,j} \text{ terms}} = \underbrace{u_j u_i u_j \cdots}_{m_{i,j} \text{ terms}} \quad \text{if } m_{i,j} \text{ is the order of } s_i s_j .$$

By the braid relation we can define $u_w = u_{i_1} \cdots u_{i_\ell}$ where $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced expression of $w \in W$. Then $\{u_w\}_{w \in W}$ form a $\mathbb{Z}[\beta]$ basis of $\mathrm{Id}_\beta(W)$.

For each $s_i \in I^X$, we define divided-difference operator $\pi_i^{(a)}$ and $\psi_i^{(a)}$ with respect to the variables $a = (a_1, a_2, \ldots)$ as follows. Assume that $R \supset \mathbb{Z}[\beta]$ is a ring with a group action of $W(X)$. We define the action of $W(X)$ on $R[a, \bar{a}] := R[a_1, a_2, \ldots, \bar{a}_1, \bar{a}_2, \ldots]$ as follows.

DEFINITION 2. The action of $s_i^{(a)} \in I^X$ on the variables $a_1, a_2, \ldots, \bar{a}_1, \bar{a}_2, \ldots$.

- If $i \geq 1$, $s_i^{(a)}(a_i) = a_{i+1}$, $s_i^{(a)}(a_{i+1}) = a_i$, $s_i^{(a)}(\bar{a}_i) = \bar{a}_{i+1}$, $s_i^{(a)}(\bar{a}_{i+1}) = \bar{a}_i$, and $s_i^{(a)}(a_k) = a_k$, $s_i(\bar{a}_k) = \bar{a}_k$ for $k \neq i, i+1$.
- $s_0^{(a)}(a_1) = \bar{a}_1$, $s_0^{(a)}(\bar{a}_1) = a_1$, and $s_0^{(a)}(a_k) = a_k$, $s_0^{(a)}(\bar{a}_k) = \bar{a}_k$ for $k > 1$.
- $s_{\hat{1}}^{(a)}(a_1) = \bar{a}_2$, $s_{\hat{1}}^{(a)}(a_2) = \bar{a}_1$, $s_{\hat{1}}^{(a)}(\bar{a}_1) = a_2$, $s_{\hat{1}}^{(a)}(\bar{a}_2) = a_1$, and $s_{\hat{1}}^{(a)}(a_k) = a_k$, $s_{\hat{1}}^{(a)}(\bar{a}_k) = \bar{a}_k$ for $k > 2$.

We write the induced action on $R[a, \bar{a}]$ by $s_i^{(a)}$. Divided difference operators $\pi_i^{(a)}$ and $\psi_i^{(a)}$ are defined as follows. For $f \in R[a, \bar{a}] = R[a_1, a_2, \ldots, \bar{a}_1, \bar{a}_2, \ldots]$,

$$\pi_i^{(a)}(f) := \frac{f - (1 + \beta\alpha_i(a)) s_i^{(a)}(f)}{\alpha_i(a)} \text{ and } \psi_i^{(a)} := \pi_i^{(a)} + \beta ,$$

where $\alpha_i(a)$ is the element in $\mathbb{Z}[\beta][a, \bar{a}]$ corresponding to the root α_i, i.e. $\alpha_i(a) = a_i \oplus \bar{a}_{i+1}$ for $i = 1, 2, \ldots$, $\alpha_0^B(a) = \bar{a}_1$, $\alpha_0^C(a) = \bar{a}_1 \oplus \bar{a}_1$ and $\alpha_{\hat{1}}(a) = \bar{a}_1 \oplus \bar{a}_2$.

$\left(\text{Formally we can think as } \alpha_i(a) = \dfrac{e^{\beta\alpha_i} - 1}{\beta}. \text{ cf. [7].}\right)$

PROPOSITION 1. *We have the following relations of operators*: (*we write* $\pi = \pi^{(a)}$, $\psi = \psi^{(a)}$ *for short.*)

$$\pi_i^2 = -\beta\pi_i\,,\ \psi_i^2 = \beta\psi_i \textit{ for all } s_i \in I^X\,,$$

$$\underbrace{\pi_i\pi_j\pi_i\cdots}_{m_{i,j}\text{ terms}} = \underbrace{\pi_j\pi_i\pi_j\cdots}_{m_{i,j}\text{ terms}},\ \underbrace{\psi_i\psi_j\psi_i\cdots}_{m_{i,j}\text{ terms}} = \underbrace{\psi_j\psi_i\psi_j\cdots}_{m_{i,j}\text{ terms}}$$

if $m_{i,j}$ *is the order of* s_is_j .

PROOF. We can check the relations by direct calculations. □

The explicit form of $\psi_i^{(a)}$ is as follows,

$$\psi_i^{(a)}(f) = \tfrac{s_i^{(a)}f - f}{a_{i+1}\ominus a_i} \text{ for } i \geq 1\ ,$$
$$\psi_{0,B}^{(a)}(f) = \tfrac{s_0^{(a)}f - f}{a_1}\ ,\ \psi_{0,C}^{(a)}(f) = \tfrac{s_0^{(a)}f - f}{a_1\oplus a_1} \text{ and } \psi_{\hat{1}}^{(a)}(f) = \tfrac{s_{\hat{1}}^{(a)}f - f}{a_1\oplus a_2}\,.$$

Similarly we can define divided difference operators $\pi_i^{(b)}$ and $\psi_i^{(b)}$ corresponding to the variables $b_1, b_2, \ldots$ using $s_i^{(b)}$ and $\alpha_i(b)$.

3. Basic Properties

We always assume that all the variables x, y or a, b commute with u_i and consider in a suitable extension of the ring of coefficients in $\mathrm{Id}_\beta(W)$. Let $h_i(x) := 1 + xu_i$. Then it follows that $h_i(x)h_i(y) = h_i(x \oplus y)$ and $h_i(x)$ is invertible with $h_i(x)^{-1} = h_i(\bar{x})$.

LEMMA 1 (Yang-Baxter relations [9]). *The following equalities hold.*

$$\begin{array}{rclr} h_i(x)h_j(y) & = & h_j(y)h_i(x) & m_{i,j} = 2 \\ h_i(x)h_j(x\oplus y)h_i(y) & = & h_j(y)h_i(x\oplus y)h_j(x) & m_{i,j} = 3 \\ h_i(x)h_j(x\oplus y)h_i(x\oplus y\oplus y)h_j(y) & = & h_j(y)h_i(x\oplus y\oplus y)h_j(x\oplus y)h_i(x) & m_{i,j} = 4 \end{array}$$

These can be proved by direct calculations. (We omit the case of $m_{i,j} = 6$ which we don't need.)

DEFINITION 3. We define the following elements in $\mathrm{Id}_\beta(W)[x]$ for $W = W(X)$ with $X = A, B, C, D$.

$$
\begin{aligned}
A_i^{(n)}(x) &:= \prod_{k=n-1}^{i} h_k(x) = h_{n-1}(x)h_{n-2}(x)\cdots h_i(x).\ \ (i=1,2,\ldots,n-1)\,,\\
F_n^B(x) &:= A_1^{(n)}(x)\ h_0(x)\ A_1^{(n)}(\bar{x})^{-1}\\
&= h_{n-1}(x)h_{n-2}(x)\cdots h_1(x)h_0(x)h_1(x)\cdots h_{n-2}(x)h_{n-1}(x)\,,\\
F_n^C(x) &:= A_1^{(n)}(x)\ h_0(x)^2\ A_1^{(n)}(\bar{x})^{-1}\\
&= h_{n-1}(x)h_{n-2}(x)\cdots h_1(x)h_0(x)^2h_1(x)\cdots h_{n-2}(x)h_{n-1}(x)\,,\\
F_n^D(x) &:= A_2^{(n)}(x)\ h_{\hat{1}}(x)h_1(x)\ A_2^{(n)}(\bar{x})^{-1}\\
&= h_{n-1}(x)\cdots h_2(x)h_1(x)h_{\hat{1}}(x)h_2(x)\cdots h_{n-1}(x)\,.
\end{aligned}
$$

For $1 \le i \le j$, we abbreviate

$$[i,j]_x := h_i(x)h_{i+1}(x)\cdots h_j(x) \text{ and } [j,i]_x := h_j(x)h_{j-1}(x)\cdots h_i(x)\,.$$

LEMMA 2. *For $1 \le i \le j$, we have $[i,j]_x[j,i]_y = [j,i]_y[i,j]_x$.*

PROOF. We will prove by induction on $j-i$. When $j-i=0$, i.e. $i=j$, it is trivial. When $j-i=1$, $[i,i+1]_x[i+1,i]_y = [i+1,i]_y[i,i+1]_x$ by Yang-Baxter relation. For $j-i=k\ge 2$, we can use induction hypothesis and Yang-Baxter relation again to get $[i,j]_x[j,i]_y = [i]_x[i+1,j]_x[j,i+1]_y[i]_y = [i]_x[j,i+2]_y[i+1]_y[i+1]_x[i+2,j]_x[i]_y = [j,i+2]_y[i]_x[i+1]_x[i+1]_y[i]_y[i+2,j]_x = [j,i+2]_y[i+1,i]_y[i,i+1]_x[i+2,j]_x = [j,i]_y[i,j]_x$. □

LEMMA 3. *We have the following equalities.*
(1) $A_i^{(n)}(x)A_i^{(n)}(y) = A_i^{(n)}(y)A_i^{(n)}(x)$,
(2) $F_n^X(x)F_n^X(y) = F_n^X(y)F_n^X(x)$ for $X = B, C, D$,
(3) $F_n^X(x)F_n^X(\bar{x}) = 1$.

PROOF. (1) As $A_i^{(n)}(x) = [n-1,i]_x$ and $A_i^{(n)}(y)^{-1} = [i,n-1]_{\bar{y}}$, it follows from Lemma 2.

(2) Using (1) and Yang-Baxter relations again, we can show the equalities as follows. For $X=B$, we have

$$
\begin{aligned}
&F_n^B(x)F_n^B(y)\\
&= [n-1,1]_x[0]_x[1,n-1]_x[n-1,1]_y[0]_y[1,n-1]_y\\
&= [n-1,1]_x[0]_x[n-1,1]_y[1,n-1]_x[0]_y[1,n-1]_y\\
&= [n-1,1]_x[n-1,2]_y[0]_x[1]_y[1]_x[0]_y[2,n-1]_x[1,n-1]_y\\
&= [n-1,1]_x[n-1,1]_y[1]_{\bar{y}}[0]_x[1]_y[1]_x[0]_y[1]_{\bar{x}}[1,n-1]_x[1,n-1]_y\\
&= [n-1,1]_y[n-1,1]_x[1]_{\bar{y}}[1]_{\bar{x}}[1]_x[0]_x[1]_x[1]_y[0]_y[1]_y[1]_{\bar{y}}[1]_{\bar{x}}[1,n-1]_y[1,n-1]_x\\
&= [n-1,1]_y[n-1,1]_x[1]_{\bar{y}}[1]_{\bar{x}}[1]_y[0]_y[1]_y[1]_x[0]_x[1]_x[1]_{\bar{y}}[1]_{\bar{x}}[1,n-1]_y[1,n-1]_x\\
&= [n-1,1]_y[n-1,2]_x[0]_y[1]_y[1]_x[0]_x[2,n-1]_y[1,n-1]_x\\
&= [n-1,1]_y[0]_y[n-1,2]_x[1]_x[1]_y[2,n-1]_y[0]_x[1,n-1]_x\\
&= [n-1,1]_y[0]_y[1,n-1]_y[n-1,1]_x[0]_x[1,n-1]_x\\
&= F_n^B(y)F_n^B(x)
\end{aligned}
$$

Similar arguments with appropriate modifications will give $X = C, D$ cases. The essential equalities to be used are

$$h_1(x \oplus \bar{y})h_0(x \oplus x)h_1(x \oplus y)h_0(y \oplus y)h_1(\bar{x} \oplus y) = h_0(y \oplus y)h_1(x \oplus y)h_0(x \oplus x)$$

and

$$h_2(x \oplus \bar{y})h_1(x)h_{\hat{1}}(x)h_2(x \oplus y)h_1(y)h_{\hat{1}}(y)h_2(\bar{x} \oplus y) = h_1(y)h_{\hat{1}}(y)h_2(x \oplus y)h_1(x)h_{\hat{1}}(x).$$

(3) This follows essentially by the relation $h_i(x)h_i(\bar{x}) = 1$. □

4. β-supersymmetric functions

DEFINITION 4. β-supersymmetric function with respect to variables $x_1, x_2, \ldots, x_n$ is a symmetric function $f(x_1, x_2, \ldots, x_n)$ on $x_1, x_2, \ldots, x_n$ which satisfies the following cancellation property:

$$f(t, \bar{t}, x_3, \ldots, x_n) = f(0, 0, x_3, \ldots, x_n) \text{ for every } t\,.$$

REMARK 1. The β-supersymmetric property is translated to usual supersymmetricity by the change of variables x_i to $\frac{e^{\beta x_i}-1}{\beta}$. If $\beta = 0$, then the β-supersymmetric property becomes usual supersymmetric property, i.e. $f(t, -t, x_3, \ldots, x_n) = f(0, 0, x_3, \ldots, x_n)$ for every t.

Let $SS_\beta(x_1, \ldots, x_n) := \{f \in \mathbb{Z}[\beta][x_1, \ldots, x_n] \mid f : \beta\text{-supersymmetric}\}$ and set $\mathbf{SS}_\beta(x) := \varprojlim_n SS_\beta(x_1, \ldots, x_n)$. Then $\mathbf{SS}_\beta(x)$ is the ring of β-supersymmetric functions. (It is denoted as $G\Gamma$ in [17].) If $\beta = 0$ this becomes the ring Γ' in [15].

4.1. K-theoretic Schur functions $GP_\lambda(x), GQ_\lambda(x)$.

In [17] β-supersymmetric functions $GP_\lambda(x), GQ_\lambda(x)$ are defined. Let $b_1, b_2, \ldots$ be indeterminates, and set $[x|b]^k = (x \oplus b_1) \cdots (x \oplus b_k)$ and $[[x|b]]^k = (x \oplus x)(x \oplus b_1) \cdots (x \oplus b_{k-1})$.

Let SP_n be the set of strict partitions of length at most n. i.e. $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0)$ is an integer sequence such that $r \leq n$. We also set $SP = \bigcup_n SP_n$.

DEFINITION 5 (Ikeda-Naruse [17]). For a strict partition $\lambda \in SP_n$,

$$GP_\lambda(x_1, \ldots, x_n|b) := \frac{1}{(n-r)!} \sum_{w \in S_n} w \left(\prod_{1 \leq i \leq r} \left([x_i|b]^{\lambda_i} \prod_{i<j\leq n} \frac{x_i \oplus x_j}{x_i \ominus x_j} \right) \right),$$

$$GQ_\lambda(x_1, \ldots, x_n|b) := \frac{1}{(n-r)!} \sum_{w \in S_n} w \left(\prod_{1 \leq i \leq r} \left([[x_i|b]]^{\lambda_i} \prod_{i<j\leq n} \frac{x_i \oplus x_j}{x_i \ominus x_j} \right) \right),$$

where $w \in S_n$ acts $x_1, \ldots, x_n$ as permutation of indices.

We also put

$$\begin{aligned}
GP_\lambda(x_1,\dots,x_n) &:= GP_\lambda(x_1,\dots,x_n|0)\,,\\
GQ_\lambda(x_1,\dots,x_n) &:= GQ_\lambda(x_1,\dots,x_n|0)\,,\\
GP_\lambda(x) &:= \varprojlim_n GP_\lambda(x_1,\dots,x_n)\,,\\
GQ_\lambda(x) &:= \varprojlim_n GQ_\lambda(x_1,\dots,x_n)\,,\\
GP_\lambda(x|b) &:= \varprojlim_n GP_\lambda(x_1,\dots,x_{2n}|b)\,, \text{ and}\\
GQ_\lambda(x|b) &:= \varprojlim_n GQ_\lambda(x_1,\dots,x_n|b)\,.
\end{aligned}$$

N.B. $GP_\lambda(x_1,\dots,x_n|b)$ has only mod 2 stability (cf. [17] Remark 3.1), and we define $GP_\lambda(x|b)$ to be the even limit as in [17].

EXAMPLE 1. The followings are some examples of GP, GQ.

$$GP_1(x_1,\dots,x_n) = x_1 \oplus x_2 \oplus \cdots \oplus x_n\,.$$
$$GQ_1(x_1,\dots,x_n) = (x_1 \oplus x_1) \oplus (x_2 \oplus x_2) \oplus \cdots \oplus (x_n \oplus x_n)\,.$$

LEMMA 4 ([17] Theorem 3.1, Proposition 3.2.). *The followings hold.*

(1) $GP_\lambda(x_1,\dots,x_n)$ *and* $GQ_\lambda(x_1,\dots,x_n)$ *are* β*-supersymmetric functions.*

(2) $\{GP_\lambda(x_1,\dots,x_n)\}_{\lambda\in SP_n}$ *forms a basis of* $SS_\beta(x_1,\dots,x_n)$ *over* $\mathbb{Z}[\beta]$.

(3) *Let* $SS^C_\beta(x_1,\dots,x_n)$ *be the* $\mathbb{Z}[\beta]$*-subspace of* $SS_\beta(x_1,\dots,x_n)$ *spanned by* $GQ_\lambda(x_1,\dots,x_n)(\lambda\in SP_n)$. *Then* $\{GQ_\lambda(x_1,\dots,x_n)\}_{\lambda\in SP_n}$ *forms a basis of* $SS^C_\beta(x_1,\dots,x_n)$ *over* $\mathbb{Z}[\beta]$.

REMARK 2. We remark that the definition of β-supersymmetry and the polynomials GP_λ, GQ_λ can be generalized in more general setting such as algebraic cobordism [26], cf. [28].

According to [17] (6.5), we define an action of $W(X)$ as follows.

DEFINITION 6. The action of Weyl group $W(X_n)$ on $SS_\beta(x_1,\dots,x_n)\otimes_{\mathbb{Z}[\beta]}\mathbb{Z}[\beta][a,\bar{a}]\otimes_{\mathbb{Z}[\beta]}\mathbb{Z}[\beta][b,\bar{b}]$ is derived from the action as follows (together with Definition 2). For $f(x)\in\mathbf{SS}_\beta(x)$,

$$s_i^{(a)}f(x) = f(x) = s_i^{(b)}f(x) \text{ for } i\ge 1\,,$$
$$s_0^{(a)}f(x) = f(a_1,x)\,,\ s_0^{(b)}f(x) = f(b_1,x)\,,$$
$$s_{\hat{1}}^{(a)}f(x) = f(a_1,a_2,x)\,,\ s_{\hat{1}}^{(b)}f(x) = f(b_1,b_2,x)\,.$$

These actions can be clarified by the change of variables explained in the second definition below (cf.§5.2 Remark 4).

LEMMA 5 ([17] Theorem 6.1 and Theorem 7.1). *$GP_\lambda(x|b)$ and $GQ_\lambda(x|b)$ are characterized by (left) divided difference relations and initial conditions. i.e. For a maximal Grassmannian element $w \in W(X)/S_\infty$ and $s_i \in I^X$,*

$$\pi_i^{(b)} GX_{\lambda(w)}(x|b) = \begin{cases} GX_{\lambda(s_i w)}(x|b) & \text{if} \quad s_i w < w \\ -\beta\, GX_{\lambda(w)}(x|b) & \text{if} \quad s_i w \geq w \end{cases}$$

and

$$GX_\emptyset(x|b) = 1\,,$$

where $GB_\lambda(x|b) = GP_\lambda(x|0,b)$, $GC_\lambda(x|b) = GQ_\lambda(x|b)$, $GD_\lambda(x|b) = GP_\lambda(x|b)$.

4.2. K-theoretic Stanley symmetric functions $\mathcal{F}_w^X(x)$, $X = B, C, D$

DEFINITION 7. For $X = B, C, D$, we define

$$\mathbf{F}_n^X(x) := F_n^X(x_1, \dots, x_n) = \prod_{i=1}^n F_n^X(x_i) \text{ and } \mathbf{F}_\infty^X(x) := \varprojlim_n \mathbf{F}_n^X(x)\,.$$

Using these we define $\mathcal{F}_w^X(x_1, \dots, x_n)$ and $\mathcal{F}_w^X(x)$ by the following expansions.

$$\mathbf{F}_n^X(x) = \sum_{w \in W(X_n)} \mathcal{F}_w^X(x_1, \dots, x_n) u_w\ ,\ \mathbf{F}_\infty^X(x) = \sum_{w \in W(X)} \mathcal{F}_w^X(x) u_w\,.$$

By definition $\mathcal{F}_w^X(x_1, \dots, x_n)$ are weakly stable but not strongly stable (cf. Definition 8). $\mathcal{F}_w^X(x)$ is a K-theoretic analogue of Stanley symmetric function of type $X = B, C, D$. (cf. [3])

LEMMA 6. *For each $w \in W(X_n)$, $\mathcal{F}_w^X(x_1, x_2, \dots, x_n)$ is a β-supersymmetric function.*

PROOF. This follows from Lemma 3 (2) and (3). □

LEMMA 7. (0) *For $X = B, C, D$, we have*

$$\mathcal{F}_{w^{-1}}^X(x_1, x_2, \dots, x_n) = \mathcal{F}_w^X(x_1, x_2, \dots, x_n)\,.$$

(1) *For $X = B, C, D$, $\mathcal{F}_w^X(x_1, x_2, \dots, x_n)$ can be expanded in $GP_\lambda(x_1, x_2, \dots, x_n)$ with coefficients in $\mathbb{Z}[\beta]$.*

(2) *For a (maximal) Grassmannian element $w \in W(X_n)$,*

$$\mathcal{F}_w^B(x_1, x_2, \dots, x_n) = GP_{\lambda_B(w)}(x_1, x_2, \dots, x_n)\,,$$

$$\mathcal{F}_w^C(x_1, x_2, \dots, x_n) = GQ_{\lambda_C(w)}(x_1, x_2, \dots, x_n)\,,$$

$$\mathcal{F}_w^D(x_1, x_2, \dots, x_n) = GP_{\lambda_D(w)}(x_1, x_2, \dots, x_n)\,.$$

PROOF. (0) This follows from the symmetry of $\mathbf{F}_n^X(x)$, i.e. if a coefficient of u_w comes from the product $u_{i_1}u_{i_2}\cdots u_{i_k}$ in $F_n^X(x_1)\cdots F_n^X(x_n)$, then the same coefficient appears in the product $u_{i_k}u_{i_{k-1}}\cdots u_{i_1}$ in $F_n^X(x_n)\cdots F_n^X(x_1)$, by picking up the symmetric positions.

(1) This follows from Lemma 4 (2), because $\mathbf{F}_n^X(x)$ is β-supersymmetric.

(2) This is Proposition 5 below with $b = 0$. □

REMARK 3. We state conjecture that the coefficients in the expansion of (1) are positive, i.e. the coefficients will be polynomials in β with nonnegative integers. This will be a consequence of K-theory analogue of "transition equation" for type B, C, D. (cf. [15])

EXAMPLE 2. Belows are some examples of $\mathcal{F}_w^X(x_1,\ldots,x_n)$.

$\mathcal{F}_{s_0}^B(x_1,\ldots,x_n) = GP_1(x_1,\ldots,x_n)$.

$\mathcal{F}_{s_0}^C(x_1,\ldots,x_n) = GQ_1(x_1,\ldots,x_n) = 2GP_1(x_1,\ldots,x_n) + \beta GP_2(x_1,\ldots,x_n)$.

$\mathcal{F}_{s_{\hat{1}}}^D(x_1,\ldots,x_n) = GP_1(x_1,\ldots,x_n)$.

5. Main results

In this section we define the main object of this paper, the double Grothendieck polynomials of classical types $\mathcal{G}_w^X(a,b;x)$, $(X = B, C, D)$, and show some of their fundamental properties.

First we recall the type A Grothendieck polynomials $\mathcal{G}_w^A(a)$ cf. [7]. These polynomials satisfy the strong stability in the following sense.

DEFINITION 8. Fix an element $w \in W(X)$ $(X = A, B, C, D)$. Suppose that for each n such that $w \in W(X_n)$ we have given a polynomial $f_w^{(n)} \in R[x_1, x_2, \ldots, x_n]$. Then

(1) $\left\{f_w^{(n)}\right\}_{n\geq 1}$ is called weakly stable (with respect to x) if for all $m > n$ we have $f_w^{(m)}|_{x_{n+1}=\cdots=x_m=0} = f_w^{(n)}$.

(2) $\left\{f_w^{(n)}\right\}_{n\geq 1}$ is called strongly stable (with respect to x) if for all $m > n$ we have $f_w^{(m)} = f_w^{(n)}$.

We set $G_{A_{n-1}}(a_1,\ldots,a_{n-1}) := A_1^{(n)}(a_1)A_2^{(n)}(a_2)\cdots A_{n-1}^{(n)}(a_{n-1})$. Then for $w \in S_n$, we define $\mathcal{G}_w^{A_{n-1}}(a)$ by the following equation.

$$G_{A_{n-1}}(a_1,\ldots,a_{n-1}) = \sum_{w\in S_n} \mathcal{G}_w^{A_{n-1}}(a)u_w\,.$$

Furthermore, we can consider $G_A(a) := \varprojlim_n G_{A_{n-1}}(a_1,\ldots,a_{n-1})$ and get $\mathcal{G}_w^A(a)$ by

$$G_A(a) = \sum_{w\in S_\infty} \mathcal{G}_w^A(a)u_w\,.$$

It is easy to see that for $w \in S_n$ and $m > n$, we have $\mathcal{G}_w^{A_{m-1}}(a) = \mathcal{G}_w^{A_{n-1}}(a) = \mathcal{G}_w^{A}(a)$, therefore the type A Grothendieck polynomials are strongly stable. Recall that the type A_{n-1} double Grothendieck polynomials $\mathcal{G}_w^{A_{n-1}}(a,b)$ are defined as follows.

$$G_{A_{n-1}}(\bar{b}_1,\dots,\bar{b}_{n-1})^{-1}G_{A_{n-1}}(a_1,\dots,a_{n-1}) = \sum_{w\in S_n}\mathcal{G}_w^{A_{n-1}}(a,b)u_w =: G_{n-1}^{A}(a,b)\,.$$

LEMMA 8. *We have the following equation.*

$$\begin{aligned}
& G_{A_{n-1}}(\bar{b}_1,\dots,\bar{b}_{n-1})^{-1}G_{A_{n-1}}(a_1,\dots,a_{n-1})\\
= \; & h_{n-1}(a_1\oplus b_{n-1})\\
& h_{n-2}(a_1\oplus b_{n-2})h_{n-1}(a_2\oplus b_{n-2})\\
& \qquad\vdots\\
& h_2(a_1\oplus b_2)h_3(a_2\oplus b_2)\cdots h_{n-1}(a_{n-2}\oplus b_2)\\
& h_1(a_1\oplus b_1)h_2(a_2\oplus b_1)\cdots h_{n-2}(a_{n-2}\oplus b_1)h_{n-1}(a_{n-1}\oplus b_1)\\
\\
= \; & h_{n-1}(a_1\oplus b_{n-1})h_{n-2}(a_1\oplus b_{n-2})\cdots h_2(a_1\oplus b_2)h_1(a_1\oplus b_1)\\
& h_{n-1}(a_2\oplus b_{n-1})h_{n-2}(a_2\oplus b_{n-2})\cdots h_2(a_2\oplus b_1)\\
& \qquad\vdots\\
& h_{n-1}(a_{n-2}\oplus b_2)h_{n-2}(a_{n-2}\oplus b_1)\\
& h_{n-1}(a_{n-1}\oplus b_1)
\end{aligned}$$

PROOF. We can use Yang-Baxter relations many times to transform the given product. □

5.1. The first definition

DEFINITION 9. We define for $X = B, C$ or D,

$$G_n^X(a,b;x) := G_{A_{n-1}}(\bar{b}_1,\dots,\bar{b}_{n-1})^{-1}\mathbf{F}_n^X(x)G_{A_{n-1}}(a_1,\dots,a_{n-1})$$

and define $\mathcal{G}_{n,w}^X(a,b;x)$ as the coefficient of u_w.

$$G_n^X(a,b;x) = \sum_{w\in W(X_n)}\mathcal{G}_{n,w}^X(a,b;x)u_w\,.$$

Furthermore, we define $\mathcal{G}_w^X(a,b;x)$ by

$$G_A(\bar{b})^{-1}\mathbf{F}_\infty^X(x)G_A(a) = \sum_{w\in W(X)}\mathcal{G}_w^X(a,b;x)u_w\,.$$

By the definition we can see $\mathcal{G}_{n,w}^X(a,b;x) \in SS_\beta(x_1,\dots,x_n)[a_1,\dots,a_{n-1},b_1,\dots,b_{n-1}]$ and $\mathcal{G}_w^X(a,b;x) \in \mathbf{SS}_\beta(x)[a_1,\dots,a_{n-1},b_1,\dots,b_{n-1}]$ for $w \in W(X_n)$, i.e. a polynomial in $a_1,\dots,a_{n-1},b_1,\dots,b_{n-1}$ with coefficients in $SS_\beta(x_1,\dots,x_n)$ or $\mathbf{SS}_\beta(x)$. When we set $\beta = 0$, $a_i = z_i$ and $b_i = -t_i$, $\mathcal{G}_w^X(a,b;x)$ becomes the double Schubert polynomial of classical type defined in [15]. The main features of these polynomials are summarized in the following.

THEOREM 1. *For $X = B, C, D$, $\mathcal{G}^X_w(a, b; x)$ satisfies the K-theoretic (double) version of the properties* (0), (1), (2), (3), (4_s) *listed in Introduction.*

PROOF. (0) follows by the definition. (We set $\deg a_i = \deg b_i = \deg x_i = 1, \deg \beta = -1$.)

(1) K-theoretic divided difference compatibility follows by Corollary 1 below.

(2) follows by Theorem 2 in the next section.

(3) follows by the definition. (Here nonnegativity means that in $\mathcal{G}^X_w(a, b; x)$ each coefficient of monomials in variables a, b, x is a polynomial in β with nonnegative integer coefficients.) For explicit combinatorial formulas see Theorem 5 and 6 in section 8.

(4_s) follows by Proposition 4 below. □

We will write $w \star v = z$ (called Demazure product) if $u_w u_v = \beta^{\ell(w)+\ell(v)-\ell(z)} u_z$. It is associative and $w \star v = wv$ when $\ell(w) + \ell(v) = \ell(wv)$.

PROPOSITION 2. *For $X = B, C, D$ and $w \in W(X)$, we have*

$$\mathcal{G}^X_w(a, b; x) = \sum_{(v_1,u,v_2)\in R(w)} \mathcal{G}^A_{v_1^{-1}}(b)\mathcal{F}^X_u(x)\mathcal{G}^A_{v_2}(a)\,,$$

where $R(w) = \{(v_1, u, v_2) \in S_\infty \times W(X) \times S_\infty \mid v_1 \star u \star v_2 = w\}$.

PROPOSITION 3. *We have*
$\pi^{(a)}_i G^X_n(a, b; x) = G^X_n(a, b; x)(u_i - \beta)$ *and* $\pi^{(b)}_i G^X_n(a, b; x) = (u_i - \beta)G^X_n(a, b; x)$.

PROOF. We will prove $\psi^{(a)}_i G^X_n(a, b; x) = G^X_n(a, b; x)u_i$. Recall the explicit formula of ψ_i after the Proposition 1. $G_{A_{n-1}}(\bar{b})^{-1}$ is invariant for the action of $s^{(a)}_i$, $s_i \in I^X$. For $i > 0$, $\psi^{(a)}_i \mathbf{F}^X_n(x) = \mathbf{F}^X_n(x)$ and $\psi^{(a)}_i G_{A_{n-1}}(a) = G_{A_{n-1}}(a)u_i$ (cf. [7]), therefore

$$\begin{aligned}
\psi^{(a)}_i \mathbf{F}^X_n(x)G_{A_{n-1}}(a) &= \mathbf{F}^X_n(x)G_{A_{n-1}}(a)u_i\,.\\
\psi^{(a)}_{0,B}(\mathbf{F}^B_n(x)G_{A_{n-1}}(a)) &= \frac{\mathbf{F}^B_n(x)F^B_n(a_1)G_{A_{n-1}}(\bar{a}_1,a_2,\ldots,a_{n-1})-\mathbf{F}^B_n(x)G_{A_{n-1}}(a))}{a_1}\\
&= \mathbf{F}^B_n(x)G_{A_{n-1}}(a)u_0\,.\\
\psi^{(a)}_{0,C}(\mathbf{F}^C_n(x)G_{A_{n-1}}(a)) &= \frac{\mathbf{F}^C_n(x)F^C_n(a_1)G_{A_{n-1}}(\bar{a}_1,a_2,\ldots,a_{n-1})-\mathbf{F}^C_n(x)G_{A_{n-1}}(a)}{a_1\oplus a_1}\\
&= \mathbf{F}^C_n(x)G_{A_{n-1}}(a)u_0\,.\\
\psi^{(a)}_{\hat{1}}(\mathbf{F}^D_n(x)G_{A_{n-1}}(a)) &= \frac{\mathbf{F}^D_n(x)F^D_n(a_1,a_2)G_{A_{n-1}}(\bar{a}_2,\bar{a}_1,\ldots,a_{n-1})-\mathbf{F}^D_n(x)G_{A_{n-1}}(a)}{a_1\oplus a_2}\\
&= \mathbf{F}^D_n(x)G_{A_{n-1}}(a)u_{\hat{1}}\,.
\end{aligned}$$

Similar arguments hold for the action of $\psi^{(b)}_i$. □

COROLLARY 1.

$$\pi_i^{(a)}\mathcal{G}_w^X(a,b;x)=\begin{cases}\mathcal{G}_{ws_i}^X(a,b;x) & \textit{if } \ell(ws_i)=\ell(w)-1\,,\\ -\beta\mathcal{G}_w^X(a,b;x) & \textit{otherwise}\end{cases}$$

and

$$\pi_i^{(b)}\mathcal{G}_w^X(a,b;x)=\begin{cases}\mathcal{G}_{s_iw}^X(a,b;x) & \textit{if } \ell(s_iw)=\ell(w)-1\,,\\ -\beta\mathcal{G}_w^X(a,b;x) & \textit{otherwise}\end{cases}.$$

PROPOSITION 4 (strong stability). *$\mathcal{G}_w^X(a,b;x)$ has strong stability with respect to a and b (cf. Definition 8), i.e. if $i_{n+1}: W(X_n)\to W(X_{n+1})$ is the natural inclusion, then*

$$\mathcal{G}_{i_{n+1}(w)}^X(a,b;x)=\mathcal{G}_w^X(a,b;x)\in \mathbf{SS}_\beta(x)[a_1,b_1,a_2,b_2,\ldots]\,.$$

This means that $\mathcal{G}_w^X(a,b;x)$ does not depend on n for $w\in W(X_n)$.

The special case of w being a Grassmannian permutation, $\mathcal{G}_w^X(a,b;x)$ is the K-theoretic analogue of factorial Schur P- or Q-function in [17].

PROPOSITION 5 (Grassmannian elements). *For a Grassmannian element $w\in W(X)$ $(X=B,C,D)$, we have the following equalities.*

$$\begin{aligned}\mathcal{G}_w^B(a,b;x)&=GP_{\lambda_B(w)}(x|0,b)\,,\\ \mathcal{G}_w^C(a,b;x)&=GQ_{\lambda_C(w)}(x|b)\,,\\ \mathcal{G}_w^D(a,b;x)&=GP_{\lambda_D(w)}(x|b)\,.\end{aligned}$$

PROOF. In [17] Corollary 7.1, the map $\Phi: G\Gamma^X\to \mathrm{Fun}(\mathcal{SP},\mathcal{R})$ is defined and indicated that it is injective. Let $w\in W(X)$ be a Grassmannian element with corresponding strict partition $\lambda=\lambda_X(w)$. Then $\mathcal{G}_w^X(a,b;x)$ is in $G\Gamma^X=\mathbf{SS}_\beta(x)\otimes\mathbb{Z}[\beta][b,\bar b]$ and satisfy the left divided difference property (Corollary 1). This means that $\mathcal{G}_w^X(a,b;x)=GX_\lambda(x|b)$ by the Theorem 7.1 of [17]. $\square$

5.2. The second definition. As in [8], we can use "change of variables" for x_i, $i=1,2.\ldots$ to define the double Grothendieck polynomial $\mathcal{G}_w^{X_n}(a,b)$ with two sets of variables a,b as follows. We just write F_n for F_n^X.

$$F_n(x_i)=\sqrt{F_n(\bar a_i)}\sqrt{F_n(\bar b_i)}$$

where

$$\sqrt{1+T}=1+\frac{T}{2}-\frac{T^2}{8}+\frac{T^3}{16}-\frac{5T^4}{128}\cdots\ \text{(Taylor expansion)}.$$

We will also write

$$\sqrt{F_n(a_1)}\sqrt{F_n(a_2)}\ldots\sqrt{F_n(a_n)}\ \text{as}\ \sqrt{F_n(a_1,a_2,\ldots,a_n)}$$

because $F_n(a_i)$ commutes with each other. Note that $\sqrt{F_n(t)} \in \mathbb{Q}[\beta][[t]] \otimes \mathrm{Id}_\beta(W_n)$.

REMARK 4. By the definition of the action of s_0 and the cancellability of F_n, we have $s_0^{(a)}(\sqrt{F_n(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n)}) = \sqrt{F_n(a_1, \bar{a}_2, \dots, \bar{a}_n)} = \sqrt{F_n(a_1, a_1, \bar{a}_1, \bar{a}_2, \dots, \bar{a}_n)} = F_n(a_1)\sqrt{F_n(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n)}$. This explains the action $s_0^{(a)}(F_n(x_1, x_2, \dots, x_n)) = F_n(a_1, x_1, x_2, \dots, x_n)$ and $s_0^{(b)}(F_n(x_1, x_2, \dots, x_n)) = F_n(b_1, x_1, x_2, \dots, x_n)$. The action of $s_{\hat{1}}^{(a)}$ and $s_{\hat{1}}^{(b)}$ are the like.

DEFINITION 10. Let $X = B, C, D$. For $w \in W_n^X$, we define $G_n^X(a)$ and $G_n^X(a, b)$ as follows.

$$G_n^X(a) := \sqrt{F_n^X(\bar{a}_1, \dots, \bar{a}_n)}G_{A_{n-1}}(a) \qquad \text{and} \qquad G_n^X(a, b) := G_n^X(\bar{b})^{-1}G_n^X(a)\,.$$

By expanding these in terms of u_w, we can define $\mathcal{G}_{n,w}^X(a)$ and $\mathcal{G}_{n,w}^X(a, b)$ by

$$G_n^X(a) = \sum_{w \in W(X)} \mathcal{G}_{n,w}^X(a)u_w \qquad \text{and} \qquad G_n^X(a, b) = \sum_{w \in W(X_n)} \mathcal{G}_{n,w}^X(a, b)u_w\,.$$

REMARK 5. This double Grothendieck polynomial $\mathcal{G}_{n,w}^X(a, b)$ is essentially the same as defined in [18]. This has weak stability. i.e. $\mathcal{G}_{n,w}^X = \mathcal{G}_{n+1,w}^X|_{a_{n+1}=b_{n+1}=0}$ for $w \in W(X_n)$. But it doesn't have strong stability.

Note that for $w \in W(X_n)$, then

$\mathcal{G}_{n,w}^X(a) \in \mathbb{Q}[\beta][[a_1, \dots, a_n, \bar{a}_1, \dots, \bar{a}_n]]$ and
$\mathcal{G}_{n,w}^X(a, b) \in \mathbb{Q}[\beta][[a_1, \dots, a_n, \bar{a}_1, \dots, \bar{a}_n, b_1, \dots, b_n, \bar{b}_1, \dots, \bar{b}_n]]\,.$

EXAMPLE 3. The followings are some examples of $\mathcal{G}_{n,w}^X(a, b)$.

$\mathcal{G}_{2,s_0}^B(a, b) = \frac{\sqrt{1+(\bar{a}_1\oplus\bar{a}_2\oplus\bar{b}_1\oplus\bar{b}_2)\beta}-1}{\beta} = \frac{\bar{a}_1\oplus\bar{a}_2\oplus\bar{b}_1\oplus\bar{b}_2}{2} - \beta\frac{(\bar{a}_1\oplus\bar{a}_2\oplus\bar{b}_1\oplus\bar{b}_2)^2}{8} + \cdots\,.$

$\mathcal{G}_{2,s_0}^C(a, b) = \bar{a}_1 \oplus \bar{a}_2 \oplus \bar{b}_1 \oplus \bar{b}_2\,,\ \mathcal{G}_{3,s_0}^C(a, b) = \bar{a}_1 \oplus \bar{a}_2 \oplus \bar{a}_3 \oplus \bar{b}_1 \oplus \bar{b}_2 \oplus \bar{b}_3\,.$

$\mathcal{G}_{3,s_{\hat{1}}}^D(a, b) = \frac{\sqrt{1+(\bar{a}_1\oplus\bar{a}_2\oplus\bar{a}_3\oplus\bar{b}_1\oplus\bar{b}_2\oplus\bar{b}_3)\beta}-1}{\beta}\,.$

PROPOSITION 6. *The following holds for $X = B, C, D$ and $s_i \in I_{X_n}$:*

$$\pi_i^{(a)}G_n^X(a, b) = G_n^X(a, b)(u_i - \beta)\,,$$
$$\pi_i^{(b)}G_n^X(a, b) = (u_i - \beta)G_n^X(a, b)\,.$$

PROOF. These are Proposition 3 with change of variables. □

6. Identification with Schubert class

6.1. Equivariant K-theory. Torus T-equivariant K-theory $K_T(X)$ of smooth algebraic variety X acted by T is defined as follows. Let $Coh_T(X)$ be the abelian category of T-equivariant coherent sheaves on X, and $K_T(X)$ be its Grothendieck group. As we assumed X to be smooth, we can give $K_T(X)$ a ring structure by defining product coming from the tensor product of T-equivariant vector bundles. The class $[\mathcal{O}_X]$ of the structure sheaf of X is the identity and for each closed T-subvariety $Z \subset X$ we can associate its T-equivariant class $[\mathcal{O}_Z] \in K_T(X)$. In particular the K-theory Schubert class $[\mathcal{O}_{X^w}]$of the structure sheaf $\mathcal{O}_{X^w}$ of the (opposite) Schubert variety $X^w = \overline{B_- wB/B} \subset X = G/B$, where B_- is the opposite Borel subgroup, i.e. a unique Borel subgroup with the property that the intersection $B \cap B_- = T$ is the maximal torus contained in B.

For a torus $T = (\mathbb{C}^*)^n$ of rank n, we have $K_T(pt) = \mathbb{Z}[e^{\pm t_1}, \ldots, e^{\pm t_n}]$. The Littlewood-Richardson coefficient $c_{u,v}^w \in K_T(pt)$ is the structure constant of $K_T(X)$ with respect to the Schubert basis $\{[\mathcal{O}_{X^w}]\}_{w\in W}$ defined by

$$[\mathcal{O}_{X^u}][\mathcal{O}_{X^v}] = \sum_{w\in W} c_{u,v}^w [\mathcal{O}_{X^w}]\,.$$

6.2. Algebraic localization map. We first define algebraic localization map. This is a K-theoretic analogue of the universal localization map constructed in [15], and extend the (maximal) Grassmannian case of [17] to the full flag case. This is a β-deformation (or connective K-theory version) of Lam-Shilling-Shimozono construction using K-NilHecke algebra. (But in our case we must treat infinite rank Kac-Moody Lie group corresponding to root system of type X_∞, for $X = A, B, C, D$.)

Let $\mathcal{R}_\beta^a := \mathbb{Z}[\beta][a_1, a_2, \ldots]$ and $\mathcal{R}^{b,\bar{b}} := \mathbb{Z}[\beta][b_1, \bar{b}_1, b_2, \bar{b}_2, \ldots]$. ($\mathcal{R}^{b,\bar{b}}$ will play the role of $K_T(pt)$ (when $\beta = -1$) for $T = \prod_{i=1}^\infty(\mathbb{C}^*)$ (as we are considering thick Schubert variety).

Let

$$P_\infty^A := \mathcal{R}_\beta^a \otimes_{\mathbb{Z}[\beta]} \mathcal{R}^{b,\bar{b}},\ \ P_\infty^B = P_\infty^D := P_\infty^A \otimes_{\mathbb{Z}[\beta]} \mathbf{SS}_\beta(x)\,.$$

For type C, let $\mathbf{SS}_\beta^C(x) = \varprojlim_n SS_\beta^C(x_1, \ldots, x_n)$ and define

$$P_\infty^C := P_\infty^A \otimes_{\mathbb{Z}[\beta]} \mathbf{SS}_\beta^C(x)\,.$$

For $X = A, B, C, D$, we define $\mathcal{R}^{b,\bar{b}}$-linear (algebraic) localization map

$$\Phi^X : P_\infty^X \to \mathrm{Fun}(W(X), \mathcal{R}^{b,\bar{b}})\,,$$

as follows.

For $X = A$, $v = [v(1), v(2), \ldots] \in W(A)$ and $f(a, b, \bar{b}) \in P_\infty^A$, we define

$$\Phi^A(f(a, b, \bar{b}))(v) := f(v(\bar{b}), b, \bar{b})\,,$$

which mean that $f(v(\bar{b}), b, \bar{b})$ is obtained from $f(a, b, \bar{b})$ by substituting each a_i with $\bar{b}_{v(i)}$. For $X = B, C, D$, $v = [v(1), v(2), \ldots] \in W(X)$ and $f(a, b, \bar{b}; x) \in P_\infty^X$, we define

$$\Phi^X(f(a, b, \bar{b}; x))(v) := f(v(\bar{b}), b, \bar{b}; v[\bar{b}]) ,$$

which mean that $f(v(\bar{b}), b, \bar{b}; v[\bar{b}])$ is obtained from $f(a, b, \bar{b}; x)$ by substituting $a_i = \bar{b}_{v(i)}$ for all i, and substituting $x_i = b_{v(i)}$ if $v(i) < 0$ and $x_i = 0$ if $v(i) > 0$. Here we have used the convention that $b_{-i} = \bar{b}_i$. These are K-theoretic analogue of the universal localization map in [15] §6.1. Let Δ_X be the root system of type $X = A, B, C, D$.

DEFINITION 11 (GKM subspace). We define the Goresky-Kottwitz-MacPherson subspace (GKM subspace for short) $\mathrm{GKM}^X \subset \mathrm{Fun}(W(X), \mathcal{R}^{b,\bar{b}})$, as follows

$$\mathrm{GKM}^X := \left\{ f \in \mathrm{Fun}(W(X), \mathcal{R}^{b,\bar{b}}) \;\middle|\; \begin{array}{c} f(v) - f(s_\alpha v) \in \alpha(b)\mathcal{R}^{b,\bar{b}} \\ \text{for all } \alpha \in \Delta_X, v \in W(X) \end{array} \right\}$$

Here we write $s_\alpha = w s_i w^{-1}$,$\alpha(b) := w(\alpha_i(b))$ if the root $\alpha \in \Delta_X$ has the form $\alpha = w(\alpha_i)$.

PROPOSITION 7. *The image of* Φ^X *has GKM property, i.e.* $\mathrm{Im}\, \Phi^X \subset \mathrm{GKM}^X$.

PROOF. For type A case it is easy. For type B, C, D case this is a consequence of supersymmetricity of $\mathrm{SS}(x)$ and $\mathrm{SS}^C(x)$. □

REMARK 6. Actually we can show that the $\mathcal{R}^{b,\bar{b}}$-linear map

$$\widetilde{\Phi}^X : \prod_{w \in W(X)} \mathcal{R}^{b,\bar{b}} \mathcal{G}_w^X \to GKM^X$$

defined by $\widetilde{\Phi}^X(\prod_{w\in W(X)} c_w \mathcal{G}_w^X) := \sum_{w \in W(X)} c_w \Phi^X(\mathcal{G}_w^X)$ is (well defined) and an isomorphism as the same reasoning in [21] Proposition 2.6. The ring $\prod_{w\in W(X)} \mathcal{R}^{b,\bar{b}} \mathcal{G}_w^X$ contains $\frac{1}{1+\beta a_i} = 1 - \beta a_i + \beta^2 a_i^2 - \beta^3 a_i^3 + \cdots$ as well as $\frac{1}{1+\beta \mathcal{G}_w^X}$.

PROPOSITION 8. *If* $f \in \mathrm{Im}(\Phi^X)$ *then* $\pi_i(f) \in \mathrm{Im}(\Phi^X)$.

We define (left) divided difference operator π_i on $\mathrm{GKM}^X \subset \mathrm{Fun}(W(X), \mathcal{R}^{b,\bar{b}})$ as follows. (cf. [17] §5.2.) For $f \in \mathrm{Im}(\Phi^X)$,

$$(\pi_i(f))(v) = \frac{f(v) - (1 + \beta\alpha_i(b)) s_i^{(b)}(f(s_i v))}{\alpha_i(b)} .$$

By the GKM property of $\mathrm{Im}(\Phi^X)$ we have $(\pi_i(f))(v) \in \mathcal{R}^{b,\bar{b}}$.

PROPOSITION 9. Φ^X *is compatible with* $\pi_i^{(b)}$ *and* π_i, *i.e.*

$$\Phi^X \pi_i^{(b)} = \pi_i \Phi^X .$$

K-theory Schubert classes are determined by the localization (Prop. 2.10 in [21]), and they are determined uniquely by 'left hand' recurrence (Remark 2.3 in [21]).

PROPOSITION 10. (Connective) *K-theory Schubert classes $(\psi^w)_{w\in W(X)}$, $\psi^w \in \mathrm{Fun}(W(X), \mathcal{R}^{b,\bar{b}})$ are uniquely determined by*

(i) $\psi^w(e) = \delta_{w,e}$

(ii) *for $v > s_i v$,*

$$\psi^w(v) = \begin{cases} s_i^{(b)}\psi^w(s_i v) & \text{if } s_i w > w \\ (1+\beta\alpha_i(b))s_i^{(b)}\psi^w(s_i v) + \alpha_i(b)s_i^{(b)}\psi^{s_i w}(s_i v) & \text{if } s_i w < w\,. \end{cases}$$

THEOREM 2. *For $X = A, B, C, D$, $\left(\psi^w = \Phi^X(\mathcal{G}_w^X)\right)_{w\in W(X)}$ satisfies the recurrence relations in Proposition* 10 *and gives the system of* (*equivariant*) *Schubert classes.*

PROOF. We use left recurrence relations.

$$\mathcal{G}_e^X = 1 \text{ and } \pi_i^{(b)}\mathcal{G}_w^X = \begin{cases} \mathcal{G}_{s_i w}^X & \text{if } s_i w < w \\ -\beta\mathcal{G}_w^X & \text{if } s_i w > w\,. \end{cases}$$

We will write $G_w|_v := \Phi^X(\mathcal{G}_w^X)(v)$. $\psi_\beta^w(v) := \Phi^X(G_w^X)(v)$ (i) If we localize the generating function at $v = e$ we will specialize $a_i = \bar{b}_i$ and $x_i = 0$ for all $i \geq 1$. This gives the result $\mathcal{G}_w|_e = \delta_{w,e}$.

(ii) By the definition of divided difference $\pi_i^{(b)}$, we have

$$(\pi_i^{(b)}G_w)|_v = \frac{G_w|_v - (1+\beta\alpha_i(b))s_i^{(b)}(G_w|_{s_i v})}{\alpha_i(b)}\,.$$

If $s_i w > w$ then

$$\frac{G_w|_v - (1+\beta\alpha_i(b))s_i^{(b)}(G_w|_{s_i v})}{\alpha_i(b)} = (-\beta)G_w|_v\,.$$

From this we get $G_w|_v = s_i^{(b)}(G_w|_{s_i v})$.
If $s_i w < w$ then

$$\frac{G_w|_v - (1+\beta\alpha_i(b))s_i^{(b)}(G_w|_{s_i v})}{\alpha_i(b)} = G_{s_i w}|_v\,.$$

From this we get

$$G_w|_v = (1+\beta\alpha_i(b))s_i^{(b)}(G_w|_{s_i v}) + \alpha_i(b)G_{s_i w}|_v\,.$$

□

COROLLARY 2. *Assume*

$$\mathcal{G}_u^X(a,b;x)\,\mathcal{G}_v^X(a,b;x) = \sum_{w\in W(X)} c_{u,v}^{w,X}(\beta)\,\mathcal{G}_w^X(a,b;x),\ c_{u,v}^{w,X}(\beta)\in\mathcal{R}^{b,\bar{b}}.$$

Then $c_{u,v}^{w,X}(\beta)|_{\beta=-1}$ *is the generalized Littlewood-Richardson coefficient* $c_{u,v}^{w}$ *for equivariant* K*-theory of type* X. (b_i *is considered as* $1-e^{t_i}$.)

REMARK 7. $c_{u,v}^{w}(0)$ is the generalized Littlewood-Richardson coefficient for equivariant cohomology if we replace b_i to $-t_i$. (cf. [15].)

EXAMPLE 4. The following is an example of the expansion.

$$\mathcal{G}_{s_0}^C(a,b;x)\,\mathcal{G}_{s_0}^C(a,b;x) = (b_1\oplus b_1)\mathcal{G}_{s_0}^C(a,b;x) + \mathcal{G}_{s_1s_0}^C(a,b;x) + \beta\mathcal{G}_{s_0s_1s_0}^C(a,b;x).$$

6.3. Explicit localization formula. Let $\mathcal{R}^{b,\bar{b}}\#\mathbb{Z}[W]$ denote the smash product of $\mathcal{R}^{b,\bar{b}}$ and group algebra $\mathbb{Z}[W]$. In this ring we have $(f\otimes v)(g\otimes w) = f v^{(b)}(g)\otimes vw$. We define $\mathcal{R}^{b,\bar{b}}$-linear map $\varepsilon : \mathcal{R}^{b,\bar{b}}\#\mathbb{Z}[W]\to\mathcal{R}^{b,\bar{b}}$ by $\varepsilon(f\otimes w) = f$.

PROPOSITION 11. *Let* $w, v\in W(X)$ *and* $v = s_{i_1}s_{i_2}\cdots s_{i_r}$ *be any reduced decomposition of* v *and set* $\mathbf{i} = (i_1, i_2, \ldots, i_r)$. *For* $c = (c_1, c_2, \ldots, c_r)\in\{0,1\}^r$ *let* $|c| = \sum_{i=1}^r c_i$. *Then*

$$\Phi^X(\mathcal{G}_w^X)(v) = \varepsilon\left(\sum_{c\in C(\mathbf{i},w)} \beta^{|c|-\ell(w)} \prod_{k=1}^{r} \begin{cases} \alpha_{i_k}^X(b)\otimes s_{i_k} & \text{if } c_k = 1\\ 1\otimes s_{i_k} & \text{if } c_k = 0\end{cases}\right),$$

where $C(\mathbf{i},w) := \left\{ c = (c_1, c_2, \ldots, c_r)\in\{0,1\}^r \;\middle|\; \prod_{k, b_k=1} u_{i_k} = \beta^{|c|-\ell(w)}u_w \right\}.$

PROOF. We can follow the proof in [21] Proposition 2.10. (By induction on $\ell(v)$ and $\ell(w)$, using left recurrence relations (i), (ii) in Proposition 10.) □

COROLLARY 3 (Vanishing property). *For* $w, v\in W(X)$ *we have*

$$\Phi^X(\mathcal{G}_w^X)(v) = 0 \text{ if } w\not\leq v.$$

$$\Phi^X(\psi_v^{(a)}\mathcal{G}_w^X)(e) = \delta_{w,v} = \Phi^X(\psi_{v^{-1}}^{(b)}\mathcal{G}_w^X)(e).$$

7. Adjoint polynomials

We can also define the adjoint polynomials $\mathcal{H}_{n,w}^X$, for each $w\in W_n^X$, (when $\beta=-1$) corresponding to the class of ideal sheaf $\mathcal{O}_{X^w}(-\partial X^w)$ of boundary ∂X^w in X^w. cf. [13, 23]. Then the pairing $\langle\cdot\,,\cdot\rangle : K_T(X)\otimes_{R(T)} K_T(X)\to R(T)$ is given (cf. [13]) by

$$\langle v_1, v_2\rangle = \chi(X, v_1\otimes v_2) \qquad \text{where} \qquad \chi(X,\mathcal{F}) = \sum_{p\geq 0}(-1)^p \mathrm{ch}\, H^p(X,\mathcal{F}).$$

Here ch M for T-module M is the formal character defined by

$$\mathrm{ch}\, M = \sum_{e^\lambda \in X(T)} \dim(M_\lambda) e^\lambda$$

where M_λ is the weight space corresponding to the weight λ. With these notations we have (cf. [13] Proposition 2.1)

$$\langle [\mathcal{O}_{X_w}], [\mathcal{O}_{X^v}(-\partial X^v)] \rangle = \delta_{w,v}\,,$$

where $X_w = \overline{BwB/B} \subset G/B$ is the usual Schubert variety. The relation between $[\mathcal{O}_{X^w}]$ and $[\mathcal{O}_{X^w}(-\partial X^w)]$ is as follows. (cf. [13] Lemma 4.2)

$$[\mathcal{O}_{X^w}(-\partial X^w)] = \sum_{w \le v \le w_0} (-1)^{\ell(v)-\ell(w)} [\mathcal{O}_{X^v}]\,.$$

We (formally) define the relative adjoint polynomial $\mathcal{H}^X_{w,v}$ for $w \le v$ by $\mathcal{H}^X_{w,v} := \psi^{(a)}_{w^{-1}v}(\mathcal{G}^X_v)$. The adjoint polynomial for $w \in W(X_n)$ is defined by $\mathcal{H}^X_{n,w} := \mathcal{H}^X_{w,w_0^{(n)}}$, where $w_0^{(n)}$ is the longest element in $W(X_n)$ (cf. [23]).

PROPOSITION 12. *For $w \in W(X_n)$, we have*

$$\mathcal{H}^X_{n,w} = \sum_{w \le v \le w_0^{(n)}} \beta^{\ell(v)-\ell(w)} \mathcal{G}^X_v\,.$$

Therefore if we specialize $\beta = -1$, $\mathcal{H}^X_{n,w}$ represents the boundary class $[\mathcal{O}_{X^w}(-\partial X^w)]$.

PROOF. We can use the property of divided difference that

$$\psi^{(a)}_w = \sum_{v \le w} \beta^{\ell(w)-\ell(v)} \pi^{(a)}_v\,.$$

□

These polynomials $\mathcal{H}^X_{n,w}$ are no longer stable but have similar properties as Grothendieck polynomials.

PROPOSITION 13. *For $w \in W(X_n)$, we have*

$$\begin{aligned}
\mathcal{H}^B_{n,e} &= \prod_{1\le i\le n-1}(1+\beta a_i)^{n-i}\prod_{1\le i\le n-1}(1+\beta b_i)^{n-i}\prod_{1\le i\le n}(1+\beta x_i)^{2n-1}\\
\mathcal{H}^C_{n,e} &= \prod_{1\le i\le n-1}(1+\beta a_i)^{n-i}\prod_{1\le i\le n-1}(1+\beta b_i)^{n-i}\prod_{1\le i\le n}(1+\beta x_i)^{2n}\\
\mathcal{H}^D_{n,e} &= \prod_{1\le i\le n-1}(1+\beta a_i)^{n-i}\prod_{1\le i\le n-1}(1+\beta b_i)^{n-i}\prod_{1\le i\le n}(1+\beta x_i)^{2n-2}
\end{aligned}$$

and

$$\mathcal{H}^X_{n,w} = (-1)^{\ell(w)} \mathcal{H}^X_{n,e} \overline{\mathcal{G}^X_{n,w}}\,,$$

where $\overline{\mathcal{G}^X_{n,w}} = \mathcal{G}^X_{n,w}(\overline{a}, \overline{b}; \overline{x})$.

We can derive these formula using generating functions. Let us define $H^X_n(a, b; x)$ as

$$H^X_n(a, b; x) := \sum_{w \in W(X_n)} (-1)^{\ell(w)} \mathcal{H}^X_{n,w}(a, b; x) u_w \,.$$

Then we get the following formula.

PROPOSITION 14. *The generating function $H^X_n(a, b; x)$ has the following factorization.*

$$H^X_n(a, b; x) = \mathcal{H}^X_{n,e} G^X_n(\bar{a}, \bar{b}; \bar{x}) \,.$$

Actually we can show the following property.

PROPOSITION 15. *For $s_i \in I^X_n$ we have*

$$\pi^{(a)}_i H^X_n(a, b; x) = H^X_n(a, b; x)(-u_i)$$
$$\pi^{(b)}_i H^X_n(a, b; x) = (-u_i) H^X_n(a, b; x) \,.$$

PROPOSITION 16 (Interpolation formula). *For $F \in \mathbf{SS}_\beta(x) \otimes_{\mathbb{Z}[\beta]} \mathcal{R}^{(a)}_\beta \otimes_{\mathbb{Z}[\beta]} \mathcal{R}^{(b)}_\beta$,*

$$F = \sum_{v \in W(X)} (\psi^{(a)}_v(F)|_e)\, \mathcal{G}^X_v(a, b; x)$$

where the summation is infinite in general and $|_e$ means the localization at e, i.e. take substitutions $a_i = \bar{b}_i$ and $x_i = 0$ for all i.

PROOF. F can be expanded as a formal sum $F = \sum_{v \in W(X)} c_v(F) \mathcal{G}^X_v(a, b; x)$. To find $c_v(F) \in \mathcal{R}^{(b)}_\beta$, we can use the vanishing property (Corollary 3), i.e.

$$\psi^{(a)}_v(\mathcal{G}^X_w(a, b; x))|_e = \delta_{w,v} \,.$$

Using the formula in the proof of Proposition 3, it follows that $\psi^{(a)}_v(\mathcal{G}^X_n(a, b; x)) = \mathcal{G}^X_n(a, b; x) u_v$ for $v \in W(X_n)$. By the localization property $(\mathcal{G}^X_v(a, b; x))|_e = \delta_{v,e}$, we have $\psi^{(a)}_w(\mathcal{G}^X_v(a, b; x))|_e = \delta_{w,v}$. From this we get the formula. □

COROLLARY 4. *The equivariant Littlewood-Richardson coefficient can be written as*

$$c^{w,X}_{u,v}(\beta) = \psi^{(a)}_w(\mathcal{G}^X_u(a, b; x) \mathcal{G}^X_v(a, b; x))|_e \,.$$

THEOREM 3. *We have the following change of parameter formula.*

$$\mathcal{G}^X_w(a, b; x) = \sum_{uv = w, u \le w} \mathcal{H}^X_{u,w}(\bar{c}, b; 0) \mathcal{G}^X_v(a, c; x) \,.$$

PROOF. This is just a consequence of Proposition 16 and the definition of $\mathcal{H}^X_{u,w}(a,b;x) = \psi^{(a)}_{u^{-1}w}(\mathcal{G}^X_w(a,b;x))$. More precisely, we introduce new set of variables $c_1, c_2, \dots$ and $d_1, d_2, \dots$ and consider

$$\mathcal{G}^X_w(a,d;x) \in \mathbf{SS}_\beta(x) \otimes_{\mathbb{Z}[\beta]} \mathcal{R}^{(a)}_\beta \otimes_{\mathbb{Z}[\beta]} \mathcal{R}^{(b)}_\beta \otimes_{\mathbb{Z}[\beta]} \mathcal{R}^{(d)}_\beta \,.$$

Proposition 16 can be extended to this case by scalar extension and we can write

$$\mathcal{G}^X_w(a,d;x) = \sum_{v \in W(X)} (\psi^{(a)}_v(\mathcal{G}^X_w(a,d;x))|_e)\, \mathcal{G}^X_v(a,b;x)\,,$$

where $\psi^{(a)}_v(\mathcal{G}^X_w(a,d;x))|_e \in \mathcal{R}^{(d)}_\beta$. As

$$\psi^{(a)}_v(\mathcal{G}^X_w(a,d;x))|_e = \mathcal{H}^X_{wv^{-1},w}(\bar{b},d;0)\,,$$

we get

$$\mathcal{G}^X_w(a,d;x) = \sum_{v \in W(X)} \mathcal{H}^X_{wv^{-1},w}(\bar{b},d;0)\, \mathcal{G}^X_v(a,b;x)\,.$$

Replacing b by c and then replacing d by b, we get the desired formula. □

REMARK 8. There is also similar formula using second version of type B, C, D double Grothendieck polynomials.

8. Combinatorial descriptions

We give in this section two kinds of combinatorial formula for the Grothendieck polynomials of classical types. Actually these are essentially the same but they have different names and descriptions.

8.1. Compatible sequence formula. In [10] S. Fomin and R. Stanley used nilCoxeter algebra to prove compatible sequence formula for type A Schubert polynomials $\mathfrak{S}_w$ and in [9] S. Fomin and A. N. Kirillov gave a compatible sequence formula for type A Grothendieck polynomials $\mathcal{G}_w$. In [3] S. Billey and M. Haiman used Edelman-Greene type bijection for type B and D, to give similar combinatorial formula for Stanley symmetric functions E_w and F_w. We use idCoxeter algebra to give compatible sequence formula for double Grothendieck polynomials $\mathcal{G}^A_w$, and K-theoretic Stanley symmetric functions $\mathcal{F}^X_w$ for $X = B, C, D$. To give the formula we need some notations and definitions.

We consider a sequence $\tilde{a} = (\tilde{a}_1, \dots, \tilde{a}_\ell) \in (I^X)^\ell$ of indices of generators in I^X. We denote by $\ell(\tilde{a})$ the length ℓ of the sequence $\tilde{a} = (\tilde{a}_1, \dots, \tilde{a}_\ell)$. For type $X = B, D$, we denote by $o^B(\tilde{a})$ the number of appearance of 0's in $\tilde{a}$, by $o^D(\tilde{a})$ the total number of appearance of 1 and $\hat{1}$ in $\tilde{a}$. For type $X = D$ case, we denote by $\tilde{\tilde{a}}$ the *flattened* word of $\tilde{a} = (\tilde{a}_1, \dots, \tilde{a}_\ell)$ which is obtained from $\tilde{a}$ by replacing all appearance of $\hat{1}$ with 1. cf. [3]. For $w \in W(X)$ we define

$$\tilde{R}(w) := \{\tilde{a} = (\tilde{a}_1, \dots, \tilde{a}_\ell) \mid u_{s_{\tilde{a}_1}} \cdots u_{s_{\tilde{a}_\ell}} = \beta^{\ell-\ell(w)} u_w\}.$$

We define

$$B(n; \ell) := \left\{ \tilde{b} = (\tilde{b}_1, \dots, \tilde{b}_\ell) \;\middle|\; \tilde{b}_i \in \mathbb{Z}, 1 \le \tilde{b}_1 \le \cdots \le \tilde{b}_\ell \le n \right\}.$$

For $\tilde{b} \in B(n;\ell)$, we denote by $|\tilde{b}|$ the number of distinct $\tilde{b}_i$'s. For $\tilde{a} = (\tilde{a}_1, \dots, \tilde{a}_\ell) \in (I^X)^\ell$ and $\tilde{b} \in B(n;\ell)$, we denote by

$$\gamma(\tilde{a}, \tilde{b}) := \#\{\, i \mid \tilde{a}_i = \tilde{a}_{i+1} \text{ and } \tilde{b}_i = \tilde{b}_{i+1}\}.$$

DEFINITION 12. For $\tilde{a} = (\tilde{a}_1, \dots, \tilde{a}_\ell) \in \tilde{R}(w)$ of $w \in W(X)$, we define the set of compatible sequences $C^X(\tilde{a})$ as follows.

$$C^{A_n}(\tilde{a}) = \left\{ \tilde{b} \in B(n;\ell) \;\middle|\; \begin{array}{c} \tilde{a}_{i-1} \le \tilde{a}_i \implies \tilde{b}_{i-1} < \tilde{b}_i \\ \text{and} \\ \tilde{b}_i \le \tilde{a}_i \end{array} \right\},$$

$$C^{B_n}(\tilde{a}) = C^{C_n}(\tilde{a}) = \left\{ \tilde{b} \in B(n;\ell) \;\middle|\; \tilde{a}_{i-1} \le \tilde{a}_i \ge \tilde{a}_{i+1} \implies \tilde{b}_{i-1} < \tilde{b}_{i+1} \right\},$$

$$C^{D_n}(\tilde{a}) = \left\{ \tilde{b} \in B(n;\ell) \;\middle|\; \begin{array}{l} \tilde{\tilde{a}}_{i-1} \le \tilde{\tilde{a}}_i \ge \tilde{\tilde{a}}_{i+1} \implies \tilde{b}_{i-1} < \tilde{b}_{i+1} \\ \qquad\qquad \text{and} \\ \left.\begin{array}{l} \tilde{a}_i = \tilde{a}_{i+1} = 1 \\ \text{or} \\ \tilde{a}_i = \tilde{a}_{i+1} = \hat{1} \end{array}\right. \implies \tilde{b}_i < \tilde{b}_{i+1} \end{array} \right\}.$$

PROPOSITION 17 (Compatible sequence formula cf. ([9], [3] Prop. 3.4 Prop. 3.10)). *For $w \in W(X_n)$, we have*

$$\begin{aligned}
\mathcal{G}^A_w(x_1, \dots, x_n; y_1, \dots, y_n) &= \sum_{\tilde{a} \in \tilde{R}(w)} \sum_{\tilde{b} \in C^{A_n}(\tilde{a})} \beta^{\ell(\tilde{a})-\ell(w)} \prod_{i=1}^{\ell(\tilde{a})} (x_{\tilde{b}_i} \oplus y_{\tilde{a}_i - \tilde{b}_i + 1}), \\
\mathcal{F}^B_w(x_1, \dots, x_n) &= \sum_{\tilde{a} \in \tilde{R}(w)} \sum_{\tilde{b} \in C^{B_n}(\tilde{a})} \beta^{\ell(\tilde{a})-\ell(w)} 2^{|\tilde{b}| - \gamma(\tilde{a},\tilde{b}) - o^B(\tilde{a})} x_{\tilde{b}}, \\
\mathcal{F}^C_w(x_1, \dots, x_n) &= \sum_{\tilde{a} \in \tilde{R}(w)} \sum_{\tilde{b} \in C^{C_n}(\tilde{a})} \beta^{\ell(\tilde{a})-\ell(w)} 2^{|\tilde{b}| - \gamma(\tilde{a},\tilde{b})} x_{\tilde{b}}, \\
\mathcal{F}^D_w(x_1, \dots, x_n) &= \sum_{\tilde{a} \in \tilde{R}(w)} \sum_{\tilde{b} \in C^{D_n}(\tilde{a})} \beta^{\ell(\tilde{a})-\ell(w)} 2^{|\tilde{b}| - \gamma(\tilde{a},\tilde{b}) - o^D(\tilde{a})} x_{\tilde{b}},
\end{aligned}$$

where we write $x_{\tilde{b}} = x_{\tilde{b}_1} \cdots x_{\tilde{b}_\ell}$, $\tilde{b} = (\tilde{b}_1, \dots, \tilde{b}_\ell) \in B(n;\ell)$.

REMARK 9. For cohomology case ($\beta = 0$), the formulas above for $X = C, D$ reduce to the formulas in [3] Proposition 3.4, 3.10 .

PROOF. These follow immediately from the expansion of the corresponding defining generating functions. More precisely we will explain as follows.

The type A_n double Grothendieck polynomials $\mathcal{G}_w^{A_n}(x, y)$ are defined as follows.

$$G_{A_n}(\bar{y}_1, \dots, \bar{y}_n)^{-1} G_{A_n}(x_1, \dots, x_n) = \sum_{w \in S_{n+1}} \mathcal{G}_w^{A_n}(x, y) u_w .$$

By Lemma 8, the left hand side can be changed as below.

$$\begin{aligned}\prod_{b=1}^{n}\left(\prod_{a=n}^{b} h_a(x_b \oplus y_{a-b+1})\right) \quad = \quad & [h_n(x_1 \oplus y_n) \cdots h_1(x_1 \oplus y_2) h_1(x_1 \oplus y_1)] \\ & [h_n(x_2 \oplus y_{n-1}) \cdots h_2(x_2 \oplus y_1)] \\ & \quad \vdots \\ & [h_n(x_n \oplus y_1)] .\end{aligned}$$

By expanding this it is easy to see that there is a one to one correspondence between the compatible sequences and the expanded terms which implies the first formula.

For $X = C$ case, $F_n^C(x_1, \dots, x_n) = F_n^C(x_1) \cdots F_n^C(x_n)$. As $F_n^C(x_b) = h_{n-1}(x_b) h_{n-2}(x_b) \cdots h_1(x_b) h_0(x_b) h_0(x_b) h_1(x_b) \cdots h_{n-2}(x_b) h_{n-1}(x_b) = \left(\prod_{a=n-1}^{0} h_a(x_b)\right)\left(\prod_{a=0}^{n-1} h_a(x_b)\right)$, each expanded term has the form $x_b^m u_{a_1} u_{a_2} \cdots u_{a_m}$ with two cases below.
(1) there exists i , $1 \le i \le m$ such that

$$n > a_1 > a_2 > \cdots > a_{i-1} > a_i < a_{i+1} < \cdots < a_m < n$$

(if $i = 1$ then we assume $a_{i-1} = n$ and if $i = m$ then we assume $a_{i+1} = n$), or
(2) there exists i , $1 \le i \le m - 1$ such that

$$n > a_1 > a_2 > \cdots > a_{i-1} > a_i = a_{i+1} < \cdots < a_m < n .$$

(if $i = 1$ then we assume $a_{i-1} = n$).
There are two possible choices of u_{a_i} for each (1) case while there is only one possibility for each case (2), which explains the factor $2^{|\tilde{b}| - \gamma(\tilde{a}, \tilde{b})}$.

For type B case, (1) has the exception of $a_i = 0$ in which case it can occur only once. This explains the factor of $2^{|\tilde{b}| - \gamma(\tilde{a}, \tilde{b}) - o^B(\tilde{a})}$.

For the case of type D is similar. For each (1) case, if $a_i = 1$ or $a_i = \hat{1}$ it corresponds to the case $\tilde{a}_i = 1$ or $\tilde{a}_i = \hat{1}$ (the flattened $\tilde{\tilde{a}}_i = 1$) of compatible sequence. The factor 2^{1-0-1} counts correctly in each of this case. The case (2) will be modified when $a_i = 1$ and $a_{i+1} = \hat{1}$. It corresponds to either $\tilde{a}_i = 1$ and $\tilde{a}_{i+1} = \hat{1}$ or $\tilde{a}_i = \hat{1}$ and $\tilde{a}_{i+1} = 1$ (the flattened word $\tilde{\tilde{a}}_i = \tilde{\tilde{a}}_{i+1} = 1$) . The corresponding factor 2^{1-0-2} appears twice which sum up to 1. So the factor $2^{|\tilde{b}| - \gamma(\tilde{a}, \tilde{b}) - o^D(\tilde{a})}$ properly counts the expanded terms. □

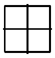
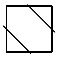
EXAMPLE 5. Type D, $n = 2$ case $w = [\bar{1}, \bar{2}] = s_1 s_{\hat{1}}$.
$\tilde{b} = (1, 1)$ is a compatible sequence for $\tilde{a} = (1, \hat{1}), (\hat{1}, 1)$.
$\tilde{b} = (1, 2)$ is a compatible sequence for $\tilde{a} = (1, \hat{1}), (\hat{1}, 1)$.
$\tilde{b} = (2, 2)$ is a compatible sequence for $\tilde{a} = (1, \hat{1}), (\hat{1}, 1)$.
$\tilde{b} = (1, 1, 2)$ is a compatible sequence for $\tilde{a} = (1, \hat{1}, 1), (\hat{1}, 1, \hat{1}), (1, \hat{1}, \hat{1}), (\hat{1}, 1, 1)$.
$\tilde{b} = (1, 2, 2)$ is a compatible sequence for $\tilde{a} = (1, \hat{1}, 1), (\hat{1}, 1, \hat{1}), (1, 1, \hat{1}), (\hat{1}, \hat{1}, 1)$.
$\tilde{b} = (1, 1, 2, 2)$ is a compatible sequence for

$$\tilde{a} = (1, \hat{1}, 1, \hat{1}), (\hat{1}, 1, \hat{1}, 1), (\hat{1}, 1, 1, \hat{1}), (1, \hat{1}, \hat{1}, 1) .$$

There are no other compatible sequences and the sum of the terms becomes

$$\mathcal{F}^D_{s_1 s_{\hat{1}}}(x_1, x_2) = x_1^2 + 2x_1x_2 + x_2^2 + 2\beta x_1^2 x_2 + 2\beta x_1 x_2^2 + \beta^2 x_1^2 x_2^2 = (x_1 \oplus x_2)^2 .$$

REMARK 10. We can use Proposition 2 and 17 to get the compatible sequence formula for double Grothendieck polynomials of type B, C, D.

8.2. Pipe dream (extended EYD) formula. There is a state sum formula called pipe dream formula for type A Schubert polynomials. It was first introduced in [1] and called RC-graph . They use two kinds of configurations to realize a pattern of given permutation $w \in S_n$ and the sum of weights for each pattern gives the Schubert polynomial S_w. Example 6 gives such a pattern, where we use  and  to realize a pattern of the permutation $w = [3, 1, 4, 2]$ (we delete useless lines). The name of pipe dream is given after the name of similar game. Here we connect i to $w(i)$ $(1 \le i \le n)$ for a given $w \in S_n$. We can extend this to type B, C, D cases as follows.

Let us recall the type A pipe dream formula for double Grothendieck polynomials. Fix a sequence Δ^A_{n-1} of simple reflections which gives a reduced expression for the longest element w_0 in S_n as follows.

$$\Delta^A_{n-1} := (s_{n-1} | s_{n-2}, s_{n-1} | \cdots | s_1, s_2, \ldots, s_{n-1}) = (d_1, d_2, \ldots, d_N) ,$$

where $N := n(n-1)/2$, i.e. if $\frac{m(m-1)}{2} < k \le \frac{m(m+1)}{2}$ then $d_k = s_{n-1-(k-\frac{m(m+1)}{2})}$. We arrange this sequence in triangular form, with coordinate (i, j) for $i + j < n$, from left to right from top to bottom as follows.

$$\begin{array}{ccccc} d_1 & & & & \\ d_2 & d_3 & & & \\ d_4 & d_5 & d_6 & & \\ & \cdots & & & \\ d_{M+1} & d_{M+2} & d_{M+3} & \cdots & d_N \end{array} \qquad \begin{array}{cccc} (1, n-1) & & & \\ (1, n-2) & (2, n-2) & & \\ (1, n-3) & (2, n-3) & (3, n-3) & \\ & \cdots & & \\ (1, 1) & (2, 1) & \cdots & (n-1, 1) \end{array}$$

where $M = \frac{(n-1)(n-2)}{2}$. Set $wt(d_k) = a_i \oplus b_j$ when d_k is in the coordinate (i, j), cf. Example 6 for $n = 4$ case.

Each term in the expansion of right hand side of

$$G_{A_{n-1}}(\bar{b}_1, \dots, \bar{b}_{n-1})^{-1} G_{A_{n-1}}(a_1, \dots, a_{n-1}) = \prod_{j=n-1}^{1} \left(\prod_{i=1}^{n-j} h_{i+j-1}(a_i \oplus b_j) \right) \tag{2}$$

corresponds to a subsequence of Δ^A_{n-1}. To give the statement uniformly for $X = A, B, C, D$, we need some notations in general.

DEFINITION 13. Given a sequence $\Delta = (d_1, d_2, \dots, d_N)$ of simple reflections in $W(X)$ and an element $w \in W(X)$ with length $\ell(w) = \ell$, let $\mathrm{Rsub}(\Delta, w)$ be the set of subsequences of Δ each element of which gives a reduced expression of w with length $\ell = \ell(w)$. i.e.

$$\mathrm{Rsub}(\Delta, w) := \{(d_{j_1}, d_{j_2}, \dots, d_{j_\ell}) \mid 1 \le j_1 < j_2 < \cdots < j_\ell \le N, d_{j_1} d_{j_2} \cdots d_{j_\ell} = w\}.$$

We will call $\mathbf{D} \in \mathrm{Rsub}(\Delta, w)$ an extended EYD. We also define the set $B(\mathbf{D})$ of backward movable positions for an element $\mathbf{D} = (d_{j_1}, d_{j_2}, \dots, d_{j_\ell}) \in \mathrm{Rsub}(\Delta, w)$, by considering $j_{\ell+1} := N + 1$,

$$B(\mathbf{D}) := \{d_j \mid j \le N, \exists p \text{ such that } j_p < j < j_{p+1}, d_{j_1} d_{j_2} \cdots d_{j_p} = (d_{j_1} d_{j_2} \cdots d_{j_p}) \star d_j\}.$$

Then we have the following extended EYD formula.

THEOREM 4. *For $w \in S_n$, we have*

$$\mathcal{G}^{A_{n-1}}_w(a, b) = \sum_{\mathbf{D} \in \mathrm{RSub}(\Delta^A_{n-1}, w)} Wt(\mathbf{D}),$$

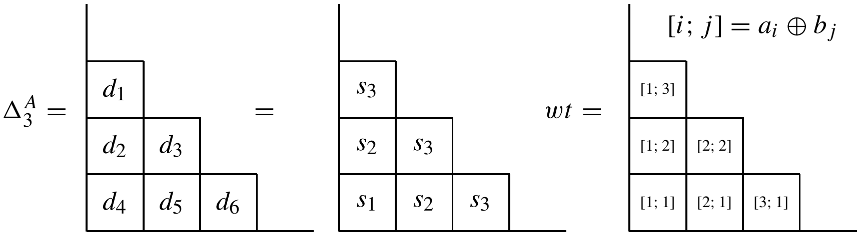
where

$$Wt(\mathbf{D}) = \prod_{\square \in \mathbf{D}} wt(\square) \times \prod_{\bigcirc \in B(\mathbf{D})} (1 + \beta wt(\bigcirc)).$$

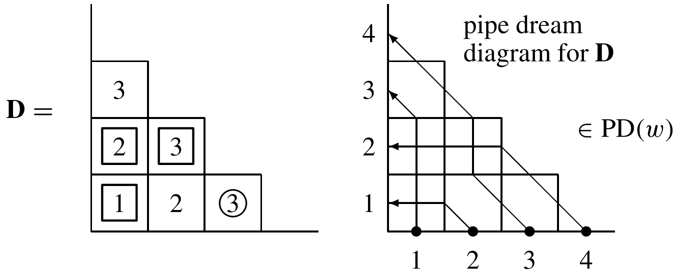
PROOF. This is just a consequence of the equation (2). $\square$

There is a one to one correspondence between $\mathrm{Rsub}(\Delta^A_{n-1}, w)$ and the set of reduced pipe dreams $PD(w)$ for $w \in S_n$. For $\mathbf{D} \in \mathrm{Rsub}(\Delta^A_{n-1}, w)$, we put two patterns on Δ^A_{n-1}. One is which corresponds to the selected box $\boxed{\square}$ in EYD configuration of $\mathbf{D}$. The other case (corresponding to unselected box $\square$) we put in the box. Each selected box $\boxed{\square}$ in $\mathbf{D}$ corresponds to a word of the reduced expression of w corresponding to $\mathbf{D}$, which appears in a subsequence of Δ^A_{n-1}. The positions of the elements in $B(\mathbf{D})$ are indicated by circles $\boxed{\bigcirc}$ in the examples below (cf. [17]).

EXAMPLE 6. Type A_3.

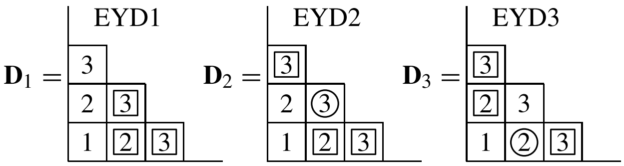


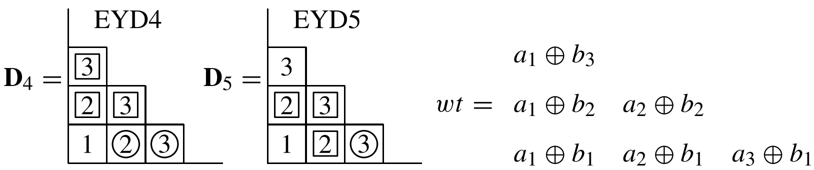
Example
of an EYD
configuration
for $w = [3, 1, 4, 2]$
$= s_2 s_3 s_1$.



$$\mathbf{D} = (d_2, d_3, d_4) = (s_2, s_3, s_1) \in \mathrm{Rsub}(\Delta_3^A, w),\ B(\mathbf{D}) = \{d_6\}$$
$$Wt(\mathbf{D}) = (a_1 \oplus b_2)(a_2 \oplus b_2)(a_1 \oplus b_1)(1 + \beta\ a_3 \oplus b_1)$$

EXAMPLE 7. Type A_3, $w = [1, 4, 3, 2] = s_3 s_2 s_3 = s_2 s_3 s_2 \in S_4$. One can show that $\mathcal{G}_w^{A_3}(a, b)|_{a=1,b=0} = 5 + 5\beta + \beta^2$.





$$\begin{aligned}
Wt(\mathbf{D}_1) &= (a_2 \oplus b_2)(a_2 \oplus b_1)(a_3 \oplus b_1)\\
Wt(\mathbf{D}_2) &= (a_1 \oplus b_3)(a_2 \oplus b_1)(a_3 \oplus b_1)(1 + \beta(a_2 \oplus b_2))\\
Wt(\mathbf{D}_3) &= (a_1 \oplus b_3)(a_1 \oplus b_2)(a_3 \oplus b_1)(1 + \beta(a_2 \oplus b_1))\\
Wt(\mathbf{D}_4) &= (a_1 \oplus b_3)(a_1 \oplus b_2)(a_2 \oplus b_2)(1 + \beta(a_2 \oplus b_1))(1 + \beta(a_3 \oplus b_1))\\
Wt(\mathbf{D}_5) &= (a_1 \oplus b_2)(a_2 \oplus b_2)(a_2 \oplus b_1)(1 + \beta(a_3 \oplus b_1))
\end{aligned}$$

From these data we get

$$\mathcal{G}^{A_3}_{s_2s_3s_2}(a,b) = Wt(\mathbf{D}_1) + Wt(\mathbf{D}_2) + Wt(\mathbf{D}_3) + Wt(\mathbf{D}_4) + Wt(\mathbf{D}_5)\,.$$

REMARK 11. Actually there is an algorithm to create all the extended EYD diagrams for a given $w \in S_n$. The algorithm is essentially written in [1]. Combinatorics related to extended EYD diagrams (including type B, C, D case) will be discussed elsewhere.

LEMMA 9. *For type B_n or C_n case, we can rewrite the generating function $\mathcal{G}^X_n(a,b;x) = \sum_{w\in W(X_n)} \mathcal{G}^X_{n,w}(a,b;x)u_w$ in Definition* 8 *as follows.*

$$\left(\prod_{j=n-1}^{1}\prod_{i=1}^{n-j} h_{i+j-1}(x_{n-i+1}\oplus b_j)\right)\left(\prod_{i=n}^{1}\prod_{j=n}^{i} h_{j-i}(x^X_{i,j})\right)\left(\prod_{i=n-1}^{1}\prod_{j=1}^{n-i} h_{i+j-1}(x_i\oplus a_j)\right) \tag{3}$$

where $x^X_{i,j} = x_i \oplus x_j$ if $i \neq j$, $x^B_{i,i} = x_i$ and $x^C_{i,i} = x_i \oplus x_i$.

PROOF. We will prove for the case of C_n. By Lemma 2 and Lemma 3, we have

$$\begin{aligned}
&F_n(x_1)F_n(x_2)\\
=\;& [n-1,1]_{x_1}[0]_{x_1\oplus x_1}[1,n-1]_{x_1}[n-1,1]_{x_2}[0]_{x_2\oplus x_2}[1,n-1]_{x_2}\\
=\;& [n-1,1]_{x_1}[0]_{x_1\oplus x_1}[n-1,1]_{x_2}[1,n-1]_{x_1}[0]_{x_2\oplus x_2}[1,n-1]_{x_2}\\
=\;& [n-1,1]_{x_1}[n-1,2]_{x_2}[0]_{x_1\oplus x_1}[1]_{x_2}[1]_{x_1}[0]_{x_2\oplus x_2}[2,n-1]_{x_1}[1,n-1]_{x_2}\\
=\;& [n-1,1]_{x_1}[n-1,2]_{x_2}[0]_{x_1\oplus x_1}[1]_{x_1\oplus x_2}[0]_{x_2\oplus x_2}[2,n-1]_{x_1}[1,n-1]_{x_2}\,.
\end{aligned}$$

Continuing this procedure we get

$$\begin{aligned}
&F_n(x_1)F_n(x_2)\cdots F(x_n)\\
=\;& [n-1,1]_{x_1}[n-1,2]_{x_2}\cdots[n-1]_{x_{n-1}}\\
&\qquad\qquad \nabla(x_1,\ldots,x_n)[n-1]_{x_2}[n-2,n-1]_{x_3}\cdots[1,n-1]_{x_n}\\
=\;& G_{A_{n-1}}(x_1,\ldots,x_{n-1})\nabla(x_1,\ldots,x_n)G_{A_{n-1}}(\bar{x}_n,\ldots,\bar{x}_2)^{-1}\,,
\end{aligned}$$

where

$$\nabla(x_1,\ldots,x_n) = \prod_{i=1}^{n}\left(\prod_{j=i}^{n} h_{i+j-1}(x_i\oplus x_j)\right)\,.$$

Then reversing the order of $x_1,\ldots,x_n$ to $x_n,\ldots,x_1$ and using Lemma 8, we get $\mathcal{G}^C_n(a,b;x) =$

$$G^A_{n-1}(x_n,\ldots,x_2,b_1,\ldots,b_{n-1})\nabla(x_n,\ldots,x_1)G^A_{n-1}(a_1,\ldots,a_{n-1},x_1,\ldots,x_{n-1})\,.$$

This is the formula (3). For type B_n case almost the same argument holds. □

For $X = B, C$, we define a sequence of simple reflections of $W_n(X)$ as follows.

$$\Delta^X_n = \Delta^A_{n-1}(s_0,s_1,s_2,\ldots,s_{n-1})^n = (d_1,d_2,\ldots,d_N)\,,$$

where $N = \frac{n(n-1)}{2} + n^2$. Note that in this case it doesn't correspond to a reduced decomposition of the longest element $w_0^{X_n}$. We arrange this sequence in trapezoidal form from left to right and from top to bottom, cf. Example 8 for the type C, $n = 3$ case. The coordinate of d_k is as follows. For $1 \leq k \leq \frac{n(n-1)}{2}$, the coordinate of d_k is $(k - \frac{m(m-1)}{2}, 2n - m)$ if $\frac{m(m-1)}{2} < k \leq \frac{m(m+1)}{2}$. For $\frac{n(n-1)}{2} < k \leq N$, the coordinate of d_k is $(s + t, n - s)$ if $k = \frac{n(n-1)}{2} + sn + t$, for some integers s, t with $0 \leq s$ and $1 \leq t \leq n$.

We also define the weight $wt_n^C(d_k) = p_i \oplus q_j$ if the coordinate of d_k is (i, j), where p_i and q_j are defined as follows.

$$p_i = x_{n+1-i} \text{ if } 1 \leq i \leq n \text{ and } p_i = a_{i-n} \text{ if } n < i\,, \tag{4}$$

$$q_j = x_j \text{ if } 1 \leq j \leq n \text{ and } q_j = b_{j-n} \text{ if } n < j\,. \tag{5}$$

For type $X = B$ case, we set $wt_n^B(d_k) = wt_n^C(d_k)$ except for the case of the coordinate of d_k is $(i, n + 1 - i)$ in which case we set $wt_n^B(d_k) = x_{n+1-i}$.

THEOREM 5. *For $w \in W_n(X)$, $X = B, C$, we have*

$$\mathcal{G}_{n,w}^X(a, b; x) = \sum_{\mathbf{D} \in \mathrm{RSub}(\Delta_n^X, w)} Wt_n^X(\mathbf{D})\,,$$

where

$$Wt_n^X(\mathbf{D}) = \prod_{\Box \in \mathbf{D}} wt_n^X(\Box) \times \prod_{\bigcirc \in B(\mathbf{D})} (1 + \beta wt_n^X(\bigcirc))\,.$$

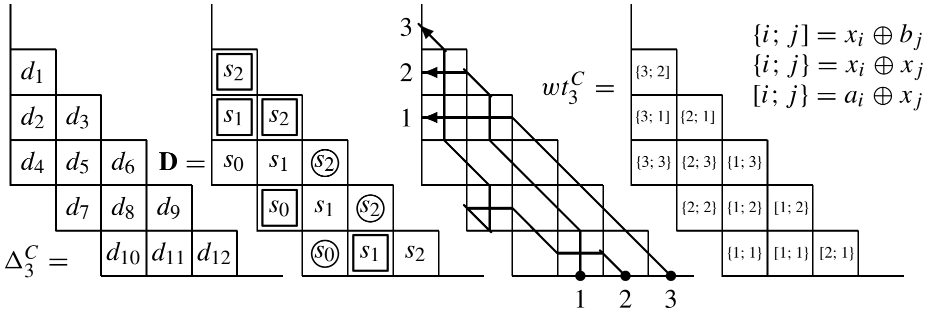
PROOF. This follows from the equation (3). Indeed using the relation $h_i(x)h_j(y) = h_j(y)h_i(x)$ for $i, j \geq 0$ s.t. $|i - j| > 1$, (3) can be rewritten as follows, from which we get the result.

$$G_n^C(a, b; x) = \left(\prod_{j=2n-1}^{n+1} \prod_{i=1}^{2n-j} h_{i+j-n-1}(p_i \oplus q_j) \right) \left(\prod_{j=n}^{1} \prod_{i=n+1-j}^{2n-j} h_{i+j-n-1}(p_i \oplus q_j) \right).$$

$$G_n^B(a, b; x) = \left(\prod_{j=2n-1}^{n+1} \prod_{i=1}^{2n-j} h_{i+j-n-1}(p_i \oplus q_j) \right) \left(\prod_{j=n}^{1} \prod_{i=n+1-j}^{2n-j} h_{i+j-n-1}(wt_n^B(i, j)) \right),$$

where $wt_n^B(i, j) = p_i \oplus q_j$ if $i + j > n + 1$ and $wt_n^B(i, n + 1 - i) = q_{n+1-i}$ for $1 \leq i \leq n$. $\Box$

EXAMPLE 8. Type C_3, $w = [2, \bar{3}, 1] = s_2 s_1 s_2 s_0 s_1$.



$$
\begin{aligned}
&\mathbf{D} = (d_1, d_2, d_3, d_7, d_{11}) = (s_2, s_1, s_2, s_0, s_1) \in \mathrm{Rsub}(\Delta_3^C, w)\,,\\
&B(\mathbf{D}) = \{d_6, d_9, d_{10}\}\,,\\
&Wt_3^C(\mathbf{D}) = (x_3 \oplus b_2)(x_3 \oplus b_1)(x_2 \oplus b_1)(x_2 \oplus x_2)(x_1 \oplus a_1)\\
&\qquad\qquad \times (1 + \beta(x_1 \oplus x_3))(1 + \beta(a_1 \oplus x_2))(1 + \beta(x_1 \oplus x_1))\,.
\end{aligned}
$$

Comparing this to the type A case, we get the following formula.

PROPOSITION 18. *For $w \in W(A_{n-1}) \subset W(B_n) = W(C_n)$, we have*

$$\mathcal{G}^B_{n,w}(a, b; x) = \mathcal{G}^C_{n,w}(a, b; x) = \mathcal{G}^{A_{2n-1}}_{1^n \times w}(x_1, \ldots, x_n, a_1, \ldots, a_{n-1}\ ,\ x_1, \ldots, x_n, b_1, \ldots, b_{n-1})\,.$$

PROOF. According to the setting of weight $wt^X(d_k)$, it is clear that $\mathcal{G}^B_{n,w}(a, b; x) = \mathcal{G}^C_{n,w}(a, b; x)$ for $w \in W(A_{n-1})$. Comparing the weights of type A_{2n-1} and C_n cases with the formula (4) and (5) , we get $\mathcal{G}^C_{n,w}(a, b; x) = \mathcal{G}^{A_{2n-1}}_{1^n \times w}(x_n, \ldots, x_1, a_1, \ldots, a_{n-1}\ ,\ x_1, \ldots, x_n, b_1, \ldots, b_{n-1})$. But in this case the first n variables of $\mathcal{G}^{A_{2n-1}}_{1^n \times w}(a_1, \ldots, a_{2n-1}, b_1, \ldots b_{2n-1})$ are commutative, because $(1^n \times w)s_i > (1^n \times w)$ for $1 \le i \le n - 1$. □

For type D_n case, we assume $n = 2m$ an even integer. For odd $n = 2m - 1$ case we can get the formula by just erasing the last variable $x_{2m} = 0$ for $n = 2m$ case.

LEMMA 10. *The generating function $G^D_n(a, b; x) = G_{A_{n-1}}(\bar{b}_1, \ldots, \bar{b}_{n-1})^{-1} F^D_n(x) G_{A_{n-1}}(a_1, \ldots, a_{n-1})$ can be rewritten as follows by using Yang-Baxter relations.*

$$\left(\prod_{j=n-1}^{1}\prod_{i=1}^{n-j} h_{i+j-1}(x_{n-i+1} \oplus b_j)\right)\left(\prod_{i=n-1}^{1}\prod_{j=n}^{i+1} h_{i,j}(x_{i,j})\right)\left(\prod_{i=n-1}^{1}\prod_{j=1}^{n-i} h_{i+j-1}(x_i \oplus a_j)\right) \tag{6}$$

where $h_{i,j}(x_{i,j}) := h_{j-i}(x_i \oplus x_j)$ if $j - i \ge 2$, $h_{i,i+1}(x_{i,i+1}) := h_{\hat{1}}(x_i \oplus x_{i+1})$ if $i = odd$ and $h_{i,i+1}(x_{i,i+1}) := h_1(x_i \oplus x_{i+1})$ if $i = even$.

PROOF. The argument is almost the same as Lemma 9 and we omit the details □

Let us define (for $n = 2m$ case) the sequence Δ_n^D of simple reflections by

$$\Delta_n^D := \Delta_{n-1}^A((s_{\hat{1}}, s_2, \ldots, s_{n-1})(s_1, s_2, \ldots s_{n-1}))^m = (d_1, d_2, \ldots, d_N)\,.$$

where $N = \frac{n(n-1)}{2} + (n-1)^n$. We arrange these in trapezoidal form with coordinate beginning from $(1, 2n-1)$ to $(2n-1, 1)$ as type C_n case, but skip the coordinate $(i, n+1-i)$ for $1 \le i \le n$. The weight is $wt_n^D(d_k) = p_i \oplus q_j$ when d_k is in the coordinate(i, j). Formally it is the same as type C_n case but we skip the position $(i, n+1-i)$, $1 \le i \le n$ for the type D_n case. See Example 9 below for $n = 4$ case.

THEOREM 6. *For $w \in W_n(D)$ $(n = 2m)$, we have*

$$\mathcal{G}_{n,w}^D(a, b; x) = \sum_{\mathbf{D} \in \mathrm{RSub}(\Delta_n^D, w)} Wt_n^D(\mathbf{D})\,,$$

where

$$Wt_n^D(\mathbf{D}) = \prod_{\square \in \mathbf{D}} wt_n^D(\square) \times \prod_{\bigcirc \in B(\mathbf{D})} (1 + \beta wt_n^D(\bigcirc))\,.$$

For $n = 2m - 1$ case we can use the above formula with $x_{2m} = 0$.

PROOF. This follows by expanding the product (6), which can be rewritten using Yang-Baxter relations as in the proof of Theorem 5 as follows.

$$G_n^D(a, b; x) = \left(\prod_{j=2n-1}^{n+1} \prod_{i=1}^{2n-j} h_{i+j-1}(p_i \oplus q_j) \right) \left(\prod_{j=n}^{1} \prod_{i=n+2-j}^{2n-j} h_{i,j}(p_i \oplus q_j) \right)$$

where p_i, q_j are defined by (4), (5), and

$$h_{i,j} = \begin{cases} h_{i+j-n-1} & \text{if } i + j > n + 2\,, \\ h_1 & \text{if } i + j = n + 2 \text{ and } j \text{ is odd}\,, \\ h_{\hat{1}} & \text{if } i + j = n + 2 \text{ and } j \text{ is even}\,. \end{cases}$$

$\square$

EXAMPLE 9. Type D_4 , $w = [\bar{2}, 4, \bar{1}, 3] = s_3 s_{\hat{1}} s_2$.

$$\Delta_4^D = \begin{array}{|c|c|c|c|c|c|c|} d_1 & & & & & & \\ d_2 & d_3 & & & & & \\ d_4 & d_5 & d_6 & & & & \\ & d_7 & d_8 & d_9 & & & \\ & & d_{10} & d_{11} & d_{12} & & \\ & & & d_{13} & d_{14} & d_{15} & \\ & & & & d_{16} & d_{17} & d_{18} \end{array} \qquad \mathbf{D} = \begin{array}{|c|c|c|c|c|c|c|} s_3 & & & & & & \\ s_2 & \boxed{s_3} & & & & & \\ s_1 & s_2 & s_3 & & & & \\ & \boxed{s_{\hat{1}}} & s_2 & s_3 & & & \\ & & s_1 & s_2 & s_3 & & \\ & & & s_{\hat{1}} & \boxed{s_2} & s_3 & \\ & & & & s_1 & s_2 & s_3 \end{array}$$

$$wt_4^D = \begin{array}{|c|c|c|c|c|c|c|} \{4;3] & & & & & & \\ \{4;2] & \{3;2] & & & & & \\ \{4;1] & \{3;1] & \{2;1] & & & & \\ & \{3;4\} & \{2;4\} & \{1;4\} & & & \\ & & \{2;3\} & \{1;3\} & [1;3\} & & \\ & & & \{1;2\} & [1;2\} & [2;2\} & \\ & & & & [1;1\} & [2;1\} & [3;1\} \end{array} \qquad \begin{aligned} \{i;\, j] &= x_i \oplus b_j \\ \{i;\, j\} &= x_i \oplus x_j \\ [i;\, j\} &= a_i \oplus x_j \end{aligned}$$

$\mathbf{D} = (d_3, d_7, d_{14}) = (s_3, s_{\hat{1}}, s_2) \in \mathrm{RSub}(\Delta_4^D, w)$, $B(\mathbf{D}) = \{d_6, d_9, d_{12}, d_{13}, d_{17}\}$.
$Wt_4^D(\mathbf{D}) = (x_3 \oplus b_2)(x_3 \oplus x_4)(a_1 \oplus x_2) \times$
$(1+\beta(x_2 \oplus b_1))(1+\beta(x_1 \oplus x_4))(1+\beta(a_1 \oplus x_3))(1+\beta(x_1 \oplus x_2))(1+\beta(a_2 \oplus x_1))$.

PROPOSITION 19. *For $w \in \langle s_2, s_3, \dots, s_{n-1}\rangle \subset W(D_n)$, we have*

$$\mathcal{G}^D_{n,w}(a,b;x) = \mathcal{G}^{A_{2n-1}}_{1^n \times w}(x_1, \dots, x_n, a_1, \dots, a_{n-1}\ ,\ x_1, \dots, x_n, b_1, \dots, b_{n-1}) .$$

REMARK 12. If w is a maximal Grassmannian element of type B_n, C_n or D_n, then the above pipe dream formula can be regarded as the excited Young diagram formula of [17] Th 9.2. Therefore the above gives a generalization of the EYD formula. Therefore we can call Theorem 4,5,6 as extended EYD formula. Note also that even in type A (Theorem 4) case the formula given in this form is a compressed form compared to compatible sequence formula (Proposition 17).

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